Strong Sharp Quantitative Stability for Crystalline Surface Tensions

By Kenneth DeMason, Nov 28 2023

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Introduction:

i) Isoperimetry

Consider the following variational problem:

$$I(v) = \inf \{ P(E) \mid E \subseteq \mathbb{R}^n, |E| = v \}$$

where $P(E) = \mathcal{H}^{n-1}(\partial E)$, the perimeter

We take the infimum over a nice class of objects.

↳ E with (piecewise) smooth boundary

↳ E a set of finite perimeter.

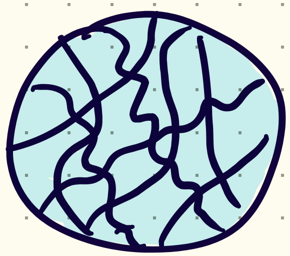
Sets of finite perimeter:

Def: We say $E \subseteq \mathbb{R}^n$ is a set of finite perimeter if

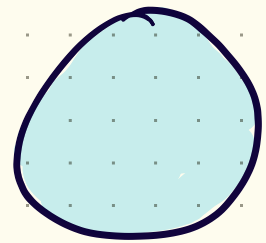
$$\sup \left\{ \int_E \operatorname{div}(T(x))\, dx \;\middle|\; \begin{array}{l} T \in C^1_c(\mathbb{R}^n; \mathbb{R}^n), \\ |T| \leq 1 \end{array} \right\}$$

is finite. In this case, $P(E)$ is the above quantity and is called the distributional perimeter

Ex. i)



vs

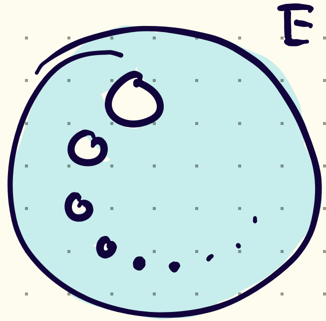


Distributional per: $P(E) = P(B)$

but $\mathcal{H}^1(\partial E) > \mathcal{H}^1(\partial B)$.

Extreme case: $F = B \cup \mathbb{Q}^2$ but $\mathcal{H}^1(\partial F) = \infty$

ii)



Is a set of finite perimeter, but is topologically complex

Observe if $E \subseteq \mathbb{R}^n$ has ∂E smooth then

$$\int_E \operatorname{div}(T(x))\,dx = \int_{\partial E} \langle T(x), \nu_E(x) \rangle \, d\mathcal{H}^{n-1}(x)$$

for every $T \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$. In particular, when $|T| \leq 1$,

$$\int_E \operatorname{div}(T(x))\,dx \leq \mathcal{H}^{n-1}(\partial E) = P(E)$$

On the other hand, $\nu_E \notin C_c^1(\mathbb{R}^n; \mathbb{R}^n)$. However, can approximate by $T_\varepsilon \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$ w/ $|T| \leq 1$. So the sup is $P(E)$.

In this spirit, to each set of finite perimeter E there exists a vector-valued Radon measure μ_E, called the Gauss-Green measure, such that

$$\int_E \operatorname{div}(T(x))\,dx = \int_{\mathbb{R}^n} \langle T(x), d\mu_E(x) \rangle$$

for all $T \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$ w/ $|T| \leq 1$

I.e., $\mu_E = -D\chi_E$ and $P(E) = |\mu_E|(\mathbb{R}^n)$.

So sets of finite perimeter are a relaxation of smooth sets still satisfying a divergence theorem. Can we recover the usual?

Requires a better understanding of ν_E.

Notion of a outer unit normal:

Def: The reduced boundary of E a set of finite perimeter, denoted $\partial^* E$, is the set of points in $\mathbb{R}^n$ such that

$$\lim_{r \to 0} \frac{\nu_E(B_r(x))}{|\nu_E|(B_r(x))} \in S^{n-1}$$

Can define a Borel map $\nu_E : \partial^* E \to \mathbb{R}^n$ as the above limit, called the measure-theoretic outer unit normal

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De Giorgi's Structure Theorem

<u>Prop</u>: For a set of finite perimeter E, its Gauss-Green measure satisfies

$$\mu_E = \nu_E \, \mathcal{H}^{n-1} \llcorner \partial^* E \quad \text{and} \quad |\mu_E| = \mathcal{H}^{n-1} \llcorner \partial^* E$$

So that $P(E) = \mathcal{H}^{n-1}(\partial^* E)$. Moreover

$$\int_E \operatorname{div}(T(x))\, dx = \int_{\partial^* E} \langle T(x), \nu_E(x) \rangle \, d\mathcal{H}^{n-1}(x)$$

for all $T \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$ w/ $|T| \leq 1$.

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Compactness of sets of finite perimeter

<u>Prop</u>: Let $\{E_j\}_{j=1}^{\infty}$ be a sequence of sets of finite perimeter. If

i) There exists $R>0$ such that
$$E_j \subset B_R(0) \text{ for all } j \in \mathbb{N}$$

ii) $$\sup_{j\in\mathbb{N}} P(E_j) < \infty$$

Then there exists a convergent subsequence $\{E_j\}_{j=1}^{\infty}$ such that $E_j \xrightarrow{L^1} E_\infty$ a set of finite perimeter.

Question: Does there exist E a set of finite per. st $P(E) = I(v)$?

Answer: Yes, via the Direct Method.

* Take a minimizing sequence $\{E_j\}$
* Use compactness to extract a limit E_∞
* Perimeter is (lower semi-) continuous, So E_∞ minimizes P.

$$I(v) \overset{\text{def}}{\leq} P(E_\infty) \overset{\text{LSC}}{\leq} \liminf_{j\to\infty} P(E_j) = I(v)$$

So, minimizers of v.p. exist, what can we say about them?

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Turns out that balls are minimizers! This fact is encoded in the isoperimetric inequality:

For any sofp E with $|E| < \infty$,

$$P(E) \geq n|B_1|^{1/n} |E|^{n-1/n} \qquad \text{(II)}$$

with equality if and only if $E = B_R$ (up to translation and measure-zero modifications)

↳ Have rigidity, so we can ask about stability.

Rigidity: Characterization of equality case

Stability: If close to equality, are you close to the equality case?

ii) Quantitative Stability.

Easy to define closeness to equality in $\boxed{\mathrm{II}}$

Introduce the isoperimetric deficit

$$\delta(E) = \frac{P(E)}{n|B_1|^{1/n}|E|^{n-1/n}} - 1 \quad (\geq 0)$$

$\rightsquigarrow$ $\delta(E) = 0$ iff $|E \Delta B_R(x_0)| = 0$ for some $R>0$, $x_0 \in \mathbb{R}^n$.

So stability asks if $\delta(E) \approx 0$, then is E "close" to B_1?

↳ Use an asymmetry index α to make precise

- Qualitative: Asymptotics

  E.g. If $\delta(E_j) \to 0$ then $\alpha(E_j) \to 0$

- Quantitative: Gives an actual bound, sharp if p cannot be improved

  E.g. $\alpha(E)^p \leq C(n)\,\delta(E)$

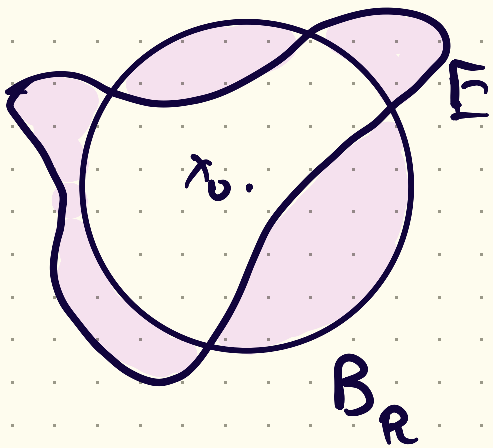
Asymmetry:

- Fraenkel asymmetry

$$\alpha(E) = \inf_{x_0 \in \mathbb{R}^n} \left\{ \frac{|E \,\Delta\, B_R(x_0)|}{|E|} \;\middle|\; |B_R| = |E| \right\}$$

Translate to maximize overlap

Normalization for scale invariance



We defined the above in terms of
$|E \Delta B_R|$, the Vitali distance (L^1)

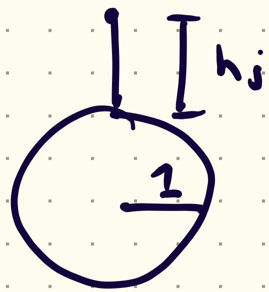
↳ Could have used $d_H(E, B_R)$, the Hausdorff distance, instead. (L^∞)

Rem: It is natural to use the L^1 versions.
Indeed, recall

$$\delta(E) = 0 \text{ iff } |E \Delta B_R(x_0)| = 0$$
for some $R > 0$, $x_0 \in \mathbb{R}^n$.

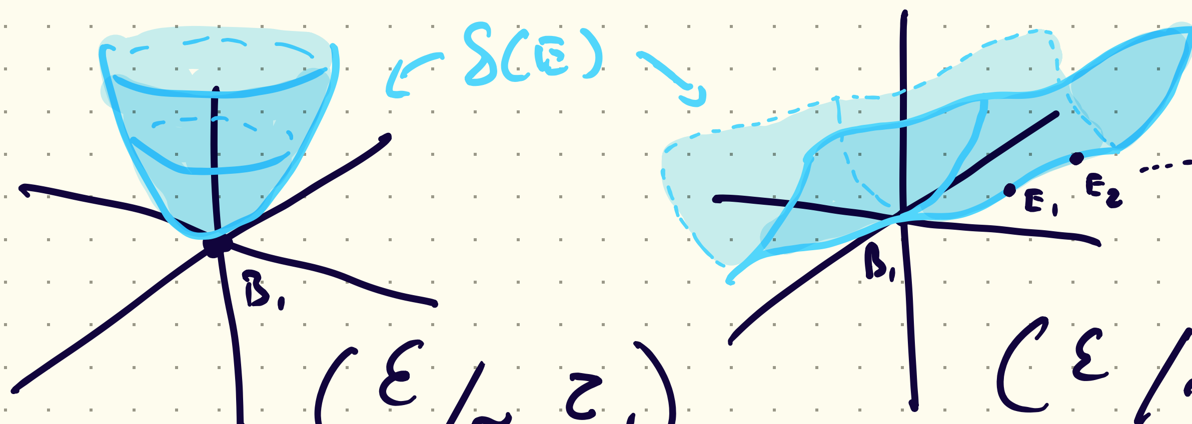
Moreover, even if $|E \Delta B_R(x_0)| = 0$,
$d_H(E, B_R(x_0))$ could be arbitrarily large.

Ex: Consider the sets E_j:

E_j 

- $P(E_j) = P(B_1)$, $\delta(E_j) = 0$
- $|E_j \Delta B_1| = 0$ but $d_{\mathcal{H}}(E_j, B_1) = h_j$

So the topology on $\mathcal{E} = \{E \subseteq \mathbb{R}^n \mid P(E) < \infty,\ 0 < |E| < \infty\}$ matters!



$(\mathcal{E}/\sim, \tau_{L^1})$ $\qquad$ $(\mathcal{E}/\sim, \tau_{d_{\mathcal{H}}})$

where $E \sim E'$ iff $E = rL(E') + x$ for some $x \in \mathbb{R}^n$, $r > 0$, $L \in \mathrm{Isom}(B_1)$.

Previous Results:

Quantitative stability of (II):

There exists $C(n)>0$ such that for every set of finite perimeter E with $|E|<\infty$,

$$\alpha(E)^p \leq C \cdot \delta(E) \qquad \text{(QS)}$$

- (Fuglede, '89) Showed (QS) with barycentric asymmetry & assumption that E is convex or star-shaped w/ small norm
- (Hall-Hayman-Weitman, '91) Showed (QS) w/ Fraenkel asymmetry and E w/ smooth boundary
- (Hall, '92) Improved previous, sharp if E is axially symmetric $(p=2)$
- (Fusco-Maggi-Pratelli, '08) Sharp (QS) w/ Fraenkel asymmetry in generic case.

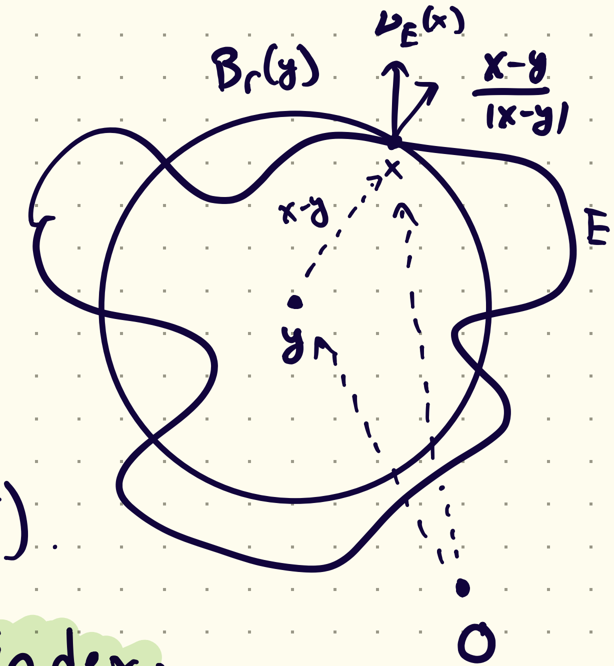
The latter three use symmetrization techniques to reduce the problem.

A new technique was used by Cicalese-Leonardi in 2012 called the selection principle.

↳ General idea is to use regularity theory of almost-minimizers of VP, to reduce to a nice case

↳ Reprove FMP '08 result by reducing to star-shaped case

In 2014, Fusco - Julin use the selection principle to prove the following strong, sharp Q.S.:



$$\alpha(E)^2 + \beta(E)^2 \leq C(n)\,\delta(E).$$

Where $\beta(E)$ is the oscillation index:

$$\beta(E)^2 := \inf_{y \in \mathbb{R}^n} \left\{ \frac{1}{n|B_1|^{1/n}|E|^{\frac{n-1}{n}}} \int_{\partial^* E} \left[1 - \frac{\langle x-y, \nu_E(x) \rangle}{|x-y|} \right] d\mathcal{H}^{n-1}(x) \right\}$$

How is β a measure of oscillation?

Cauchy-Schwarz: $\left\langle \frac{x-y}{|x-y|}, \nu_E(x) \right\rangle \leq 1$

with equality iff $x-y \parallel \nu_E(x)$

Moreover β measures closeness in H^1-sense:

F-J: There exists $C(n)$ so st for any set of finite perimeter $E \subseteq \mathbb{R}^n$ with $0 < |E| < \infty$ then

$$\alpha(E) + \delta(E)^{1/2} \le c(n) \cdot \beta(E)$$

Poincaré like

If $\partial E = \{x + u(x) \cdot x \mid x \in \partial B_1\}$ with $u \in C^1(\partial B_1)$, $\|u\|_{W^{1,\infty}}$ small

then

$$\|u\|^2_{H^1(\partial B_1)} \lesssim \delta(E) \lesssim \beta(E)^2$$

Also in this case,

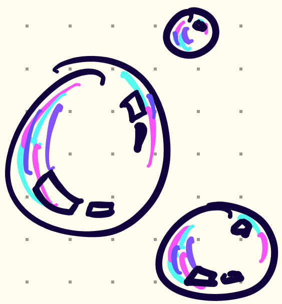
$$n|B_1| \cdot \beta(E)^2 \le \|u\|^2_{H^1(\partial B_1)}$$

by testing at $y=0$.

iii) Anisotropy:

Volume constrained perimeter minimization explains the formation of bubbles

Tries to minimize surface tension



Many physical phenomena can be described in terms of energy minimization.

What about e.g. salt crystals?

Thermodynamics:

Crystals at equilibrium should minimize Gibbs free energy

Area of i^{th} face

Wulff

$$\Delta G := \sum_i \gamma_i A_i = \lambda \sum h_i A_i$$

for crystals

surface energy per unit area

Lagrange mult.

dist. to face

Def: A **Wulff shape** is an open, bounded, convex subset $K \subseteq \mathbb{R}^n$ containing the origin.

There are two important functions associated to K, the **support function** $f : \mathbb{R}^n \to [0, \infty)$ and the **gauge function** $f_* : \mathbb{R}^n \to [0, \infty)$. Both are convex and 1-homogeneous ($g(\lambda x) = \lambda g(x)$, $\lambda > 0$).

$$\begin{tikzcd}
 & K \arrow[dl, bend right, "\text{Minkowski functional}"'] \arrow[dr, bend left, "*"] & \\
f_* \arrow[ur, bend right, "\text{unit Ball}"'] \arrow[rr, leftrightarrow, "\text{Legendre Transform}"'] & & f \arrow[ul, bend right, "\text{Convex hull of } \{\langle x, \nu \rangle < f(\nu)\}"']
\end{tikzcd}$$

[usually view $f : S^{n-1} \to (0, \infty)$]

f and f_* are semi-norms on $\mathbb{R}^n$, norms if K is symmetric about the origin.

When f_* is a norm, f is the corresponding dual norm.

Ex. $f_* = \|\cdot\|_{\ell^p}$ then $f = \|\cdot\|_{\ell^q}$

In fact:

$$f(\nu) = \sup\{\langle x, \nu\rangle \mid f_*(x) \le 1\}$$
$$f_*(x) = \sup\{\langle x, \nu\rangle \mid f(\nu) \le 1\}$$

Hence, for $x \in \mathbb{R}^n$ and $\nu \in S^{n-1}$

$$\langle x, \nu\rangle \le f_*(x) f(\nu)$$

with equality iff Σ_ν supports K at x.

Question: For what energy Φ are the volume-constrained minimizers some prescribed Wulff shape K? (up to translation and appropriate scaling)

i.e. Find Φ such that

$$E \in \operatorname{argmin}\{\Phi(F) \mid |F| = v\} \text{ iff } E = rK + x_0, \quad |rK| = v.$$

Def: The Anisotropic perimeter (associated to K) is

$$\Phi(E) = \int_{\partial^* E} f(\nu_E(x))\, d\mathcal{H}^{n-1}$$

The isotropic case is recovered with $f = \|\cdot\|_{\ell^2}$

For polyhedra $f(\nu_i) = h_i$!

The anisotropic isoperimetric inequality holds:

$$\Phi(E) \geq n|K|^{1/n}|E|^{n-1/n} \qquad (AII)$$

with equality iff $|E \Delta (rK+x_0)| = 0$ for some $x_0 \in \mathbb{R}^n$, $r>0$

This is the same as (II) with $P \to \Phi$, $B_1 \to K$.

What about quantitative stability?

$\hookrightarrow$ Introduce anisotropic variants

Deficit: $$\delta_{\Phi}(E) := \frac{\Phi(E)}{n|K|^{1/n}|E|^{n-1/n}} - 1$$

Asymmetry: $$\alpha_{\Phi}(E) := \inf_{x_0 \in \mathbb{R}^n} \left\{ \frac{|E \Delta (rK + x_0)|}{|E|} \;\middle|\; |rK| = |E| \right\}$$

Osc: $$\beta_{\Phi}(E) := \inf_{y \in \mathbb{R}^n} \left\{ \frac{1}{n|K|^{1/n}|E|^{n-1/n}} \int_{\partial^* E} f(\nu_E(x)) - \frac{\langle x-y, \nu_E(x) \rangle}{f_*(x-y)} \, d\mathcal{H}^{n-1}(x) \right\}$$

Note that the integrand of β_{Φ} comes from the Fenchel inequality:

$$\langle x-y, \nu \rangle \leq f_*(x-y) f(\nu).$$

Previous results:

- (Figalli - Maggi - Pratelli, '12) sharp Q.S. for A.I.I. w/ Fraenkel asymmetry
- (Neumayer '16) strong sharp Q.S. for A.I.I. when K is uniformly convex or K is a polygon in $n=2$ (crystalline)

$$\alpha_{\Phi}(E)^2 + \beta_{\Phi}(E)^2 \leq C\, \delta_{\Phi}(E)$$

Unif conv: $C(n, \ldots, \|\nabla^2 f\|_{C^0(\partial K)})$

$\hookrightarrow$ Can get $C(n)$ if exp is $4n/n+1$

$n=2$ Crystalline: $C(K)$

Main Result: When K is a polytope.

THM [D. '23] There exists $C(n,K)>0$ such that for any $E \subseteq \mathbb{R}^n$ a set of finite perimeter with $|E|<\infty$,

$$\alpha_\Phi(E)^2 + \beta_\Phi(E)^2 \leq C(n,K)\,\delta_\Phi(E).$$

The exponents are sharp.

This generalizes Neumayer's $n=2$ crystalline result.

Sketch of main THM:

1. Prove the result for parallel polytopes

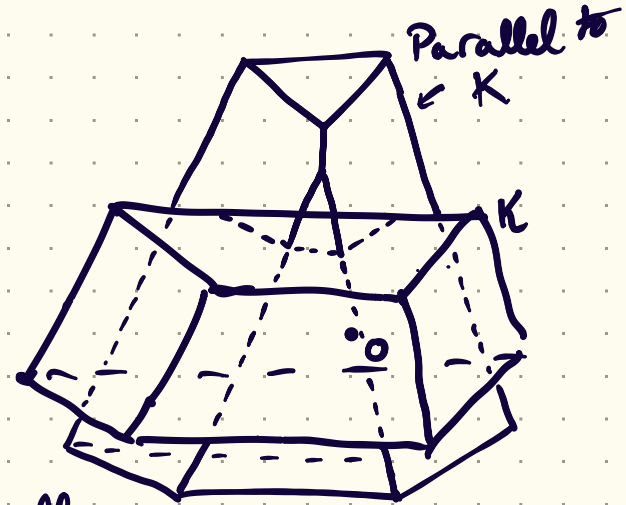
2. Prove for E satisfying uniform density estimates

↳ Upgrade L^1 control to Hausdorff

a) Project E onto a special polytope parallel to K

b) Show this projection decreases deficit & increases oscillation

c) Apply 1.



$$\beta_{\Phi}(E)^2 \underset{2b)}{\leq} \beta_{\Phi}(K^*)^2 \underset{1)}{\lesssim} \delta_{\Phi}(K^*) \underset{2b)}{\lesssim} \delta_{\Phi}(E)$$

Sketch of main THM:

3. Selection principle argument

a) Supposing not generate minimizing sequence $\{E_j\}$

b) Replace w/new sequence $\{F_j\}$

↳ satisfy UDEs by regularity of almost minimizers

- Need to show $\{F_j\}$ satisfy same properties as $\{E_j\}$
- Need control on oscillation & anisotropic perimeter to replace

c) Apply 2 $\lightning$.

Thanks for coming!

Questions?