Strong Sharp Quantitative Stability for Crystalline Surface Tensions

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Introduction:

i) Isoperimetry

Consider the following variational problem:

$$I(v) = \inf \{ P(E) \mid E \subseteq \mathbb{R}^n, |E| = v \}$$

where $P(E) = \mathcal{H}^{n-1}(\partial E)$, the perimeter

We take the infimum over a nice class of objects

↳ E with (piecewise) smooth boundary

↳ E a set of finite perimeter.

Question: Does there exist E w/ these properties st $P(E) = I(v)$?

Answer: Yes, via the Direct Method.

* ⋆ Take a minimizing sequence $\{E_j\}$
* ⋆ Use compactness* to extract a limit E_∞
* ⋆ Perimeter is (lower semi-) continuous, so E_∞ minimizes P.

Aside: Same method works on any compact smooth manifold; Can replace compactness assumption by an "asymptotically flat" condition.

So, minimizers of v.p. exist, what can we say about them?

Turns out that balls are minimizers! This fact is encoded in the isoperimetric inequality:

For any sofp E with $|E| < \infty$,

$$P(E) \geq n|B_1|^{1/n} |E|^{n-1/n} \qquad \text{(II)}$$

With equality if and only if $E = B_R$ (up to translation and measure-zero modifications)

$\hookrightarrow$ Have rigidity, so we can ask about stability.

Rigidity: Characterization of equality case

Stability: If close to equality, are you close to the equality case?

ii) Quantitative Stability.

Easy to define closeness to equality in $\boxed{\text{II}}$

Introduce the isoperimetric deficit

$$\delta(E) = \frac{P(E)}{n|B_1|^{1/n}|E|^{n-1/n}} - 1 \quad (\geq 0)$$

$\rightsquigarrow$ $\delta(E) = 0$ iff $|E \Delta B_R(x_0)| = 0$ for some $R>0$, $x_0 \in \mathbb{R}^n$.

So stability asks if $\delta(E) \approx 0$, then is E "close" to B_1?

↳ Use an asymmetry index α to make precise

- Qualitative: Asymptotics
  E.g. If $\delta(E_j) \to 0$ then $\alpha(E_j) \to 0$
- Quantitative: Gives an actual bound
  E.g. $\alpha(E)^p \leq C(n)\,\delta(E)$

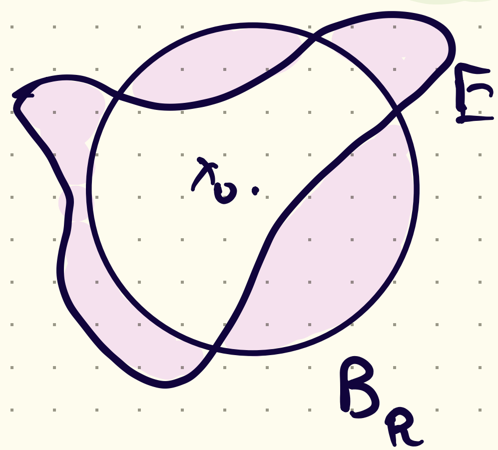
Asymmetry:

- Fraenkel asymmetry

$$\alpha(E) = \inf_{x_0 \in \mathbb{R}^n} \left\{ \frac{|E \,\Delta\, B_R(x_0)|}{|E|} \;\middle|\; |B_R| = |E| \right\}$$

Translate to maximize overlap

Normalization for scale invariance



- Barycentric

$$\alpha_b(E) = \frac{|E \,\Delta\, B_R(x_E)|}{|E|}$$

where $|B_R| = |E|$ and x_E is the barycenter of E: $x_E = \frac{1}{|E|} \int_E x \, dx$.

We defined the above in terms of $|E \Delta B_R|$, the Vitali distance

Could have used $d_{\mathcal{H}}(E, B_R)$, the Hausdorff distance, instead.

↳ More of an L^∞ distance.

Rem: It is more natural to study the L^1 versions. Indeed recall

$$\delta(E) = 0 \text{ iff } |E \Delta B_R(x_0)| = 0$$

for some $R>0$, $x_0 \in \mathbb{R}^n$.

Moreover, even if $|E \Delta B_R(x_0)| = 0$, $d_{\mathcal{H}}(E, B_R(x_0))$ could be arbitrarily large.

Previous results:

- (Fuglede, '89) Q.S. w/ Hausdorff barycentric asymmetry when E is convex or star-shaped w/ small norm
- (Hall-Hayman-Weitman, '91) Q.S. w/ Fraenkel asymmetry, not sharp.
- (Hall, '92) Q.S. w/ Fraenkel asymmetry, sharp under assumption that E is axially symmetric
- (Fusco-Maggi-Pratelli, '08) Sharp Q.S. w/ Fraenkel asymmetry:

  There exists $C(n) > 0$ st for any E sofp w/ $|E| < \infty$,

$$\alpha(E)^2 \leq C(n)\,\delta(E)$$

The latter three use symmetrization techniques to reduce the problem.

A new technique was used by Cicalese-Leonardi in 2012 called the selection principle.

↳ General idea is to use regularity theory of almost-minimizers of VP, to reduce to a nice case

↳ Reprove FMP '08 result by reducing to star-shaped case

In 2014, Fusco - Julin use the selection principle to prove the following strong, sharp Q.S. result:

$$\alpha(E)^2 + \beta(E)^2 \leq C(n)\,\delta(E).$$

Where $\beta(E)$ is the oscillation index:

$$\beta(E)^2 := \inf_{y \in \mathbb{R}^n} \left\{ \frac{1}{n|B_1|^{1/n}|E|^{\frac{n-1}{n}}} \int_{\partial^* E} \left[1 - \frac{\langle x-y, \nu_E(x) \rangle}{|x-y|} \right] d\mathcal{H}^{n-1}(x) \right\}$$

How is β a measure of oscillation?

Cauchy-Schwarz: $\left\langle \frac{x-y}{|x-y|}, \nu_E(x) \right\rangle \leq 1$

with equality iff $x-y \parallel \nu_E(x)$

So if $\beta(E)=0$, then $E = B_R(y)$ w/ $y \in \mathbb{R}^n$ attaining the infimum.

↳ β measures closeness of E to a ball via its unit normals, a first order quantity.

↳ More formally, when E is star-shaped w/ barycenter at the origin and $|E| = |B_1|$, can parametrize ∂E by $x + u(x) \cdot x$, where $u \in C^1(\partial B)$. Then,

$$n\,|B_1|\,\beta(E)^2 \leq \|u\|^2_{H^1(\partial B)}$$

iii) Anisotropy

Not same "direction"

What happens if we weight the perimeter according to ν_E?

Given $f : \mathbb{R}^n \setminus \{0\} \to (0, \infty)$ 1-homogeneous and convex, can define the anisotropic perimeter

$$\Phi(E) = \int_{\partial^* E} f(\nu_E(x))\, d\mathcal{H}^{n-1}(x)$$

The weight f is known as the surface tension and defines the Wulff shape

$$K = \bigcap_{\nu \in S^{n-1}} \{ \langle x, \nu \rangle < f(\nu) \}$$

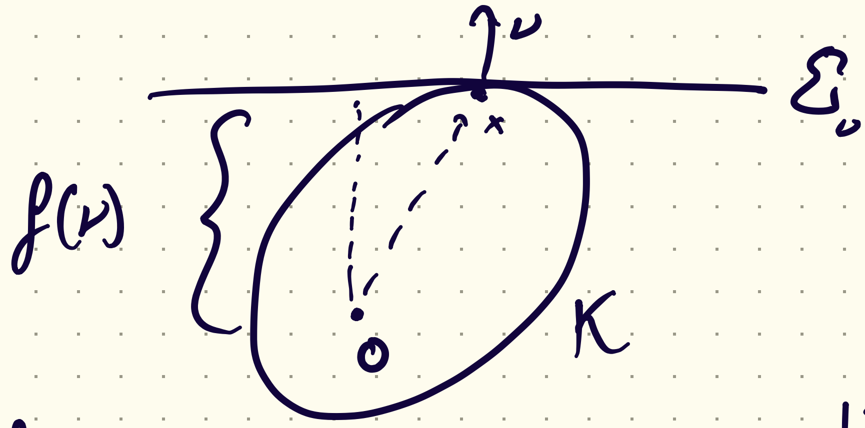
(open, bounded, convex, contains 0)

Conversely, given a Wulff shape we can extract the corresponding surface tension by

$$f(\nu) = \sup_{x \in K} \langle x, \nu \rangle$$

for $\nu \in S^{n-1}$ and extend 1-homogeneously.

$f(\nu)$ gives the distance from the origin to the supporting hyperplane Σ_ν of K w/unit normal ν.



Along with f is another naturally associated function to K, the gauge

$$f_*(x) = \sup_{\nu \in S^{n-1}} \{ \langle x, \nu \rangle \mid f(\nu) \leq 1 \}$$

which is "dual" to f in the sense

$$\langle x, \nu \rangle \leq f_*(x) f(\nu)$$

with equality iff Σ_ν supports K at x.

More concretely,

$$f_*(x) = \inf \{ \lambda > 0 \mid \tfrac{x}{\lambda} \in K \},$$

So K is characterized as the unit ball in f_*

The isotropic case is recovered with $f_* = \|\cdot\|_{\ell^2}$

↳ Is Φ the "correct" anisotropic analogue of P?

The anisotropic isoperimetric inequality holds:

$$\Phi(E) \geq n|K|^{1/n}|E|^{n-1/n} \qquad (AII)$$

with equality iff $|E \Delta (rK + x_0)| = 0$ for some $x_0 \in \mathbb{R}^n$, $r > 0$

This is the same as the II with $P \to \Phi$ and $B_1 \to K$

What about quantitative stability?

↳ Introduce anisotropic variants

Deficit: $$\delta_{\Phi}(E) := \frac{\Phi(E)}{n|K|^{1/n}|E|^{n-1/n}} - 1$$

Asymmetry: $$\alpha_{\Phi}(E) := \inf_{x_0 \in \mathbb{R}^n} \left\{ \frac{|E \,\Delta\, (rK + x_0)|}{|E|} \;\middle|\; |rK| = |E| \right\}$$

Osc: $$\beta_{\Phi}(E) := \inf_{y \in \mathbb{R}^n} \left\{ \frac{1}{n|K|^{1/n}|E|^{n-1/n}} \int_{\partial^* E} f(\nu_E(x)) - \frac{\langle x-y, \nu_E(x) \rangle}{f_*(x-y)} \, d\mathcal{H}^{n-1}(x) \right\}$$

Note that the integrand of β_{Φ} comes from the Fenchel inequality:

$$\langle x-y, \nu \rangle \leq f_*(x-y) f(\nu).$$

Previous results:

- (Figalli - Maggi - Pratelli, '12) sharp Q.S. for A.I.I. w/ Fraenkel asymmetry
- (Neumayer '16) strong sharp Q.S. for A.I.I. when K is uniformly convex or K is a polygon in $n=2$

$$\alpha_{\Phi}(E)^2 + \beta_{\Phi}(E)^2 \le C\, \delta_{\Phi}(E)$$

Unif Conv: $C(n, \dots, \|\nabla^2 f\|_{C^0(\partial K)})$

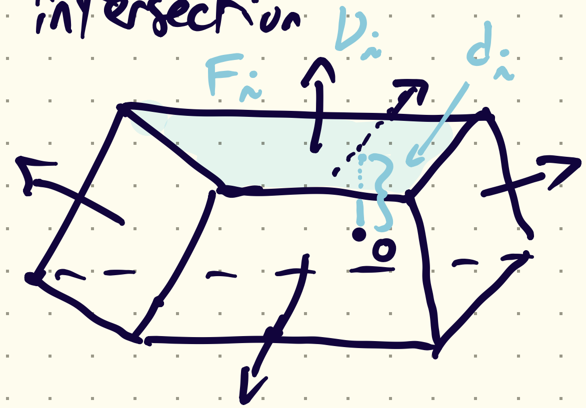
$\hookrightarrow$ Can get $C(n)$ if exp is $4n/n+1$

$n=2$ Crystalline: $C(K)$

Main Result:

Extension of Neumayer's result. The main object of interest is a polytope

Def: A polytope P is the intersection of finitely many half-spaces



Gauge & support function:

$$f_*(x) = \max_{i \in I} \frac{\langle x, \nu_i \rangle}{d_i} \qquad f(\nu) = \max_{j \in J} \langle \nu, x_j \rangle$$

- ν_i - unit normals
- d_i - distance to facet w/ normal ν_i

We call the corresponding surface tension crystalline

Throughout, K is a fixed polytope containing the origin.

THM [D. '23] There exists $C(n,K) > 0$ such that for any $E \subseteq \mathbb{R}^n$ a set of finite perimeter with $|E| < \infty$,

$$\alpha_\Phi(E)^2 + \beta_\Phi(E)^2 \leq C(n,K)\, \delta_\Phi(E).$$

The exponents are sharp.

This generalizes Neumayer's $n=2$ crystalline result.

Sketch of main THM:

1. Prove the result for parallel polytopes

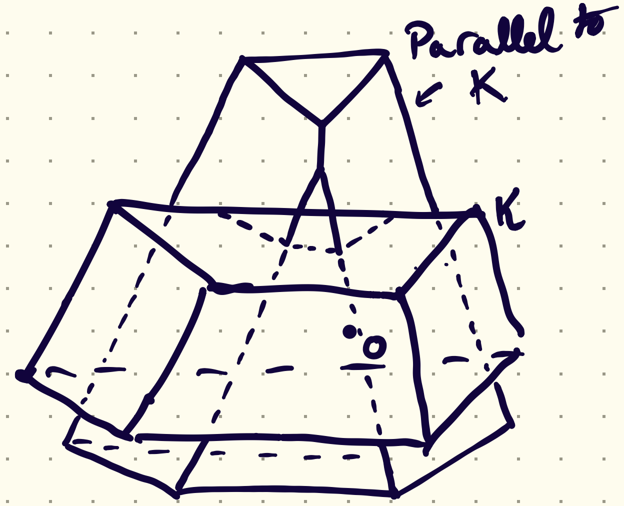
2. Prove for E satisfying uniform density estimates

↳ Upgrade L^1 control to Hausdorff

a) Project E onto a special polytope parallel to K

b) Show this projection increases deficit & decreases oscillation

c) Apply 1.



Sketch of main THM:

3. Selection principle argument

a) Supposing not generate minimizing sequence $\{E_j\}$

b) Replace w/new sequence $\{F_j\}$

↳ satisfy UDEs by regularity of almost minimizers

- Need to show $\{F_j\}$ satisfy same properties as $\{E_j\}$
- Need control on oscillation & anisotropic perimeter to replace

c) Apply 2 $\lightning$.

Thanks for Coming!

Questions?