Kenneth DeMason, April 4, 2021

The following is a compilation of all full credit HW problems from M382C - Algebraic Topology, taken Fall 2020 at UT Austin. This also includes problems which almost received full credit; such problems have a small addendum pointing out the original mistake. Problems are organized by homework. Most problems are from Hatcher, but some (not included in the table of contents) are not; I included these anyways because they seemed interesting.

Contents

HW1

Problem 1. This problem is for when you think you understand what the result of some gluing is; if you are right then this result lets you prove it. Suppose X, Y are compact Hausdorff spaces and $f : X \to Y$ is continuous and onto. Define $\sim$ as the equivalence relation on X given by: $x_1 \sim x_2$ if and only if $f(x_1) = f(x_2)$.

a) Prove that the quotient space $X/\sim$ is Hausdorff.
b) Use this to prove that the induced map $X/\sim \to Y$ is a homeomorphism.
c) The simplest example, but still very useful: identifying the ends of the interval gives S^1.
d) Give a cooler example.

Solution:

a) We prove a) and b) simultaneously. First, we can show that f is a quotient map. To do so, we show that $C \subset Y$ is closed iff $f^{-1}(C)$ is closed. Clearly one direction follows from continuity, so we need only check that if $f^{-1}(C)$ is closed, then C is closed. By hypothesis, $f^{-1}(C)$ is a closed subset of a compact set X. Hence, $f^{-1}(C)$ is compact. By continuity of f, $f(f^{-1}(C))$ is compact in Y. But f is surjective, so $C = f(f^{-1}(C))$. Finally, Y is Hausdorff so that any compact subset is closed.

Next, we show that f induces a homeomorphism between $X/\sim$ and Y. First we may define $X/\sim$ set theoretically as

$$X/\sim := \{f^{-1}(y) \mid y \in Y\}.$$

Let $p : X \to X/\sim$ be defined by $p(x) = f^{-1}(f(x))$. Endow $X/\sim$ with the quotient topology induced by p. Then, $X/\sim$ is the quotient space defined in the problem.

Let $\tilde{x} \in X/\sim$. Choose some $\overline{x} \in \tilde{x}$. Let $y = f(\overline{x})$. Then $p^{-1}(\tilde{x}) = \{x \in X \mid f(x) = f(\overline{x}) = y\}$. Then $f(p^{-1}(\tilde{x})) = y$ so that f is constant on each fiber $p^{-1}(\tilde{x})$. Define $h : X/\sim \to Y$ by $h(\tilde{x}) = f(p^{-1}(\tilde{x}))$. That is to say, $h(\{x \in X \mid f(x) = y\}) = y$. It is easily seen that $h(p(x)) = f(x)$.

We show that h is continuous. Let U be an open set in Y. Then $f^{-1}(U)$ is open by continuity of f. But then $p^{-1}(h^{-1}(U)) = f^{-1}(U)$ is open. By definition of the quotient topology, this set is open iff $h^{-1}(U)$ is open. Hence h is continuous.

We now show h is bijective. Since $f = h \circ p$ and f is surjective, we see that h is surjective. Suppose now that $h(\tilde{x}_1) = h(\tilde{x}_2) = y$ for $\tilde{x}_1, \tilde{x}_2 \in X/\sim$. Then $\tilde{x}_1 = \{x \in X \mid f(x) = y\} = \tilde{x}_2$. Thus h is injective.

We finally show that h is an open map. To do so, we show h is a quotient map. Let $U \subset Y$ and suppose $h^{-1}(U)$ is open. Then by continuity of p, $p^{-1}(h^{-1}(U))$ is open. But $p^{-1}(h^{-1}(U)) = f^{-1}(U)$. We showed first that f is a quotient map, hence U is open in Y. Let $V \subset X/\sim$ be open. We wish to show $h(V)$ is open. Set $U = h(V)$, then we need to show $h^{-1}(U)$ is open. But by injectivity of h, $h^{-1}(h(V)) = V$, which is open.

To see that $X/\sim$ is Hausdorff, simply note that Y is Hausdorff and h is a homeomorphism.

b) See a)

c) Let $f : I \to S^1$ be given by $f(t) = (\cos(2\pi t), \sin(2\pi t))$. Then $f(0) = f(1) = (1, 0)$, and is otherwise bijective. So $X/\sim$ consists of singletons $\{t\}$ for $t \in (0, 1)$ and $\{0, 1\}$. *Addendum: Since I and S^1 are compact, Hausdorff, we can apply b).* Hence, f acts by identifying endpoints of the interval.

d) Define $f : S^2 \to \mathbb{R}^4$ by

$$f(x, y, z) = (x^2 - y^2, xy, xz, yz).$$

Suppose $f(x_1, y_1, z_1) = f(x_2, y_2, z_2)$. Then we have

$$\begin{aligned} x_1^2 - y_1^2 &= x_2^2 - y_2^2 \\ x_1y_1 = x_2y_2,\ x_1z_1 &= x_2z_2,\ y_1z_1 = y_2z_2 \end{aligned}$$

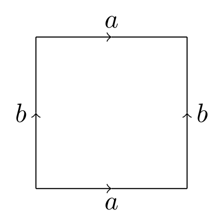
Suppose first that $x_1 \neq 0$ while $y_1 = z_1 = 0$. This implies that $x_2 \neq 0$ by the first equation – if both x_2 and y_2 are zero, then the RHS is zero, if $x_2 = 0$ and $y_2 \neq 0$, then the RHS is negative while the LHS is positive. Looking now at the first equation in the bottom row, we see that $x_2y_2 = 0$, implying $y_2 = 0$. The same thing occurs with $x_1z_1 = x_2z_2$, implying $z_2 = 0$. Finally, we have that $x_1^2 = x_2^2$, implying that $(x_1, y_1, z_1) = \pm(x_2, y_2, z_2)$.

One can do the same thing in all other cases, and conclude that $(x_1, y_1, z_1) = \pm(x_2, y_2, z_2)$ in all. That is, $\sim$ identifies antipodal points on S^2. By restricting f to its image, we see that $f : S^2 \to \mathbb{RP}^2$.

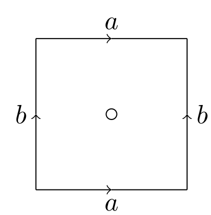
Hatcher Chapter 0, problems 1, 3ab, 17:

0.0.1. Construct an explicit deformation retraction of the torus with one point deleted onto a graph consisting of two circles intersecting in a point, namely, longitude and meridian circles of the torus.

Solution: We may view the torus by identifying opposite sides of the square $[-1, 1] \times [-1, 1]$ in parallel directions, i.e.



So, the punctured torus may be viewed WLOG as the above figure without, say, the origin.



Let $I = [-1, 1]$ and let $S = I^2 \setminus \{0\}$. We can now find a deformation retraction of S onto ∂I^2 as follows: take any point p on the boundary ∂I^2 and look at the line segment from 0 to p. All these points (minus 0) get mapped to p. Explicitly, what we do is define a norm on $\mathbb{R}^2$ as

$$\|\cdot\| = \inf\{\lambda > 0 \mid x/\lambda \in I^2\}$$

and note that ∂I^2 is simply all the points in $\mathbb{R}^2$ with $\|x\| = 1$. Of course, this norm ends up being l^∞. Then a retraction of S onto ∂I^2 is given by $r(x) = x/\|x\|$ restricted to S.

To show that $r : \mathbb{R}^2 \setminus \{0\} \to S^1$ is continuous, first note that $r(x) = (r_1(x), r_2(x))$ where $r_i(x) = x_i/\max(|x_1|, |x_2|)$. Both x_i and $\max(|x_1|, |x_2|)$ are continuous, and the latter is nonzero. So $r_i(x)$ is the quotient of two continuous functions, and hence is continuous. Since all the components of r are continuous, it is continuous. Finally, r is continuous in the l^∞ topology, but we want it to be continuous in the l^2 topology. This is not an issue since all norms are equivalent on a finite dimensional space.

Define a homotopy $H(x, t)$ by $H(x, t) = (1 - t)\mathrm{id}_S + tr$, here now r is already restricted to S so that $r : S \to \partial I^2$. Set $f_t(x) = H(x, t)$, where t is fixed. We must check four things: $H(x, t)$ is continuous, f_0 is the identity on S, $f_1(S) = \partial I^2$, and $f_t|_{\partial I^2}$ is the identity on ∂I^2.

i) Since r is the restriction of a continuous map, it too is continuous. The identity on S is obviously continuous. For fixed x, $H(x, t)$ is a polynomial in t, and hence is continuous. Clearly then $H(x, t)$ is continuous.

ii) By definition, $f_0(x) = (1 - 0)\mathrm{id}_S + 0r = \mathrm{id}_S$, so f_0 is the identity on S.

iii) Setting $t = 1$, $f_1(x) = (1 - 1)\mathrm{id}_S + r = r$. Then, $f_1(S) = r(S) = \partial I^2$. To see that r is surjective, first note that $\partial I^2 = \{x \in \mathbb{R}^2 \mid \|x\| = 1\}$. Then we have that $r|_{\partial I^2} = x/\|x\| = x$. Therefore $r|_{\partial I^2} = \mathrm{id}|_{\partial I^2}$ and $r(\partial I^2) = \partial I^2$. But $\partial I^2 \subset S$, and r maps into ∂I^2, so that $\partial I^2 \subset r(S) \subset r(\partial I^2) = \partial I^2$.

iv) Fix a $t \in [0,1]$. Then $f_t(x) = (1-t)\mathrm{id}_S + tr$. Restricting this to ∂I^2 gives $f_t|_{\partial I^2}(x) = (1-t)\mathrm{id}|_{\partial I^2} + tr|_{\partial I^2}$, since $\partial I^2 \subset S$. But we saw in iii) that $r|_{\partial I^2} = \mathrm{id}|_{\partial I^2}$ so that

$$f_t|_{\partial I^2}(x) = (1-t)\mathrm{id}|_{\partial I^2} + t\mathrm{id}|_{\partial I^2} = \mathrm{id}|_{\partial I^2}.$$

The remaining space is the wedge sum of two circles, namely the longitudinal and meridian circles of the torus.

It is worthwhile to mention that the above deformation retraction works for any convex set K containing the origin by replacing I^2 with K.

0.0.3.

a) Show that the composition of homotopy equivalences $X \to Y$ and $Y \to Z$ is a homotopy equivalence $X \to Z$. Deduce that homotopy equivalence is an equivalence relation.
b) Show that the relation of homotopy among maps $X \to Y$ is an equivalence relation.
c) Show that a map homotopic to a homotopy equivalence is a homotopy equivalence. (*Not necessary)

Solution:

a) Recall that $f : X \to Y$ is a homotopy equivalence if there exists a $\tilde{f} : Y \to X$ such that $\tilde{f} \circ f \simeq \mathrm{id}_X$ and $f \circ \tilde{f} \simeq \mathrm{id}_Y$. Let $g : Y \to Z, \tilde{g} : Z \to Y$ be homotopy inverses. We now need the following lemma.

Lemma: If $f \simeq g$ where $f, g : X \to Y$ then $f \circ h \simeq g \circ h$, where $h : Z \to X$ is continuous. Similarly if $h : X \to Z$ then $h \circ f \simeq h \circ g$.

Proof: Let $H(x,t)$ be a homotopy such that $H(x,0) = f$ and $H(x,1) = g$. Define $\tilde{H} : Z \times I \to Y$ by $\tilde{H}(z,t) = H(h(z),t)$. Then $\tilde{H}$ is continuous as a composition of continuous functions. Moreover, $\tilde{H}(z,0) = H(h(z),0) = f(h(z))$, and $\tilde{H}(z,1) = H(h(z),1) = g(h(z))$. So $\tilde{H}$ is a homotopy from $f \circ h$ to $g \circ h$. The other case is similar, and is achieved by defining $\tilde{H} : X \times I \to Z$ by $\tilde{H}(x,t) = h(H(x,t))$.

We wish to show $g \circ f$ is a homotopy equivalence from $X \to Z$ with homotopy inverse $\tilde{f} \circ \tilde{g}$. First, by associativity of composition,

$$(\tilde{f} \circ \tilde{g}) \circ (g \circ f) = \tilde{f} \circ (\tilde{g} \circ g) \circ f.$$

By application of the lemma, we see that

$$(\tilde{g} \circ g) \circ f \simeq \mathrm{id}_Y \circ f = f.$$

Once more, by application of the lemma

$$\tilde{f} \circ (\tilde{g} \circ g) \circ f \simeq \tilde{f} \circ f \simeq \mathrm{id}_X,$$

where in the last step we used the fact that $f, \tilde{f}$ are homotopy inverses.

Similarly,

$$(g \circ f) \circ (\tilde{f} \circ g) = g \circ (f \circ \tilde{f}) \circ \tilde{g} \simeq g \circ \mathrm{id}_Y \circ \tilde{g} = g \circ \tilde{g} \simeq \mathrm{id}_Z.$$

Thus, $g \circ f$ is a homotopy equivalence from $X \to Z$ with homotopy inverse $\tilde{g} \circ \tilde{f}$.

We now show homotopy equivalence is an equivalence relation. First, two spaces X, Y are homotopy equivalent if there exist maps $f : X \to Y$ and $\tilde{f} : Y \to X$ such that $f, \tilde{f}$ are homotopy inverses. Transitivity was shown above, so we need only show reflexivity and symmetry.

- Reflexive: Clearly X is homotopy equivalent to itself by taking $f = \mathrm{id}_X$.
- Symmetry: If $X \simeq Y$ then there exist $f, \tilde{f}$ such that $\tilde{f} \circ f \simeq \mathrm{id}_X$ and $f \circ \tilde{f} \simeq \mathrm{id}_Y$. By reversing the roles of $f, \tilde{f}$ we see that $Y \simeq X$.

b) We show that homotopy is an equivalence relation.

- Reflexive: Let $f : X \to Y$. Define $H(x,t) = f(x)$, the constant homotopy. Then $H : X \times I \to X$ is a continuous map. Clearly $H(x,0) = f(x)$ and $H(x,1) = f(x)$ for all x, so H is a homotopy from f to f. Thus $f \simeq f$.

- Symmetric: Let $f, g : X \to Y$ be homotopic. Then there exists $H(x,t)$ such that $H(x,0) = f(x)$ for all x and $H(x,1) = g(x)$ for all x. Define $\tilde{H}(x,t) = H(x, 1-t)$. Then $\tilde{H}(x,0) = H(x,1) = g(x)$ and $\tilde{H}(x,1) = H(x,0) = f(x)$ for all x. Since $1-t$ is continuous, $\tilde{H}$ is a composition of continuous maps, and hence is continuous. Thus $\tilde{H}$ is a homotopy from g to f, and $g \simeq f$.
- Transitive: Suppose $f \simeq g$ and $g \simeq h$, via the homotopies $H_1(x,t)$ and $H_2(x,t)$. Define $H(x,t)$ as follows:

$$H(x,t) = \begin{cases} H_1(x, 2t) & \text{for } (x,t) \in X \times \left[0, \frac{1}{2}\right] \\ H_2(x, 2t-1) & \text{for } (x,t) \in X \times \left[\frac{1}{2}, 1\right] \end{cases}$$

Both $H_1(x,2t)$ and $H_2(x,2t-1)$ are continuous as compositions of continuous functions. Now recall the pasting lemma, which states if $U, V \subset_{cl} X$ and $u : U \to Y$, $v : V \to Y$ are continuous and agree on $U \cap V$, then $h : U \cup V \to Y$ defined by

$$w(x) = \begin{cases} u(x) & \text{for } x \in U \\ v(x) & \text{for } x \in V \end{cases}$$

is continuous.
Clearly both $X \times [0, 1/2]$ and $X \times [1/2, 1]$ are closed in $X \times I$. Moreover, their intersection is $X \times \{1/2\}$. Now,

$$H_1(x, 2(1/2)) = H_1(x,1) = g(x) = H_2(x,0) = H_2(x, 2(1/2) - 1)$$

so that $H_1(x,2t)$ and $H_2(2t-1)$ agree on $X \times \{1/2\}$. Therefore $H(x,t)$ is continuous. Finally, $H(x,0) = H_1(x,2(0)) = H_1(x,0) = f(x)$ and $H(x,1) = H_2(x,2(1)-1) = H_2(x,1) = h(x)$. So, H is a homotopy from f to h, and $f \simeq h$.

c) Suppose that $f, \tilde{f}$ are homotopy inverses and $h \simeq f$. We show there exists a homotopy inverse $\tilde{h}$ to h. By the lemma in part a), using $f \simeq h$ we have $f \circ \tilde{f} \simeq h \circ \tilde{f}$. But, $f \circ \tilde{f} \simeq \mathrm{id}_Y$. So, $h \circ \tilde{f} \simeq \mathrm{id}_Y$. Once more applying the lemma, we see that $\mathrm{id}_X \simeq \tilde{f} \circ f \simeq \tilde{f} \circ h$. So, $\tilde{f}$ is a homotopy inverse of h too.

0.0.17.

a) Show that the mapping cylinder of every map $f : S^1 \to S^1$ is a CW complex.
b) Construct a 2 dimensional CW complex that contains both an annulus $S^1 \times I$ and a Möbius band as deformation retracts.

Solution:

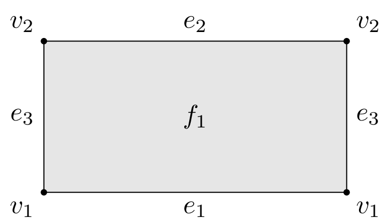
a) Let x_0 in S^1 and take two 0-cells corresponding to x_0 and $f(x_0)$. Then attach 1-cells e_1 and e_2 to each of these to form two circles. Finally, attach a 1-cell e_3 connecting the two 0-cells.

We imagine this construction in $\mathbb{R}^3$. Consider two circles given by $(\cos 2\pi t, \sin 2\pi t, 0)$ and $(\cos 2\pi t, \sin 2\pi t, 1)$ for $t \in [0,1]$. The point $(1,0,0)$ is identified with the 0-cell corresponding to x_0, while the point $(1,0,1)$ is identified with the 0-cell corresponding to $f(x_0)$. Let n be the degree/winding number of the map f. We view the third 1-cell e_3 as a helix wrapping around the "cylinder" n times. Explicitly, it is parameterized by

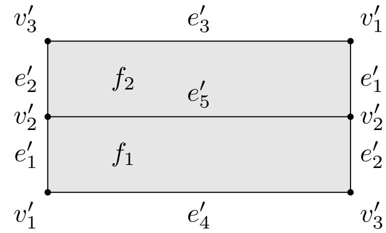
$$\gamma(t) = (\cos 2n\pi t, \sin 2n\pi t, t).$$

We now attach a 2-cell E as follows. Regard E as the square $[0,1] \times [0,1]$. The attaching map φ is defined on each edge. First, define $\varphi(0,t) = (\cos 2\pi t, \sin 2\pi t, 0)$ for $t \in [0,1]$. That is, the left-hand vertical edge is mapped to the lower S^1 of the cylinder. Next, $\varphi(1,t) = (f(\cos 2\pi t, \sin 2\pi t), 1)$. That is, we map the right-hand vertical side to the upper S^1 via f. Finally, we define the attaching map on the horizontal sides. For $i = 0, 1$ define $\varphi(t,i) = (\cos 2n\pi t, \sin 2n\pi t, t)$. That is, we identify the upper and lower horizontal sides with the helix joining x_0 to $f(x_0)$.
Addendum: The above does not work when f is a constant. In this case, we define $\varphi(t,i)$ by $\varphi(t,i) = \varphi(f(0), t)$. In this case, we should get a cone, where the top $S^1 \times I$ was crushed to $f(0)$

b) The cylinder $S^1 \times I$ and the Möbius band M can each be regarded as CW complexes depicted below.



$S^1 \times I$



M

Here, we use the convention that v stands for 0-cells, e for 1-cells, and f for 2-cells. Now we can consider a CW complex with the above as subcomplexes. So, identify v_1 with v'_2 and e_1 with e'_5.

We now construct deformation retractions. We regard the CW complex structure for $S^1 \times I$ above as the square $[0,1] \times [0,1]$ (regarding v_1 as $(0,0)$ and $(1,0)$, e_1 as $[0,1] \times \{0\}$, etc.). We now construct a deformation retraction of $S^1 \times I$ onto its lower circle. Define $H(x,t) = H((x_1,x_2),t)$ by

$$H(x,t) = (x_1, (1-t)x_2).$$

H is continuous since it is continuous in each component, $H(x,0) = (x_1,x_2)$ which is the identity, and H fixes $[0,1] \times \{0\}$ throughout the homotopy. But this is regarded as subcomplex with e_1 attached to v_1 at both endpoints. Hence H is a deformation retraction of $S^1 \times I$ onto its lower circle.

Now similarly regard the CW complex above for M as the square $[0,1] \times [0,1]$. We construct a deformation retraction onto its central circle $[0,1] \times \{1/2\}$ by

$$H(x,t) = (x_1, t/2 + (1-t)x_2).$$

Similarly, H is continuous since it is continuous in each component, $H(x,0) = (x_1,x_2)$, and H fixes $[0,1] \times \{1/2\}$ during the homotopy (since $x_2 = 1/2$, and $t/2 + (1-t)/2 = 1/2$).

In our final CW complex, we may perform each of these deformation retractions separately on the subcomplexes above. For example, deformation retracting the subcomplex associated with $S^1 \times I$ by above brings it into the central circle of the Möbius band subcomplex. By extending this deformation retraction to the entire complex by the identity elsewhere (this is possible since the two subcomplexes intersect on the central circle, a closed subset, which is fixed in both deformation retractions).

Hatcher Chapter 1, problems 3, 6, 7, 16:

1.1.3. For a path-connected space X, show that $\pi_1(X)$ is abelian iff all basepoint-change homomorphisms β_h depend only on the endpoints of the path h.

Solution: First, since X is path connected, all its points lie in the same path component. Hence, given x_0 and x_1 in X, we can always find a path h from x_0 to x_1. The basepoint-change homomorphism β_h is given by $\beta_h(f) = [h * f * \tilde{h}]$, where $\tilde{h}(t) = h(1-t)$. I will use $*$ to denote both concatenation of paths and the group operation of $\pi_1(X)$. That is, for loops f, g based at x_0, $[f] * [g] = [f * g]$. Note: I perform my path concatenations from left to right. So, $f * g$ means to go around g first, then around f.

Suppose $\pi_1(X)$ is abelian, and let h, h' be two paths from x_0 to x_1. We wish to show $\beta_h = \beta_{h'}$. By direct computation,

$$\begin{aligned}
\beta_h(f) &= [h * f * \tilde{h}] = [h] * [f] * [\tilde{h}] \\
&= ([h] * [f] * [\tilde{h}']) * ([h'] * [\tilde{h}]) \\
&= ([h'] * [\tilde{h}]) * ([h] * [f] * [\tilde{h}']) \\
&= [h'] * [f] * [\tilde{h}'] = [h * f * \tilde{h}'] = \beta_{h'}(f)
\end{aligned}$$

where we have used associativity and commutivity of $*$ freely. Now suppose $\beta_h(f) = \beta_{h'}(f)$ for all paths h from x_0 to x_1. Let f, g be loops at x_0 and set $h' = h * g$, where h is a path from x_0 to x_1. Then h' is a path from x_0 to x_1 and

$$\beta_{h'}(f) = [h * g] * [f] * [h * g]^{-1} = [h] * ([g] * [f] * [\tilde{g}]) * [\tilde{h}].$$

On the other hand,

$$\beta_{h'}(f) = \beta_h(f) = [h] * [f] * [\tilde{h}].$$

It follows that

$$[g] * [f] * [g]^{-1} = [f],$$

so that $\pi_1(X)$ is abelian.

1.1.6. We can regard $\pi_1(X, x_0)$ as the set of basepoint-preserving homotopy classes of maps $(S^1, s_0) \to (X, x_0)$. Let $[S^1, X]$ be the set of homotopy classes of maps $S^1 \to X$, with no conditions on basepoints. Thus there is a natural map $\Phi : \pi_1(X, x_0) \to [S^1, X]$ obtained by ignoring basepoints. Show that Φ is onto if X is path-connected, and that $\Phi([f]) = \Phi([g])$ iff $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$. Hence Φ induces a one-to-one correspondence between $[S^1, X]$ and the set of conjugacy classes in $\pi_1(X)$ when X is path-connected.

Solution: Choose a basepoint $x_0 \in X$. Let $[f] \in [S^1, X]$. Then the image of $f : S^1 \to X$ is some loop, passing through say $x_1 = f((1, 0))$. Define $g : [0, 1] \to X$ by setting $g(t) = f(\cos 2\pi t, \sin 2\pi t)$. Then $x_1 = g(0) = g(1)$, and g is a loop based at x_1 (this transformation is only necessary because we need the domain to be $[0, 1]$). Now since X is path connected there exists a path $h : I \to X$ such that $h(0) = x_0$ and $h(1) = x_1$. Then $\tilde{h} * g * h$ is a loop based at x_0.

Now we can construct a basepoint-ignoring map Φ for each basepoint in X – in particular for x_0 and x_1. We denote these by Φ_{x_0} and Φ_{x_1}. Observe that $g * h * \tilde{h}$ is a loop based at x_1, and moreover the image of this loop is the same as the image of the loop $\tilde{h} * g * h$. Hence, upon ignoring the basepoint, we have the same loop. Alternatively, each of $\tilde{h} * g * h$ and $g * h * \tilde{h}$ can be regarded as a map $S^1 \to X$, each of which can be obtained by precomposing with some rotation of S^1. In any case, it follows that

$$\Phi_{x_0}([\tilde{h} * g * h]) = \Phi_{x_1}([g * h * \tilde{h}]).$$

On the other hand, $[g * h * \tilde{h}] = [g]$. We constructed g from f by identifying a basepoint x_1, hence $\Phi_{x_1}[g] = [f]$. In total,

$$\Phi_{x_0}([\tilde{h} * g * h]) = \Phi_{x_1}([g * h * \tilde{h}]) = \Phi_{x_1}[g] = [f]$$

so that Φ_{x_0} is surjective.

We now drop the subscript notation on Φ. Suppose f, g are loops based at x_0. Set $h = \tilde{g} * f * g$. Then trivially $[h]$ and $[f]$ are conjugate since $[h] = [g]^{-1} * [f] * [g]$. Then,

$$\Phi([\tilde{g} * f * g]) = \Phi([f * g * \tilde{g}]) = \Phi([f]).$$

Hence $\Phi([h]) = \Phi([f])$.

Now let f, h be loops at x_0 such that $\Phi([f]) = \Phi([h])$. We show that there exists a loop g at x_0 such that $[h] = [g]^{-1} * [f] * [g]$ so that $[f], [h]$ are conjugate. That $\Phi([f]) = \Phi([h])$ means that there exists a homotopy $H : S^1 \times I \to X$ with $H(-, 0) = f$ and $H(-, 1) = h$ (since they are in the same homotopy equivalence class as maps from $S^1 \to X$). Here we regard f, h as maps from S^1 to X, where $f(0) = f(1) = f(1, 0) = x_0$, and similarly for h. Let $g : I \to X$ be defined by $g(t) = H((1, 0), t)$. Note that g is actually a loop based at x_0 since $H((1, 0), 0) = H((1, 0), 1) = x_0$. To conclude, since $f(\mathrm{id}_{S^1}) = f$ (similarly for h) we have

$$H_{0*}[\mathrm{id}_{S^1}] = [H_0(\mathrm{id}_{S^1})] = [f] \quad \text{and} \quad H_{1*}[\mathrm{id}_{S^1}] = [H_1(\mathrm{id}_{S^1})] = [h]$$

Then, by Hatcher Lemma 1.19, we have

$$H_{0*} = \beta_g H_{1*},$$

and so $[f] = \beta_g[h] = [g] * [h] * [g]^{-1}$. Hence f, h are conjugate.

1.1.7. Define $f : S^1 \times I \to S^1 \times I$ by $f(\theta, s) = (\theta + 2\pi s, s)$, so f restricts to the identity on the two boundary circles of $S^1 \times I$. Show that f is homotopic to the identity by a homotopy f_t that is stationary on one of the boundary circles, but not by any homotopy f_t that is stationary on both boundary circles. [Consider what f does to the path $s \mapsto (\theta_0, s)$ for fixed $\theta_0 \in S^1$.]

Solution: Define $H(x, t) = H((\theta, s), t)$ by

$$H((\theta, s), t) = (\theta + (1 - t)2\pi s, s).$$

Continuity is obvious, $H((\theta, s), 0) = (\theta + 2\pi s, s) = f(\theta, s)$, and $H((\theta, s), 1) = (\theta, s) = \mathrm{id}_{S^1 \times I}$. So H is a homotopy from f to the identity. We also want to show that H_t is stationary on one of the boundary circles. It is stationary on the bottom circle, since

$$H((\theta, 0), t) = (\theta + (1 - t)2\pi(0), 0) = (\theta, 0)$$

and the lower circle is parameterized by $(\theta, 0)$ for $\theta \in S^1$.

Now can there exist a homotopy H such that H_t fixes both boundary circles? Consider the path $\gamma(s) = (0, s)$. Then each $H_t \circ \gamma$ is a path satisfying

$$(H_t \circ \gamma)(0) = H_t(0, 0) = (0, 0) \quad \text{and} \quad (H_t \circ \gamma)(1) = H_t(0, 1) = (0, 1)$$

since H_t fixes both boundary circles. Moreover,

$$H_0 \circ \gamma = f \circ \gamma, \quad \text{and} \quad H_1 \circ \gamma = \mathrm{id}_{S^1 \times I} \circ \gamma.$$

So, $H_t \circ \gamma = H(\gamma, t)$ is a homotopy of paths in $S^1 \times I$ from $f \circ \gamma$ to γ. Note that

$$f \circ \gamma(s) = f(0, s) = (2\pi s, s),$$

which wraps around the cylinder once.

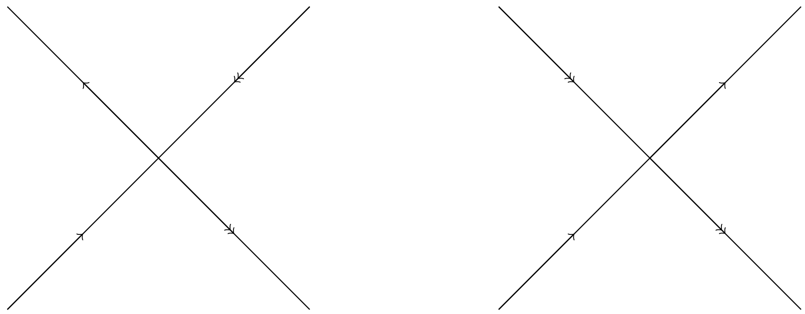
Consider now the projection $\pi : S^1 \times I \to S^1 \times \{0\}$. Then $\pi \circ \gamma$ is the constant path at $(0, 0)$ (note: this is in polar coordinates) while $\pi \circ (f \circ \gamma)$ wraps around the circle once. Thus they are not in the same homotopy class. But, by the Lemma proven in 3a, we have that $\pi \circ \gamma \simeq \pi \circ (f \circ \gamma)$ since $\gamma \simeq f \circ \gamma$.

1.1.16. Show that there are no retractions $r : X \to A$ in the following cases:

a) $X = \mathbb{R}^3$ with A any subspace homeomorphic to S^1.
b) $X = S^1 \times D^2$ with A its boundary torus $S^1 \times S^1$.
c) $X = S^1 \times D^2$ and A the circle shown in the figure.
d) $X = D^2 \vee D^2$ with A its boundary $S^1 \vee S^1$.
e) X a disk with two points on its boundary identified and A its boundary $S^1 \vee S^1$.
f) X the Möbius band and A its boundary circle.

Solution:

a) Suppose such a retraction exists. Then the homomorphism induced by the inclusion $i : A \to X$, denoted $i_* : \pi_1(A) \to \pi_1(X)$ is injective. But $\pi_1(\mathbb{R}^3)$ is trivial, whereas $\pi_1(A) \simeq \mathbb{Z}$. So there can be no injective map $\mathbb{Z} \to \{0\}$.
b) Suppose a retraction exists. Then i_* is injective. But $\pi_1(A) = \mathbb{Z} \times \mathbb{Z}$ whereas $\pi_1(X) = \mathbb{Z}$. To calculate $\pi_1(X)$ one appeals to the fact that $\pi_1(X \times Y) \simeq \pi_1(X) \times \pi_1(Y)$ when X, Y are path-connected. Let $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be any group homomorphism. We show that $\mathrm{Ker}(h) \neq \{(0, 0)\}$, and hence h cannot be injective. Suppose $h(1, 0) = a$ and $h(0, 1) = b$. Then $h(m, n) = am + bn$. It follows that $h(-b, a) = (0, 0)$, and hence $\mathrm{Ker}(h) \neq \{(0, 0)\}$.
c) Once more supposing that we have a retraction, then i_* is an injective homomorphism. Now take a loop γ in A that traverses it once. This is also a loop in the solid torus. We show that $[\gamma]$ is trivial in $\pi_1(X)$. To see this, simply observe that we may unlink the knot during a homotopy – that is, for each fixed t, it does not need to be that $H(x, t)$ is an embedding. The only possible issue may be continuity of $H(x, t)$. So long as we do not change the "direction" of our path when it self intersects, this should not be an issue. I.e., we probably do not want something like the below to occur



Since $[\gamma]$ is not trivial in $\pi_1(A)$, we conclude that i_* cannot be injective.

d) By van Kampen's theorem, we know that $\pi_1(X)$ is trivial whereas $\pi_1(S^1 \vee S^1) \simeq \mathbb{Z} * \mathbb{Z}$. Assuming the existence of a retraction, we conclude that i_* is an injective homomorphism. But no injective map from $\mathbb{Z} * \mathbb{Z}$ to $\{0\}$ can exist.

e) Let a and b be two distinct points on ∂D^2. We show that D^2 deformation retracts onto the minor arc connecting a and b. To see this, let c be the midpoint of a and b along the minor arc, and rotate D^2 so that c lies at $(0,-1)$. Denote the lower half-arc by C. Define $H : D^2 \times I \to C$ by

$$H((x_1, x_2), t) = \left(x_1, (1-t)x_2 - t\sqrt{1-x_1^2}\right).$$

This simply takes (x_1, x_2) and pulls it down along the line $x = x_1$ until it reaches C. First,

$$H((x_1, x_2), 0) = (x_1, x_2) = \mathrm{id}_{D^2}.$$

Next,

$$H((x_1, x_2), 1) = \left(x_1, -\sqrt{1-x_1^2}\right),$$

which parameterizes C. Thus H maps onto C. Finally,

$$H\left(\left(x_1, -\sqrt{1-x_1^2}\right), t\right) = \left(x_1, (1-t)\left(-\sqrt{1-x_1^2}\right) - t\sqrt{1-x_1^2}\right) = \left(x_1, -\sqrt{1-x_1^2}\right).$$

Thus H fixes C during the homotopy. It follows that H is a deformation retraction of D^2 onto C.

Next we construct a deformation retraction of C onto the minor arc connecting a and b, denoted C_{ab}. Without loss of generality suppose $a = (a_1, -\sqrt{1-a_1^2})$ and $b = (b_1, -\sqrt{1-b_1^2})$ where $a_1 < b_1$ (actually, we know that $a_1 = -b_1$ and $\sqrt{1-a_1^2} = \sqrt{1-b_1^2}$ by our positioning of the midpoint at $(0,-1)$). Now define $H(x,t)$ a homotopy from C to C_{ab} piecewise as follows:

$$H\left(\left(x_1, \sqrt{1-x_1^2}\right), t\right) = \begin{cases} \left((1-t)x_1 + ta_1, -\sqrt{1-((1-t)x_1+ta_1)^2}\right) & -1 \le x_1 \le a_1 \\ \left(x_1, -\sqrt{1-x_1^2}\right) & a_1 \le x_1 \le b_1 \\ \left((1-t)x_1 + tb_1, -\sqrt{1-((1-t)x_1+tb_1)^2}\right) & b_1 \le x_1 \le 1 \end{cases}$$

which simply takes the endpoints $(-1,0)$ and $(1,0)$ and brings them along C to a and b respectively (note: this is definitely not the cleanest way to do it, I wanted to see if I could do it this way though. Probably easier to take C into an interval, deform it there, then bring it back). The pasting lemma guarantees that this is continuous. One can easily verify that this is a deformation retraction (clearly it's the identity on C_{ab} by the piecewise definition).

So, we have that D^2 is a deformation retraction onto the minor arc connecting a and b. By identifying a and b, we see that X is a deformation retraction onto S^1, and therefore $\pi_1(X) \simeq \mathbb{Z}$. On the other hand, $\pi_1(S^1 \vee S^1) \simeq \mathbb{Z} * \mathbb{Z}$. But $\mathbb{Z} * \mathbb{Z}$ is not abelian, while $\mathbb{Z}$ is, and the image of a group under an injective homomorphism is a subgroup. More explicitly, if $[a]$ and $[b]$ are two noncommuting elements of $\pi_1(S^1 \vee S^1)$ then

$$i_*([a] * [b]) = [a] * [b] = [b] * [a] = i_*([b] * [a]),$$

a contradiction.

f) Observe that the boundary of the Möbius band, ∂M, can be regarded as a circle which wraps around twice. That is, consider a loop on the boundary which wraps around once. Then, bringing it to the center circle of M yields a loop which wraps around twice. But $\pi_1(M) \simeq \mathbb{Z}$, since M deformation retracts onto its center circle (shown in 0.0.17b). Hence $\pi_1(\partial M) \simeq 2\mathbb{Z}$. As discussed in Hatcher, if X retracts onto A we may identify $\pi_1(A)$ with its image under i_*. Then, $r_* : \pi_1(X) \to \pi_1(A)$ is such that r_* restricts to the identity on $\pi_1(A)$. Finally, by identifying $2 \in \mathbb{Z}$ with the loop which wraps around the center circle twice, and $2 \in 2\mathbb{Z}$ with the loop which wraps around the boundary ∂M once in the same direction, we see that $r_*(2) = 2$. But this implies that $r_*(1) = 1$ since r_* is a homomorphism. Yet, $1 \notin 2\mathbb{Z}$, a contradiction.

HW2

Problem 1. An n-dimensional *manifold with boundary* means a Hausdorff space M, such that every $x \in M$ has a neighborhood U such that the pair (U, x) is homeomorphic to either $(\mathbb{R}^n, 0)$ or $(\mathbb{R}^n \times [0, \infty), 0)$, where in both cases 0 means $(0, ..., 0)$. We call x an interior or boundary point according to which of these holds. Note that this is *not* the usual use of "interior" and "boundary" from point-set topology. The set of boundary points is written ∂M. Prove that the inclusion $M \setminus \partial M \hookrightarrow M$ is a homotopy equivalence. You may use without proof the fact that no point can be both an interior and a boundary point. Assume further that M has compact boundary.

Remarks: informally, I think of $M \setminus \partial M$ as a sort of deformation-retract of M. But it is easy to see that if $\partial M \neq \emptyset$ then M does not actually deformation retract to $M \setminus \partial M$. Also, without the extra hypotheses, the only solution I know uses something you probably have not seen: topological dimension, which lets you build an open cover with good overlap properties.

Solution: A strategy for the proof: We show that a closed subset V of M and $M \setminus \partial M$ is a deformation retraction of M and $M \setminus \partial M$.

By Proposition 3.42 of Hatcher, every compact manifold with boundary has a collar neighborhood of ∂M in M. Meaning, there exists an open neighborhood $U \subset M$ of ∂M such that U is homeomorphic to $\partial M \times [0, 1)$, where ∂M is taken to $\partial M \times \{0\}$. Let $h : \partial M \times [0, 1) \to U$ be such a homeomorphism (that is, $h(x, 0) = x$ for $x \in \partial M$). The set $[0, 1/2)$ is open in $[0, 1)$ since $[0, 1)$ inherits the subspace topology on $\mathbb{R}$, hence $\partial M \times [0, 1/2)$ is an open subset of $\partial M \times [0, 1)$. It follows that $h(\partial M \times [0, 1/2))$ is some open set – let V be its complement in M, which is closed.

Let $x \in M$ and consider $h^{-1}(x)$. Then $h^{-1}(x) = (\tilde{x}, t)$ for some $\tilde{x} \in \partial M$ and $t \in [0, 1)$. Define $\pi_0 : \partial M \times [0, 1) \to \partial M$ by $\pi_0(\tilde{x}, t) = \tilde{x}$ and $\pi_1 : \partial M \times [0, 1) \to [0, 1)$ by $\pi_1(\tilde{x}, t) = t$. Since h is bijective, it follows that

$$h(\pi_0(h^{-1}(x)), \pi_1(h^{-1}(x))) = h(\pi_0(\tilde{x}, t), \pi_1(\tilde{x}, t)) = h(\tilde{x}, t) = x$$

for any $x \in h(\partial M \times [0, 1))$.

Note that we need to take a complement of a smaller collar neighborhood, because it could be that $h(\partial M \times [0, 1))^c$ is empty. For example, take the annulus $M = \{x \in \mathbb{R}^2 \mid 1/2 < |x| \leq 1\}$. Then $\partial M = S^1$ and a collar neighborhood of S^1 in M is M itself, since

$$h(x, t) = (1 - t/2)x$$

is a homeomorphism of $S^1 \times [0, 1)$ and M.

We now define a homotopy $H : M \times [0, 1] \to M$ as follows:

$$H(x, s) = \begin{cases} (1 - s)\mathrm{id}_M(x) + sh(\pi_0(h^{-1}(x)), 1/2) & \pi_1(h^{-1}(x)) \in [0, 1/2] \\ \mathrm{id}_M(x) & x \in V \end{cases}$$

This is continuous since each piecewise component is continuous. Their domains overlap when $\pi_1(h^{-1}(x)) = 1/2$, since $V = h(\partial M \times [1/2, 1)) \cup [M \cap U^c]$. But, in this case we have that

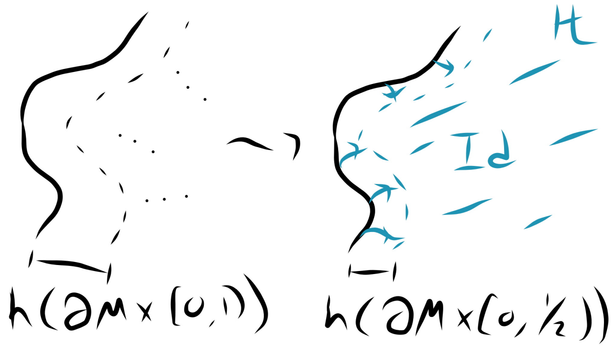
$$h(\pi_0(h^{-1}(x)), 1/2) = h(\pi_0(h^{-1}(x)), \pi_1(h^{-1}(x))) = \mathrm{id}_M(x).$$

Then, for any $s \in [0, 1]$,

$$(1 - s)\mathrm{id}_M(x) + s\mathrm{id}_M(x) = \mathrm{id}_M(x),$$

so that each piecewise definition agrees on the overlap. Hence, H is continuous.

Observe further that $H(x, 0) = \mathrm{id}_M(x)$ and $H(x, 1) = V$. Clearly, $\mathrm{id}_M(V) = V$ so that $V \subset H(x, 1)$. But, using the first piecewise definition, we must include also $h(\pi_0(h^{-1}(x)), 1/2)$ for all x with $\pi_1(h^{-1}(x)) \in [0, 1/2]$. But, V contains all the points $h(\tilde{x}, 1/2)$ for $\tilde{x} \in \partial M$. So, these are mapped into V as well. Finally, for all s we have that $H(v, s) = \mathrm{id}_M(v)$ where $v \in V$. So, H is a deformation retraction of M to V. The following picture depicts the homotopy.



Morally, we look at the flow lines $\varphi(t) = h_x(t)$ which push $x \in \partial M$ into the collar neighborhood. These lines extend until $t = 1/2$, and we collapse the lines onto their endpoint.

We can define a similar deformation retraction of $M \setminus \partial M$ to V by disregarding the case when $\pi_1(h^{-1}(x)) = 0$ (since $\partial M\{0\}$ is mapped to ∂M under h). Equivalently, we restrict H to $(M \setminus \partial M) \times [0,1]$. Hence, we have found a deformation retraction of M to V and $M \setminus \partial M$ to V. It follows that M and $M \setminus \partial M$ are homotopy equivalent.

Another possible solution is to take finitely many open neighborhoods U_i covering the boundary, each of which is homeomorphic to $\mathbb{R}^{n-1} \times [0,\infty)$. Then, look at a particular half space H_i homeomorphic to U_i. We may possibly have overlap with other U_j – look at the overlaps in the half space H_i and define a homotopy H_i to be the identity there, and push the boundary in on the complement (where there is no overlap). Bringing these homotopies into the manifold and applying them one after another should give a homotopy which pushes the boundary in, as above.

Problem 2 (A "bad" group action). Let $X = \mathbb{R}^2 \setminus 0$ where 0 is the origin. Let G be the group of homeomorphisms of X generated by the transformation $(x,y) \mapsto (2x, y/2)$. Let Y be the quotient space X/G.

a) Prove that every orbit is discrete. This is meant as a stepping stone to the more general result (b).
b) Prove that G's action on X satisfies the hypothesis of the theorem from class about $\pi_1(X/G) \simeq G$, namely: every $x \in X$ has a neighborhood U such that $U \cap g(U) = \emptyset$ for every $g \in G \setminus \{1\}$.
c) Prove that Y is a manifold, except for the fact that it is *not* Hausdorff.

(When working on a theorem involving a group action, if I wonder whether some hypothesis can be omitted, checking it for this single example usually reveals the answer.)

Solution:

a) First observe that the orbit of a point $(x,y) \in \mathbb{R}^2 \setminus 0$ consists of all the points $(2^n x, 2^{-n} y)$ for all $n \in \mathbb{Z}$. Thus, each orbit $O(x,y)$ is countable. It suffices to show that, for a fixed $n \in \mathbb{Z}$ the singleton $\{(2^n x, 2^{-n} y)\}$ is open in $O(x,y)$. This suffices since a set is discrete iff all its subsets are open, and the countable union of open sets is open. The orbit $O(x,y)$ inherits the subspace topology from $\mathbb{R}^2 \setminus 0$. Thus, we need only find an open set U in $\mathbb{R}^2 \setminus 0$ such that $\{(2^n x, 2^{-n} y)\} = U \cap O(x,y)$.

   To do this, first note that all points in $O(x,y)$ lie on the curve $\gamma(t) = (2^t x, 2^{-t} y)$ for $t \in (-\infty, \infty)$. Define Γ to be the image of γ. Then there exists an inverse map $\gamma^{-1} : \Gamma \to \mathbb{R}$ defined by
$$\gamma^{-1}(2^t x, 2^{-t} y) = t.$$
   Now note that Γ is the image of a monotone curve, so the intersection of any vertical slice $\{a < x < b\}$ in $\mathbb{R}^2$ with Γ is path-connected. A basis for the open sets of Γ consists of $U(a,b) = \{(2^t x, 2^{-t} y) \mid a < t < b\} = \Gamma \cap \{2^a x < t < 2^b x\}$ for $a < b \in \mathbb{R}$. Because

$\gamma(a,b) = U(a,b)$, we see that γ is an open map and hence γ^{-1} is continuous. Finally, γ^{-1} is bijective so that it is a homeomorphism.

By construction, $\gamma^{-1}(O(x,y)) = \mathbb{Z}$, with $\gamma^{-1}(x,y) = 0$. Take an interval $(-c,c)$ such that $0 < c < 1$, so that $(-c,c) \cap \mathbb{Z} = \{0\}$. Then, $\gamma(-c,c) = U(-c,c)$ is an open neighborhood disjoint from $O(x,y)$ except at (x,y). Note that each $U(a,b)$ is open in $\mathbb{R}^2 \setminus \{0\}$ since it is the intersection of Γ (open in $\mathbb{R}^2 \setminus \{0\}$ because it is homeomorphic to $(-\infty,\infty)$) and an open vertical strip.

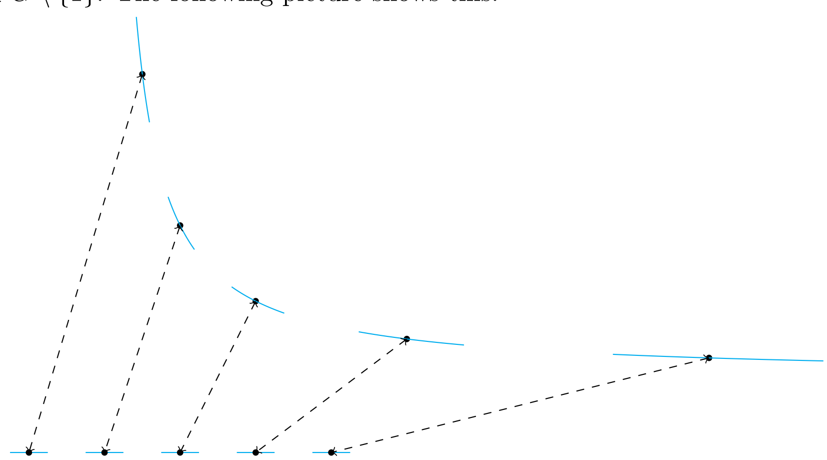
b) Consider the neighborhood $U(-c,c)$ of (x,y) as given previously. Fix a $g \in G$, and look at $g(U(-c,c))$. Since g is a homeomorphism $(x,y) \mapsto (2^n x, 2^{-n} y)$ for some $n \in \mathbb{Z}$, we see that

$$\begin{aligned} g(U(-c,c)) &= \{(2^n 2^t x, 2^{-n} 2^{-t} y) \mid -c < t < c\} \\ &= \{(2^t x, 2^{-t} y) \mid n-c < t < n+c\} = U(n-c, n+c). \end{aligned}$$

Hence, $\gamma^{-1}(g(U(-c,c))) = (n-c, n+c) = (-c,c) + n$. It follows that

$$\bigcup_{g \in G} \gamma^{-1}(g(U(-c,c))) = \bigcup_{n \in \mathbb{Z}} (-c,c) + n,$$

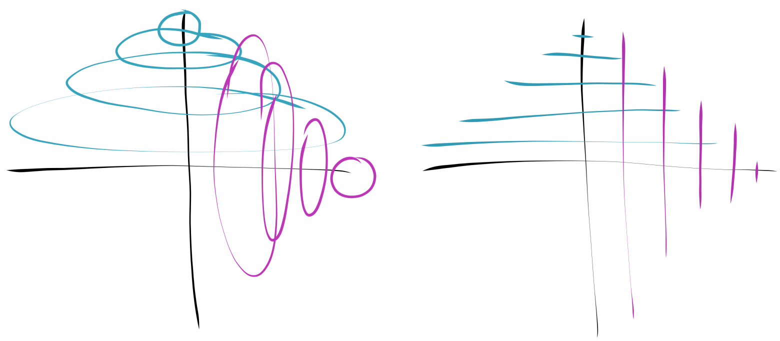
a union of translates of the interval $(-c,c)$. Thus, we obtain a disjoint union of intervals provided $0 < c \leq 1/2$ (otherwise, for example, $(-2/3, 2/3)$ intersects $(-2/3+1, 2/3+1) = (1/3, 5/3)$). Since γ is a homeomorphism, it follows that $U(-1/2,1/2) \cap g(U(-1/2,1/2)) = \emptyset$ for all $g \in G \setminus \{1\}$. The following picture shows this.



One can also use a suitably small square centered at (x,y), but I find the above example to be nicer.

c) Let $x \in X/G$, which is the orbit of some point $\tilde{x}$ in X. Let $q : X \to X/G$ be the quotient map. Observe first that the only fixed point of X under any of the homeomorphisms would be the origin, but this is excluded. Hence G acts freely on X. Let $x \in X$. By b), we can find an open set U of x such that $g(U) \cap U \neq \emptyset$ only for $g = \{1\}$. Each of these is mapped homeomorphically into one another (since g is always a homeomorphism). Now, $q^{-1}(q(U))$ is precisely $\cup_g g(U)$. Hence, $q(U)$ is open, and contains x. Since all the $g(U)$ are disjoint, it follows that q is a homeomorphism from $q(U)$ to any of the $g(U)$, in particular a homeomorphism from $q(U)$ to U. We may take U small enough so that it does not contain the origin. Then, U is homeomorphic to $\mathbb{R}^2$, and hence X/G is locally homeomorphic to $\mathbb{R}^2$.

To show that X/G is not Hausdorff, we can consider the orbits of $(1,0)$ and $(0,1)$. The former is a sequence of points converging to the origin along the horizontal axis, the latter along the vertical. Suppose that there exist disjoint U and V in X/G containing these orbits respectively. We can consider $q^{-1}(U)$ and $q^{-1}(V)$. Each of these produces a series of open sets around each point in the orbits; take U,V small enough so that each of these is disjoint. Let $r_x, r_y > 0$ and small enough so that $B_{r_x}(1,0) \subset q^{-1}(U)$ and $B_{r_y}(0,1) \subset q^{-1}(V)$; we can do this since each $q^{-1}(U), q^{-1}(V)$ is open. Now, we have that the images of each ball under each $g \in G$ are contained in $q^{-1}(U), q^{-1}(V)$ respectively. What are these images? Let g be a homeomorphism $(x, y) \mapsto (2^k x, 2^{-k} y)$ for some integer k. Then $g(B_{r_x}(1,0))$ is an ellipse centered at $(2^k, 0)$ containing the line segment $\{2^k\} \times [-2^{-k} r_x, 2^{-k} r_x]$. To compute the endpoints of the line segment, look at the orbit of $(r_x, 0)$ and $(-r_x, 0)$. Similarly, $g(B_{r_y}(0,1))$ is an ellipse centered at $(0, 2^k)$ containing the line segment $[-2^{-k} r_y, 2^{-k} r_y] \times \{2^k\}$. The picture below shows this process:



We now take k towards $-\infty$, so that $(2^k, 0) \to (0,0)$ and $(0, 2^k) \to (0,0)$. Do $g(B_{r_x}(1,0))$ and $g(B_{r_y}(0,1))$ ever intersect? For this to happen, we would need $2^{-k} r_y \geq 2^k$ and $2^{-k} r_x \geq 2^k$. One can easily find a k such that $2^{-k} r = 2^k$, where $r > 0$. Namely, take $k = \text{floor}(-\ln(r)/\ln(4))$. Since each r_x and r_y may produce a different k, we take the minimum of them. Thus, the neighborhoods $B_{r_x}(1,0)$ and $B_{r_y}(0,1)$ eventually intersect in the orbit. This is a contradiction, since any intersection point should be mapped into both U and V by the quotient map.

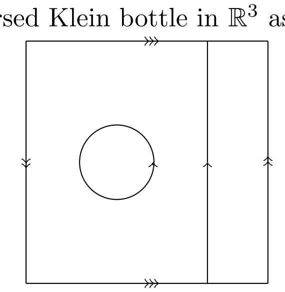
Hatcher Chapter 0, problems 9, 20:

0.0.9. Show that a retract of a contractible space is contractible.

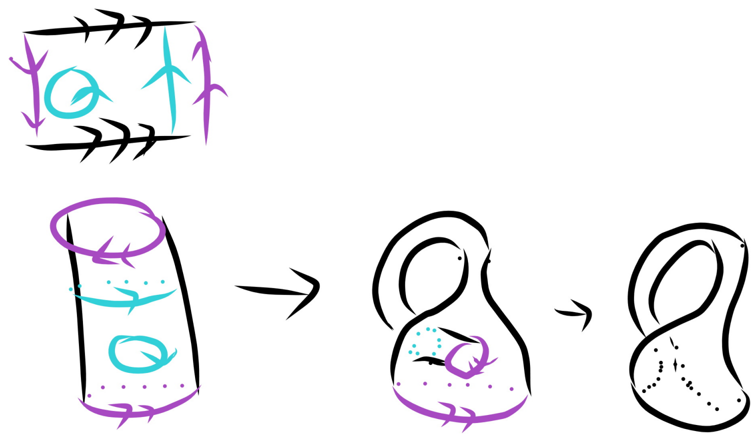
Solution: Suppose X is contractible and A is a retract of X, meaning there exists a retraction $r : X \to A$. We wish to show for some $a_0 \in A$ that $\text{id}_A \simeq f$, where $f : A \to A$ given by $f(a) = a_0$ is the constant map. Since X is contractible, we have that $\text{id}_X \simeq g$, where $g : X \to X$ given by $g(x) = x_0$ is a constant map in X. Thus, there exists a homotopy $H : X \times I \to X$ such that $H(x, 0) = \text{id}_X(x)$ and $H(x, 1) = x_0$ for all $x \in X$. Define $\tilde{H} : A \times I \to A$ by $\tilde{H}(a, t) = r(H(a, t))$. Then $\tilde{H}(a, 0) = r(H(a, 0)) = r(\text{id}_A(a)) = \text{id}_A(a)$ since $\text{id}_A(a) \in A$ for all $a \in A$, and $r|_A = \text{id}_A$. Moreover, $\tilde{H}(a, 1) = r(H(a, 1)) = r(x_0) \in A$ since $r : X \to A$. Since $r(x_0)$ is a constant, we have that id_A is homotopic to a constant map, namely $f = r(x_0)$.

0.0.20. Show that the subspace $X \subset \mathbb{R}^3$ formed by a Klein bottle intersecting itself in a circle, as shown in the figure, is homotopy equivalent to $S^1 \vee S^1 \vee S^2$.

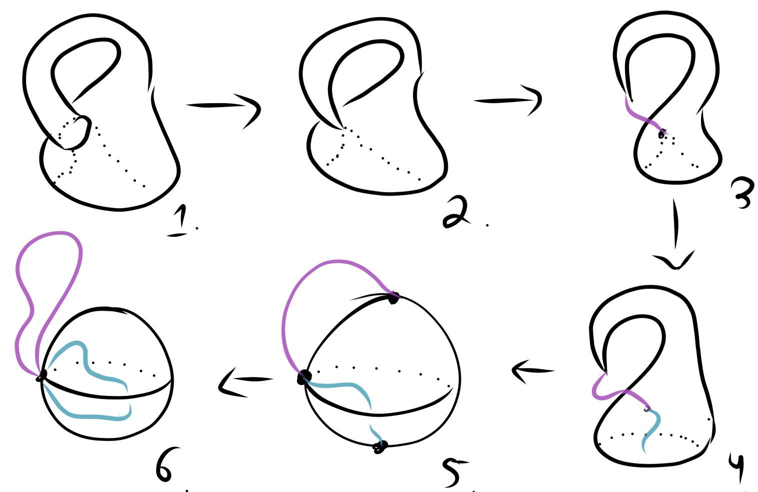
Solution: We can describe the immersed Klein bottle in $\mathbb{R}^3$ as the following CW complex:



In the above, each region is filled in with a 2-cell. The picture below shows how this CW-complex gives the Klein bottle.



In the CW complex structure above, we have a contractible subcomplex – namely the disc and the line segment identified with its boundary. As discussed in Hatcher, we then have that the quotient map $X \to X/A$ is a homotopy equivalence. We can now perform the following sequence of homotopies:



First note that 4 and 5 are homeomorphic, so they have the same homotopy type.

To see that 2, 3, and 4 are homotopy equivalent, it helps to remember example 0.0.8 from Hatcher. There, we take S^2 and attach an arc A to the north and south poles. We let B be an arc on S^2 from the north to south pole. Then X/A (the pinched sphere) and X/B ($S^2 \vee S^1$) are homotopy equivalent, since we collapse contractible subcomplexes. Here, we have something completely analogous but with two pinch points instead of one. Take S^2 and consider an arc A_1 outside of S^2 connecting the north and west poles, and A_2 an arc inside of S^2 connecting the west and south poles. Call this space Y – observe that Y is homeomorphic to spaces 4 and 5.

On one hand, we can take arcs B_1 and B_2 on S^2 connecting the north and west, and south and west poles respectively. Then, $Y/(B_1 \cup B_2)$ is homeomorphic to 6, and is obtained by collapsing a contractible subcomplex. Hence 4,5,6 have the same homotopy type.

On the other hand, we can take Y/A_2 (collapsing along the blue line segment), which is homeomorphic to 3 – call this $\tilde{Y}$. Then we can take Y/A_1 (collapse along the remaining purple line segment), which is homeomorphic to 2. Each of these also collapses a contractible subcomplex, so 2,3,4 have the same homotopy type.

Hatcher Chapter 1.1, problems 17, 18, 20:

1.1.17. Construct infinitely many nonhomotopic retractions $S^1 \vee S^1 \to S^1$.

Solution: Let $S^1 \vee S^1$ be the wedge of two unit circles centered at $(-1,0)$ and $(1,0)$ in the complex plane – call these A and B respectively. Define $r_n : S^1 \vee S^1 \to S^1$ by

$$r^{(n)}(z) = \begin{cases} -\overline{((z+1)^n - 1)} & |z+1| = 1 \\ z & |z-1| = 1 \end{cases}$$

The set of complex numbers z satisfying $|z+1| = 1$ is evidently A, and satisfying $|z-1| = 1$ is B. The map $z \mapsto (z+1)^n - 1$ on B has the action of taking z to the unit circle, rotating it by its polar angle $n-1$ times, then translating it back to B. Afterwards, we reflect this over the vertical axis onto A. Informally, we have the following chain of compositions:

$$(-1+\cos(t), \sin t) \mapsto (-1+\cos(nt), \sin(nt)) \mapsto (1-\cos(nt), \sin(nt))$$

where we have parameterized B by $(-1+\cos(t), \sin t)$ for $t \in [0, 2\pi)$. In total, what we do is the following: As we trace along B counterclockwise from 0, we wrap around A clockwise n times, starting at 0.

We clearly have that $r^{(n)}|_A$ is the identity. The intersection of A and B is the origin $z = 0$, which gets mapped to 0 under $-\overline{((z+1)^n - 1)}$ and z. Thus, by the pasting lemma, $r^{(n)}$ is continuous. It follows that $r^{(n)}$ is a retraction of $S^1 \vee S^1 \to S^1$ for each n.

Suppose $f, g : X \to Y$ are homotopic. Then there exists a continuous $H : X \times I \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all x. Now consider some $A \subset X$. Define $\tilde{f} := f|_A$ and similarly for g. Are $\tilde{f}$ and $\tilde{g}$ homotopic? Define $\tilde{H} : A \times I \to Y$ by $\tilde{H}(a,t) = H(a,t)$ for all $a \in A$ (that is, restrict the first component of H to A). Then, $\tilde{H}(a,0) = H(a,0) = f(a) = \tilde{f}(a)$, and similarly $\tilde{H}(a,1) = \tilde{g}(a)$ for all $a \in A$. It follows that $\tilde{H}$ is a homotopy from $\tilde{f}$ to $\tilde{g}$.

Suppose now that $n \neq m$ and $r^{(n)}$ and $r^{(m)}$ are homotopic. Then there exists some homotopy $H : S^1 \vee S^1 \times I \to S^1$ such that $H(x,0) = r^{(n)}(x)$ and $H(x,1) = r^{(m)}(x)$. In light of our construction, we view $H : A \vee B \times I \to A$. Now restrict H to B so that we get a homotopy $\tilde{H} : B \times I \to A$ such that $\tilde{H}_0 = r^{(n)}|_B$ and $\tilde{H}_1 = r^{(m)}|_B$. It follows that $r^{(n)}|_B$ and $r^{(m)}|_B$ are homotopic. But, each of these is a loop in A based at 0 which wraps around A clockwise n, m times respectively. Hence, if $n \neq m$, we have $[r^{(n)}|_B] \neq [r^{(m)}|_B]$, a contradiction.

1.1.18. Using Lemma 1.15, show that if a space X is obtained from a path-connected subspace A by attaching a cell e^n with $n \geq 2$, then the inclusion $A \hookrightarrow X$ induces a surjection on π_1. Apply this to show:

a) The wedge sum $S^1 \vee S^2$ has fundamental group $\mathbb{Z}$.
b) For a path-connected CW complex X the inclusion map $X^1 \hookrightarrow X$ of its 1 skeleton induces a surjection $\pi_1(X^1) \to \pi_1(X)$. [For the case that X has infinitely many cells, see Proposition A.1 in the Appendix.]

Solution: I assume that X is T_1 so that singletons are closed. Let $\Phi : e^n \to X$ be the characteristic map of e^n. Let $x_1, x_2 \in e^n \setminus \partial e^n$ be distinct. Let $A_1 = X \setminus \{x_1\}$ and $A_2 = X \setminus \{x_2\}$. Since A is assumed path-connected, $e^n \setminus \{x_1, x_2\}$ is path-connected, and Φ is continuous, we see that $A_1 \cap A_2 = X \setminus \{x_1, x_2\}$ is path-connected. Each of A_1 and A_2 is open, so the lemma applies. Let $a_0 \in \Phi(\partial e^n)$, so that a_0 is in the part of A which e^n attaches to. Since each x_1, x_2 is not in ∂e_n, it follows that $a_0 \in A_1, A_2$. Now let f be a loop in X based at a_0. Using the lemma, f is homotopic to a product of loops $\{\alpha_i\}_{i=1}^n$ in A_1 based at a_0 and a product of loops $\{\beta_j\}_{j=1}^m$ in A_2 based at a_0. For each α_i, we can look at the portion of α_i inside $\Phi(e_n)$ and push it to $\Phi(\partial e_n) \subset A$ (to do this, one might pull back this part of α_i under Φ and look at the homotopy there; the point is that $e_n \setminus \{x_1\}$ deformation retracts onto its boundary). By doing this for each α_i, and accordingly for each β_j by the same process, f is homotopic to a path entirely lying in A. Hence, the induced homomorphism of the inclusion $A \hookrightarrow X$ is surjective.

a) We can form $S^1 \vee S^2$ by taking S^1 and attaching a 2-cell e^2 via an attaching map that sends ∂e^2 to a single point a_0 in S^1. Since S^1 is path connected, we can apply the preceeding theorem and show that i_*, the homomorphism induced by the inclusion of S^1 into $S^1 \vee S^2$, is a surjection of $\pi_1(A, a_0)$ to $\pi_1(X, a_0)$. By the first isomorphism theorem, it follows that $\pi_1(X, a_0) \simeq \pi_1(A, a_0)/\ker(i_*)$. Clearly $S^1 \vee S^2$ retracts onto S^2 by taking the retraction to be the identity on S^1 and crushing S^2 to a_0. Then i_* is injective, so the kernel is trivial and $\pi_1(X, a_0) \simeq \pi_1(A, a_0)$.

b) The CW complex X is formed from X^1 by attaching n-cells, $n \geq 2$. As done in the prehomework, if X is path-connected then X^1 is too. Suppose that X is finite, so that $X = X^k$ for some k. Note that we obtain each skeleton X^i from X^{i-1} by attaching i-cells with $i \geq 2$. Moreover, each X^{i-1} is path-connected for $i \geq 1$. Hence, we can apply the preceeding theorem. Let $i_{j*} : \pi_1(X^j) \to \pi_1(X^{j+1})$ be the homomorphism induced by the inclusion $i : X^j \to X^{j+1}$. By the theorem, each of these is surjective. Thus,

$$i_* = (i_{k-1} \circ i_{k-2} \circ ... \circ i_2 \circ i_1)_* = i_{(k-1)*} \circ i_{(k-2)*} \circ ... \circ i_{2*} \circ i_{1*}$$

is surjective, where $i : X^1 \hookrightarrow X$.

In the case that X has infinitely many cells, we first let $[f] \in \pi_1(X)$ and take some representative element of this equivalence class, namely $f : I \to X$. But f is continuous on a compact set, so $f(I)$ is compact. By Prop A.1 in Hatcher, each compact subspace of a CW complex is contained in a finite subcomplex. Let A denote a finite path-connected subcomplex containing $f(I)$ (a path-connected one is possibly obtained by removing unnecessary cells). Denote the natural inclusion $A^1 \hookrightarrow A$ by j. Now, using the finite case, we know that j_* is surjective. Moreover, f lies entirely in A, so we can view f as a loop in A. By surjectivity of j_*, there exists some loop $\tilde{f}$ in A^1 such that $j_*[\tilde{f}] = [f]$. We then have that $\tilde{f}$ is a loop in X^1, so that $[\tilde{f}] \in \pi_1(X^1)$. We show now that $i_*[\tilde{f}] = f$, where $i : X^1 \hookrightarrow X$. Since $A^1 \subset X^1$ and $\tilde{f}$ lies entirely in A^1, we have $i \circ \tilde{f} = j \circ \tilde{f}$. Then by definition of the induced homomorphism,

$$i_*[\tilde{f}] = [i \circ \tilde{f}] = [j \circ \tilde{f}] = j_*[\tilde{f}] = [f].$$

Hence i_* is surjective.

1.1.20. Suppose $f_t : X \to X$ is a homotopy such that f_0 and f_1 are each the identity map. Use Lemma 1.19 to show that for any $x_0 \in X$, the loop $f_t(x_0)$ represents an element of the center of $\pi_1(X, x_0)$. [One can interpret the result as saying that a loop represents an element of the center of $\pi_1(X)$ if it extends to a loop of maps $X \to X$.]

Solution: The center of a group G is defined as $Z(G) := \{z \in G \mid zg = gz \text{ for all } g \in G\}$. Let $z(t) = f_t(x_0)$. By Lemma 1.19 we have that

$$f_{0*} = \beta_z f_{1*}.$$

Let $[g] \in \pi_1(X, x_0)$. Then, since f_0 and f_1 are the identity maps,

$$[g] = f_{0*}[g] = \beta_z f_{1*}[g] = \beta_z[g] = [z] * [g] * [z]^{-1},$$

which implies for any $[g] \in \pi_1(X, x_0)$ that $[g] * [z] = [z] * [g]$. Thus $[z] = [f_t(x_0)]$ is in $Z(\pi_1(X, x_0))$.

Hatcher Chapter 1.2, problems 2, 4:

1.2.2. Let $X \subset \mathbb{R}^m$ be the union of convex open sets $X_1, ..., X_n$ such that $X_i \cap X_j \cap X_k \neq \emptyset$ for all i, j, k. Show that X is simply-connected.

Solution: For X to be simply-connected, it must be path-connected and have trivial fundamental group. Since each X_i is convex, they are path-connected. Now X is the union of (at least) pairwise non-disjoint path-connected sets, and hence is path-connected.

We now show that X has trivial fundamental group. Since each X_i is convex, it has trivial fundamental group. Moreover, the intersections $X_i \cap X_j \cap X_k$ are not disjoint, and so is also path connected. Finally, denote by $i_{jk} : \pi_1(X_j \cap X_k) \to \pi_1(X_j)$ the induced homomorphism by the inclusion $X_j \cap X_k \hookrightarrow X_j$. Then by van Kampen we have $\pi_1(X) \simeq *_i \pi_1(X_i)/N$, where N is the normal subgroup generated by $i_{jk}(w)i_{kj}(w)^{-1}$ where $w \in \pi_1(X_j \cap X_k)$. But there is only one such element for each pair j, k, namely the homotopy class of the constant map. Since the identity gets mapped to the identity under a homomorphism, $i_{jk}(w)i_{kj}(w)^{-1}$ is simply the product of the identities of $\pi_1(X_j)$ and $\pi_i(X_k)$. It follows that N is trivial, and hence $\pi_1(X) \simeq *_i \pi_1(X_i) \simeq \{1\}$. *Addendum: The above is not quite correct since each the sets X_i may not all share a common basepoint. Instead, we should apply VK to each triplet X_i, X_j, X_k and conclude that $X_i \cup X_j \cup X_k$ has trivial fundamental group. Then, repeatedly apply VK with the triplets until we reach X.*

1.2.4. Let $X \subset \mathbb{R}^3$ be the union of n lines through the origin. Compute $\pi_1(\mathbb{R}^3 \setminus X)$.

Solution: Define $f : \mathbb{R}^3 \setminus X \to S^2 \setminus \{x_1, x_2, ..., x_{2n}\}$ by $f(x) = x/|x|$. This is the standard deformation retraction of $\mathbb{R}^3 \setminus \{0\}$ onto S^2. Since we have removed n lines, each of which passes through two antipodal points of S^2, f deformation retracts $\mathbb{R}^3 \setminus X$ to S^2 without $2n$ points. WLOG we may choose one of these to be the north pole, and look at the stereographic projection onto $\mathbb{R}^2$. It follows that $S^2 \setminus \{x_1, ..., x_{2n}\}$ is homeomorphic to $\mathbb{R}^2 \setminus \{y_1, ..., y_{2n-1}\}$ (all other points get projected, but the north pole "goes to infinity"). We can WLOG assume that $y_k = (2k, 0)$ so that $\mathbb{R}^2 \setminus \{y_1, ..., y_{2n-1}\}$ deformation retracts onto $2n-1$ copies of S^1, each intersecting pairwise at a point (except the first and last copies, which do not intersect). Hence, it deformation retracts onto $S^1 \vee ... \vee S^1$ (with $2n-1$ copies of S^1). Thus,

$$\pi_1(\mathbb{R}^3 \setminus X) \simeq \pi_1(S^1 \vee ... \vee S^1) \simeq \mathbb{Z} * ... * \mathbb{Z},$$

where there are $2n-1$ copies of $\mathbb{Z}$.

HW3

Hatcher Chapter 1.2, problems 1, 10, 14, 16, 21:

1.2.1. Show that the free product $G * H$ of nontrivial groups G and H has trivial center, and that the only elements of $G * H$ of finite order are the conjugates of finite-order elements of G and H .

Solution: Suppose that $w \in Z(G * H)$ is some reduced word of length n, say $a_1 a_2 ... a_n$. We may assume that $a_1 \in G$. Let $h \in H$ be nontrivial. Then hw is some reduced word of length $n+1$. What about wh? If a_n is also in G, then wh is also a reduced word of length $n+1$. But, the first letter is in G, whereas the first letter of hw is in H, so the two cannot be the same. Hence $w \notin Z(G * H)$. If instead $a_n \in H$, then $wh = a_1 a_2 ... a_{n-1} \tilde{a}_n$, where $\tilde{a}_n = a_n h$. So wh has at most n letters, and cannot be hw which has $n+1$ letters. We can repeat the same argument if $a_1 \in H$ with some nontrivial $g \in G$. It follows that the only word in $Z(G * H)$ is the empty word.

Now let $w = a_1 a_2 ... a_n$ be a reduced word of order m, so that

$$(a_1 a_2 ... a_n)(a_1 a_2 ... a_n)...(a_1 a_2 ... a_n) = 1,$$

where we have m copies of w on the LHS. WLOG suppose that $a_1 \in G$. We look at $a_n a_1$. Since the product reduces, it must be that $a_n \in G$. If $a_n a_1 \neq 1$, then we could not continue reducing the word, and it follows that $w^m \neq 1$. So, we must have that $a_n a_1 = 1$. If w has even length, then it must be the empty word (since, if w starts with $a_1 \in G$, it must end with $a_n \in H$, but by the above $a_n = a_1^{-1} \in G$). So, suppose w has odd length. Set $k = (n-1)/2$. We can continue reducing, and see that

$$w = a_1 a_2 ... a_{n-1} a_n = a_1 a_2 ... a_k a a_k^{-1} ... a_2^{-1} a_1^{-1} = \tilde{w} a \tilde{w}^{-1},$$

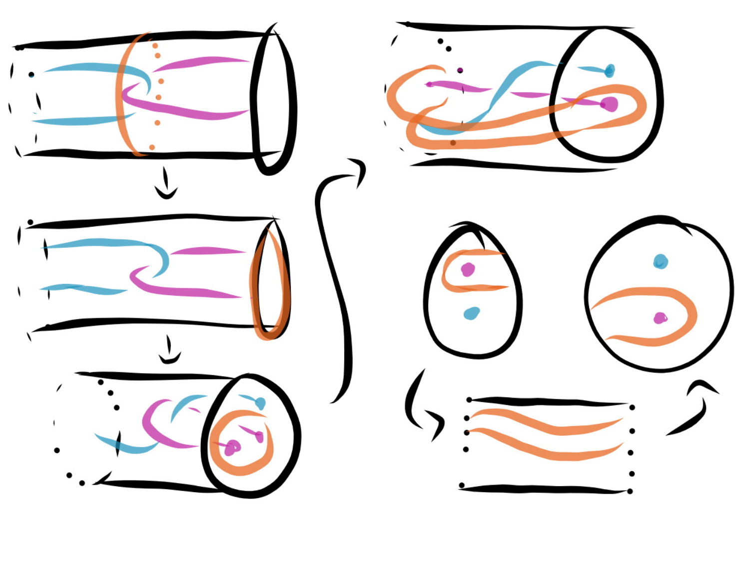
where $\tilde{w} = a_1 a_2 ... a_k$. Now, a (the middle letter) can be in either G or H; suppose it is in G. Then,

$$1 = w^m = (\tilde{w} a \tilde{w}^{-1})(\tilde{w} a \tilde{w}^{-1})...(\tilde{w} a \tilde{w}^{-1}) = \tilde{w} a^m \tilde{w}^{-1}.$$

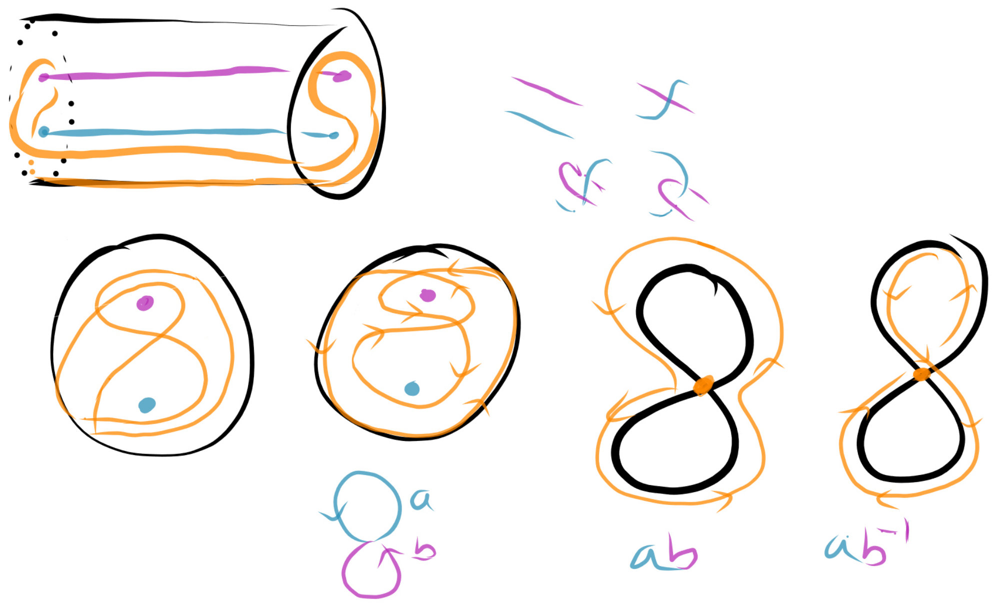
It follows that $a^m = 1$ so that a has finite order in G. The same conclusion is reached if we assume $a \in H$.

1.2.10. Consider two arcs α and β embedded in $D^2 \times I$ as shown in the figure. The loop γ is obviously nullhomotopic in $D^2 \times I$, but show that there is no nullhomotopy of γ in the complement of $\alpha \cup \beta$.

Solution: We move around the endpoints of α and β so that they become two parallel lines. We carry around γ throughout. This process is shown below



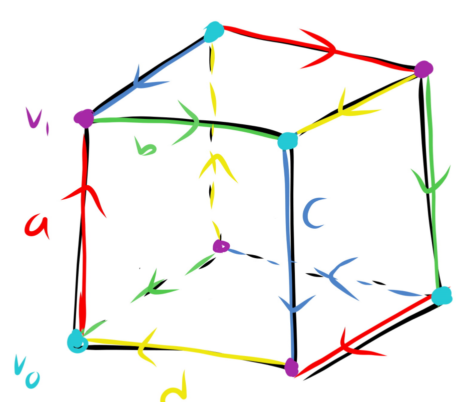
The last pictures show the final product from various angles. We are not quite done yet since the arcs α and β are not parallel. Once we do this, we can collapse the cylinder (where α and β become points) and look at what happens to γ. Doing this gives



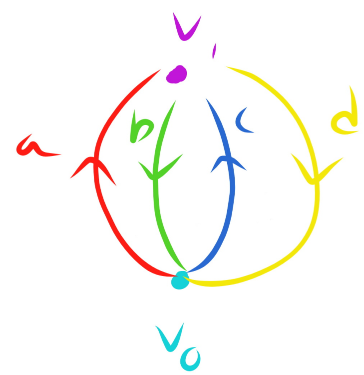
Now since α, β were removed, we have D^2 without two points. We can deformation retract this to the wedge of two circles. Carrying γ along gives the concatenation of the two loops at the bottom right. Clearly $abab^{-1}$ is not nullhomotopic. Even though we changed the basepoint throughout, this can be neglected since all nullhomotopic loops are in the same conjugacy class, and this is independent of basepoint (from HW1). *Addendum: I got full credit here, but I think I drew my second set of pictures incorrectly. It might be $ab^{-1}a^{-1}b$ instead. Either way, the resulting loop is not nullhomotopic.*

1.2.14. Consider the quotient space of a cube I^3 obtained by identifying each square face with the opposite square face via the right-handed screw motion consisting of a translation by one unit in the direction perpendicular to the face combined with a one-quarter twist of the face about its center point. Show this quotient space X is a cell complex with two 0 cells, four 1 cells, three 2 cells, and one 3 cell. Using this structure, show that $\pi_1(X)$ is the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$, of order eight.

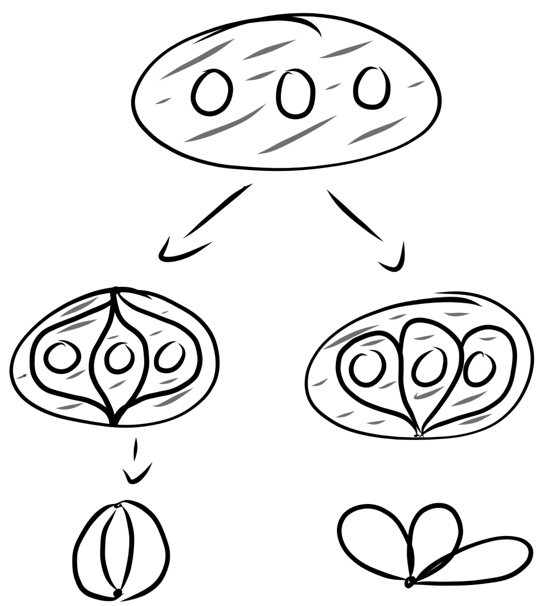
Solution: We get the following picture after identifying the left hand, forward, and bottom with their opposite faces by a right-handed screw motion.



The one skeleton is given as follows, and can be seen by looking at one face.



We compute $\pi_1(X^1)$. First, we note that X^1 and $S^1 \vee S^1 \vee S^1$ are deformation retractions of the same space. This is visualized below.



It follows that $\pi_1(X^1) \simeq \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$. The generators are given by $i = [ab]$, $j = [ac^{-1}]$, and $k = [ad]$.

We now attach the 2-cells. Each loop around a face in the quotient space can be regarded as a loop in X^1 and a loop in the attached 2-cell. By Van Kampen's theorem, their homotopy classes must be set equal to each other. For example, the 2-cell corresponding to the front face has boundary loop $d^{-1}c^{-1}b^{-1}a^{-1}$. The homotopy class for this is $[d^{-1}c^{-1}b^{-1}a^{-1}]$, which must be equal to the homotopy class of the loop corresponding to 1 in $\pi_1(S^1) \simeq \mathbb{Z}$. So, by some algebraic manipulation,

$$1 = [d^{-1}c^{-1}b^{-1}a^{-1}] = [d^{-1}a^{-1}ac^{-1}b^{-1}a^{-1}] = [(ad)^{-1}ac^{-1}(ab)^{-1}] = k^{-1}ji^{-1}.$$

Hence, $j = ki$. Similarly, the boundary loops for the other two faces are given by $b^{-1}dca^{-1}$ (left face) and $d^{-1}a^{-1}cb$ (bottom face). We can also simplify these as

$$\begin{aligned} 1 &= [b^{-1}dca^{-1}] = [b^{-1}a^{-1}adca^{-1}] = [(ab)^{-1}ad(ac^{-1})^{-1}] = i^{-1}kj^{-1} \\ 1 &= [d^{-1}a^{-1}cb] = [d^{-1}a^{-1}ca^{-1}ab] = [(ad)^{-1}(ac^{-1})^{-1}ab] = k^{-1}j^{-1}i. \end{aligned}$$

So $k = ij$ and $i = jk$. It follows that

$$\pi_1(X^2) = \frac{\pi_1(X)}{\langle k^{-1}ji^{-1}, i^{-1}kj^{-1}, k^{-1}j^{-1}i\rangle} = \langle i, j, k \mid i = jk, j = ki, k = ij\rangle.$$

Then,

$$i^2 = i(jk) = ijk \quad , \quad k^2 = (ij)k = ijk$$
$$j = ki = k(jk) = kjk \quad , \quad j^2 = jkjk = ijk$$

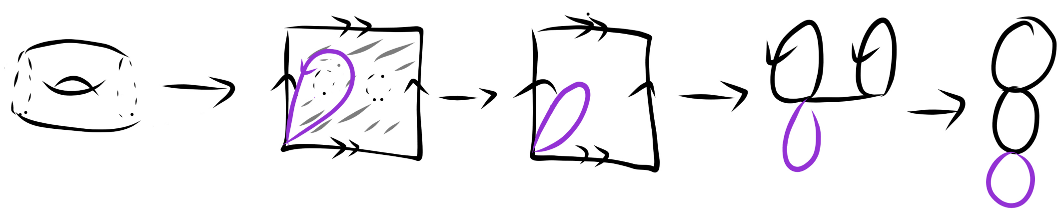
It follows that $i^2 = j^2 = k^2 = ijk$, and

$$\pi_1(X^2) = \langle i, j, k \mid i^2 = j^2 = k^2 = ijk\rangle,$$

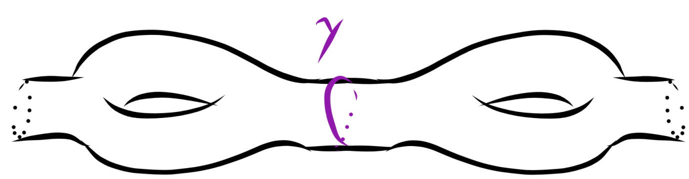
which is the representation of the quaternions. Since adding 3-cells in general does not change π_1, it follows that $\pi_1(X) = \pi_1(X^2)$.

1.2.16. Show that the fundamental group of the surface of infinite genus shown below is free on an infinite number of generators

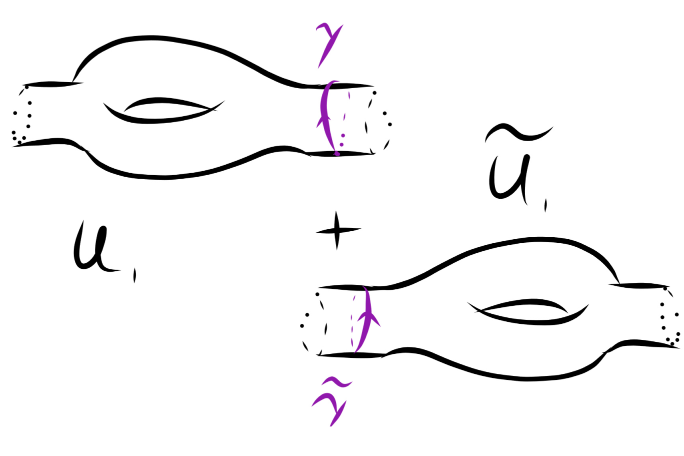
Solution: Consider the torus with two discs removed, denoted U_1. The fundamental group can be computed via the following deformation retraction.



So $\pi_1(U_1) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$. Let α, β, γ be generators, and consider another copy of U_1 denoted $\tilde{U}_1$ with generators $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$. Let U_2 be the following double torus



We have two embeddings of U_1 and $\tilde{U}_1$ into U_2, seen below



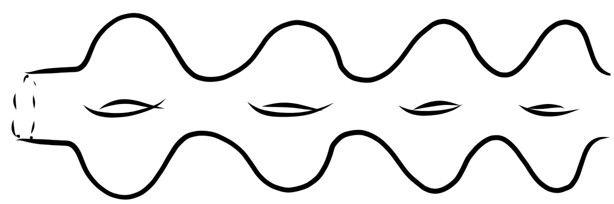
Here, we have depicted two generators γ and $\tilde{\gamma}$. In U_2, these correspond to the same loop. The intersection is a cylinder, which is path connected, and each $U_1, \tilde{U_1}$ is path connected. So, by Van Kampen's theorem:

$$\pi_1(U_2) \simeq \langle \alpha, \beta, \gamma, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} \mid \gamma = \tilde{\gamma} \rangle = \langle \alpha, \beta, \gamma, \tilde{\alpha}, \tilde{\beta} \rangle.$$

Thus $\pi_1(U_2) \simeq \mathbb{Z} * ... * \mathbb{Z}$, where we have five copies of $\mathbb{Z}$. In general,

$$\pi_1(U_g) \simeq \mathbb{Z} * ... * \mathbb{Z},$$

where there are $2g+1$ copies of $\mathbb{Z}$, and U_g is a surface with genus g and two discs removed. Continuing in this process, we get the following set U:



which is an infinite genus surface with one disc removed and extending in the opposite direction. We wish to compute $\pi_1(U)$. In some sense, it should be the limit of $\pi_1(U_g)$ as $g \to \infty$. Note that there are obvious inclusions $i_g : U_g \hookrightarrow U$. It follows that

$$\bigcup_{g=1}^{\infty} U_g = U.$$

By construction, $U_g \subset U_{g+1}$ so that $U_i \cap U_j \cap U_k = U_g$, where $g = \min\{i, j, k\}$. Since all U_g are path-connected, it follows that $U_i \cap U_j \cap U_k$ is always path-connected. So, we may apply Van Kampen's theorem.

Consider the free group on infinite generators, and label these generators as $\gamma, \alpha_1, \beta_1, \alpha_2, \beta_2, ...$ We can view γ as a loop in U going around the cut out disc, and α_1, β_1 are the other two generators coming from U_1. For each torus we add on, we add two more generators α_i, β_i. Base all of these loops at some $x_0 \in U_1 \subset U$. We show that any loop in $\pi_1(U, x_0)$ can be written as a product of these generators, and that any nullhomotopic loop in $\pi_1(U, x_0)$ is such that each letter is nullhomotopic (the former shows surjectivity of the "limit" while the latter shows injectivity). Hence,

$$\pi_1(U, x_0) \simeq \mathbb{Z} * \mathbb{Z} * \mathbb{Z}...$$

where there are infinitely many copies of $\mathbb{Z}$.

So, let f be a loop in U based at x_0. Since $f : S^1 \to U$, it follows that the image of f in U is compact. Hence, $f \in U_g$ for some g. We know that any loop in $\pi_1(U_g, x_0)$ can be written as a product of the above generators. In particular, f can.

Now let f be a nullhomotopic loop in U based at x_0. Then there exists some null homotopy $H : S^1 \times I \to U$. Since S^1 and I are compact, the image of H is compact, and hence lies in some U_g. Since $\pi_1(U_g, x_0)$ is free, the conclusion follows.

Finally, we apply Van Kampen with two copies of U, exactly the same way that we did with U_1 and $\tilde{U_1}$. Label the generators of U as $u_1, u_2, u_3, ...$ where u_1 is a loop around the removed disc, and similarly for $\tilde{U}$. Then $X = U \cup \tilde{U}$ with $U \cap \tilde{U}$ a path-connected cylinder and u_1 identified with $\tilde{u}_1$. The representation is thus

$$\pi_1(X) = \langle u_1, \tilde{u}_1, u_2, \tilde{u}_2, ... \mid u_1 = \tilde{u}_1 \rangle,$$

which is the free group on infinitely many generators.

1.2.21. Show that the join $X * Y$ of two nonempty spaces X and Y is simply-connected if X is path-connected.

Solution: The join of two spaces $X * Y$ is defined as

$$X * Y = (X \times Y \times I)/\sim$$

where $\sim$ collapses $X \times Y \times \{0\}$ to X and $X \times Y \times \{1\}$ to Y. Intuitively speaking, we imagine the join of X, Y as connecting all of the points in X to Y by line segments.

First, let us divide Y into its path components Y_i. It follows that $X * Y = \cup_i X * Y_i$. Since X and Y_i are path-connected, it follows that $X \times Y_i \times I$ is path-connected. Since quotient spaces of path-connected spaces are path-connected, we see that $X * Y_i$ is path-connected.

Observe now that $X \times Y_i \times \{1/2\} \simeq X \times Y_i$, which is path-connected. Letting $U = (X \times Y_i \times [0, 3/4))/\sim$ and $V = (X \times Y_i \times (1/4, 1])/\sim$, we see that each of U, V is path-connected and open, and $U \cap V$ deformation retracts to $X \times Y_i \times \{1/2\} \simeq X \times Y_i$. So, we may apply Van Kampen's theorem.

Notice that U deformation retracts onto X (retracting $[0, 3/4)$ to 0, then applying the identification – we imagine pushing in all the line segments into X) while V deformation retracts onto Y_i (similar reasoning). Now notice that since U deformation retracts onto X, and V deformation retracts onto Y_i, then there exits isomorphisms $\pi_1(X) \simeq \pi_1(U)$ and similarly for Y_i and V. Because of this, if we take a loop $f \in U \cap V$, when we regard it as loops in U and V separately, we can actually regard it as a loop in X or Y_i. Van Kampen tells us that these need to be the same, so

$$\pi_1(X * Y_i) = \pi_1(U) * \pi_1(V)/N \simeq \pi_1(X) * \pi_1(Y_i)/N,$$

where N is the normal subgroup generated by ab^{-1}, where a is a loop in X and b is a loop in Y_i. Hence, $N = \pi_1(X) * \pi_1(Y_i)$. It follows that $\pi_1(X * Y_i) \simeq 1$.

Now let $Z_i = X * Y_i$. It follows that $Z_i \cap Z_j$ is X (since $Y_i \cap Y_j = \emptyset$), which is path connected. The same conclusion holds for triple intersections. So, by Van Kampen,

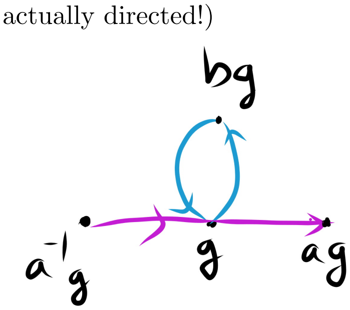
$$\pi_1(X * Y) = \pi_1(Z_1) * \pi_1(Z_2) * .../N,$$

but all the $\pi_1(Z_i)$ are trivial so that $\pi_1(X * Y)$ is trivial.

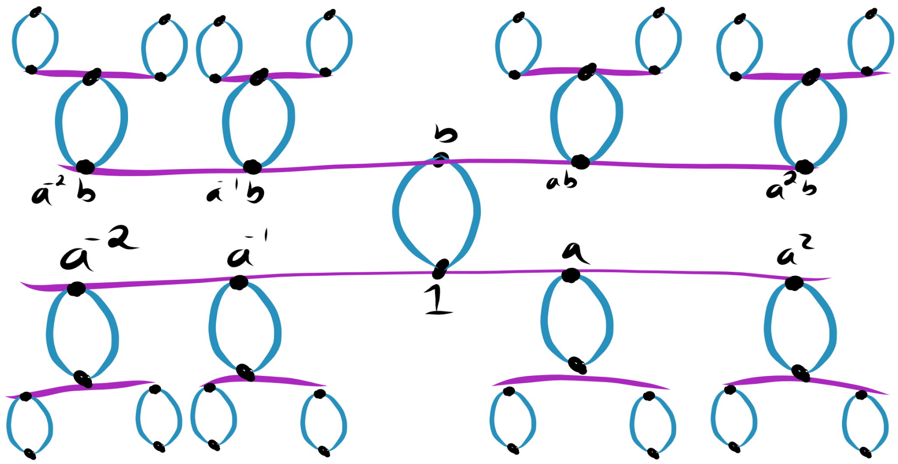
Hatcher Chapter 1.3, problem 30:

1.3.30. Draw the Cayley graph of the group $\mathbb{Z} * \mathbb{Z}_2 = \langle a, b \mid b^2 \rangle$.

Solution: To construct a Cayley graph X_G of a group G, we let the elements of G be vertices in the graph. To construct the edges, we fix a $g \in G$ and for each generator g_α of G, we draw an edge from g to $g_\alpha g$. Then, repeat this process for all g. Here, the generators are a, b. Since $b^2 = 1$, for a fixed $g \in G$ the degree of g in X_g is four. We have two edges connecting g to ag and bg respectively. Now, bg is connected to g since $bbg = b^2 g = g$. Furthermore, $a^{-1}g$ is connected to g since $aa^{-1}g = g$. No other edges are connected to g – for this to occur, we would need some $h \in G$ where $ahg = g$ or $bhg = g$. These imply that $ah = 1$, so $h = a^{-1}$, or $bh = 1$ so $h = b^{-1} = b$. We get the following (note that it is not actually directed!)



Now to continue adding edges, once at bg, we must multiply by either a or a^{-1}. Since, if we multiply by b, then we simply return to g. We then obtain the following Cayley graph. We do not add any 2-cells since we are not asked for a Cayley complex.



Hatcher Chapter 1.A, problem 5:

1.A.5. Construct a connected graph X and maps $f, g : X \to X$ such that $fg = \text{Id}_X$ but f and g do not induce isomorphisms on π_1. [Note that $f_* g_* = \text{Id}_{\pi_1(X)}$ implies that f_* is surjective and g_* is injective.]

Solution: Let X be an infinite boquet of circles. So, X is a CW complex with one 0-cell and countably many 1-cells where each endpoint is mapped to the 0-cell. We can enumerate the loops as $S_1, S_2, ...$ and let $\alpha_1, \alpha_2, ...$ be loops such that α_n traverses S_n exactly once. Hence, there is a bijection between S_n and S^1 given by α_n. Observe that $\alpha_n \alpha_n^{-1} = \text{id}_{S_n}$ and $\alpha_n^{-1} \alpha_n = \text{id}_{S^1}$ (here α_n^{-1} does not mean to traverse S_n in the opposite direction! It is the inverse map to α_n). We define $f, g : X \to X$ by their action on S_n for each n. For $x \in S_n$, define $g(x)$ by

$$g(x) = \alpha_{2n} \alpha_n^{-1}(x).$$

Intuitively, this takes the loop S_n exactly onto S_{2n}. If n is odd, we define $f|_{S_n} = \text{id}_{S_n}$. If n is even, for $x \in S_n$ we define $f(x)$ by

$$f(x) = \alpha_n \alpha_{2n}^{-1}(x).$$

Now, for any $x \in X$ we have that

$$fg(x) = \alpha_n \alpha_{2n}^{-1} \alpha_{2n} \alpha_n^{-1}(x) = \alpha_n \text{Id}_{S^1} \alpha_n^{-1}(x) = x,$$

since $g(X)$ maps entirely into $\cup_{n \in \mathbb{N}} S_{2n}$, so we only "activate" the even part of f. It follows that f_* is surjective and g_* is injective. We now show that f_* is not injective and g_* is not surjective. Evidently g_* is not surjective since S_1 is not contained in the image of g. So, g_* cannot map the homotopy class of any loop f to the homotopy class of α_1. Now, f_* cannot be injective since, for odd n we have $f(\alpha_{2n}(x)) = \alpha_n(x) = f(\alpha_n(x))$. It follows that

$$f_*[\alpha_{2n}] = [\alpha_n] = f_*[\alpha_n].$$

HW4

Hatcher Chapter 1.2, problems 8, 11, 13:

1.2.8. Compute the fundamental group of the space obtained from two tori $S^1 \times S^1$ by identifying a circle $S^1 \times \{x_0\}$ in one torus with the corresponding circle $S^1 \times \{x_0\}$ in the other torus.

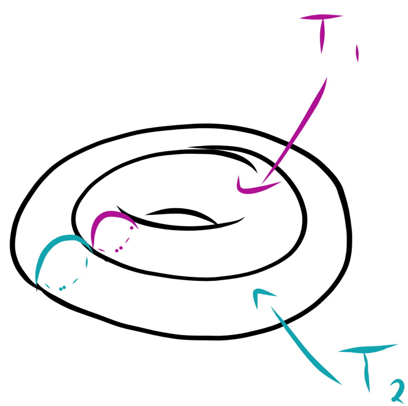
Solution: Represent the fundamental groups of each tori as follows

$$\pi_1(T_1) = \langle a, b \mid [a, b] = 1 \rangle \quad \text{and} \quad \pi_1(T_1) = \langle c, d \mid [c, d] = 1 \rangle$$

where we may identify the generators a, c with $S^1 \times \{x_0\}$. So, in applying VK we take all the generators and relations above, and add the relation $a = c$. Formally, we need to take open neighborhoods around each $S^1 \times \{x_0\}$, but we can go ahead and use T_1 and T_2. This gives

$$\pi_1(X) = \langle a, b, d \mid aba^{-1}b^{-1} = ada^{-1}d^{-1} = 1 \rangle$$

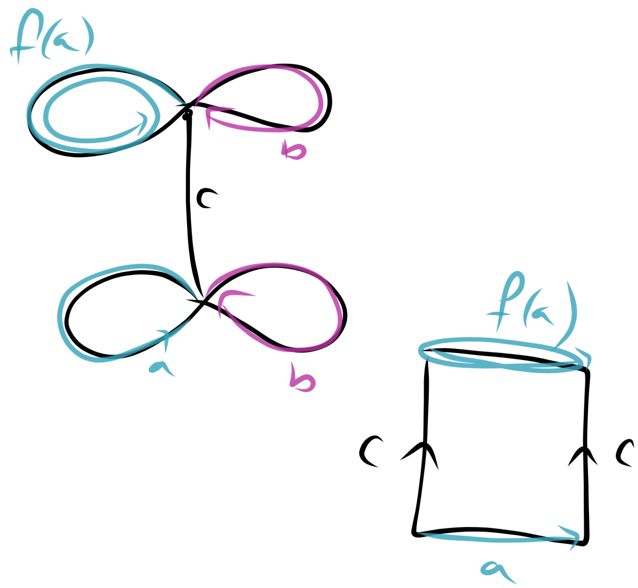
which presents $\mathbb{Z} \oplus (\mathbb{Z} * \mathbb{Z})$. We can view the identified space X as below, which makes this presentation obvious.



Another way of understanding this presentation: before taking the quotient, a, b commute, and c, d commute, but the pairs do not interact. After identification, we get one generator a which commutes with b and d, but b, d do not interact. So, b, d act like letters in a word, and form the generators of $\mathbb{Z} * \mathbb{Z}$.

1.2.11. The mapping torus T_f of a map $f : X \to X$ is the quotient of $X \times I$ obtained by identifying each point $(x, 0)$ with $(f(x), 1)$. In the case $X = S^1 \vee S^1$ with f basepoint-preserving, compute a presentation for $\pi_1(T_f)$ in terms of the induced map $f_* : \pi_1(X) \to \pi_1(X)$. Do the same when $X = S^1 \times S^1$. [One way to do this is to regard T_f as built from $X \vee S^1$ by attaching cells.]

Solution: Let us first see how to build T_f from $S^1 \vee S^1$. When we take $(S^1 \vee S^1) \times I$ and identify via $\sim$, we get a loop at the basepoint – this forms a third S^1. So, the 1-skeleton of T_f is $S^1 \vee S^1 \vee S^1$, where the third S^1 is wedged at the basepoint. Let c be a loop winding around this S^1 once. Let a, b be generators for the original $S^1 \vee S^1$. These get mapped to two loops under f. We attach two 2-cells – one has its boundary identified as below. The other is the same with a replaced by b.



By VK, we must add the relations $acf_*(a)^{-1}c^{-1}$ and $bcf_*(b)^{-1}c^{-1}$ (one for each 2-cell attached). Here, I am using $f_*(a)$ to denote $f_*([a])$ in the same way we use a to denote $[a]$. We thus get

$$\begin{aligned}\pi_1(T_f) &= \pi_1(S^1 \vee S^1 \vee S^1)/\langle acf_*(a)^{-1}c^{-1}, bcf_*(b)^{-1}c^{-1}\rangle \\ &= \langle a, b, c \mid acf_*(a)^{-1}c^{-1} = bcf_*(b)^{-1}c^{-1} = 1\rangle\end{aligned}$$

Observe that this agrees with our answer for 1.2.8. Namely, if $f = \text{id}$, then $f_*(a) = a$ and $f_*(b) = b$. Hence,

$$\pi_1(T_f) = \langle a, b, c \mid [a, c] = [b, c] = 1\rangle \simeq (\mathbb{Z} * \mathbb{Z}) \oplus \mathbb{Z}.$$

If $a \mapsto a^n$ and $b \mapsto b^m$, we should get something like $(\mathbb{Z}/n\mathbb{Z} * \mathbb{Z}/m\mathbb{Z}) \oplus \mathbb{Z}$.

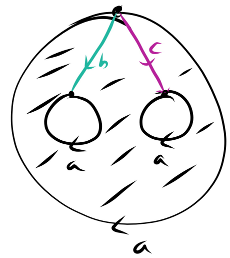
In the case $X = S^1 \times S^1$, we first look briefly at the general construction of T_f for a CW complex X. Assume f is cellular (n-skeleta are taken to n-skeleta). For simplicity, assume that we only have one 0-cell. We start by incorporating the 0-skeleton of X into T_f by noting that $X \times \{0\} \hookrightarrow T_f$. For the 0-cell v in X, we get an interval $v \times I$ in T_f with the ends identified. Hence, the 0-cell produces a 1-cell in T_f, with one end glued to $(v, 0)$ and the other end glued to $(f(v), 1) = (v, 1)$, producing a loop. Similarly, for every 1-cell e in X, we get a 2-cell $e \times [0, 1]$. Since there is only one 0-cell v, each 1-cell e in X is a loop (both ends are glued to v). Hence, the 2-cell is attached along $e \times \{0\}$, followed by $v \times I$, followed by $f(e) \times \{1\}$ backwards, followed by $v \times I$ backwards. This is analogous to what we did previously. Now to keep building T_f we need to add a $k+1$-cell for every k-cell in X. But if $k \geq 2$, we can ignore these when computing π_1, since by VK they do not affect π_1. Since $S^1 \times S^1$ has the same 0- and 1-skeleton as $S^1 \vee S^1$, we get

$$\pi_1(T_f) = \langle a, b, c \mid acf_*(a)^{-1}c^{-1} = bcf_*(b)^{-1}c^{-1} = [a, b] = 1\rangle,$$

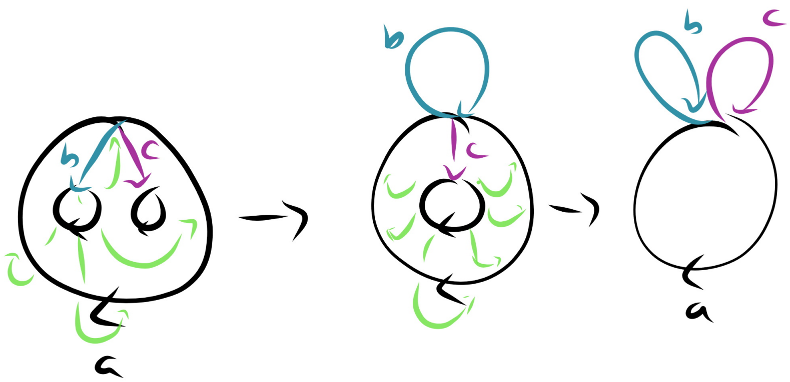
where the last relation comes from the usual construction of the torus. Technically we should also add $[f_*(a), f_*(b)] = 1$, but this is implied by $[a, b] = 1$ and the fact that f_* is a homomorphism.

1.2.13. The space Y in 1.2.12 can be obtained from a disk with two holes by identifying its three boundary circles. There are only two essentially different ways of identifying the three boundary circles. Show that the other way yields a space Z with $\pi_1(Z)$ not isomorphic to $\pi_1(Y)$. [Abelianize the fundamental groups to show they are not isomorphic.]

Solution: A CW structure for Z is shown below, consisting of one 0-cell (the basepoint), three 1-cells (the generators a, b, c, where three copies of a are identified), and one 2-cell.



We first compute $\pi_1(Z^1)$, where Z^1 is the 1-skeleton of Z. This is shown below.



We imagine moving the inner copies of a "below" the plane, and bringing them back up to coincide with the outer a. Keeping track of how b and c move gives the end result, which is $S^1 \vee S^1 \vee S^1$. So, $\pi_1(Z^1) \simeq \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$. To obtain Z, we glue on a 2-cell. The effect of this is to identify the boundary loop of the glued D^2 with the boundary it's glued to. Hence, we add the relation $1 = abab^{-1}cac^{-1}$ – the path is shown below.

So, we get that

$$\pi_1(Z) \simeq \langle a, b, c \mid abab^{-1}cac^{-1} = 1 \rangle$$

Abelianizing this allows for all the generators to commute (by adding the relations $[a, b] = 1, [b, c] = 1, [a, c] = 1$). Applying this to the given relation shows that

$$1 = abab^{-1}cac^{-1} = aaabb^{-1}cc^{-1} = a^3$$

Hence the abelianization G is

$$G = \langle a, b, c \mid [a, b] = 1, [b, c] = 1, [a, c] = 1, a^3 = 1 \rangle \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}.$$

Abelianizing $\pi_1(Y)$ from 1.2.12 gives

$$\pi_1(Y) = \langle a, b, c \mid aba^{-1}b^{-1}cbc^{-1} = 1, [a, b] = 1, [b, c] = 1, [a, c] = 1 \rangle$$

Since a, b, c can commute, we see that $1 = aba^{-1}b^{-1}cbc^{-1} = aa^{-1}bb^{-1}bcc^{-1} = b$. So $b = 1$, then our abelianized group is

$$H = \langle a, c \mid [a, c] = 1 \rangle \simeq \mathbb{Z} \oplus \mathbb{Z}.$$

Hatcher Chapter 1.3, problems 1, 4:

1.3.1. For a covering space $p : \tilde{X} \to X$ and a subspace $A \subset X$, let $\tilde{A} = p^{-1}(A)$. Show that the restriction $p : \tilde{A} \to A$ is a covering space.

Solution: Recall that $\tilde{X}$ is a covering space of X if for every $x \in X$ there exists an open neighborhood U of x in X such that

$$p^{-1}(U) = \bigcup_{\alpha \in I} V_\alpha$$

where each $V_\alpha \subset \tilde{X}$ is disjoint and open in $\tilde{X}$, and $p|_{V_\alpha}$ is a homeomorphism of V_α onto U. We call such neighborhoods evenly covered.

Let $A \subset X$ and set $\tilde{A} = p^{-1}(A)$. Let $q = p|_{\tilde{A}}$. We show that $\tilde{A}$ together with q is a covering space.

Let $x \in A$. Since $\tilde{X}$ covers X, there exists an evenly covered open neighborhood U of x in X. Now consider $V = U \cap A$, which is an open neighborhood of x in A. Then,

$$q^{-1}(V) = p^{-1}(V) \cap \tilde{A} = p^{-1}(U) \cap \tilde{A} = \bigcup_{\alpha \in I} (V_\alpha \cap \tilde{A})$$

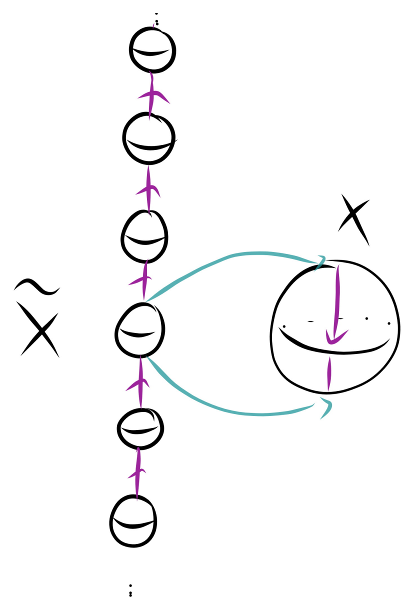
Since each V_α is open in $\tilde{X}$, their restriction to $\tilde{A}$ is open in $\tilde{A}$. Finally, p is a covering map, hence it is surjective. Thus,

$$q(V_\alpha) = p(V_\alpha \cap \tilde{A}) = p(V_\alpha) \cap p(\tilde{A}) \simeq U \cap A,$$

and q is a homeomorphism from V_α to V, as desired.

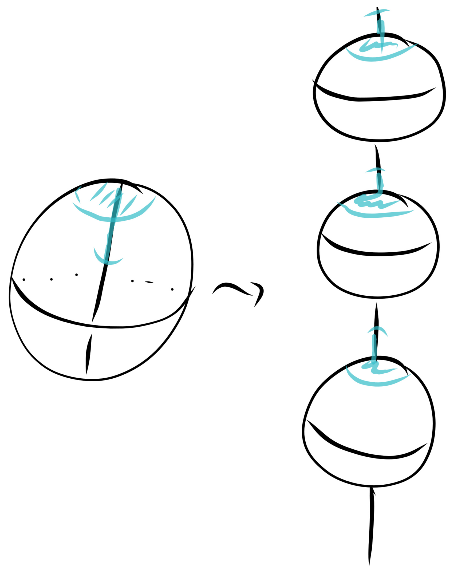
1.3.4. Construct a simply-connected covering space of the space $X \subset \mathbb{R}^3$ that is the union of a sphere and a diameter. Do the same when X is the union of a sphere and a circle intersecting it in two points.

Solution: For X the union of a sphere with a diameter, we assume WLOG that the diameter connects the north and south poles. Consider $\tilde{X}$ which consists of infinitely many copies of S^2, stacked vertically in a line, with the south pole of one connected to the north pole of the previous. This is depicted below.

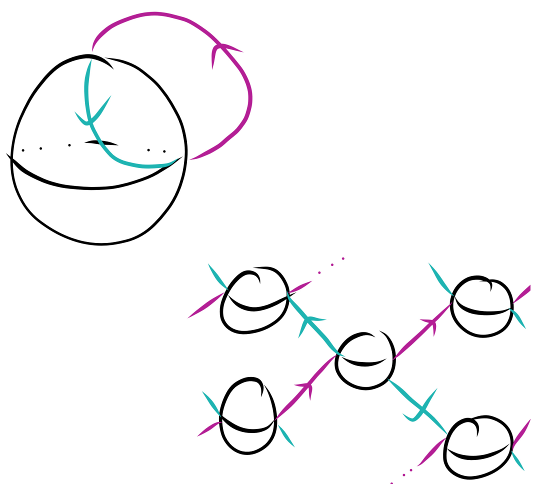


This is simply connected since we can deformation retract each line segment to a point. This gives an infinite wedge of spheres, which is simply connected by VK. Furthermore, $\tilde{X}$ is path connected. Consider $p : \tilde{X} \to X$ mapping each sphere in $\tilde{X}$ identically onto the spherical part of X. The line segment gets mapped to the diameter as follows: travelling from south pole to north pole in $\tilde{X}$ traverses the diameter from the north pole to south pole. It is clear that any open set contained in the spherical part of X is evenly covered (by sets that look exactly the same on each sphere in $\tilde{X}$) –

the same holds for open intervals on the diameter. For open sets around the poles, we have a region on the sphere, and a region extending into the diameter. Then we get something like this, which is also evenly covered



Now for X a union of sphere and a circle intersecting it in two points, we can suppose WLOG that the circle goes through the north and east poles.

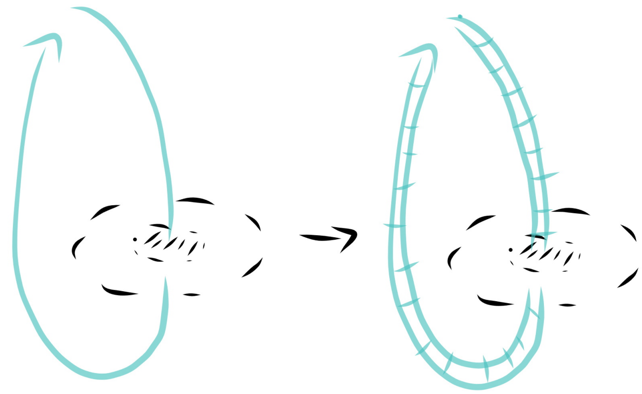


The covering space consists of a grid of spheres with four line segments emanating from each sphere. Two come from one point, while the other two come from the others. The covering map takes each sphere onto that in X, and the line segments onto the components of S^1 w/orientation. This is a covering space in the same way that the previous was. Moreover, it is simply connected since we can contract each segment, and once more get an infinite wedge of spheres. This has trivial fundamental group. Finally, $\tilde{X}$ is path-connected.

Problem 1. Prove that the Dehn presentation gives $\pi_1(\mathbb{R}^3 \setminus L)$.

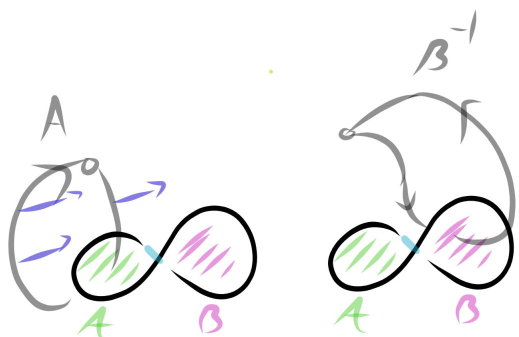
Solution: We start with $\pi_1(\mathbb{R}^3 \setminus X)$, where X was constructed in the problem description. What we can do is deformation retract each vertical segment. This doesn't change the homotopy type. Now observe that we get the projected link with its filled in regions. This is homeomorphic to a disc. Hence, $\pi_1(\mathbb{R}^3 \setminus X) \simeq 0$, since $\mathbb{R}^3 \setminus D^2$ deformation retracts to S^2.

Now we add the regions. Consider the following picture, which is to be used as a motivating example. We have a cut out disc (inside $\mathbb{R}^3$) and we add a circular region R into it.

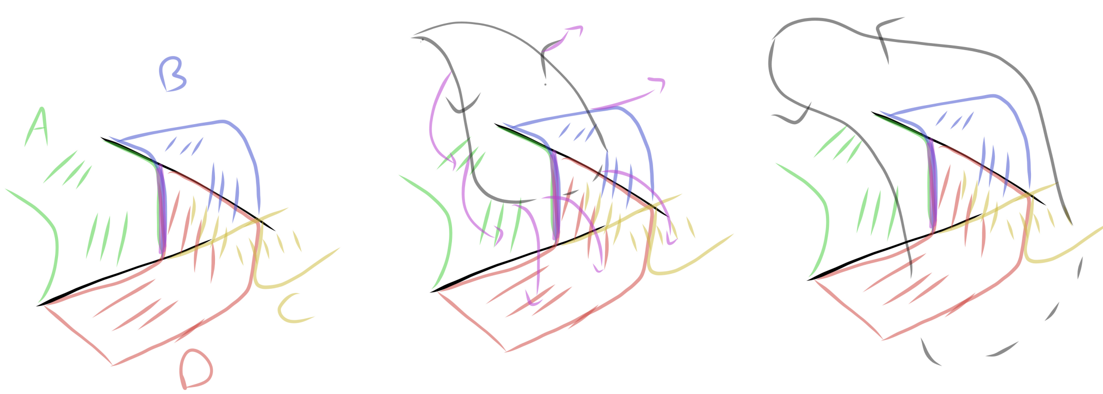


Imagine taking a a loop (in green) through R. We can get an open solid cylinder C around it (still passing through R). Let $U = C \cup R$, which deformation retracts to the prescribed loop. So $\pi_1(U) \simeq \mathbb{Z}$. Let $V = \mathbb{R}^3 \setminus X$. We've shown that $\pi_1(V) \simeq 0$. The intersection $U \cap V$ is $C \setminus R$, a cut cylinder. This deformation retracts (by pulling each end towards the basepoint) to a point. Hence $\pi_1(U \cap V) \simeq 0$. It follows by VK that $\pi_1(U \cup V) \simeq \mathbb{Z}$. The same process shows that we add a generator for each region.

For the relations, we need to add back the vertical segments. Doing so allows for loops to pass through. For example, consider the following:



Here we can move the generator A (passing through the region A) past through the vertical segment (in light blue) over to B. Doing so ends up inverting the direction, so we get the relation $A = B^{-1}$. Doing something similar for a general crossing gives



So that $AB^{-1} = DC^{-1}$, thus $AB^{-1}CD^{-1} = 1$. This is precisely the relation used in the Dehn presentation. Note: The above doesn't show how the rest of the regions are glued, but what matters is what happens at the crossings. Let S be the vertical segment at this crossing. I think the open sets to be used here are $U = \mathbb{R}^3 \setminus (X^1 \cup C \cup D) \cup S$ and $V = \mathbb{R}^3 \setminus (X^1 \cup A \cup B) \cup S$. Then $U \cap V = \mathbb{R}^3 \setminus X^1 \cup S$, and we can homotope AB^{-1} and DC^{-1} so that they lie entirely in $U \cap V$. So these must be considered equal.

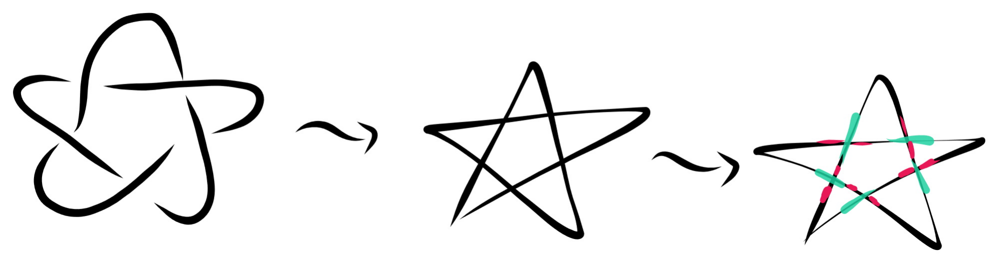
Adding in all the vertical segments gives $\mathbb{R} \setminus L$. Hence, $\pi_1(\mathbb{R}^3 \setminus L)$ has generators given by the regions (where the unbounded is the identity), and the relations are given as in the problem.

Problem 2 (Dehn presentation example).

a) Use the Dehn presentation to present $\pi_1(\mathbb{R}^3) \setminus K$ where K is the $(2, n)$ torus knot (in particular, n is odd).
b) You'll get a presentation with $n + 1$ generators. Simplify it down to 2 generators. How similar is the result to the elegant presentation $\langle x, y \mid x^2 = y^n \rangle$ from the book?

Solution:

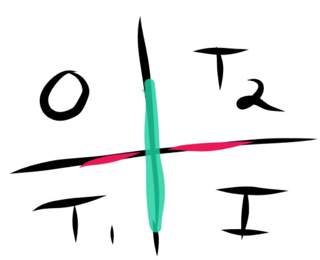
a) I will use the $(2, 5)$ knot as a first example. A picture is given below, along with a projection into a plane below it.



Observe that the projection consists of a regular pentagon with 5 regular triangles glued to it: each regular triangle is glued to one edge, along that edge. There is a 5-fold rotational symmetry. In particular, as shown to the far right, the arrangement of each crossing is preserved under rotation. So, we really need to only compute the relation for one crossing. Then, with a systematized way of labelling each face, we can extend this to all crossings.

Note that each crossing comes into contact with four regions: the inner region of the pentagon, the outer region, and two neighboring triangular regions. This holds in general – with a $(2, n)$ torus knot, we still have four regions: the inner n-gon, outer region, and two neighboring triangles.

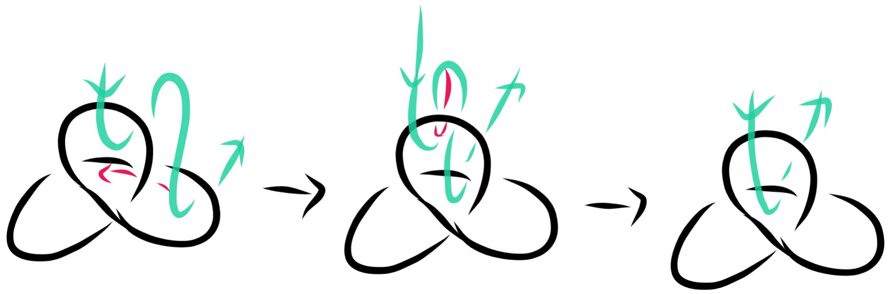
Label the inner region as I and the outer region as O. We will use O to generate the identity element. Pick a crossing, and orient it like below:



Label the bottom-left triangle T_1 and the top-right triangle T_2. We then have the following relation: $OT_2^{-1}IT_1^{-1} = 1$. Upon rotating by $2\pi/n$, we see that O, I remain fixed but T_2 moves to the bottom-left. Label the top-right triangle T_3, and continue labelling clockwise. We thus obtain the relation $OT_{i+1}^{-1}IT_i^{-1} = 1$ for all $i = 1, ..., n$ where $T_{n+1} = T_1$ by convention. Finally, we have the relation $O = 1$. The generators are thus $I, T_1, ..., T_n$ with the relations $T_{i+1}^{-1}IT_i^{-1} = 1$. We can rearrange this to $I = T_{i+1}T_i$, hence I can be written in terms of all the other generators, and we may remove it. Thus

$$\pi_1(\mathbb{R}^3 \setminus K) \simeq \langle T_1, ..., T_n \mid T_{i+1}T_i = 1\rangle$$

The relation actually says something geometrically neat – each triangular region defines a lobe in the torus knot, indexed clockwise. The relations say that entering one from above, then exiting to the unbounded region, then coming back in through the next one clockwise is equivalent to just going through the inner region.



b) Let's return to the representation that still has I in it, which has $n+1$ generators:

$$\pi_1(\mathbb{R}^3 \setminus K) \simeq \langle I, T_1, ..., T_n \mid T_{i+1}^{-1} I T_i^{-1} = 1\rangle.$$

Let's consider going around each lobe – set $V = T_n T_{n-1} ... T_2 T_1$. We saw that $T_{i+1}T_i = I$ so that

$$V^2 = (T_n T_{n-1})...(T_3T_2)(T_1T_n)(T_{n-1}T_{n-1})...(T_2T_1) = I^n$$

Hence,

$$\pi_1(\mathbb{R}^3 \setminus K) \simeq \langle V, I \mid V^2 = I^n\rangle,$$

which is the same presentation.

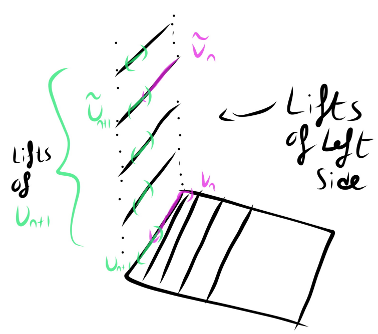
HW5

Hatcher Chapter 1.3, problems 5, 9, 16, 20:

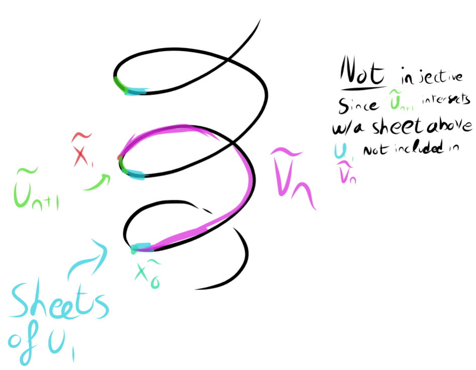
1.3.5. Let X be the subspace of $\mathbb{R}^2$ consisting of the four sides of the square $[0,1] \times [0,1]$ together with the segments of the vertical lines $x = 1/2, 1/3, 1/4, ...$ inside the square. Show that for every covering space $\tilde{X} \to X$ there is some neighborhood of the left edge of X that lifts homeomorphically to $\tilde{X}$. Deduce that X has no simply-connected covering space.

Solution: Let $\tilde{X} \to X$ be a covering space. Then for every $x \in X$, there exists an evenly covered neighborhood U_x. Since X is path-connected, each the sheets of $\tilde{X}$ over U_x are nonempty. Consider $\{U_x\}$, an open cover of X. But, X is compact so that we can extract a finite subcover, denoted $\{U_\alpha\}$. One of these sets contains, say, the top left corner of X, denoted x_0 – call this U_1. Let $\tilde{x}_0$ be a point above x_0 in $\tilde{X}$. Then one of the sheets of U_1 contains $\tilde{x}_0$ – call this $\tilde{U}_1$. We now construct our open set inductively. The idea is to lift sets that start by intersecting with U_1 along the left edge. We can choose sheets that start by intersecting with $\tilde{U}_1$. Stitching these together gives our desired neighborhood.

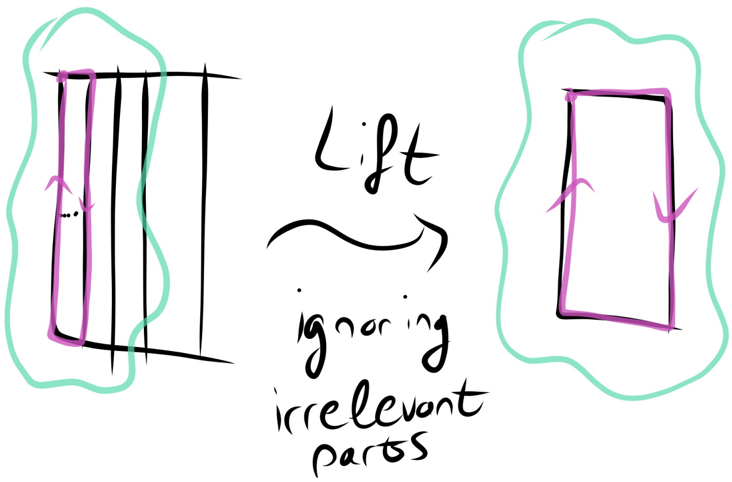
Assume there is some subcollection $\{U_k\}_{k=1}^n$ of $\{U_\alpha\}$ such that there exists a $\tilde{V}_n$ that is mapped homeomorphically to $V_n = \cup_{k=1}^n U_k$. If $\{U_k\}$ does not cover the left edge, then there exists some set in $\{U_\alpha\}$ which intersects both V_n and the complement of V_n in the left edge. Meaning, we can always enlargen our cover by adding in a new set which intersects nontrivially with a U_k. If not, then $\{U_k\}$ and $\{U_\alpha\} \setminus \{U_k\}$ form a separation of the left edge, a contradiction since it is connected. Call this set U_{n+1}.



Now let $x_1 \in U_{n+1} \cap V_n$. Then there exists a $\tilde{x}_1 \in \tilde{V}_n$. In particular, $\tilde{x}_1$ is in a sheet of U_{n+1}, denoted $\tilde{U}_{n+1}$. This is mapped homeomorphically onto U_{n+1}. Let V_{n+1} be $\tilde{V}_n \cup \tilde{U}_{n+1}$, so that $\tilde{V}_{n+1}$ maps homeomorphically onto V_{n+1}. This follows by the gluing lemma, and the fact that the restrictions of p to $\tilde{U}_{n+1}$ is a homeomorphism onto U_{n+1}, while the restriction of p to $\tilde{V}_n$ is a homeomorphism onto V_n. Intuitively, the only reason why this might not be injective is if the sheet $\tilde{U}_{n+1}$ extends from $\tilde{V}_n$ into a sheet above some U_k not included in $\tilde{V}_n$. But, we should be able to take our U_k small enough to avoid this. This process must terminate since $\{U_\alpha\}$ is finite.



To show that X has no simply-connected covering space is more or less the same as showing the Hawaiian earring does not. Consider an evenly covered neighborhood of the left edge. This must contain all but finitely many of the line segments $x = 1/2, 1/3, 1/4, \ldots$. Choose one of them. Now, we have a copy of a box (with no interior). Since the neighborhood is evenly covered, we have a similar picture in some sheet in $\tilde{X}$. Take a loop going around the boundary of the box once based at $\tilde{x_0}$. Then, this is mapped to a nontrivial loop under p, and thus could not have been trivial in $\pi_1(\tilde{X}, \tilde{x_0})$ since p_* is a homomorphism.



1.3.9. Show that if a path-connected, locally path-connected space X has $\pi_1(X)$ finite, then every map $X \to S^1$ is nullhomotopic. [Use the covering space $\mathbb{R} \to S^1$.]

Solution: Let $f : X \to S^1$, and choose an $x_0 \in X$. Let $s_0 = f(x_0)$ and choose some $r_0 \in \mathbb{R}$ such that $p(r_0) = s_0$. Then, $f_*(\pi_1(X, x_0)) \simeq 1$ since the only finite subgroup of $\pi_1(S^1, s_0) \simeq \mathbb{Z}$ is the trivial subgroup 1. Since $\mathbb{R}$ is simply-connected, we see that

$$f_*(\pi_1(X, x_0)) \subset p_*(\pi_1(\mathbb{R}, r_0)).$$

Hence, a lift $\tilde{f} : (X, x_0) \to (\mathbb{R}, r_0)$ of f exists. Since $\mathbb{R}$ is contractible, $\tilde{f}$ is nullhomotopic – in particular homotopic to the constant map at r_0, since $\mathbb{R}$ deformation retracts onto r_0. Thus, there exists a homotopy $\tilde{H} : X \times I \to \mathbb{R}$ such that $\tilde{H}(x, 0) = \tilde{f}(x)$ and $\tilde{H}(x, 1) = r_0$. Since $\tilde{f}$ is a lift, we have that $p \circ \tilde{f} = f$. It follows by continuity of p that $H = p \circ \tilde{H}$ is a homotopy of f to $p(r_0) = s_0$. So, f is nullhomotopic.

I probably don't have to worry about the basepoint as much as I did.

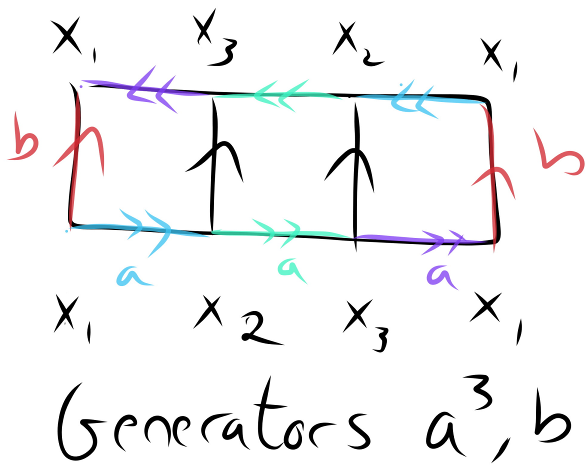
1.3.16. Given maps $X \to Y \to Z$ such that both $Y \to Z$ and the composition $X \to Z$ are covering spaces, show that $X \to Y$ is a covering space if Z is locally path-connected, and show that this covering space is normal if $X \to Z$ is a normal covering space.

Solution: Let $p : X \to Y$ and $q : Y \to Z$ be the given maps. Then the covering maps are q and $q \circ p : X \to Z$. Let $y \in Y$, then there exists a neighborhood U of y such that $U \to q(U)$ is a homeomorphism (since q is a covering map). Since Z is locally path-connected, we can pass to a smaller neighborhood and assume that $q(U)$ is path-connected and evenly covered by $q \circ p$. Path-connectedness is necessary here for surjectivity reasons (the open sets in covers are determined by extending paths downstairs, but if you can't reach a point...) Now consider $(q \circ p)^{-1}(q(U)) = p^{-1}(q^{-1}(q(U))) = p^{-1}(U)$. This is a disjoint collection of open neighborhoods each homeomorphic to $q(U)$. But $q(U)$ is homeomorphic to U, so that we have an even covering of U by p.

If $X \to Z$ is normal, then choosing a basepoint $z \in Z$ and lifts $y \in Y$ and $x \in X$, we have that $(q \circ p)_*(\pi_1(X, x))$ is a normal subgroup of of $\pi_1(Z, z)$ and $q_*(\pi_1(Y, y))$ is a subgroup of $\pi_1(Z, z)$. For brevity, I will denote these image subgroups as H_X and H_Y, and $\pi_1(Z,z)$ as G. So, H_X is a normal subgroup of G and H_Y is a subgroup of G. Then, $gH_Xg^{-1} = H_X$ for all $g \in G$ by normality. In particular, this is true for all $g \in H_Y \leq G$. Since q_* is a homomorphism, it follows that $p_*(\pi_1(X,x))$ is a normal subgroup of $\pi_1(Y,y)$.

1.3.20. Construct nonnormal covering spaces of the Klein bottle by a Klein bottle and by a torus.

Solution: The fundamental group of the Klein bottle K^2 is presented by $\langle a, b \mid aba = b\rangle$. Consider the subgroup H_1 generated by a^3, b. This subgroup is non-normal. We can view this as a 3-sheeted covering of K^2 by Klein bottles as follows. Only colored arrows are identified.



This reminds me of a 1/3 Dehn twist. The points x_1, x_2, x_3 are the lifts of the basepoint in K^2. The generators are found by taking two different loops based at x_1. Recall that normal subgroups are invariant under conjugation. In the cover, conjugation amounts to a change in basepoint. Consider the loop b with endpoints at x_1. We can also view b as a path from x_2 to x_3, which is not a loop. Hence $b \in p_*(\pi_1(X,x_1))$ and $b \notin p_*(\pi_1(X,x_2))$. But, $\pi_1(X,x_2)$ is obtained from $\pi_1(X,x_1)$ by conjugation by a. So H_1 is not normal.

Now consider the subgroup H_2 of this generated by b^6, ab^{-2}. Suppose this is normal. Then $a(ab^{-1})a^{-1} \in H_2$ and $b^{-1}ab^{-2}b = b^{-1}ab^{-1} \in H_2$. Now, a and b^2 commute since the relation says $ab = ba^{-1}$ and (equivalently) $ba = a^{-1}b$. Then,

$$ab^2 = bb^{-1}(ab)b = bb^{-1}ba^{-1}b = b(a^{-1}b) = bba = b^2a$$

(for terms in parenthesis, we use the relations to convert). So a and b^2 commute. Doing some algebra gives

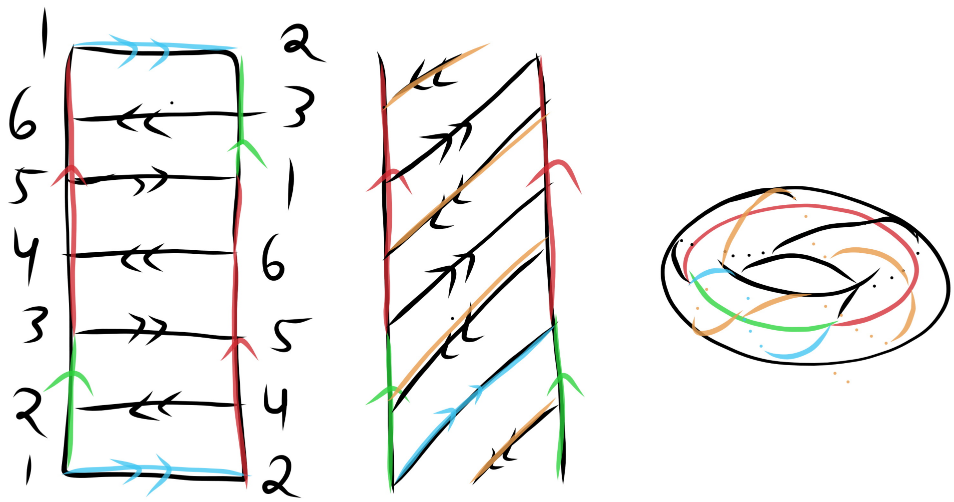
$$\begin{aligned} b^{-1}ab^{-1} &= b^{-1}b^{-1}bab^{-1} = b^{-2}(ba)b^{-1} \\ &= b^{-2}a^{-1}bb^{-1} = b^{-2}a^{-1} = (ab^2)^{-1} \end{aligned}$$

which we concluded was in H_1 by normality. Thus, $ab^2 \in H_1$. Hence also

$$(aab^{-2}a^{-1})(ab^2) = a^2$$

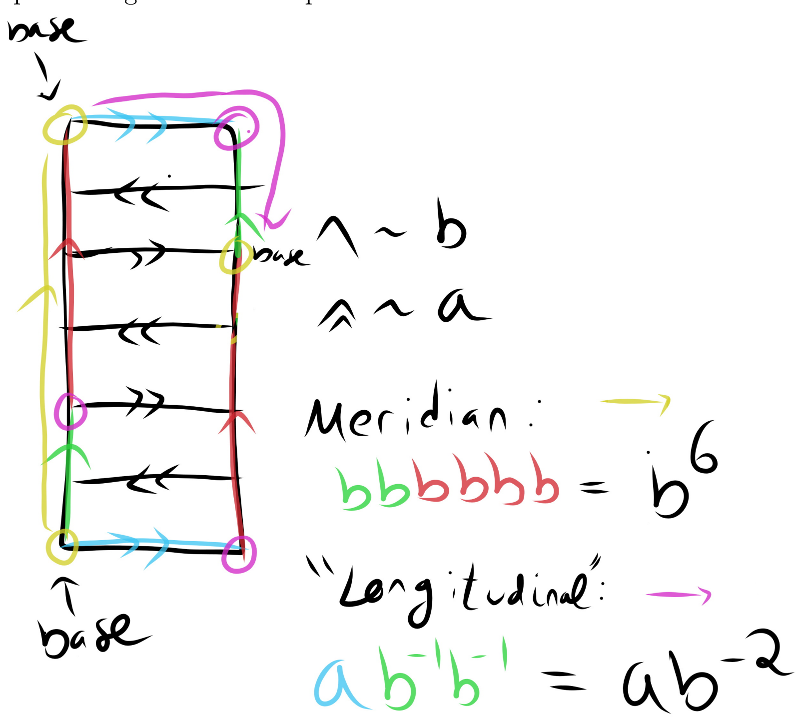
is in H_1. I claim this cannot occur. The only way to get two copies of a is to use two copies of ab^{-2} and then cancel out the b terms somehow. If we could combine everything nicely, we would get a^2b^{-4}. Even so, there is no way to isolate a^2 – we would need b^4, but we only have b^6. This logic isn't perfectly clear, but I hope it gets the point across.

We can view this covering space as follows. Here, the black arrows are not identified. They are merely used to show how the covering map works. We map each "copy" of the Klein bottle here onto K^2, so we get a 6-sheeted covering. The lifts of the basepoint are denoted $1-6$ to help visualize this.



The red, green, and blue tell us how to glue this space together. We get a torus, shown to the right. Note that we get two different loops around the torus (I've shown one in orange, the other comes from the blue and black going in the other direction). All deck transformations send the orange loop to itself – hence, there is none sending it to the other loop. We conclude that this covering space is also non-normal. Finally we can consider the loop ab^{-2} and change the basepoint to 6 (conjugate by b^{-1}). This produces a path $(6 \to 4 \to 3 \to 2)$, hence it cannot be normal either.

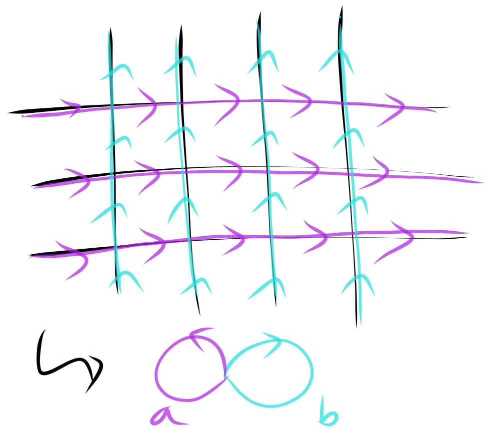
To see that this actually corresponds to the group we want, see the below picture. We trace out two loops starting from the basepoint 1.



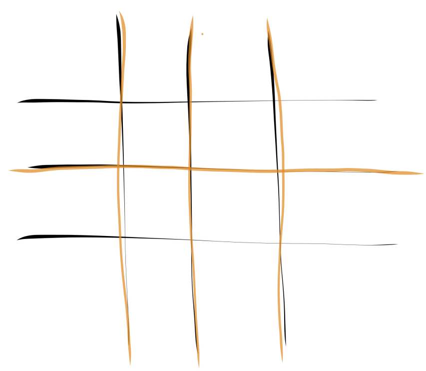
Hatcher Chapter 1.A, problems 6, 7:

1.A.6. Let F be the free group on two generators and let F' be its commutator subgroup. Find a set of free generators for F' by considering the covering space of the graph $S^1 \vee S^1$ corresponding to F'.

Solution: Let a, b be generators of F. Then $F' = \langle aba^{-1}b^{-1} \rangle$. Consider the following graph X with edges identified:



We see that this is a covering space for $S^1 \vee S^1$ since each vertex has one a and b edge coming in, one a and b edge going out. The covering map p sends a around one copy of S^1 once and b around the other copy of S^1 once, and maps all vertices to the wedge point. We can take the following spanning tree.



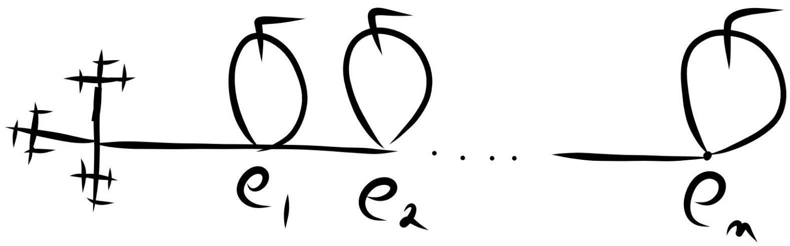
This contains the x-axis and all vertical lines. Observe that there are still infinitely many edges (horizontal ones) remaining. Each of these determines a loop f, and $\pi_1(X)$ is a free group with basis determined by the f (See Prop 1.A.2). Hence, there are infinitely many generators. To actually find them, we go along a path in the tree, go along the edge, and back through a path in the tree. For an edge $(m, n) - -(m + 1, n)$ with m, n integers, and $n \neq 0$, we first travel along the x axis to m, then up the line $x = m$ to (m, n), across the edge, down the line $x = m + 1$, and back across the x-axis to the origin. Doing this amounts to a^m (for the first horizontal walk), then b^n (for the first vertical walk), then a, then b^{-n} (for the second vertical walk), then $a^{-(m+1)}$ (for the second horizontal walk). In total,

$$a^m b^n a b^{-n} a^{-(m+1)}$$

generate F' with m, n integers and $n \neq 0$. I think there's probably a cleaner representation where the generators are $[a^m, b^n]$, but I'm unsure.

1.A.7. If F is a finitely generated free group and N is a nontrivial normal subgroup of infinite index, show, using covering spaces, that N is not finitely generated.

Solution: Suppose F has n generators, and let $X = S^1 \vee ... \vee S^1$ (with n copies). Let $\tilde{X}$ be a covering space of X corresponding to N. Suppose $\pi_1(\tilde{X})$ is finitely generated with m elements. Since N has infinite index, we know that $\tilde{X}$ covers X with infinitely many sheets. So, $\tilde{X}$ consists of an infinite tree (for the infinitely many sheets) and m loops $e_1, ..., e_m$ (for the m generators).



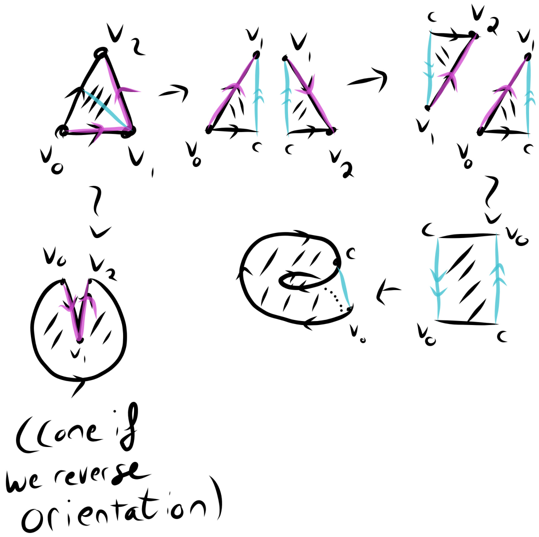
Choose a basepoint x_0 in X and a nontrivial path γ. Consider a lift $\tilde{\gamma}$ based at $\tilde{x}_0$. Since $\tilde{X}$ is normal, for each pair of lifts of x_0, there exists a deck transformation taking one to the other. Choose another $\tilde{x}_1$ above x_0, and consider a deck transformation taking $\tilde{x}_0$ to $\tilde{x}_1$. We may choose $\tilde{x}_1$ so that it is "farther away" from the loops $e_1, ..., e_m$. By "farther away" we mean further in the tree (see pg. 85 of Hatcher). Let $f : \tilde{X} \to \tilde{X}$ be a deck transformation sending $\tilde{x}_0$ to $\tilde{x}_1$. Then $f(\tilde{\gamma})$ contains more edges (since its endpoint is pulled further into the tree). Each edge in $\tilde{X}$ corresponds to a loop in X. So, f cannot be a deck transformation, since the projections of $\tilde{\gamma}$ and $f(\tilde{\gamma})$ do not agree (the latter has more loops in X). One might also be able to look at deck transformations for one of the vertices with a loop e_i. Descending this into the tree carries the loop into the tree – hence each vertex has a loop attached to it, contradicting the fact that N is finitely generated. Another way to see that this space is not normal is just by noting that it has no nice symmetry.

HW6

Hatcher Chapter 2.1, problems 1, 2, 3, 5, 24:

2.1.1. What familiar space is the quotient Δ-complex of a 2-simplex $[v_0, v_1, v_2]$ obtained by identifying the edges $[v_0, v_1]$ and $[v_1, v_2]$, preserving the ordering of vertices?

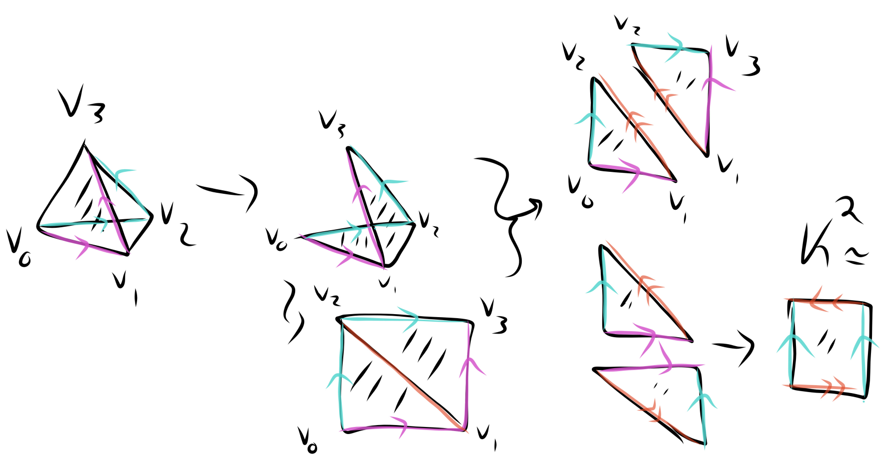
Solution: This is a Möbius strip, which can be seen by performing the following cuts and pastes (colored edges are glued together with proper orientation).



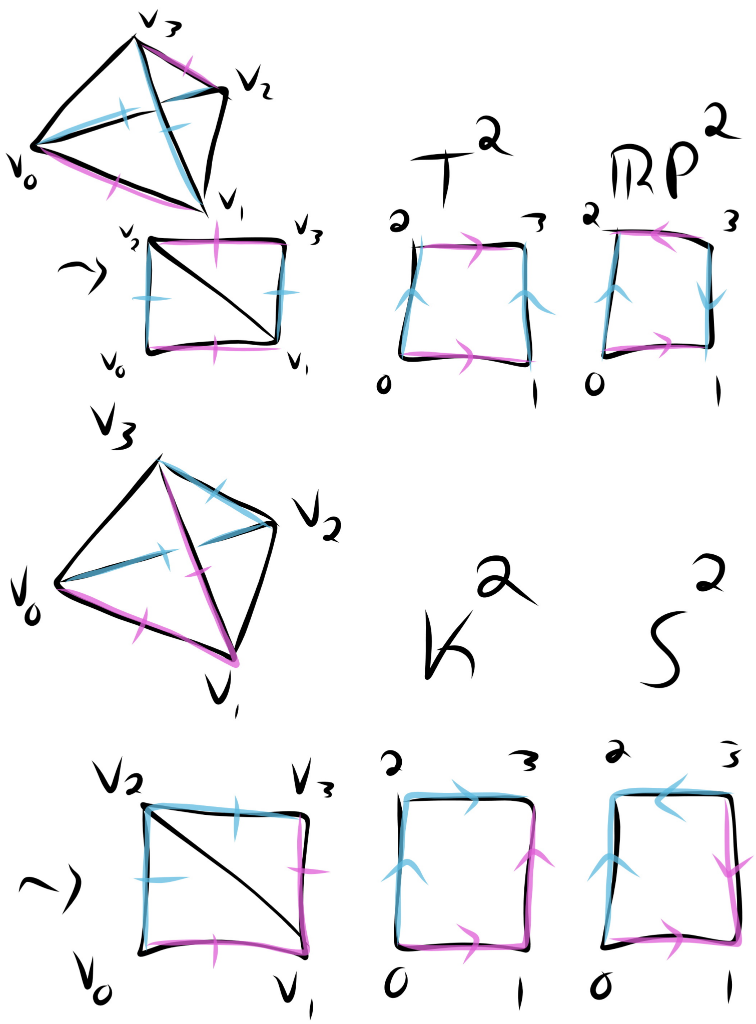
Observe that if we reverse one of the orientations and glue, we get a "cone". (I imagine this in a similar way as constructing the dunce cap, but without gluing all three edges together).

2.1.2. Show that the Δ-complex obtained from Δ^3 by performing the order-preserving edge identifications $[v_0, v_1] \sim [v_1, v_3]$ and $[v_0, v_2] \sim [v_2, v_3]$ deformation retracts onto a Klein bottle. Also, find other pairs of identifications of edges that produce Δ-complexes deformation retracting onto a torus, a 2-sphere, and $\mathbb{RP}^2$.

Solution: We can imagine deformation rectracting the tetrahedron Δ^3 onto a square with edges $[v_0, v_1], [v_1, v_3], [v_0, v_2], [v_2, v_3]$ as follows.



Then, by cutting along the diagonal (highlighted in orange), rearranging, and gluing, we get the standard fundamental polygon of the Klein bottle. To get the other spaces, we do something similar. See the figure below.



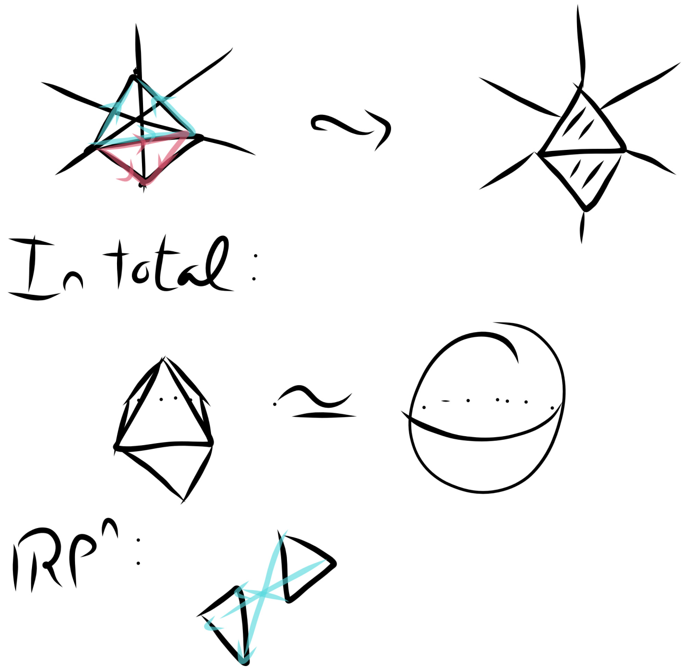
Here, we identify same colored edges. The perpendicular line on each edge signals that we haven't specified exactly how the edges are glued (in what direction), just that those edges are glued. The proper edge identifications are

i) T^2: $[v_0, v_2] \sim [v_1, v_3]$ and $[v_0, v_1] \sim [v_2, v_3]$
ii) S^2: $[v_0, v_2] \sim [v_3, v_2]$ and $[v_0, v_1] \sim [v_3, v_1]$
iii) $\mathbb{RP}^2$: $[v_0, v_2] \sim [v_3, v_1]$ and $[v_0, v_1] \sim [v_3, v_2]$

which need not preserve vertex orientations.

2.1.3. Construct a Δ-complex structure on $\mathbb{RP}^n$ as a quotient of a Δ-complex structure on S^n having vertices the two vectors of length 1 along each coordinate axis in $\mathbb{R}^{n+1}$.

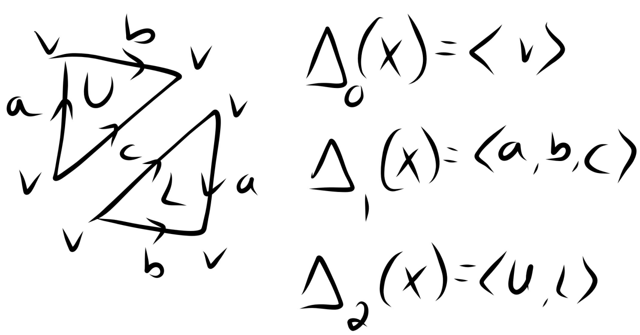
Solution: Let $\{e_i\}_{i=0}^n$ be the canonical orthonormal basis for $\mathbb{R}^{n+1}$. Let $v_i^+ = e_i$ and $v_i^- = -e_i$. We can form 2^n simplices in the "quadrants" of $\mathbb{R}^{n+1}$ as $[v_0^\pm, v_1^\pm, ..., v_n^\pm]$, where each simplex is given by a choice of $\pm$ for the v_i. We glue these together along shared $n-1$ subsimplices – for example, in $n = 2$ we glue $[v_1^+, v_2^+, v_3^+]$ and $[v_1^+, v_2^+, v_3^-]$ along $[v_1^+, v_2^+]$, but we would not glue either to $[v_1^-, v_2^-, v_3^+]$, since this does not share any $n-1$ subsimplices with the previous two. This gives a Δ-complex structure on S^n (each simplex covers a quadrant of S^n).



To get $\mathbb{RP}^n$ we perform the usual identification – namely, we identify opposite faces of the Δ-complex. So, we identify $[v_0^{\pm}, v_1^{\pm}, ..., v_n^{\pm}]$ with $[v_0^{\mp}, v_1^{\mp}, ..., v_n^{\mp}]$.

2.1.5. Compute the simplicial homology groups of the Klein bottle using the Δ-complex structure described at the beginning of this section.

Solution: We have that $\Delta_0(X) = \langle v \rangle, \Delta_1(X) = \langle a, b, c\rangle$, and $\Delta_2(X) = \langle U, L\rangle$. Since there are no higher dimensional simplices, $\Delta_n(X) = 0$ for $n \geq 3$. Now consider the following picture



It is easy to see that $\partial_2 U = a + b - c$ and $\partial_2 L = a - b + c$. Then

$$\partial_2(nU + mL) = (n + m)a + (n - m)(b - c)$$

Suppose that $nU + mL$ lies in $\text{Ker}(\partial_2)$. Then it must be that $n + m = 0$ and $n - m = 0$. Hence $n = m = 0$ and $\text{Ker}(\partial_2) = 0$. Since $\Delta_3(X) = 0$, we have $\text{Im}(\partial_3) = 0$. It is also easy to see that $\partial_1 a = \partial_1 b = \partial_1 c = 0$ since these are loops. Thus $\partial_1 = 0$ and $\text{Ker}(\partial_1) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ and $\text{Im}(\partial_1) = 0$. Finally, $\partial_0 = 0$ by definition so that $\text{Ker}(\partial_0) = \mathbb{Z}$. In total,

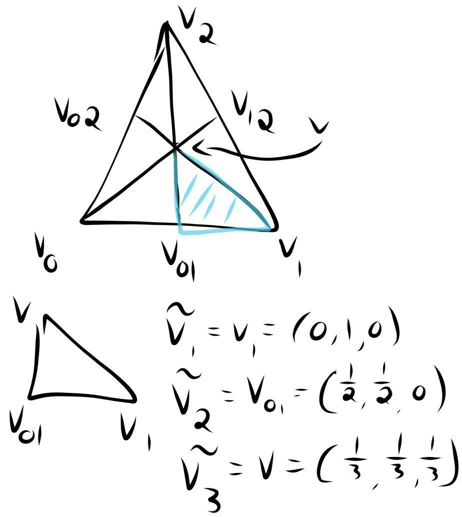
$$H_n^{\Delta}(X) = \begin{cases} \text{Ker}(\partial_0)/\,\text{Im}(\partial_1) = \mathbb{Z} & n = 0 \\ \text{Ker}(\partial_1)/\,\text{Im}(\partial_2) = \langle a, b, c \mid a + b - c, a - b + c\rangle & n = 1 \\ \text{Ker}(\partial_2)/\,\text{Im}(\partial_3) = 0 & n = 2 \\ 0 & n \geq 3 \end{cases}$$

We can find a better presentation for $H_1^\Delta(X)$. First, a different basis for $\Delta_1(X)$ is $a, b, a+b-c$. Next, solving for c in $a+b-c=0$ gives $c=a+b$. Substituting this into the second relation gives $2a=0$. Hence,

$$H_1^\Delta(X) = \langle a, b, a+b-c \mid a+b-c=0, 2a=0\rangle = \mathbb{Z}_2 \oplus \mathbb{Z}.$$

2.1.24. Show that each n-simplex in the barycentric subdivision of Δ^n is defined by n inequalities $t_{i_0} \leq t_{i_1} \leq ... \leq t_{i_n}$ in its barycentric coordinates, where $(i_0, ..., i_n)$ is a permutation of $(0, ..., n)$.

Solution: First consider the case of a 2-simplex. This will provide intuition towards the general idea. The barycentric subdivision is given as seen below



Here we have labelled the vertices of Δ^2 in the canonical way. The midpoints of each edge are labelled v_{01}, v_{12}, v_{02}, and are the barycenters of the edges. The point $v = (1/3, 1/3, 1/3)$ is the barycenter of Δ^2. We loosely call all of these the points in the barycentric subdivision. Note that all the vertices have one nonzero component, which is 1. All the barycenters of the edges have two nonzero components, each of which is 1/2. The barycenter has three nonzero components, each is 1/3.

Consider the highlighted 2-simplex. We relabel the vertices $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3$ as indicated. We start at 1 so as to line up the index with the denominator of the nonzero components. Observe that each 2-simplex in the barycentric subdivision consists of an original vertex, a barycenter of an edge, and the barycenter. From this, we can obtain an ordering on the components. In this particular example, we see that $t_1 \geq t_0 \geq t_2$. We determine i_2 via $\tilde{v}_1$ and i_1 via $\tilde{v}_2$. We know that $\tilde{v}_1$ only has one nonzero component, so we define this to be i_2 (thus, here $i_2 = 1$). Obviously then $\tilde{v}_1$ satisfies both $t_1 \geq t_2 \geq t_0$ and $t_1 \geq t_0 \geq t_2$, since its other components are just zero. Now $\tilde{v}_2$ determines i_1. We have that one of the 1/2's will be in position i_2. So we find the other one, and set its position as i_1 – thus here $i_1 = 0$. This determines i_0, since it has to be the only remaining one.

Hence, this 2-simplex – call it $\tilde{\Delta}^2$ – should be defined by the above inequalities. Consider now any point w in $\tilde{\Delta}^2$. Then w is a convex combination of $\tilde{v}_i$, e.g. $w = \lambda_1\tilde{v}_1 + \lambda_2\tilde{v}_2 + \lambda_3\tilde{v}_3$ with $\lambda_1 + \lambda_2 + \lambda_3 = 1$ and $\lambda_i \geq 0$. Then,

$$w = \left(\frac{\lambda_2}{2} + \frac{\lambda_3}{3}, \lambda_1 + \frac{\lambda_2}{2} + \frac{\lambda_3}{3}, \frac{\lambda_3}{3}\right)$$

which obviously satisfies the inequalities since $(w)_0 - (w)_2 = \lambda_2/2 \geq 0$ and $(w)_1 - (w)_0 = \lambda_1 \geq 0$; $(w)_i$ indicates the t_i component of w.

Note that $\tilde{\Delta}^2$, which is one of the n-simplices in the barycentric subdivision, is precisely the convex hull of $\tilde{v}_1, \tilde{v}_2, \tilde{v}_2$. We have shown that any point in this simplex satisfies the above inequalities, but we have not shown yet that no other point in Δ^2 does.

So, suppose we have some $w = (t_0, t_1, t_2)$ with $t_1 \geq t_0 \geq t_2$ and $t_0 + t_1 + t_2 = 1$. We wish to show that w is in $\tilde{\Delta}^2$. Set $\lambda_3 = 3t_2$, $\lambda_2 = 2(t_0 - t_2)$ and $\lambda_1 = t_1 - t_0$. Then,

$$\begin{aligned} \lambda_3/3 &= t_2 \\ \lambda_2/2 + \lambda_3/3 &= t_0 - t_2 + t_2 = t_0 \\ \lambda_1 + \lambda_2/2 + \lambda_3/3 &= t_1 - t_0 + t_0 = t_1 \end{aligned}$$

so that $w = \lambda_1 \tilde{v}_1 + \lambda_2 \tilde{v}_2 + \lambda_3 \tilde{v}_3$. Now what is the sum of λ_i? Adding the three above equations says that the sum is precisely $t_0 + t_1 + t_2 = 1$. Hence, $w \in \tilde{\Delta}^2$.

In general, the barycentric coordinates of vertices of Δ^n are given by $v_0 = (1, 0, ..., 0)$, $v_1 = (0, 1, 0, ..., 0)$, ..., $v_n = (0, ..., 0, 1)$. That is $(v_i)_j = \delta_{ij}$ where $(v_i)_j$ is the j-th coordinate of v_i, and $i, j = 0, ..., n$. Suppose we are given an inequality on the coordinates as $t_{i_0} \leq t_{i_1} \leq ... \leq t_{i_n}$. We use this to define the vertices of a subsimplex as follows:

$$\tilde{v}_k = \frac{1}{k} \left(\sum_{j=n+1-k}^{n} v_{i_j} \right)$$

where now $k = 1, ..., n+1$ to make things cleaner, and $j = 0, ..., n$. Observe that

$$(\tilde{v}_k)_{i_j} = \frac{1}{k}$$

if $n + 1 - k \leq j$. Otherwise, we have $(\tilde{v}_k)_{i_j} = 0$. Accordingly, we have that each $\tilde{v}_k$ obeys $t_{i_0} \leq t_{i_1} \leq ... \leq t_{i_n}$. Let $\tilde{\Delta}^n$ be the n-simplex spanned by these vertices (i.e. the convex hull in $\mathbb{R}^{n+1}$). Note that the $\tilde{v}_k$ are points in the barycentric subdivision, with $\tilde{v}_1$ a vertex, $\tilde{v}_2$ a barycenter of an edge, $\tilde{v}_3$ a barycenter of a face, etc. Hence $\tilde{\Delta}^n$ is one of the n-simplices in the barycentric subdivision.

Is it such that all points in $\tilde{\Delta}^n$ satisfy these inequalities? Consider

$$w = \sum_{k=1}^{n+1} \lambda_k \tilde{v}_k$$

for $\lambda_1 + ... + \lambda_{n+1} = 1$ and $\lambda_k \geq 0$. We see that

$$(w)_{i_{n-k}} = \sum_{j=k+1}^{n+1} \frac{\lambda_j}{j}.$$

This is because $(\tilde{v}_j)_{i_{n-k}}$ is $1/j$ if $n + 1 - j \leq n - k$ and 0 otherwise. Clearly $n + 1 - j \leq n - k$ for $j = n+1, n, n-1, ..., k+1$.

As k decreases (we go from i_0 to i_n), we add on more terms (since the lower limit is decreasing). Each of these is of the form $\lambda_j / j \geq 0$, so that we're increasing the sum. Hence, w is such that $t_{i_0} \leq t_{i_1} \leq ... \leq t_{i_n}$.

Now take any $w = (t_0, ..., t_n) \in \Delta^n$ obeying the inequalities. Set $\lambda_{n+1} = (n+1)t_{i_0}, \lambda_n = n(t_{i_1} - t_{i_0})$,..., $\lambda_1 = t_{i_n} - t_{i_{n-1}}$. In general, $\lambda_k = k(t_{i_{n+1-k}} - t_{i_{n-k}})$ with $t_{i_{-1}} = 0$. Now define $\tilde{w}$ as $\sum \lambda_k \tilde{v}_k$. Then,

$$(\tilde{w})_{i_{n-k}} = \sum_{j=k+1}^{n+1} \frac{\lambda_j}{j} = \sum_{j=k+1}^{n+1} t_{i_{n+1-j}} - t_{i_{n-j}} = \sum_{j=0}^{n-k} t_{i_j} - t_{i_{j-1}} = t_{i_{n-k}} - t_{i_{-1}} = t_{i_{n-k}}.$$

Hence $\tilde{w} = w$, and $w \in \tilde{\Delta}^n$.

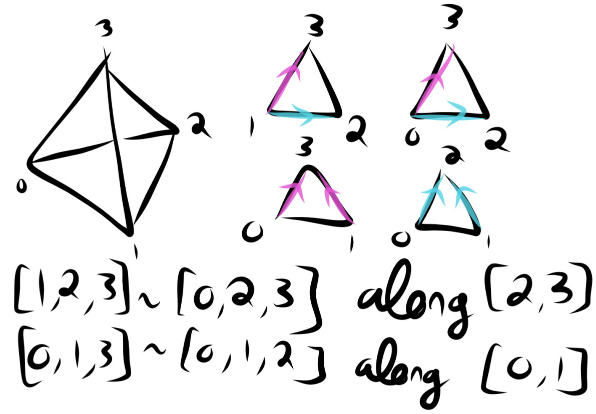
So, given an inequality $t_{i_0} \leq ... \leq t_{i_n}$, it defines one of the n-simplices in the barycentric subdivision. Since there are $n!$ of these inequalities (given by the permutations of $(0, ..., n)$), it follows that there is a bijective correspondence between the inequalities $t_{i_0} \leq ... \leq t_{i_n}$ and the n-simplices in the barycentric subdivision. Alternatively, one can consider choosing an n-simplex and label the vertices as we did in the explicit example with the 2-simplex. Then one can form the inequalities by using $\tilde{v}_1$ to define i_n, $\tilde{v}_2$ to define i_{n-1}, and so on.

HW7

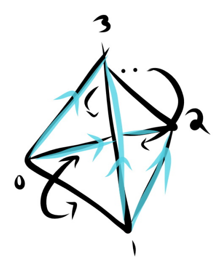
Hatcher Chapter 2.1, problems 7, 9, 11, 12, 14, 30:

2.1.7. Find a way of identifying pairs of faces of Δ^3 to produce a Δ-complex structure on S^3 having a single 3 simplex, and compute the simplicial homology groups of this Δ complex.

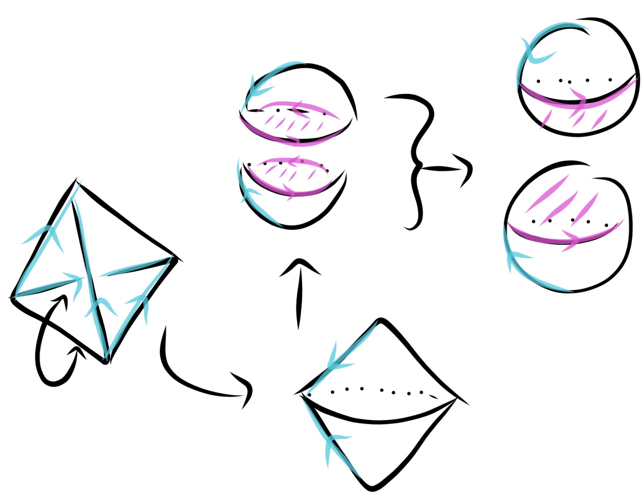
Solution: We must glue the faces of the tetrahedron together. We do this as follows: $[1, 2, 3] \sim [0, 2, 3]$ along $[2, 3]$ and $[0, 1, 3] \sim [0, 1, 2]$ along $[0, 1]$. This results in the following



The first relation tells us that, in the two faces drawn at the top right, same colored edges are identified. However, the second relation tells us that, in the two faces drawn below that, opposite colored edges are identified. Hence, all colored edges are identified.



The arrows tell us how to identify the faces. Now let us identify the front and bottom faces. This produces (after rotating the figure a bit), a double cone with two edges identified.



We can split this double cone in half, identifying the pink regions. Remember that the white regions on the boundary (but not the interior) are also identified (since they correspond to the remaining faces from the tetrahedron). Thus we have two 3-balls whose boundaries are identified – this is homemomorphic to S^3.

As for computing homology, we have one 3-simplex, two 2-simplices, three 1-simplices, and two 0-simplices. Note that $v_0 \sim v_1$ and $v_2 \sim v_2$. This may produce some loops in the chain groups, for example $[v_0, v_1]$. Thus

$$\begin{aligned} \Delta_3(X) &= \langle [v_0, v_1, v_2, v_3] \rangle \simeq \mathbb{Z} \\ \Delta_2(X) &= \langle [v_1, v_2, v_3], [v_0, v_1, v_2] \rangle \simeq \mathbb{Z} \oplus \mathbb{Z} \\ \Delta_1(X) &= \langle [v_0, v_1], [v_2, v_3], [v_0, v_2] \rangle \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \\ \Delta_0(X) &= \langle [v_0], [v_2] \rangle \simeq \mathbb{Z} \oplus \mathbb{Z} \end{aligned}$$

Let $\mathscr{A} = [v_0, v_1, v_2, v_3]$. Then

$$\begin{aligned} \partial_3(\mathscr{A}) &= [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2] \\ &= [v_0, v_2, v_2] - [v_0, v_2, v_2] + [v_0, v_0, v_2] - [v_0, v_0, v_2] = 0 \end{aligned}$$

which follows since $v_0 \sim v_1$ and $v_2 \sim v_3$ by our identifications. So $\text{Ker}(\partial_3) = \Delta_3(X) \simeq \mathbb{Z}$ and $\text{Im}(\partial_3) \simeq 0$. Since there are no n-simplices for $n \geq 4$, we have $\text{Im}(\partial_4) = 0$. Thus $H_3(X) \simeq \mathbb{Z}/0 \simeq \mathbb{Z}$.

Now let $A = [v_1, v_2, v_3]$ and $B = [v_0, v_1, v_2]$. Then,

$$\begin{aligned} \partial_2(A) &= [v_2, v_3] - [v_1, v_3] + [v_1, v_2] \\ &= [v_2, v_3] - [v_0, v_2] + [v_0, v_2] = [v_2, v_3] \end{aligned}$$

$$\begin{aligned} \partial_2(B) &= [v_1, v_2] - [v_0, v_2] + [v_0, v_1] \\ &= [v_1, v_1] - [v_0, v_2] + [v_0, v_1] = -[v_0, v_2] \end{aligned}$$

So $\text{Ker}(\partial_2) \simeq 0$ and $\text{Im}(\partial_2) \simeq \mathbb{Z} \oplus \mathbb{Z}$. It follows that $H_2(X) \simeq 0/0 \simeq 0$.

Let $a = [v_0, v_1]$, $b = [v_2, v_3]$, and $c = [v_0, v_2]$. Then,

$$\begin{aligned} \partial_1(a) &= [v_1] - [v_0] = [v_0] - [v_0] = 0 \\ \partial_1(b) &= [v_3] - [v_2] = [v_2] - [v_2] = 0 \\ \partial_1(c) &= [v_2] - [v_0] = [v_2] - [v_0] \end{aligned}$$

So $\text{Ker}(\partial_1) \simeq \mathbb{Z} \oplus \mathbb{Z}$ and $\text{Im}(\partial_1) \simeq \mathbb{Z}$. Then $H_1(X) \simeq \mathbb{Z} \oplus \mathbb{Z}/\mathbb{Z} \oplus \mathbb{Z} \simeq 0$. Finally $\text{Ker}(\partial_0) \simeq \mathbb{Z} \oplus \mathbb{Z}$ since $\partial_0 = 0$. Hence $H_0(X) \simeq \mathbb{Z} \oplus \mathbb{Z}/\mathbb{Z} \simeq \mathbb{Z}$. In summary, $H_n(X) = \mathbb{Z}$ for $n = 0, 3$ and $H_n(X) = 0$ else. This is precisely the homology of S^3 (though other spaces have the homology of S^3).

2.1.9. Compute the homology groups of the Δ-complex X obtained from Δ^n by identifying all faces of the same dimension. Thus X has a single k simplex for each $k \leq n$.

Solution: Let X^n denote the space given by Δ^n with all faces of the same dimension identified. I claim that the simplicial homology is given by

$$H_k^{\Delta}(X^n) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z} & k = n, \ n \text{ odd} \\ 0 & \text{else} \end{cases}$$

To give some intuition, it is easy to see that for $n = 0$ we have a point, $n = 1$ we have S^1.

Since there are no k-simplices for $k > n$, we see that $\Delta_k(X^n) \simeq 0$ for $k > n$. Since there is only one k-simplex for each $k \leq n$, we see that $\Delta_k(X^n) \simeq \mathbb{Z}$ for $k \leq n$. Choose a generator σ_k for each of these. Note that the restriction of σ_k to a $k-1$ dimensional face will just be σ_{k-1}. Thus,

$$\partial_k \sigma_k = \sum_{i=1}^{k+1} (-1)^i \sigma_{k-1}.$$

Clearly there are $k+1-1+1 = k+1$ many terms in the sum. Hence if k is even, the sum does not vanish, whereas if k is odd, it does. Summarized,

$$\partial_k \sigma_k = \begin{cases} 0 & k = 0 \\ 0 & k \leq n, \ k \text{ odd} \\ \sigma_{k-1} & k \leq n, \ k \text{ even} \\ 0 & k > n \end{cases}$$

where, by definition, the boundary map for $k = 0$ is just the zero map. Moreover, the case $k > n$ follows since X^n has no k-simplices when $k > n$. Hence

$$\text{Ker}(\partial_k) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z} & k \leq n, \ k \text{ odd} \\ 0 & k \leq n, \ k \text{ even} \\ 0 & k > n \end{cases} \qquad \text{Im}(\partial_k) = \begin{cases} 0 & k = 0 \\ 0 & k \leq n, \ k \text{ odd} \\ \mathbb{Z} & k \leq n, \ k \text{ even} \\ 0 & k > n \end{cases}$$

since each ∂_k is a homomorphism $\mathbb{Z} \to \mathbb{Z}$. If you write out the kernels and images as sequences, the parity of n determines the last group before a sequence of zeroes. This matters when computing the homology at $k = n$ since the kernel of ∂_k will depend on the parity of k (hence n), but the image will not! It follows that the homology groups are

$$H_k^{\Delta}(X^n) = \text{Ker}(\partial_k)/\,\text{Im}(\partial_{k+1}) = \begin{cases} \mathbb{Z}/0 & k = 0 \\ \mathbb{Z}/\mathbb{Z} & k < n, \ k \text{ odd} \\ 0/0 & k < n, \ k \text{ even} \\ \mathbb{Z}/0 & k = n, \ n \text{ odd} \\ 0/0 & k = n, \ n \text{ even} \\ 0/0 & k > n \end{cases}.$$

Restated in a cleaner way, this precisely says

$$H_k^{\Delta}(X^n) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z} & k = n, \ n \text{ odd} \\ 0 & \text{else} \end{cases}$$

2.1.11. Show that if A is a retract of X then the map $H_n(A) \to H_n(X)$ induced by the inclusion $A \subset X$ is injective.

Solution: Recall the functoriality of homology, which gives us the following

i) If $f : X \to Y$ and $g : Y \to Z$ then $(gf)_* = g_* f_*$.
ii) $(\text{Id}_X)_* = \text{Id}_{H_n(X)}$.

Now let A be a retract of X via a retraction $r : X \to A$. Then, with $i : A \to X$ the inclusion of A, we have $ri = \text{Id}_A$. Hence,

$$(ri)_* = r_* i_* = (\text{Id}_A)_* = \text{Id}_{H_n(A)}.$$

It follows that $i_* : H_n(A) \to H_n(X)$ is injective, since $\text{Id}_{H_n(A)}$ is injective. The important thing to note is that this occurs solely because of functoriality – so the same proof works no matter what functor we have!

2.1.12. Show that chain homotopy of chain maps is an equivalence relation.

Solution: We say that two chain maps $f : C_n \to C'_n$ and $g : C_n \to C'_n$ are chain homotopic if there exists a map $P : C_n \to C'_{n+1}$ such that $\partial P + P\partial = g - f$. Note that these maps are defined on C_n for all n, not just a particular fixed n. We show now that chain homotopy is an equivalence relation

- Reflexivity: Let f be a chain map. Choose P to be the zero map. Then, since ∂ is a homomorphism,
$$\partial P + P\partial = 0 = f - f.$$
So $f \sim f$.
- Symmetry: Let f, g be chain maps such that $f \sim g$. Then there exists a chain homotopy P such that $\partial P + P\partial = g - f$. Now consider $-P$. Since ∂ is a homomorphism,
$$\partial(-P) + (-P)\partial = -\partial P - P\partial = -(g - f) = f - g.$$
So, $g \sim f$ via $-P$.
- Transitivity: Let f, g, h be chain maps such that $f \sim g$ and $g \sim h$. Then there exist chain homotopies P, Q between f, g and g, h respectively. Consider $P + Q$. Once more, since ∂ is a homomorphism,
$$\partial(P + Q) + (P + Q)\partial = \partial P + \partial Q + P\partial + Q\partial = g - f + h - g = h - f.$$
So, $f \sim h$.

2.1.14. Determine whether there exists a short exact sequence $0 \to \mathbb{Z}_4 \to \mathbb{Z}_8 \oplus \mathbb{Z}_2 \to \mathbb{Z}_4 \to 0$. More generally, determine which abelian groups A fit into a short exact sequence $0 \to \mathbb{Z}_{p^m} \to A \to \mathbb{Z}_{p^n} \to 0$ with p prime. What about the case of short exact sequences $0 \to \mathbb{Z} \to A \to \mathbb{Z}_n \to 0$?

Solution: Consider a short exact sequence
$$0 \to A \to B \to C \to 0$$
with homomorphisms $\alpha : A \to B$ and $\beta : B \to C$ and A, B, C finite abelian groups. Since this sequence is exact at B we have that α is injective, β is surjective, and $\text{Ker}(\beta) = \text{Im}(\alpha)$. By the first isomorphism theorem, $C = \text{Im}(\beta) \simeq B/\text{Ker}(\beta) = B/\text{Im}(\alpha)$. By another application of the first isomorphism theorem, we see that $\text{Im}(\alpha) \simeq A/\text{Ker}(\alpha) \simeq A$ since α is injective. Thus we can write $C \simeq B/A$. In particular, this says that $|B| = |A||C|$. This gives a necessary criteria on A, B, C for a sequence $0 \to A \to B \to C \to 0$ to be exact.

Note that $|\mathbb{Z}_4| = 4$ and $|\mathbb{Z}_8 \oplus \mathbb{Z}_2| = 16$, so that we have $|\mathbb{Z}_8 \oplus \mathbb{Z}_2| = |\mathbb{Z}_4||\mathbb{Z}_4|$. So there is potential to find a short exact sequence. Let us start by finding α, which needs to be injective. We have the following small lemma from algebra

Lemma: If $\phi : G \to H$ is an injective homomorphism then ϕ preserves the order of elements.

This follows from the first isomorphism theorem since $\text{Im}(\alpha) \simeq A$. Now any homomorphism from $\mathbb{Z}_4$ is determined by its action on a generator, e.g. 1. Now 1 has order 4, so it must also be mapped to something of order 4 in $\mathbb{Z}_8 \oplus \mathbb{Z}_2$. This amounts to finding an order 4 element of $\mathbb{Z}_8$ (since the order in the direct sum is just the lcm of the orders, and anything in $\mathbb{Z}_2$ has either order 1 or 2). Take, for example, 2. So we can define α by $\alpha(1) = (2, 1)$ (or $(2, 0)$, both give something injective). Now we must find β, which needs to be surjective and such that $\text{Ker}(\beta) = \text{Im}(\alpha)$. The image of α is precisely $\{(2, 1), (4, 0), (6, 1), (0, 0)\}$. Then, the quotient $\mathbb{Z}_8 \oplus \mathbb{Z}_2/\text{Im}(\alpha)$ is a group of order 4. Consider the element $(1, 0) + \text{Im}(\alpha)$. This has order 4 since $4(1, 0) = (4, 0) \in \text{Im}(\alpha)$. So, it generates $\mathbb{Z}_8 \oplus \mathbb{Z}_2/\text{Im}(\alpha)$, and hence $\mathbb{Z}_8 \oplus \mathbb{Z}_2/\text{Im}(\alpha) \simeq \mathbb{Z}_4$. We can take β to be the quotient map composed with an isomorphism. E.g. $\beta(a, b) = a + 2b$.

Recall that we could choose $\alpha(1) = (2, 0)$. Can we get a valid β in this case? The image of α would become $\text{Im}(\alpha) = \{(0, 0), (2, 0), (4, 0), (6, 0)\}$. Then $(1, 0) + \text{Im}(\alpha)$ has order 2 while $(1, 1) + \text{Im}(\alpha)$ also has order 2. These two elements generate all four cosets, hence $\mathbb{Z}_8 \oplus \mathbb{Z}_2/\text{Im}(\alpha) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$. So the choice of α matters!

For the more general exact sequence $0 \to \mathbb{Z}_{p^m} \to A \to \mathbb{Z}_{p^n} \to 0$ it must be that $|A| = p^{m+n}$. Since A is a finite abelian group, we can write it as
$$A \simeq \mathbb{Z}_{p^{m_1}} \oplus \mathbb{Z}_{p^{m_2}} \oplus ... \oplus \mathbb{Z}_{p^{m_k}}$$

for some $k \geq 1$ where $m_1 + m_2 + ... + m_k = m + n$ and $m_i \geq 0$ for all i (there are no copies of $\mathbb{Z}$ since A is a finite group). We first determine k. To do so, consider some $x \in A$. Then either $x \in \text{Im}(\alpha)$ or not. Set $a = \alpha(1)$ and $b \in A$ such that $\beta(b) = 1$ (which exists by surjectivity). If $x \in \text{Im}(\alpha)$ then it is some multiple of a. If $x \notin \text{Im}(\alpha)$, then since $\text{Im}(\alpha) = \text{Ker}(\beta)$ we know that $\beta(x) \neq 0$. Hence, there exists a $s \neq 0$ such that $\beta(x) = s$. Since β is a homomorphism, $\beta(x) = s\beta(b) = \beta(sb)$. But then $\beta(x - sb) = s - s = 0$, so that $x - sb \in \text{Ker}(\beta) = \text{Im}(\alpha)$. It follows that $x = ta + sb$ for some t, and x is generated by a, b. We cannot have that $a = b$ since $a \in \text{Im}(\alpha) = \text{Ker}(\beta)$, which would imply that $\beta(b) = 0 \neq 1$, a contradiction. Thus there are two generators of A and

$$A \simeq \mathbb{Z}_{p^{m+n-t}} \oplus \mathbb{Z}_{p^t}.$$

for $0 \leq t \leq m + n$ are possible. But we also know that $\mathbb{Z}_{p^m}$ must inject into A, so some element of A will have order p^m. This only occurs if $\max\{m + n - t, t\} \geq m$. Hence $0 \leq t \leq \min\{m, n\}$.

Now define α_t by $\alpha_t(1) = (p^{n-t}, 1)$. Analogously to the previous example, we show that $(1, 0) + \text{Im}(\alpha_t)$ has order p^n in $A_t/\text{Im}(\alpha_t)$. Suppose we have a coset represented by (x, y). Then $(x, y) - y(p^{n-t}, 1)$ represents the same coset (since the difference of the two representatives is in $\text{Im}(\alpha_t)$). On the other hand,

$$(x, y) - y(p^{n-t}, 1) = (x - yp^{n-t}, 0) = (x - yp^{n-t})(1, 0)$$

which implies that each coset is represented as a multiple of $(1, 0)$! In particular, we get a unique coset for each $0 \leq t \leq n - 1$ so that $(1, 0) + \text{Im}(\alpha_t)$ has order p^n. So $(1, 0) + \text{Im}(\alpha_t)$ generates the quotient, because the quotient has order p^n. Thus $A_t/\text{Im}(\alpha_t) \simeq \mathbb{Z}_{p^n}$, and we may take β_t as the composition of the quotient map and an isomorphism.

For the third sequence, we similarly apply the fundamental theorem and obtain

$$A \simeq \mathbb{Z} \oplus \mathbb{Z}_{p_1^{m_1}} \oplus ... \oplus \mathbb{Z}_{p_k^{m_k}}.$$

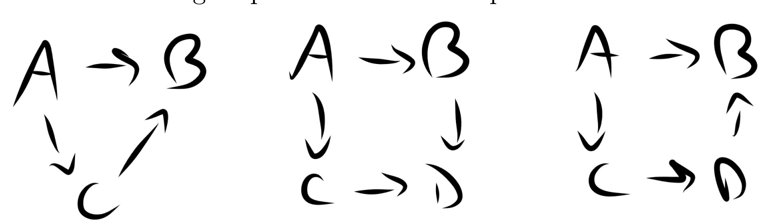
where $p_1^{m_1}...p_k^{m_k} = n$. By the exact same logic as previously, we can show that A has two generators. Hence $k = 1$. Let $A_d = \mathbb{Z} \oplus \mathbb{Z}_d$. We show that the sequence is exact only if d divides n. Define $\alpha_d : \mathbb{Z} \to A_d$ by $\alpha_d(1) = (q, 1)$ where $q = n/d$. It follows that α is injective since $\alpha_d(a) = (aq, \text{mod}(a, d)) = 0$ iff $a = 0$. Now define $\beta_d : A_d \to \mathbb{Z}_n$ by $\beta_d(a, b) = \text{mod}(a - bq, n)$. It is clear that β_d is surjective since $\beta_d(1, 0) = 1$.

Is it that $\text{Ker}(\beta_d) = \text{Im}(\alpha_d)$? We can write $\text{mod}(a, d)$ as $a - kd$ for some integer k. Then, $\beta_d(\text{Im}(\alpha_d)) = \text{mod}(aq - (a - kd)q, n) = \text{mod}(kn, n) = 0$ by choice of q. So $\text{Im}(\alpha_d) \subset \text{Ker}(\beta_d)$. Now let $(a, b) \in \text{Ker}(\beta_d)$. Then $a - bq = kn$ for some integer k. Hence

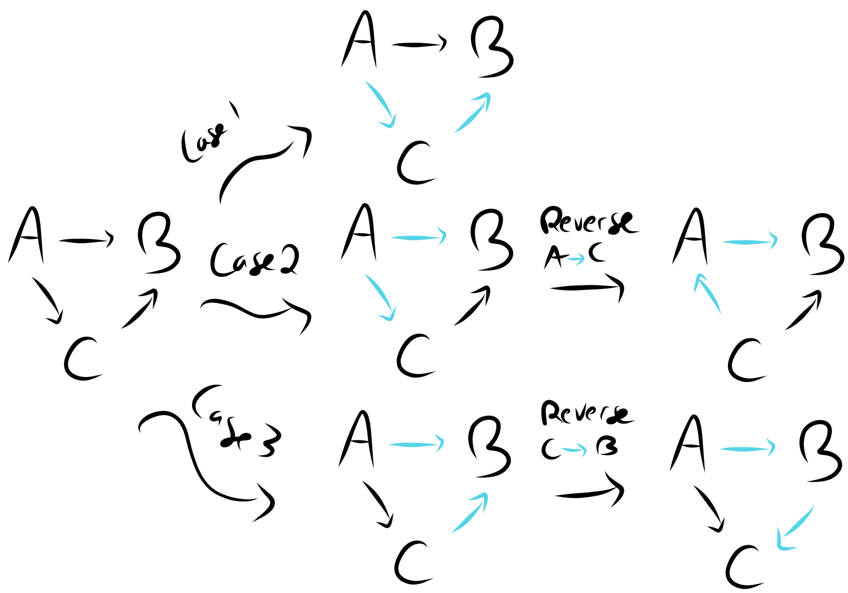
$$(a, b) = (bq + kn, b) = b(q, 1) + (kn, 0) = b(q, 1) + k(qd, d) = (b + kd)(q, 1)$$

since $\text{mod}(d, d) = 0$ and by choice of q. Thus (a, b) is a multiple of $(q, 1)$ and is in the image of α_d. It follows that A_d for each d dividing n admits an exact sequence.

2.1.30. In each of the following commutative diagrams assume that all maps but one are isomorphisms. Show that the remaining map must be an isomorphism as well.

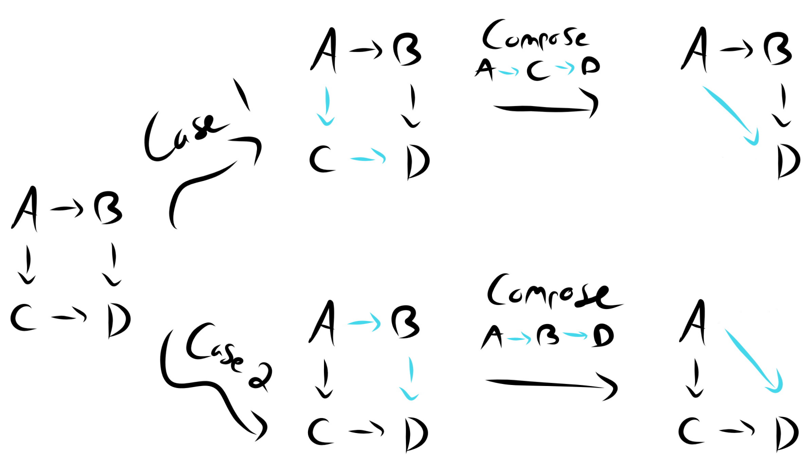


Solution: We refer to the three diagrams above as the first, second, and third diagrams respectively. Let us look at the first commutative diagram. There are three cases to consider, each corresponding to assuming that all but one of the maps are isomorphisms. In the figure, the isomorphisms are colored blue.



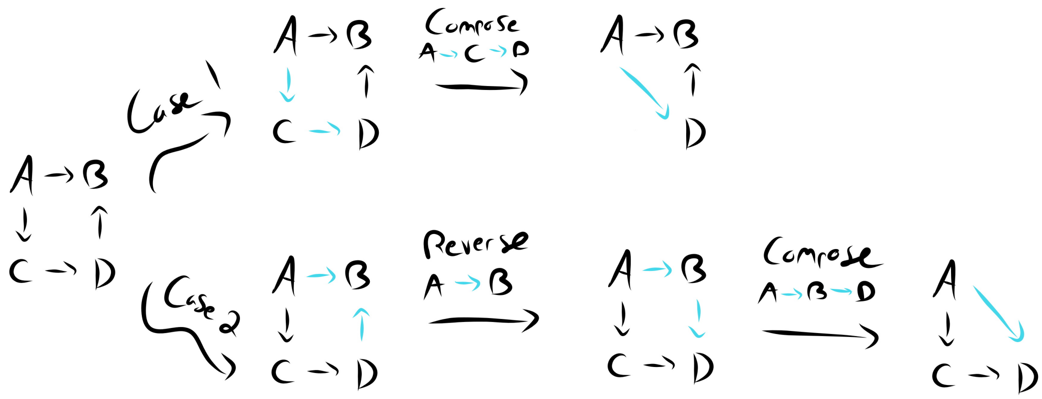
Now note that we can reverse arrows in cases 2 and 3 (namely, $A \to C$ and $C \to B$ respectively). These give two diagrams which are identical to that in case 1 (up to renaming). So, we need only prove that the remaining map $A \to B$ in case 1 is an isomorphism. But, since the diagram commutes, this is precisely the composition $A \to C \to B$, which is a composition of isomorphisms. Hence $A \to B$ is an isomorphism.

Let us now look at the four cases of the second diagram. We can group these into two cases: each where two maps are isomorphisms and the two remaining are undetermined; one of these two is an isomorphism, while the other isn't. As before, isomorphisms are blue.

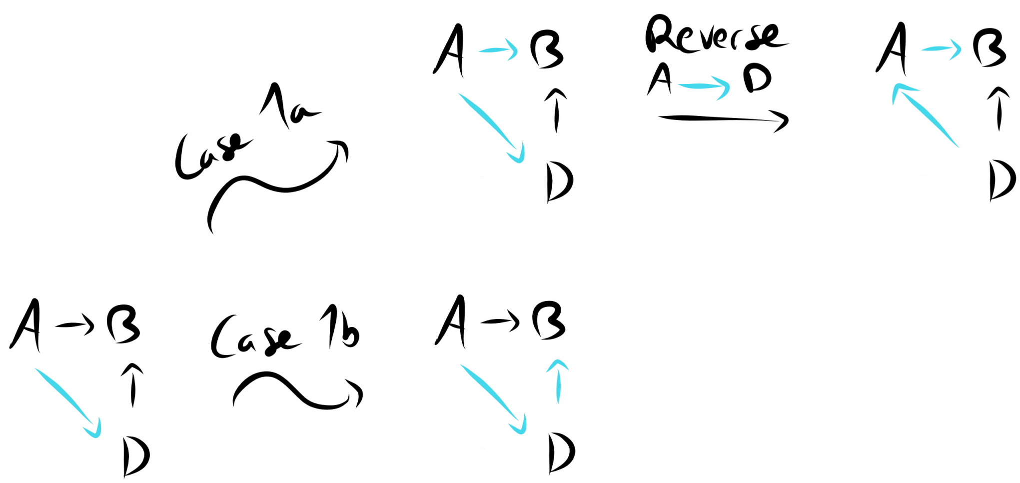


We can compose the two isomorphisms in each case to get commutative diagrams only involving three groups, as in the first diagram. Each remaining commutative diagram corresponds to two cases. E.g., the first corresponds to the cases where $A \to D$ and $A \to B$ or $B \to D$ are isomorphisms. These are two cases of the first diagram. But, we proved that all the cases of the first diagram result in isomorphisms for the remaining map.

Finally, let us investigate the third diagram, which is the same as the second except for one reversed map. We can perform the same operations (reversing and composing isomorphisms), resulting in the following.



Once more, the final diagram in case 2 reverts to the first diagram, so there is nothing to do. As for the diagram in case 1, we cannot simplify it further. However, analyzing the two cases it produces allows us to continue.



which produces two cases from the first diagram. So, there is nothing left to prove.

HW8

Hatcher Chapter 2.1, problems 15, 16, 20, 21, 22, 29:

2.1.15. For an exact sequence $A \to B \to C \to D \to E$ show that $C = 0$ iff the map $A \to B$ is surjective and $D \to E$ is injective. Hence for a pair of spaces (X, A), the inclusion $A \hookrightarrow X$ induces isomorphisms on all homology groups iff $H_n(X, A) = 0$ for all n.

Solution: Label the maps $\alpha : A \to B$, $\beta : B \to C$, $\gamma : C \to D$, and $\delta : D \to E$. Suppose first that $C = 0$. There is only one homomorphism from and to the trivial group, namely the zero map. hence $\beta = \gamma = 0$. Thus we get two short exact sequences

$$A \to B \to 0 \qquad 0 \to D \to E$$

which precisely say that $A \to B$ is surjective and $D \to E$ is injective.
Now suppose $A \to B$ is surjective and $D \to E$ is injective. Since $D \to E$ is injective we see that $\text{Ker}(\delta) = 0$. Hence, $\text{Im}(\gamma) = 0$. Since γ is the zero map, $\text{Ker}(\gamma) = C$. But $\text{Im}(\beta) = \text{Ker}(\gamma) = C$. OTOH, α is surjective so that $\text{Im}(\alpha) = B$. By exactness at B, $\text{Ker}(\beta) = B$. So, β sends all of B to 0, while simultaneously being onto C. It follows that C is the trivial group.

By Theorem 2.16, we have that

$$... \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \to ...$$

is exact with $i : A \hookrightarrow X$, $j : X \to X/A$ the quotient, and ∂ the connecting map. Supposing that $H_n(X, A) = 0$ for all n we have an exact sequence

$$... \to 0 \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} 0 \to ...$$

which exactly means that i_* is an isomorphism. Now assume i_* is an isomorphism. Then,

$$... \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \to ...$$

Then, $i_* : H_n(A) \to H_n(X)$ is surjective while $i_* : H_{n-1}(A) \to H_{n-1}(X)$ is injective. It follows by the above, since the sequence is exact, that $H_n(X, A) = 0$ for all $n > 0$. For the $n = 0$ case we have

$$... \to 0 \to H_0(A) \xrightarrow{i_*} H_0(X) \xrightarrow{j_*} H_0(X, A) \xrightarrow{\partial} 0$$

Since i_* is an isomorphism, $\text{Im}(i_*) = H_0(X)$. By exactness at $H_0(X)$, we see that $\text{Ker}(j_*) = H_0(X)$ so that j_* is the zero map. We know that ∂ is the zero map, since this is the only homomorphism to the trivial group. Exactness then implies that $H_0(X, A) = \text{Ker}(\partial) = \text{Im}(j_*) = 0$.

2.1.16.

a) Show that $H_0(X, A) = 0$ iff A meets each path-component of X.
b) Show that $H_1(X, A) = 0$ iff $H_1(A) \to H_1(X)$ is surjective and each path-component of X contains at most one path-component of A.

Solution:

a) Let us consider the long exact sequence formed by the pair (X, A):

$$... \to H_1(X, A) \xrightarrow{\partial} H_0(A) \xrightarrow{i_*} H_0(X) \xrightarrow{j_*} H_0(X, A) \to 0$$

Then, $H_0(X, A) = 0$ iff $\text{Im}(i_*) = \text{Ker}(j_*) = H_0(X)$. That is iff i_* is onto. The group $H_0(X)$ is exactly free abelian with basis generated by the path components, since we can always decompose H_0 into a direct sum of H_0 of the path components. Now i_* acts on equivalence classes in $H_0(A)$ by $i_*[a] = [i(a)]$ where a is a cycle. For i_* to be surjective, this means that for each $[b] \in H_0(X)$ there exists a cycle a such that $i(a) = b$. Hence, A has nontrivial intersection with the cycle b in X. Taking each b corresponding to a path component gives the result.

b) We consider the same long exact sequence, but a little farther up:

$$... \to H_2(X,A) \xrightarrow{\partial} H_1(A) \xrightarrow{i_*} H_1(X) \xrightarrow{j_*} H_1(X,A) \to H_0(A) \to ...$$

By the same logic as above, $H_1(X,A) = 0$ implies i_* is surjective. But, by looking at the end of the sequence we also see that $H_0(A) \to H_0(X)$ is injective. Conversely, if $H_0(A) \to H_(X)$ is injective and $H_1(A) \to H_1(X)$ is surjective then problem 2.1.15 implies $H_1(X,A) = 0$.

We now show i_* injective iff each path-component of X contains at most one path-component of A. The free abelian group $H_0(A)$ is also generated by a cycle for each path component. Suppose two path components A_1, A_2 of A are included in a path component X_k of X. Let a_1, a_2 be the cycles corresponding to these path components. Then, $i(a_1), i(a_2)$ both map into X_k. It follows that $[i(a_1)] = [i(a_2)]$, hence by injectivity of i_* that $[a_1] = [a_2]$.

Suppose $i_* : H_0(A) \to H_0(X)$ is not injective. Then we can find some element $[a] \neq 0 \in H_0(A)$ such that $i_*[a] = 0$. Let I index the path components and a_k a cycle in each A_k. Then,

$$[a] = \sum_{k \in I} c_k [a_k]$$

where the $c_k \in \mathbb{Z}$. Now, $i_*[a]$ simply says to view all the $[a_k]$ in X rather than in A. By hypothesis, $i_*[a] = 0$. If all the $[a_k]$ represented different path components, we would necessarily have that $c_k = 0$ for all k. Hence, $[a] = 0$ a contradiction.

2.1.20. Show that $\tilde{H}_n(X) \simeq \tilde{H}_{n+1}(SX)$ for all n, where SX is the suspension of X. More generally, thinking of SX as the union of two cones CX with their bases identified, compute the reduced homology groups of the union of any finite number of cones CX with their bases identified.

Solution: The suspension SX of X is defined to be the disjoint union of two cones of X with their bases identified. We will write this as $SX = C_1X \cup C_2X$, where it is understood that we identify the bases of the two cones. Since C_2X is contractible we see that

$$... \to 0 \to \tilde{H}_n(SX) \to \tilde{H}_n(SX, C_2X) \to 0 \to ...$$

implies $\tilde{H}_n(SX) \simeq \tilde{H}_n(SX, C_2X) \simeq H_n(SX, C_2X)$ for all n. Since C_1X and C_2X share a base, and that base is homeomophic to X, we have $C_1X \cap C_2X = X$. Now apply excision to $(B, A \cap B) = (C_1X, X)$ and $(X, A) = (SX, C_2X)$. Then $H_n(SX, C_2X) \simeq H_n(C_1X, X)$ for all n. (Technically here we may need to take a slightly enlarged copy of C_1X and C_2X so that the interiors cover SX. But, these enlarged copies deformation retract to C_1X and C_2X respectively, and the intersection deformation retracts to X, so there should be nothing to worry about). Finally, the long exact sequence for the pair (C_1X, X) reads

$$... \to \tilde{H}_{n+1}(C_1X) \to \tilde{H}_{n+1}(C_1X, X) \to \tilde{H}_n(X) \to \tilde{H}_n(C_1X)...$$

and by applying contractibility of C_1X we obtain $H_{n+1}(C_1X, X) \simeq \tilde{H}_{n+1}(C_1X, X) \simeq \tilde{H}_n(X)$. Thus, combined with the previous, $\tilde{H}_{n+1}(SX) \simeq \tilde{H}_n(X)$.

Notice that the above argument worked since C_1X and C_2X were contractible and glued along a common base. Let S_kX denote the suspension of X with k copies of CX glued along a common base. It is clear that $S_{k+1}X = S_kX \cup C_{k+1}X$ where the two are glued along a base. We can do exactly the same work as before with C_2X replaced by $C_{k+1}X$ and C_1X replaced by S_kX, until we make use of contractibility of C_1. Indeed, because $\tilde{H}_{n+1}(SX) \simeq \tilde{H}_n(X)$, it need not be that S_kX is contractible. So, we have

$$\tilde{H}_{n+1}(S_{k+1}X) \simeq H_{n+1}(S_kX, X).$$

On the other hand, we know that $CX/X \simeq SX$. One way to see this is as follows: view CX as the join $X * \{p\}$ where p is a point not in X. For the join $X * Y$ of two spaces, we always obtain a homeomorphic copy of $X \times Y$ embedded inside at the halfway point $X \times Y \times \{1/2\}$. With this in mind, we see that CX/X crushes the base to a point, but there will still be a homeomorphic copy of X inside (at the halfway point). So $CX/X \simeq SX$. Now, crushing the identified base of several cones to a point makes it so that the newly formed tips are wedged at a point. So $S_kX/X \simeq SX \vee SX \vee ... \vee SX$ with k copies. Hence,

$$\tilde{H}_{n+1}(S_kX, X) \simeq \tilde{H}_{n+1}(S_kX/X) \simeq \tilde{H}_{n+1}(SX \vee ... \vee SX) \simeq \bigoplus_{i=1}^{k} \tilde{H}_{n+1}(SX) \simeq \bigoplus_{i=1}^{k} \tilde{H}_n(X)$$

for all n, where we have applied the previous result.

2.1.21. Making the preceding problem more concrete, construct explicit chain maps $s : C_n(X) \to C_{n+1}(SX)$ inducing isomorphisms $\tilde{H}_n(X) \simeq \tilde{H}_{n+1}(SX)$.

Solution: Consider the following sequence of chain complexes:

$$C_n(X) \simeq C_n(X, *) \to C_{n+1}(CX, X) \to C_{n+1}(SX, *) \simeq C_{n+1}(SX).$$

Recall that in the computation in 2.1.20, we showed that $\tilde{H}_n(X) \simeq \tilde{H}_{n+1}(CX, X) \simeq \tilde{H}_{n+1}(SX)$, so this is a natural sequence to consider a posteriori. What are the maps above given by? The first takes any singular simplex $\sigma : \Delta^n \to X$ to $C\Delta^n \to CX$. Note that the cone of an n-simplex Δ^n is an $n+1$ simplex, so we may view this as mapping to $\Delta^{n+1} \to CX$. For any point $(t_0, ..., t_{n+1})$ in Δ^{n+1}, we define $C\sigma : \Delta^{n+1} \to CX$ by

$$(t_0, ..., t_{n+1}) \mapsto \sum_{i=0}^{n} t_i \sigma(v_i) + t_{n+1} p$$

where p is cone point of CX. It is basically just taking the simplex σ, inserting it into the copy of X at the base of CX, and then producing the cone of the simplex afterwards. Notice that this definition immediately implies

$$(C\sigma)|_{[v_0,...,\hat{v}_i,...,v_{n+1}]} = C(\sigma|_{[v_0,...,\hat{v}_i,...,v_{n+1}]}),\ 0 \le i \le n \quad (C\sigma)|_{[v_0,...,v_n,\hat{v}_{n+1}]} = \sigma$$

since removing v_i more or less just says to take $t_i = 0$. Computing $\partial(C\sigma)$ gives

$$\begin{aligned} \partial(C\sigma) &= \sum_{i=0}^{n+1} (-1)^i (C\sigma)|_{[v_0,...,\hat{v}_i,...,v_{n+1}]} \\ &= \sum_{i=0}^{n} (-1)^i C(\sigma|_{[v_0,...,\hat{v}_i,...,v_{n+1}]}) + (-i)^{n+1}\sigma \\ &= C(\partial\sigma) + (-i)^{n+1}\sigma \end{aligned}$$

We are able to move the sum inside the argument of C since C is linear over chains.

Consider now the long exact sequence formed by the triple $(CX, X, *)$:

$$... \to \tilde{H}_{n+1}(CX, *) \to \tilde{H}_{n+1}(CX, X) \xrightarrow{\partial} \tilde{H}_n(X, *) \to \tilde{H}_n(CX, *) \to ...$$

But CX is contractible so

$$... \to 0 \to \tilde{H}_{n+1}(CX, X) \xrightarrow{\partial} \tilde{H}_n(X, *) \to 0 \to ...$$

implies ∂ is an isomorphism by exactness. Let f be the linear extension of the map defined above taking $\sigma \to C\sigma$. We show that $f_* \circ \partial$ is the identity. For a relative cycle $\alpha \in C_{n+1}(CX, X)$ we have that $\partial\alpha$ lies in $C_n(X)$. But $f_*[\partial\alpha] = [f(\partial\alpha)]$, and f acts by extending $\partial\alpha$ to a cone. This precisely regenerates α. So, we have shown

$$f_*\partial[\alpha] = f_*[\partial\alpha] = [\alpha]$$

and thus $f_* \circ \partial$ is the identity. Since ∂ is an isomorphism we know that f_* is also an isomorphism.

We showed in 2.1.20 that $CX/X \simeq SX$. Recall that the quotient map $q : (X, A) \to (X/A, A/A)$ for a good pair (X, A) induces an isomorphism $q_* : H_*(X, A) \to H_*(X/A, A/A) \simeq H_*(X/A)$. We know that the quotient map is a chain map, and (CX, X) is a good pair since if $CX = X \times [0, 1]/(X \times \{1\})$ then $X \times [0, 1/2)$ deformation retracts to X. So we have a chain map $(CX, X) \to (CX/X, X) \simeq (CX, *)$. Moreover, q_* is an isomorphism. Thus the composition $q_* \circ f_*$ is an isomorphism. Our chain map will be $q \circ f$. This is because q quotients out $C_n(X)$, and so the σ produced by taking $\partial(C\sigma)$ is neglected.

2.1.22. Prove by induction on dimension the following facts about the homology of a finite-dimensional CW complex X, using the observation that X_n/X_{n-1} is a wedge sum of n spheres:

a) If X has dimension n then $H_i(X) = 0$ for $i > n$ and $H_n(X)$ is free.
b) $H_n(X)$ is free with basis in bijective correspondence with the n cells if there are no cells of dimension $n-1$ or $n+1$.
c) If X has k n-cells, then $H_n(X)$ is generated by at most k elements.

Solution:

a) First we see that

$$H_i(X_k, X_{k-1}) \simeq H_i(X_k/X_{k-1}) \simeq H_i\left(\bigvee_\alpha S^k\right) \simeq \bigoplus_\alpha H_i(S^k) = \begin{cases} \oplus_\alpha \mathbb{Z} & i = k \\ 0 & \text{else} \end{cases}$$

where α indexes the number of k-cells in X. Fix n the dimension of X. Suppose $i > n$ and consider now the long exact sequence

$$... \to H_i(X_{n-1}) \to H_i(X_n) \to H_i(X_n, X_{n-1}) \to ...$$

Since X has dimension n, we know that $X_n = X$. We also know that $H_i(X, X_{n-1}) \simeq 0$ since $i \neq k = n$. To proceed we need some information about $H_i(X_{n-1})$. Since X_{n-1} is an $n-1$ dimensional CW complex, and $i > n$, then $i > n-1$. So we use induction: suppose the claim holds up to dimension $n-1$. Then we have $H_i(X_{n-1}) \simeq 0$, and therefore

$$... \to 0 \to H_i(X_n) \to 0 \to ...$$

is exact at $H_i(X_n)$. It follows that $H_i(X_n) \simeq 0$. This is the inductive step, but we still need the base case. However, the base case is just if X is a 0-dimensional CW complex, i.e. a disjoint union of points α. It follows that

$$H_i(X) = \begin{cases} \oplus_\alpha \mathbb{Z} & i = 0 \\ 0 & \text{else} \end{cases}$$

This also shows that $H_n(X)$ is free for $n = 0$. Returning to the general case, let us look at the following part of our long exact sequence

$$... \to H_n(X_{n-1}) \to H_n(X_n) \to H_n(X_n, X_{n-1}) \to ...$$

by applying what we've shown so far,

$$... \to 0 \to H_n(X_n) \xrightarrow{j_*} \bigoplus_\alpha \mathbb{Z} \to ...$$

where α indexes the n-cells. By exactness at $H_n(X_n)$, we see that j_* is injective. Therefore $H_n(X) = H_n(X_n)$ is a subgroup of $\oplus_\alpha \mathbb{Z}$ and hence is free. We see that $H_n(X)$ is generated by at most the number of n-cells in X. We only have this result when n is the dimension of X – we will see in part c) that this holds for the general homology group.

b) Throughout, n is fixed and not necessarily the dimension of X. Since there are no $n-1$ cells, we see that $X_{n-1} = X_{n-2}$. Hence the exact sequence

$$... \to H_n(X_{n-1}) \to H_n(X) \to H_n(X, X_{n-1}) \to H_{n-1}(X_{n-2}) \to ...$$

gives, by an application of a),

$$... \to 0 \to H_n(X) \to H_n(X, X_{n-1}) \to 0 \to ...$$

Thus $H_n(X) \simeq H_n(X, X_{n-1})$.

We finish if $H_n(X, X_{n-1}) \simeq H_n(X_n, X_{n-1})$ since $H_n(X_n, X_{n-1})$ is the free group generated by the n-cells. To this end, we can form a long exact sequence using the triple (X, X_n, X_{n-1}) (there is a short exact sequence $0 \to C_*(X_n, X_{n-1}) \to C_*(X, X_{n-1}) \to C_*(X, X_n) \to 0$ of the chain complexes), which is

$$... \to H_{n+1}(X, X_n) \to H_n(X_n, X_{n-1}) \to H_n(X, X_{n-1}) \to H_n(X, X_n) \to ...$$

Since there are no $n+1$ cells, we have $X_n = X_{n+1}$. All that remains is to show the two outside groups, $H_{n+1}(X, X_{n+1})$ and $H_n(X, X_n)$ are trivial.

To achieve this, recall that $H_i(X_n, X_{n-1}) = 0$ unless $i = n$, in which case it is free. We thus see that exactness of

$$... \to H_{i+1}(X_n, X_{n-1}) \to H_i(X_{n-1}) \to H_i(X_n) \to H_i(X_n, X_{n-1}) \to ...$$

implies $H_i(X_{n-1}) \to H_i(X_n)$ is injective if $i \neq n-1$ and $H_i(X_{n-1}) \to H_i(X_n)$ is surjective if $i \neq n$. One way to see these restrictions is to look at the indices on the homology group and the skeleton in the domain (for the injections) and the range (for the surjections). Let us look at the injections. In particular, we can find an injection $H_i(X_{n-1}) \to H_i(X_n)$ for any $n-1 > i$. Then,

$$H_{n-1}(X_n) \hookrightarrow H_{n-1}(X_{n+1}) \hookrightarrow ... \hookrightarrow H_{n-1}(X)$$

and there is an injection $H_{n-1}(X_n) \hookrightarrow H_{n-1}(X)$. Accordingly, for the surjcections, we can do so for $n > i-1$. Then,

$$H_n(X_n) \twoheadrightarrow H_n(X_{n+1}) \twoheadrightarrow ... \twoheadrightarrow H_n(X)$$

and there is a surjection $H_n(X_n) \twoheadrightarrow H_n(X)$. Finally, let us look at the long exact sequence for the pair (X, X_n).

$$... \to H_n(X_n) \twoheadrightarrow H_n(X) \to H_n(X, X_n) \to H_{n-1}(X_n) \hookrightarrow H_{n-1}(X) \to ...$$

By 2.1.15., we see that $H_n(X, X_n) \simeq 0$.

c) As a consequence of the injections and surjections from b), we see that $H_n(X) \simeq H_n(X^k)$ for $k > n$. In particular, $H_n(X) \simeq H_n(X_{n+1})$. Now consider the long exact sequence for the pair (X_n, X_{n+1}):

$$... \to H_{n+1}(X_{n+1}, X_n) \to H_n(X_n) \to H_n(X_{n+1}) \to H_n(X_{n+1}, X_n) \to ...$$

We then obtain

$$... \to H_{n+1}(X_{n+1}, X_n) \to H_n(X_n) \xrightarrow{j_*} H_n(X) \to 0 \to ...$$

so that j_* is surjective. But by a), since X_n has dimension n, $H_n(X_n)$ is generated by at most the number of n-cells.

2.1.29. Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups in all dimensions, but their universal covering spaces do not.

Solution: We have already computed the homology of the torus: namely, it is

$$H_n(T^2) = \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ 0 & \text{else} \end{cases}$$

So we compute the homology of $S^1 \vee S^1 \vee S^1$. By Corollary 2.25 we have that for a wedge sum $\bigvee_\alpha X_\alpha$, if the basepoints $x_\alpha \in X_\alpha$ are such that (X_α, x_α) is a good pair then

$$\bigoplus_\alpha \tilde{H}_n(X_\alpha) \simeq \tilde{H}_n\left(\bigvee_\alpha X_\alpha\right).$$

We can always put a CW complex structure on S^1, S^2 so that, with a 0-cell, they form good pairs. For example, we can write S^1 as a 0-cell v with a 1-cell e attached so that both ends are glued to v. We can then extend v to a slightly larger neighborhood (homeomorphic to an interval) inside S^1, which contracts to v. Hence,

$$\tilde{H}_n(S^1 \vee S^1 \vee S^2) \simeq \tilde{H}_n(S^1) \oplus \tilde{H}_n(S^1) \oplus \tilde{H}_n(S^2).$$

Recall that the reduced homology of a sphere is

$$\tilde{H}_i(S^n) = \begin{cases} \mathbb{Z} & i = n \\ 0 & \text{else} \end{cases}$$

Thus,

$$\tilde{H}_n(S^1 \vee S^1 \vee S^2) \simeq \begin{cases} \mathbb{Z} \oplus \mathbb{Z} \oplus 0 & n = 1 \\ 0 \oplus 0 \oplus \mathbb{Z} & n = 2 \\ 0 \oplus 0 \oplus 0 & \text{else} \end{cases} \simeq \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ \mathbb{Z} & n = 2 \\ 0 & \text{else} \end{cases}$$

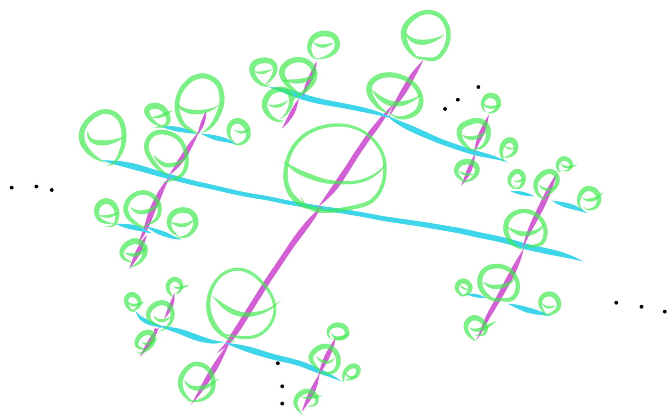
Hence, converting from reduced homology,

$$H_n(S^1 \vee S^1 \vee S^2) \simeq \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ 0 & \text{else} \end{cases} \simeq H_n(T^2)$$

as was to be shown.

A universal cover of the torus is just $\mathbb{R}^2$ with the covering map p identifying $(x, y) \sim (x + m, y + n)$ for integers n, m (this splits $\mathbb{R}^2$ into a grid of unit squares, identifies all the squares, then identifies opposite edges with the correct orientation). But $\mathbb{R}^2$ is contractible and hence has homology of a point.

The universal cover of $S^1 \vee S^1 \vee S^2$ can be obtained as follows: We can form $S^1 \vee S^1 \vee S^2$ by wedging S^2 at the wedge point of $S^1 \vee S^2$. The effect of this is to put a copy of S^2 at each vertex in the universal cover of $S^1 \vee S^1$. This is shown below.



The covering map is obvious; it takes the different colored line segments and maps them appropriately to the different copies of S^1. The spheres to to the copy of S^2.

Contracting along the line segments in the trees gives a countable wedge sum of spheres, which certainly has nonzero second homology.

HW9

Hatcher Chapter 2.1, problem 28:

2.1.28. Let X be the cone on the 1-skeleton of Δ^3, the union of all line segments joining points in the six edges of Δ^3 to the barycenter of Δ^3. Compute the local homology groups $H_n(X, X \setminus \{x\})$ for all $x \in X$. Define ∂X to be the subspace of points x such that $H_n(X, X \setminus \{x\}) = 0$ for all n, and compute the local homology groups $H_n(\partial X, \partial X \setminus \{x\})$. Use these calculations to determine which subsets $A \subset X$ have the property that $f(A) \subset A$ for all homeomorphisms $f : X \to X$.

Solution: First, recall that excision says if open sets A and B form an open cover of X then

$$H_n(B, A \cap B) \simeq H_n(X, A)$$

for all n. Let $x \in X$ and U be a neighborhood of x in X. Consider $B = U$ and $A = X \setminus \{x\}$, which is certainly an open cover of X if X is Hausdorff. Then, $A \cap B = U \setminus \{x\}$ and excision gives

$$H_n(U, U \setminus \{x\}) \simeq H_n(X, X \setminus \{x\}).$$

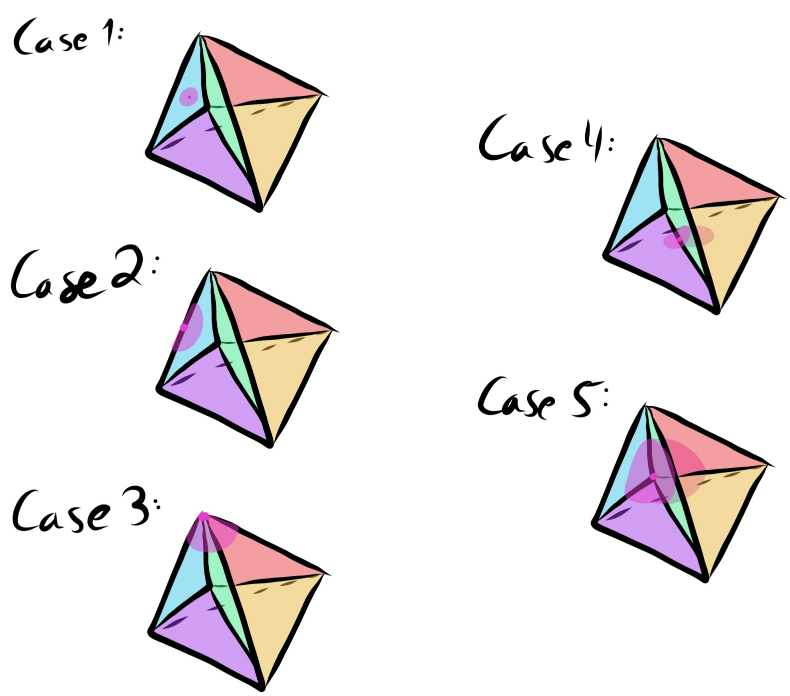
In particular, if U is contractible then using the long exact sequence for the pair $(U, U \setminus \{x\})$ we get

$$... \to \tilde{H}_n(U) \simeq 0 \to H_n(U, U \setminus \{x\}) \to \tilde{H}_{n-1}(U \setminus \{x\}) \to \tilde{H}_{n-1}(U) \simeq 0 \to ...$$

so that $H_n(U, U \setminus \{x\}) \simeq \tilde{H}_{n-1}(U \setminus \{x\})$ for all n; for $n = 0$ we have $H_0(U, U \setminus \{x\}) \simeq 0$ since the end of the long exact sequence is

$$... \to \tilde{H}_0(U) \simeq 0 \to H_0(U, U \setminus \{x\}) \to 0.$$

Hence, we have reduced the problem to computing $\tilde{H}_n(U \setminus \{x\})$ if for each $x \in X$ we can find a contractible neighborhood U. But, we can certainly do this: some illustrations are shown below.



Each of these is formed by viewing X as a subset of $\mathbb{R}^3$ and intersecting small enough balls around each point. The last few are hard to draw. In case 3, imagine three planar sectors glued along a common edge. In case 4, imagine three half-discs glued along the common edge. In case 5, we image six planar sectors glued together; the top three share a common edge while the bottom three pairwise share edges, which forms a triangulated disc. It is obvious how to contract the sectors/discs (independently) in each case, so each gives a contractible neighborhood. We now compute $\tilde{H}_n(U \setminus \{x\})$ in each case.

(1) U is homeomorphic to a disc in $\mathbb{R}^2$ so that $U \setminus \{x\}$ is homotopy equivalent to S^1. Hence,

$$\tilde{H}_n(U \setminus \{x\}) \simeq \tilde{H}_n(S^1) = \begin{cases} \mathbb{Z} & n = 1 \\ 0 & \text{else} \end{cases}$$

(2) U is homeomorphic to a half disc, which in turn is homormorphic to a disc in $\mathbb{R}^2$. Then $U \setminus \{x\}$ is homeomorphic to a disc with a boundary point removed, which is just a disc. Hence

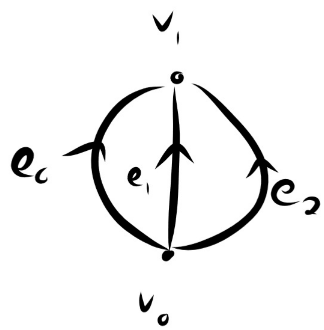
$$\tilde{H}_n(U \setminus \{x\}) \simeq 0.$$

(3) Upon removing the vertex point, we can contract the sector pieces onto their arcs. This gives something like

Then, by retracting the arcs we get a point. If follows that $U \setminus \{x\}$ deformation retracts to a point, and hence

$$\tilde{H}_n(U \setminus \{x\}) \simeq 0.$$

(4) Observe that $U \setminus \{x\}$ deformation retracts to three semicircles with their endpoints identified. Flattening this, we get the following, which has two vertices and three edges.



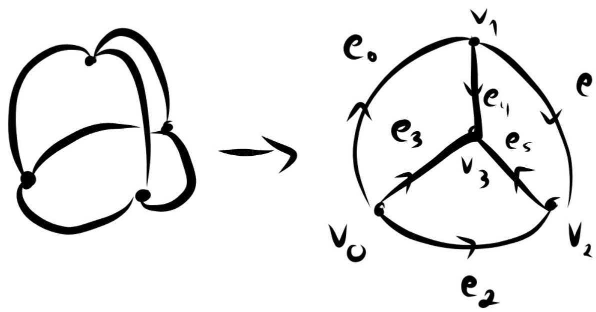
From this, we see that $\partial e_0 = \partial e_1 = \partial e_2 = v_1 - v_0$. Computing the boundary of a general 1-chain gives

$$\partial(a_0 e_0 + a_1 e_1 + a_2 e_2) = (a_0 + a_1 + a_2)(v_1 - v_0).$$

This is zero iff $a_0 + a_1 + a_2 = 0$. Choosing a_0 and a_1 determines a_2 so that the kernel has basis two elements. For example, a basis is $\{e_0 - e_1, e_1 - e_2\}$. Clearly $U \setminus \{x\}$ is path-connected so that $\tilde{H}_0(U \setminus \{x\}) = 0$. Finally, there are no n-simplices for $n \geq 2$ so that $\tilde{H}_n(U \setminus \{x\}) = 0$ for $n \geq 2$. Hence,

$$\tilde{H}_n(U \setminus \{x\}) = \begin{cases} \mathbb{Z}^2 & n = 1 \\ 0 & \text{else} \end{cases}$$

(5) We can deformation retract $U \setminus \{x\}$ to each of the six arcs from the six sectors. This has four vertices v_0, v_1, v_2, v_3 and six edges $e_0, ..., e_5$. See below for a sketch



This is essentially the 1-skeleton of Δ^3. We now have the following relations:

$$\begin{aligned} \partial e_0 &= v_1 - v_0 & \partial e_2 = v_2 - v_0 && \partial e_4 = v_3 - v_1 \\ \partial e_1 &= v_2 - v_1 & \partial e_3 = v_3 - v_0 && \partial e_5 = v_3 - v_2 \end{aligned}$$

Let us compute the boundary of a general 1-chain (generated by $e_0, ..., e_5$)

$$\begin{aligned} \partial(a_0e_0 + ... + a_5e_5) &= a_0(v_1 - v_0) + a_1(v_2 - v_1) + a_2(v_2 - v_0) \\ &+ a_3(v_3 - v_0) + a_4(v_3 - v_1) + a_5(v_3 - v_2) \\ &= (-a_0 - a_2 - a_3)v_0 + (a_0 - a_1 - a_4)v_1 \\ &+ (a_1 + a_2 - a_5)v_2 + (a_3 + a_4 + a_5) \end{aligned}$$

So, we want to compute the nullity of

$$\begin{pmatrix} -1 & 0 & -1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Putting this into reduced row echelon form gives

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

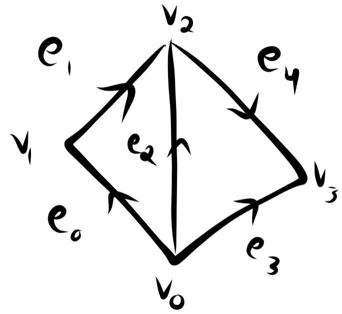
There are three free variables, hence the nullspace has dimension 3. It follows that a basis for Ker ∂ has three elements, an example is $\{e_0 + e_4 - e_3, e_1 + e_5 - e_4, e_2 + e_5 - e_3\}$. It follows that $\tilde{H}_1(U \setminus \{x\}) = \text{Ker}(\partial) \simeq \mathbb{Z}^3$. There are no simplices of dimension $n \geq 2$, and the space is path connected so that

$$\tilde{H}_n(U \setminus \{x\}) = \begin{cases} \mathbb{Z}^3 & n = 1 \\ 0 & \text{else} \end{cases}$$

From all this, we see that

1. $H_n(X, X \setminus \{x\}) = \begin{cases} \mathbb{Z} & n = 2 \\ 0 & \text{else} \end{cases}$
2. $H_n(X, X \setminus \{x\}) = 0$
3. $H_n(X, X \setminus \{x\}) = 0$
4. $H_n(X, X \setminus \{x\}) = \begin{cases} \mathbb{Z}^2 & n = 2 \\ 0 & \text{else} \end{cases}$
5. $H_n(X, X \setminus \{x\}) = \begin{cases} \mathbb{Z}^3 & n = 2 \\ 0 & \text{else} \end{cases}$

where the numeral indicates which case x falls into. In the context of this problem, we see that ∂X consists of $x \in X$ in cases 2 and 3 – i.e. the 1-skeleton of Δ^3. Now we have two cases, first if x is in the interior of an edge of ∂X, second if x is a vertex of ∂X. In the first case, we can deformation retract $\partial X \setminus \{x\}$ by deformation retracting this edge to its endpoints. Doing so yields



The above has four vertices and five edges. Computing ∂ of the edges gives

$$\begin{aligned} \partial e_0 &= v_1 - v_0 \qquad \partial e_2 = v_2 - v_0 \qquad \partial e_4 = v_3 - v_2 \\ \partial e_1 &= v_2 - v_1 \qquad \partial e_3 = v_3 - v_0 \end{aligned}$$

Then for a general 1-chain we get

$$\begin{aligned} \partial(a_0 e_0 + ... + a_4 e_4) &= a_0(v_1 - v_0) + a_1(v_2 - v_1) + a_2(v_2 - v_0) + a_3(v_3 - v_0) + a_4(v_3 - v_2) \\ &= (-a_0 - a_2 - a_3)v_0 + (a_0 - a_1)v_1 + (a_1 + a_2 - a_4)v_2 + (a_3 + a_4)v_3 \end{aligned}$$

Assuming the above is zero, choosing a_0 and a_4 immediately determines a_1, a_2, and a_3. Alternatively, we have

$$\begin{pmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Converting this into reduced row echelon form gives

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which also shows there are two free variables, and that a basis for the kernel is $\{e_0 + e_1 - e_2, -e_3 + e_4 + e_2\}$. Hence,

$$H_n(\partial X, \partial X \setminus \{x\}) = \begin{cases} \mathbb{Z}^2 & n = 2 \\ 0 & \text{else} \end{cases}$$

using the same logic as before. Now, if x is a vertex, then in $\partial X \setminus \{x\}$ we can deformation retract the edges that were connected to x to their other endpoints. This results in the 1-skeleton of Δ^2, i.e. S^1. It follows that

$$H_n(X, X \setminus \{x\}) = \begin{cases} \mathbb{Z} & n = 2 \\ 0 & \text{else} \end{cases}$$

To find which subsets $A \subset X$ are such that $f(A) \subset A$ for all homeomorphisms X, we first observe that local homology is an invariant under homeomorphism. So, the barycenter must be sent to the barycenter, edges connecting vertices to the barycenter to themselves, faces to themselves. We cannot a priori distinguish vertices and edges connecting vertices using local homology. But, points in ∂X must be sent to points in ∂X. Hence any homeomorphism $f : X \to X$ restricts to a homeomorphism $\partial X \to \partial X$. We can therefore use local homology of ∂X to distinguish between the two cases. So, vertices are sent to vertices, and edges connecting vertices to themselves.

Now, of course if A contains a vertex, it must contain all other vertices. We could have that f is a rotation of some type, so that the vertices are permuted. Since $f(A) \subset A$ for all homeomprhisms f, we must include all the vertices. The same is true for all other cases. Meaning, we have the following collections which, unioning all the sets inside, are such that $f(A) \subset A$:

1.$\{f_1, ..., f_6\}$
2.$\{v_0, v_1, v_2, v_3\}$
3.$\{[v_0, v_1], [v_1, v_2], ..., [v_2, v_3]\}$
4.$\{[v_0, b], ..., [v_3, b]\}$
5.$\{b\}$

In the above, $f_1, ..., f_6$ denote the faces of X, $v_0, ..., v_2$ the vertices, $[v_i, v_j]$ the edges connecting vertices, $[v_i, b]$ the edges connecting a vertex to the barycenter, and b the barycenter. Since functions preserve unions, intersections, and complements, any combination of the above five sets using these operations will be such that $f(A) \subset A$.

Hatcher Chapter 2.2, problems 2, 4, 8, 28:

2.2.2. Given a map $f : S^{2n} \to S^{2n}$, show that there is some point $x \in S^{2n}$ with either $f(x) = x$ or $f(x) = -x$. Deduce that every map $\mathbb{R}\mathrm{P}^{2n} \to \mathbb{R}\mathrm{P}^{2n}$ has a fixed point. Construct maps $\mathbb{R}\mathrm{P}^{2n-1} \to \mathbb{R}\mathrm{P}^{2n-1}$ without fixed points from linear transformations $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ without eigenvectors.

Solution: On one hand, if there are no fixed points (i.e., $f(x) \neq x$ for all x) then by property g) we have $\deg(f) = (-1)^{n+1} = -1$. On the other hand, suppose there are no antipodal points (i.e. $f(x) \neq -x$ for all x). Then we can define a homotopy in a similar way by

$$f_t(x) = \frac{(1-t)f(x) + tx}{\|(1-t)f(x) + tx\|}.$$

Observe that $f_0(x) = f(x)$ and $f_1(x) = x$. Hence there is a homotopy from f to the identity, and $\deg(f) = \deg(\mathrm{id}_{S^n}) = 1$. One can also apply property g) to $-f$.

Let $f : \mathbb{R}\mathrm{P}^{2n} \to \mathbb{R}\mathrm{P}^{2n}$. The universal cover of $\mathbb{R}\mathrm{P}^{2n}$ is S^{2n} with the covering map $p : S^{2n} \to \mathbb{R}\mathrm{P}^{2n}$ sending $x \mapsto [x]$. Hence by the lifting criterion, we obtain a map $\tilde{f} : S^{2n} \to S^{2n}$ such that $p \circ \tilde{f} = f \circ p$. We have shown that any map $\tilde{f} : S^{2n} \to S^{2n}$ either has a fixed point or an antipodal point. Let x be either type, then $p(\tilde{f}(x)) = [\tilde{f}(x)] = [x] = p(x)$ (since $[-x] = [x]$). Then,

$$f[x] = f(p(x)) = p(\tilde{f}(x)) = [x]$$

so that f has a fixed point.

Consider a rotation of $\mathbb{R}^{2n}$. Let $x \in \mathbb{R}^{2n}$ and A the rotation. Then $\|Ax\|^2 = \langle Ax, Ax\rangle = \langle x, x\rangle = \|x\|^2$ since A is an isometry. It follows that $A : S_r^{2n-1} \to S_r^{2n-1}$ where S_r denotes a sphere of radius r. In even dimension, rotations have no axis of rotation and the eigenvalues come in complex conjugates. We can restrict A to $A : S^{2n-1} \to S^{2n-1}$. Suppose there was a fixed point or an antipodal point x. Then either $Ax = x$ or $Ax = -x$. But, in both cases x would be an eigenvector with real eigenvalue, a contradiction. So A has no fixed points nor antipodal points. We then obtain a map $\mathbb{R}\mathrm{P}^{2n-1} \to \mathbb{R}\mathrm{P}^{2n-1}$ without any fixed points.

One easy way to see this is to use the canonical identification of $\mathbb{R}^{2n}$ with $\mathbb{C}^n$. Then we can simply use the rotation given by multiplication by i.

2.2.4. Construct a surjective map $S^n \to S^n$ of degree zero, for each $n \geq 1$.

Solution: We show that the suspension of a surjective function is surjective. First, the suspension of a function $f : X \to Y$ is given by $Sf : SX \to SY$, where $SX = X \times I/\sim$ and $\sim$ is the equivalence relation given by $(x_0, 0) \sim (x_1, 0)$ and $(x_0, 1) \sim (x_1, 1)$ for all $x_0, x_1 \in X$. Thus elements of SX are equivalence classes $[x, t]$. The suspension of a function Sf is defined by $Sf[x, t] = [f(x), t]$. Now assume f is surjective and let $[y, t] \in SY$. Choose a representative $y \in Y$ and $t \in I$. Then, since $y \in Y$ and f is surjective there exists an $x \in X$ such that $f(x) = y$. Hence, $Sf[x, t] = [f(x), t] = [y, t]$.

It follows that Sf is surjective.

We will make use of proposition 2.33, which says that $\deg(f) = \deg(Sf)$. Thus if we find a surjective function $f : S^1 \to S^1$ with degree zero, we can apply repeatedly apply this to obtain a surjective map $S^{n-1}f : S^n \to S^n$ with degree zero.

Finally, one can show that the winding number of a map $f : S^1 \to S^1$ coincides with its degree. So, we need only find a surjective path in S^1 with winding number 0. Such a map is given explicitly below:

$$f(t) = \begin{cases} (\cos(2t), \sin(2t)) & t \in [0, \pi] \\ (\cos(2t), -\sin(2t)) & t \in [\pi, 2\pi] \end{cases}$$

which wraps around S^1 once counterclockwise, then back on itself clockwise once. The map is clearly surjective and has winding number, hence degree, zero.

We can imagine Sf for $f : S^1 \to S^1$ above as extending f to S^2 in the following way: slice S^2 by horizontal planes $x_3 = c$. Each slice is a copy of S^1 with radius $\sqrt{1-c^2}$. So Sf acts on each of these slices in the same way that f acts on S^1.

2.2.8. A polynomial $f(z)$ with complex coefficients, viewed as a map $\mathbb{C} \to \mathbb{C}$, can always be extended to a continuous map of one-point compactifications $\hat{f} : S^2 \to S^2$. Show that the degree of $\hat{f}$ equals the degree of f as a polynomial. Show also that the local degree of $\hat{f}$ at a root of f is the multiplicity of the root.

Solution: First, note that any holomorphic function $f : \mathbb{C} \to \mathbb{C}$ can be written as

$$f(z) = (z - z_i)^{m_i}(c_0 + c_1(z - z_i) + c_2(z - z_i)^2 + ...)$$

in some neighborhood of a root z_i. Here the coefficients $c_0, c_1, ...$ are nonzero and m_i is the multiplicity of the root m_i. Note that if we have a polynomial f, then the coefficients above cannot be read off easily. For example, in the real case consider $f(x) = 1 - 3x^2 + 3x^3 - x^4$. This has a root at 1, and the expansion around 1 is $f(x) = -(x-1)(1 + (x-1)^2 + (x-1)^3)$.

Now, near a root z_i we have that $f(z) \sim c_0(z - z_i)^{m_i}$. I.e., for sufficiently small circles around z_i, $f(z)$ will wrap the circles onto themselves m_i times. Hence the local degree of f at z_i is m_i. Let f be a polynomial of degree n with distinct roots $z_1, ..., z_r$ each with multiplicity m_i. By the fundamental theorem of algebra we have that $m_1 + ... + m_r = n$. Consider $h : S^2 \to \overline{\mathbb{C}}$ given by the stereographic projection, where the north pole is sent to ∞ and the south pole is sent to 0. Let $p_i = h^{-1}(z_i)$ and p the south pole. By definition, $\overline{f} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, the continuous extension of f to to $\overline{\mathbb{C}}$ is such that $\hat{f} = h^{-1} \circ \overline{f} \circ h$. Extending f to $\overline{f}$ does not introduce any new roots. Then $\hat{f}^{-1}(p) = h^{-1}(f^{-1}(0))$ since $h(p) = 0$. The set $f^{-1}(0)$ consists of finitely many points $\{z_1, ..., z_r\}$. So we need to look at disjoint open neighborhoods U_i of p_i in S^2. We can choose these small enough so that the north pole is not included. Using h, we can map U_i into a homeomorphic open set V_i of z_i in $\mathbb{C}$. Now consider a small loop in V_i. By our previous analysis, f will wrap this loop onto itself m_i times. After possibly shrinking U_i, we see that every point in U_i is hit m_i times by $\hat{f}$. Hence the local homology of $\hat{f}$ at p_i is m_i. Thus,

$$\deg(\hat{f}) = \sum_{i=1}^{r} \deg(\hat{f})|_{p_i} = m_1 + ... + m_r = n.$$

Formally, I think we look at a neighborhood U of p in S^2 and U_i of p_i such that $\hat{f}(U_i) \subset U$. The local degree at p_i is found by $\hat{f}_* : H_2(U_i, U_i \setminus \{p_i\}) \to H_2(U, U \setminus \{p\})$. But these U_i are homeomorphic to V_i of z_i in $\mathbb{C}$, and U is homeomorphic to V of 0 in $\mathbb{C}$. The local degree of f at z_i is found by $f_* : H_2(V_i, V_i \setminus \{p_i\}) \to H_2(V, V \setminus \{p\})$. Since $U_i \simeq V_i$ and $U \simeq V$, the local degree of f at z_i is equal to the local degree of $\hat{f}$ at p_i. Perhaps another way to see this is that h is a homeomorphism so that its degree is ± 1. Computing the local degree, and using $\hat{f} = h^{-1} \circ f \circ h$ in $\mathbb{C}$, we see that $\deg(\hat{f})|_{p_i} = (\pm 1)^2 \deg(f)|_{z_i} = \deg(f)|_{z_i}$.

2.2.28.

a) Use the Mayer-Vietoris sequence to compute the homology groups of the space obtained from a torus $S^1 \times S^1$ by attaching a Möbius band via a homeomorphism from the boundary circle of the Möbius band to the circle $S^1 \times \{x_0\}$.
b) Do the same for the space obtained by attaching a Möbius band to $\mathbb{R}\mathrm{P}^2$ via a homeomorphism of its boundary circle to the standard $\mathbb{R}\mathrm{P}^1 \subset \mathbb{R}\mathrm{P}^2$.

Solution:

a) Let X be the space given, a torus T^2 with a glued Möbius band M. We can find a neighborhood U of the identified circle $S^1 \times \{x_0\}$ – it looks like a neighborhood of $S^1 \times \{x_0\}$ together with a neighborhood of the boundary of the Möbius strip. The neighborhood U can be taken so that it deformation retracts to $S^1 \times \{x_0\}$. Let $A = M \cup U$ and $B = T^2 \cup U$. So, A is the Möbius band with a slight extension onto T^2 around the identified circle, while B is the torus with a slight extension into M around its boundary. The homologies of these are given by

$$\tilde{H}_n(A) = \tilde{H}_n(M) = \begin{cases} \mathbb{Z} & n = 1 \\ 0 & \text{else} \end{cases}$$

$$\tilde{H}_n(B) = \tilde{H}_n(T^2) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ \mathbb{Z} & n = 2 \\ 0 & \text{else} \end{cases}$$

since M deformation retracts onto its central circle. Now, $A \cap B = U$ which deformation retracts onto $S^1 \times \{x_0\}$. So, $\tilde{H}_n(A \cap B) \simeq \tilde{H}_n(A)$. The Mayer-Vietoris sequence gives $\tilde{H}_n(X) = 0$ for $n \geq 3$ (alternatively, since there are no n-cells for $n \geq 3$). The low dimensional part of the sequence is

$$... \to \tilde{H}_2(A \cap B) \to \tilde{H}_2(A) \oplus \tilde{H}_2(B) \to \tilde{H}_2(X) \to \tilde{H}_1(A \cap B) \to ...$$

$$... \to \tilde{H}_1(A) \oplus \tilde{H}_1(B) \to \tilde{H}_1(X) \to \tilde{H}_0(A \cap B) \to ...$$

which, after applying our knowledge of these groups, gives

$$... \to 0 \to \mathbb{Z} \to \tilde{H}_2(X) \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \to \tilde{H}_1(X) \to 0 \to ...$$

Let b be a generator of $\tilde{H}_1(A \cap B)$, which is represented by the 1-cell in $S^1 \times \{x_0\}$. Similarly, generators for $\tilde{H}_1(B)$ are b and c, where b is the same as before and c is represented by the 1-cell in $\{x_0\} \times S^1$. Finally, a generator a for $\tilde{H}_1(A)$ is represented by the central circle. I will interchangeably refer to b as a representative, and the same for the other generators.

Notice that b wraps onto a twice. The map $\varphi : C_n(A \cap B) \to C_n(A) \oplus C_n(B)$ is defined by $\varphi(x) = (x, -x)$. Note that b is a cycle in both $C_n(A)$ and $C_n(B)$, and thus $\varphi(b) = (b, -b)$. We view the first b as in $C_n(A)$, which is then $2a$. We view the second $-b$ as in $C_n(B)$, which just remains as $-b$. So,

$$\Phi(b) = (2a, -b)$$

By identifying $\tilde{H}_1(A) \simeq \tilde{H}_1(A \cap B) \simeq \mathbb{Z}$ and $\tilde{H}_1(B) \simeq \mathbb{Z} \oplus \mathbb{Z}$ we can rewrite this as

$$\Phi(1) = (2, -(1, 0)).$$

It follows that Φ is injective, so $\mathrm{Ker}(\Phi) = 0$. Hence, the image of $\tilde{H}_2(X) \to \mathbb{Z}$ is zero. Then the image of $\mathbb{Z} \to \tilde{H}_2(X)$ is $\tilde{H}_2(X)$, and is therefore surjective. But it is also injective since we have

$$0 \to \mathbb{Z} \to \tilde{H}_2(X) \to ...$$

Thus $\tilde{H}_2(X) \simeq \mathbb{Z}$.
Finally, we can view $\tilde{H}_1(A) \oplus \tilde{H}_1(B)$ as $\mathbb{Z}^3$, and write $\Phi(1)$ as $(2,-1,0) = 2a - b$. By the first isomorphism theorem, it follows that

$$\tilde{H}_1(X) \simeq \langle a,b,c\rangle/\langle 2a-b=0\rangle = \langle a,c\rangle \simeq \mathbb{Z}\oplus\mathbb{Z}.$$

Since X is path-connected, we see that $\tilde{H}_0(X) = 0$. Actually, since X is path-connected we can verify our computation for $\tilde{H}_1(X)$. For path connected spaces, $\tilde{H}_1(X)$ is isomorphic to the abelianization $\pi_1(X)$. The Möbius band is homotopy equivalent to $\mathbb{Z}$, and has generator one of the loops $S^1 \times \{x_0\}$. But this loop is also a generator for $\pi_1(T^2)$, so $\pi_1(X) \simeq \mathbb{Z} * \mathbb{Z}$. The abelianization of this gives $\tilde{H}_1(X)$ as expected.
In summary,

$$\tilde{H}_n(X) = \begin{cases} \mathbb{Z}\oplus\mathbb{Z} & n=1 \\ \mathbb{Z} & n=2 \\ 0 & \text{else} \end{cases}$$

b) One can show that $\mathbb{RP}^1 \simeq S^1$. To see this, note that $\mathbb{RP}^1$ is obtained from S^1 by identifying antipodal points. Traversing along the upper arc of S^1, we never encounter two antipodal points until we traverse π radians. At this point, we are back at the beginning by the identification, hence we have traversed a circle.

We define A and B similarly as above, except with T^2 replaced by $\mathbb{RP}^2$. The relevant homology groups are

$$\begin{aligned} \tilde{H}_n(A) &\simeq \tilde{H}_n(A\cap B) = \begin{cases} \mathbb{Z} & n=1 \\ 0 & \text{else} \end{cases} \\ \tilde{H}_n(B) &= \tilde{H}_n(\mathbb{RP}^2) = \begin{cases} \mathbb{Z}_2 & n=1 \\ 0 & \text{else} \end{cases} \end{aligned}$$

As before, $\tilde{H}_n(X) = 0$ for $n \geq 3$. The relevant part of the Mayer-Vietoris sequence is

$$\ldots \to 0 \to \tilde{H}_2(X) \to \mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z}_2 \to \tilde{H}_1(X) \to 0 \to \ldots$$

Let b be a generator for $\tilde{H}_1(A\cap B)$, given by the S^1. We similarly have b is a generator for $\tilde{H}_1(B)$. Finally, let a be a generator for $\tilde{H}_1(A)$ (representing the center circle). Then, as before, b wraps onto a twice. Hence,

$$\Phi(b) = (2a,-b) = (2a,-b) \Rightarrow \Phi(1) = (2,-1)$$

after making the proper identifications. We must have that Φ is injective so that $\text{Ker}(\Phi) = 0$. By the same logic as in a), $\tilde{H}_2(X) \simeq 0$. Furthermore,

$$\tilde{H}_1(X) \simeq \langle a,b\rangle/\langle 2a-b, 2b\rangle = \langle a,b\rangle/\langle 2a-b, 4a\rangle = \mathbb{Z}_4$$

Since X is path-connected, $\tilde{H}_0(X) \simeq 0$. In total,

$$\tilde{H}_n(X) = \begin{cases} \mathbb{Z}_4 & n=1 \\ 0 & \text{else} \end{cases}$$

HW10

Hatcher Chapter 2.2, problem 40:

2.2.40. From the long exact sequence of homology groups associated to the short exact sequence of chain complexes $0 \to C_i(X) \xrightarrow{n} C_i(X) \to C_i(X; \mathbb{Z}_n) \to 0$ deduce immediately that there are short exact sequences

$$0 \to H_i(X)/nH_i(X) \to H_i(X; \mathbb{Z}_n) \to n\text{-Torsion}(H_{i-1}(X)) \to 0$$

where n-Torsion(G) is the kernel of the map $G \xrightarrow{n} G$, $g \mapsto ng$. Use this to show that $\tilde{H}_i(X; \mathbb{Z}_p) = 0$ for all i and all primes p iff $\tilde{H}_i(X)$ is uniquely p-divisible for every prime p.

Solution: The relevant long exact sequence induced by the short exact sequence of chain complexes is

$$... \to H_i(X) \xrightarrow{\tilde{n}} H_i(X) \xrightarrow{\varphi} H_i(X; \mathbb{Z}_n) \xrightarrow{\phi} H_{i-1}(X) \to ...$$

Here, the first map $\tilde{n}$ takes an equivalence class $[\sigma]$ and sends it to $n[\sigma]$. Note that any homomorphism $\phi : G \to H$ induces a short exact sequence

$$0 \to \text{Ker}(\phi) \to G \to \text{Im}(\phi) \to 0$$

the fist map is just inclusion while the second map is ϕ. This is clearly exact since the image of the inclusion map is just Ker(ϕ). Now, applying this with ϕ in the LES above and $G = H_i(X; \mathbb{Z}_n)$ gives

$$0 \to \text{Ker}(\phi) \to H_i(X; \mathbb{Z}_n) \to n\text{-Torsion}(H_{i-1}(X)) \to 0$$

where we have used exactness at $H_{i-1}(X)$ to conclude that $\text{Im}(\phi) = \text{Ker}(\tilde{n}) = n\text{-Torsion}(H_{i-1}(X))$. Once more using exactness, in combination with the first isomorphism theorem, we have

$$\text{Ker}(\phi) = \text{Im}(\varphi) \simeq H_i(X)/\text{Ker}(\varphi) = H_i(X)/\text{Im}(\tilde{n}) = H_i(X)/nH_i(X).$$

Hence, we get a short exact sequence

$$0 \to H_i(X)/nH_i(X) \to H_i(X; \mathbb{Z}_n) \to n\text{-Torsion}(H_{i-1}(X)) \to 0.$$

Now let G be a uniquely p-divisible group. We show that $G/pG \simeq 0$ and p-Torsion$(G) = 0$. First, every group satisfies $pG \subset G$. If $a \in G$ then there exists a unique $x \in G$ such that $px = a$. Hence, $a \in pG$ and $G \subset pG$. It follows that $G = pG$ so that $G/pG \simeq 0$. Next, p-Torsion$(G) = \text{Ker}(\phi)$ where $\phi(x) = px$. To say that G is p-torsion free amounts to saying that ϕ is injective. But, this is clear by uniqueness.

Conversely, suppose that G is such that $G/pG \simeq 0$ and p-torsion free. As seen before, being p-torsion free means that if a solution to $px = a$ exists, it is unique. So, we just need to show existence. But $G/pG \simeq 0$ implies that $G = pG$, which says that every $a \in G$ can be expressed as px for some $x \in G$. In total, G is uniquely p-divisible iff $G/pG \simeq 0$ and p-Torsion$(G) = 0$.

Now assume that $\tilde{H}_i(X; \mathbb{Z}_p) \simeq 0$ for all i and primes p. Then by the obtained short exact sequence, we conclude that $\tilde{H}_i(X)/p\tilde{H}_i(X) \simeq 0$ and p-Torsion$(\tilde{H}_{i-1}(X)) \simeq 0$. Since this holds for all i, it implies that $\tilde{H}_i(X)$ is uniquely p-divisible.

Conversely, suppose that $\tilde{H}_i(X)$ is uniquely p-divisible for every prime p and all i. Then $\tilde{H}_i(X)/p\tilde{H}_i(X) \simeq 0$ and p-Torsion$(\tilde{H}_{i-1}(X)) = 0$, and it follows from the short exact sequence that $\tilde{H}_i(X; \mathbb{Z}_p) \simeq 0$ for all i and primes p.

Hatcher Chapter 2.B, problems 1, 2, 8:

2.B.1. Compute $\tilde{H}_i(S^n \setminus X)$ when X is a subspace of S^n homeomorphic to $S^k \vee S^l$ or to $S^k \coprod S^l$.

Solution: Suppose first that X is homeomorphic to $S^k \coprod S^l$. Denote the two connected components of X by X^k and X^l. Then there exist homeomorphisms $h_k : S^k \to X^k$ and $h_l : S^l \to X^l$. By proposition 2B.1.b we have that

$$\tilde{H}_i(S^n \setminus X^k) \simeq \begin{cases} \mathbb{Z} & i = n - k - 1 \\ 0 & \text{else} \end{cases}$$

and similarly for l. Let A be an open set deformation retracting to $S^n \setminus X^k$ and similarly for B, with X^l Then $A \cup B = S^n$ while $A \cap B$ deformation retracts to $S^n \setminus X$. Then applying Mayer-Vietoris yields

$$... \to \tilde{H}_{i+1}(S^n) \to \tilde{H}_i(S^n \setminus X) \to \tilde{H}_i(S^n \setminus X^k) \oplus \tilde{H}_i(S^n \setminus X^l) \to \tilde{H}_i(S^n) \to ...$$

Since $k, l < n$ we have that $\tilde{H}_i(S^n \setminus X^k) \simeq \tilde{H}_i(S^n \setminus X^l) \simeq 0$ for $i = n$. Thus,

$$... \to 0 \to \tilde{H}_n(S^n \setminus X) \to 0 \to \mathbb{Z} \to ...$$

Hence $\tilde{H}_n(S^n \setminus X) \simeq 0$. The same logic holds for $i \geq n$, so that no matter what values k and l take

$$\tilde{H}_i(S^n \setminus X) \simeq 0 \quad i \geq n$$

Thus, we just need to investigate the low dimensional cases.

Consider the case when $k, l \neq 0$. Then

$$... \to 0 \to \tilde{H}_i(S^n \setminus X) \to \tilde{H}_i(S^n \setminus X^k) \oplus \tilde{H}_i(S^n \setminus X^l) \to 0 \to ...$$

where $i < n-1$. So, $\tilde{H}_i(S^n \setminus X) \simeq \tilde{H}_i(S^n \setminus X^k) \oplus \tilde{H}_i(S^n \setminus X^k)$. The only relevant parts here are when $i = n-k-1$ and $i = n-l-1$, since these yield $\mathbb{Z}$'s in the sequence. Notice that this is why we chose $k, l \neq 0$. Otherwise we'd have to investigate the $i = n-1$ case, which is more complicated since it involves $\tilde{H}_{i+1}(S^n) = \tilde{H}_n(S^n) \simeq \mathbb{Z}$. In any case, at the positions $i = n-k-1, n-l-1$ we get a copy of $\mathbb{Z}$. At all other positions, except $i = 0, n-1$ we just get zeroes. Now at $i = n-1$ we have

$$... \to 0 \to \mathbb{Z} \to \tilde{H}_{n-1}(S^n \setminus X) \to 0 \to ...$$

since $k, l \neq 0$, and from what we know about $\tilde{H}_i(S^n \setminus X^k)$ (and X^l). Hence, $\tilde{H}_{n-1}(S^n \setminus X) \simeq \mathbb{Z}$. Finally, for the case $i = 0$ we have

$$... \to 0 \to \tilde{H}_0(S^n \setminus X) \to \tilde{H}_0(S^n \setminus X^k) \oplus \tilde{H}_0(S^n \setminus X^l) \to 0.$$

so that $\tilde{H}_0(S^n \setminus X) \simeq \tilde{H}_0(S^n \setminus X^k) \oplus \tilde{H}_0(S^n \setminus X^l)$. In total, when $k, l \neq 0$,

$$H_i(S^n \setminus X) \simeq \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z} & i = n-k-1, n-l-1, n-1 \\ 0 & \text{else} \end{cases}$$

if $k = l$, we interpret the case $i = n-k-1$ as $\mathbb{Z}^2$. The same thing occurs if one of k or l is $n-1$; we adjoin a $\mathbb{Z}$ to the $i = 0$ case. We do something similar when $n = 1$, since this forces $k = l = n-1 = 0$. In this event, we remove two copies of S^0 from S^1, which amounts to just removing four points. This splits S^1 into four path components, which is exactly what you'd get by summing all of the $\mathbb{Z}$ from the four cases.

Now consider the case $k \neq 0$, $l = 0$. Our analysis remains unchanged except for $i = n-l-1 = n-1$. The Mayer-Vietoris sequence becomes

$$... \to 0 \to \mathbb{Z} \to \tilde{H}_{n-1}(S^n \setminus X) \to \mathbb{Z} \to 0 \to ...$$

where $\tilde{H}_{n-1}(S^n \setminus X^k) \simeq 0$ since $k \neq 0$. The first $\mathbb{Z}$ comes from $\tilde{H}_n(S^n)$ while the second comes from $\tilde{H}_{n-1}(S^n \setminus X^l)$. Since $\mathbb{Z}$ is free, the sequence splits, and we get

$$\tilde{H}_i(S^n \setminus X) \simeq \mathbb{Z} \oplus \mathbb{Z}.$$

Note then that $H_i(S^n \setminus X)$ is given as previous, where we understand that if $l = 0$ we merge the cases $n-l-1$ and $n-1$, and get a $\mathbb{Z} \oplus \mathbb{Z}$.

Finally, if $k = l = 0$ then the case $i = n-1$ becomes

$$... \to 0 \to \mathbb{Z} \to \tilde{H}_{n-1}(S^n \setminus X) \to \mathbb{Z} \oplus \mathbb{Z} \to 0 \to ...$$

which also splits since $\mathbb{Z}\oplus\mathbb{Z}$ is free. Hence, $\tilde{H}_{n-1}(S^n\setminus X)\simeq\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$. Once more, this is given from our previous formula with the given interpretation. So, in total,

$$H_i(S^n\setminus X)\simeq\begin{cases}\mathbb{Z} & i=0,n-k-1,n-l-1,n-1\\ 0 & \text{else}\end{cases}$$

where if any of $0,n-k-1,n-l-1,n-1$ are equal, we merge the $\mathbb{Z}$'s together into a direct sum.

Consider now when X is homeomorphic to $S^k\vee S^l$. Once more, there exists a homeomorphism $h_k:S^k\to X^k\subset X$ and a homeomorphism $h_l:S^l\to X^l\subset X$. Then, we still have

$$\tilde{H}_i(S^n\setminus X^k)\simeq\begin{cases}\mathbb{Z} & i=n-k-1\\ 0 & \text{else}\end{cases}$$

We can use the same A and B as before. Doing so gives $A\cup B=S^n\setminus p$ while $A\cap B$ deformation retracts to $S^n\setminus X$. Here, p is the wedge point; so, $A\cup B\simeq\mathbb{R}^n$ which is contractible. Thus, Mayer-Vietoris gives

$$...\to 0\to\tilde{H}_i(S^n\setminus X)\to\tilde{H}_i(S^n\setminus X^k)\oplus\tilde{H}_i(S^n\setminus X^l)\to 0\to ...$$

for all i, using contractibility of $\mathbb{R}^n$. Hence, $\tilde{H}_i(S^n\setminus X)\simeq\tilde{H}_i(S^n\setminus X^k)\oplus\tilde{H}_i(S^n\setminus X^l)$ for all i. Alternatively,

$$H_i(S^n\setminus X)\simeq\begin{cases}\mathbb{Z} & i=0,n-k-1,n-l-1\\ 0 & \text{else}\end{cases}$$

2.B.2. Show that $\tilde{H}_i(S^n\setminus X)\simeq\tilde{H}_{n-i-1}(X)$ when X is homeomorphic to a finite connected graph. [First do the case that the graph is a tree.]

Solution: Note that every tree is contractible so that $\tilde{H}_{n-i-1}(T)\simeq 0$. Hence we must show $\tilde{H}_i(S^n\setminus X)\simeq 0$ where X is homeomorphic to a tree via a homeomorphism h. We show this inductively. Let k be the number of vertices in T. If $k=1$ then T, hence X, is a single point and $S^n\setminus X\simeq\mathbb{R}^n$. Thus $\tilde{H}_i(S^n\setminus X)\simeq 0$. If $k=2$, then T is homeomorphic to D^1. It too follows that $\tilde{H}_i(S^n\setminus X)\simeq 0$ by applying proposition 2B.1.a.

Now suppose $\tilde{H}_i(S^n\setminus X)\simeq 0$ for $X\simeq T$ with k or fewer vertices. Suppose X is homeomorphic to a tree T with $k+1$ vertices by h. Since T is a tree, there must be some vertex with only one edge connected to it. Call this edge f and the *other* vertex w (so, w may have multiple edges connected to it, but one edge is f and terminates immediately) and let $e=h(f)$ and $v=h(w)$. Now, by deformation retracting e to v, we deformation retract X to a space homeomorphic to a tree T' with k vertices. Let $Y=h(T')$, so that by the inductive hypothesis $\tilde{H}_i(S^n\setminus Y)\simeq 0$. Let $A=S^n\setminus e$ and $B=S^n\setminus Y$. Then $A\cup B=S^n\setminus v$ and $A\cap B=S^n\setminus X$. By applying Mayer-Vietoris we obtain

$$...\to\tilde{H}_{i+1}(S^n\setminus v)\to\tilde{H}_i(S^n\setminus X)\to\tilde{H}_i(S^n\setminus e)\oplus\tilde{H}_i(S^n\setminus Y)\to\tilde{H}_i(S^n\setminus v)\to ...$$

Applying the inductive hypothesis yields

$$...\to 0\to\tilde{H}_i(S^n\setminus X)\to 0\to 0\to ...$$

since v,e,Y are homeomorphic to trees with $1,2,$ and k vertices respectively. Hence $\tilde{H}_i(S^n\setminus X)\simeq 0$.

Now assume X is homeomorphic to a finite graph G. Let T be a maximal subtree of G. If $G=T$, then we are done by the above. If not, we proceed by induction on the number of edges in $G\setminus T$. Assume $\tilde{H}_i(S^n\setminus Y)\simeq\tilde{H}_{n-i-1}(Y)$ where Y is homeomorphic to a graph G with m edges not in a maximal subtree. Suppose G has $m+1$ edges not in a maximal subtree T. Write G as $G=G'\cup f$ where f is one of the $m+1$ edges not in T, and G' is a graph with m edges not in T. Let $h:G\to X$ be a homeomorphism so that $h':G'\to Y$, the restriction of h to G', is a homeomorphism onto $Y\subset X$. It follows from the inductive hypothesis that $\tilde{H}_i(S^n\setminus Y)\simeq\tilde{H}_{n-i-1}(Y)$. We can actually compute the second group. Since Y is a graph with m edges not in a maximal subtree, we can contract the subtree and get a wedge of m circles. Hence,

$$\tilde{H}_{n-i-1}(Y) \simeq \begin{cases} \mathbb{Z}^m & n-i-1=1 \\ 0 & \text{else} \end{cases}$$

Now, since G is a graph and not a multigraph, it has no loops. Hence $G' \cap f$ consists of two points w_0 and w_1, the endpoints of f (both of these are in G' since $T \subset G'$ and the maximal tree must contain all the vertices). Let $e = h(f)$ and $v_i = h(w_i)$ so that v_i are the endpoints of e. We apply MV once more with $A = S^n \setminus e$ and $B = S^n \setminus Y$. Then $A \cup B = S^n \setminus \{v_0, v_1\}$ and $A \cap B = S^n \setminus X$. Note that $A \cup B \simeq S^{n-1}$. From MV,

$$... \to \tilde{H}_{i+1}(S^{n-1}) \to \tilde{H}_i(S^n \setminus X) \to \tilde{H}_i(S^n \setminus e) \oplus \tilde{H}_i(S^n \setminus Y) \to \tilde{H}_i(S^{n-1}) \to ...$$

Because e is homeomorphic to a tree, we get that $\tilde{H}_i(S^n \setminus e) \simeq 0$ from previous results. Hence,

$$... \to \tilde{H}_{i+1}(S^{n-1}) \to \tilde{H}_i(S^n \setminus X) \to \tilde{H}_i(S^n \setminus Y) \to \tilde{H}_i(S^{n-1}) \to ...$$

Now if $i \neq n-1, n-2$ then

$$... \to 0 \to \tilde{H}_i(S^n \setminus X) \to \tilde{H}_i(S^n \setminus Y) \to 0 \to ...$$

so that $\tilde{H}_i(S^n \setminus X) \simeq \tilde{H}_i(S^n \setminus Y)$. But $\tilde{H}_i(S^n \setminus Y) \simeq 0$ except in the case $i = n-2$, which we are not in. It follows that $\tilde{H}_i(S^n \setminus X) \simeq 0$ for all $i \neq n-1, n-2$. Next consider $i = n-1$. Then the relevant part of MV is

$$... \to 0 \to \tilde{H}_{n-1}(S^n \setminus X) \to \tilde{H}_{n-1}(S^n \setminus Y) \to \mathbb{Z} \to ...$$

But, since $\tilde{H}_{n-1}(S^n \setminus Y) \simeq 0$, we have that $\tilde{H}_{n-1}(S^n \setminus X) \simeq 0$. All that remains is to check the case $i = n-2$. In this case we have

$$... \to 0 \to \tilde{H}_{n-1}(S^{n-1}) \to \tilde{H}_{n-2}(S^n \setminus X) \to \tilde{H}_{n-2}(S^n \setminus Y) \to 0 \to ...$$

owing to the fact that $\tilde{H}_{n-1}(S^n \setminus Y) \simeq \tilde{H}_{n-2}(S^{n-1}) \simeq 0$. But, we know what the two outer groups are

$$...0 \to \mathbb{Z} \to \tilde{H}_{n-2}(S^n \setminus X) \to \mathbb{Z}^m \to 0 \to ...$$

Since $\mathbb{Z}^m$ is free, the sequence splits. It follows that $\tilde{H}_{n-2}(S^n \setminus X) \simeq \mathbb{Z} \oplus \mathbb{Z}^m \simeq \mathbb{Z}^{m+1}$. In total,

$$\tilde{H}_i(S^n \setminus X) \simeq \begin{cases} \mathbb{Z}^{m+1} & i = n-2 \\ 0 & \text{else} \end{cases} \simeq \tilde{H}_{n-i-1}(X)$$

where the last isomorphism comes from crushing a maximal subtree of G to a point, and noting that the remaining $m+1$ edges become a wedge of $m+1$ circles.

2.B.8. Show that $\mathbb{R}^{2n+1}$ is not a division algebra over $\mathbb{R}$ if $n > 0$ by considering how the determinant of the linear map $x \mapsto ax$ given by the multiplication in a division algebra structure would vary as a moves along a path in $\mathbb{R}^{2n+1} \setminus \{0\}$ joining two antipodal points.

Solution: Suppose we do have a division algebra structure. Choose a path $\gamma : [0,1] \to \mathbb{R}^{2n+1}$ such that $\gamma(0) = a$ and $\gamma(1) = -a$ but $\gamma(t) \neq 0$ for all t. Let A be a matrix representing the linear transformation $x \mapsto ax$ and similarly Γ_t a matrix representing the linear transformation $x \mapsto \gamma(t)x$. Hence $\Gamma_0 = A$ and $\Gamma_1 = -A$. Note that, since $2n+1$ is odd, we have $\det(\Gamma_0) = \det(A)$ while $\det(\Gamma_1) = \det(-A) = (-1)^{2n+1}\det(A) = -\det(A)$. Since we have a division algebra, these linear transformations are injective (equivalently surjective) and the determinant is nonzero. Now the determinant is a continuous function, and since our path is continuous there must be some t such that $\det(\Gamma_t) = 0$ (since it varies between $\det(A)$ and $-\det(A)$ both of which are nonzero). Thus the map $x \mapsto \gamma(t)x$ is not injective/surjective, contradicting our assumption of having a division algebra.

Note that in the case $n = 0$ we cannot find a path $\gamma(t)$ avoiding zero, so the above construction fails. For all other n we can consider a great arc joining a to $-a$ on S^{2n}_r where $r = |a|$.

You can probably do the same thing using degrees by considering the maps $x \mapsto \pm ax/\| \pm ax\|$

from $S^{2n} \to S^{2n}$. These maps should have nonzero degree varying from k to $-k$. An explicit homotopy can be given via $(x, t) \mapsto \gamma(t)x/\|\gamma(t)x\|$ with γ defined above. But, this contradicts the assumption that the the maps change sign. This sign change is a consequence of $\mathbb{R}^{2n+1}$ having odd dimension, since the antipodal map has degree $(-1)^{2n+1}$.

HW11

Hatcher Chapter 2.2, problems 9, 10, 11, 33, 42:

2.2.9. Compute the homology groups of the following 2-complexes:

a) The quotient of S^2 obtained by identifying north and south poles to a point.
b) $S^1 \times (S^1 \vee S^1)$.
c) The space obtained from D^2 by first deleting the interiors of two disjoint subdisks in the interior of D^2 and then identifying all three resulting boundary circles together via homeomorphisms preserving clockwise orientations of these circles.
d) The quotient space of $S^1 \times S^1$ obtained by identifying points in the circle $S^2 \times \{x_0\}$ that differ by $2\pi/m$ rotation and identifying points in the circle $\{x_0\} \times S^1$ that differ by $2\pi/n$ rotation.

Solution: Note that since each of the above is a 2-complex, $\tilde{H}_i(X) = 0$ for all $i \geq 3$; there are no cells in these dimensions.

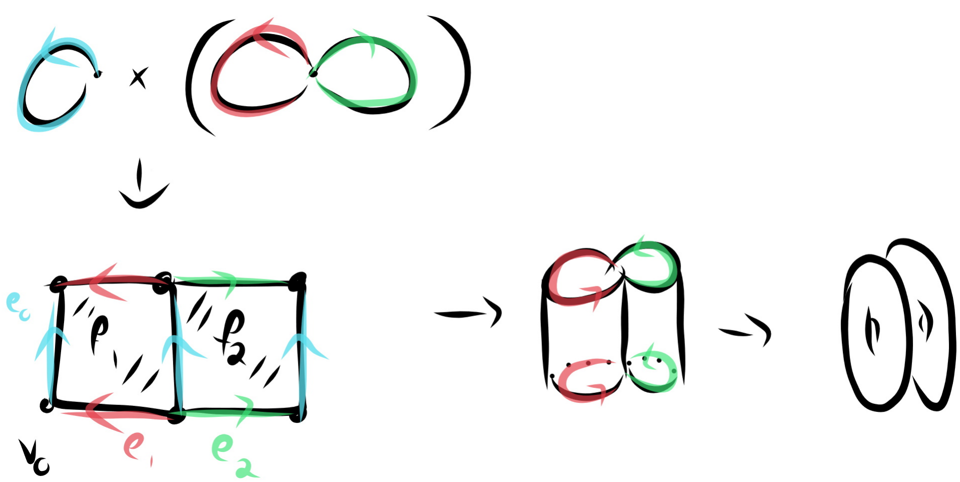
a) This is homotopy equivalent to $S^2 \vee S^1$, as shown on a previous homework. Since $(S^2, \{p\})$ and $(S^1, \{p\})$, where p is the wedge point, are good, we have that

$$\tilde{H}_i(S^2 \vee S^1) \simeq \tilde{H}_i(S^2) \oplus \tilde{H}_i(S^1).$$

Hence,

$$H_i(X) = \begin{cases} \mathbb{Z} & i = 0, 1, 2 \\ 0 & \text{else} \end{cases}$$

b) This is equivalent to $I \times (S^1 \vee S^1)$ with the endpoints $\{0\} \times (S^1 \vee S^1)$ and $\{1\} \times (S^1 \vee S^1)$ identified. This forms torus of sorts with outer circles identified. This is shown below, along with a CW structure



The chain complex is

$$\ldots \to 0 \to \mathbb{Z}^2 \to \mathbb{Z}^3 \to \mathbb{Z} \to 0$$

since there are two 2-cells, three 1-cells, and one 0-cell. Now we need to find the maps $\mathbb{Z}^2 \to \mathbb{Z}^3$ and $\mathbb{Z}^3 \to \mathbb{Z}$. To do so, we look at the attaching maps of the 2-cells and 1-cells, respectively.

Let us look at the 2-cells first. The 2-cell f_1 is attached to the 1-cells e_0 and e_0. Hence $\partial[f_1] = a[e_0] + b[e_1]$. To compute the coefficients a and b, we need to know the degree of the restriction of the attaching map to e_0 and e_1 respectively. First crushing all cells except e_0 in the 1-skeleton to a point yields S^1 consisting of a 0-cell and a 1-cell, e_0.

The attaching map for f_1 takes the bottom and top edges of $I \times I$ onto e_1 in the obvious way and the left and right edges onto e_0 in the obvious way. Now we travel along the boundary of $I \times I$ and look at the image of this path in the crushed 1-skeleton. This wraps around e_0 in one direction, stays at the 0-cell (since e_1 is crushed there), wraps around e_0 in the opposite direction once, and then stays at the 0-cell. Hence, the degree is zero, and $a = 0$. The exact same computation shows that $b = 0$. Essentially the same computation shows that $\partial[f_2] = 0$ (using symmetry). Hence $\partial_2 = 0$.

Now for ∂_1, we actually have the following. If X is path-connected and has a single 0-cell, then $\partial_1 : C_1(X) \to C_0(X)$ is the 0 map. Since X is path-connected, $H_0(X) = C_0/\operatorname{Im}(\partial_1) = \mathbb{Z}$. On the other hand, $C_0(X) = \mathbb{Z}$ since there is only one 0-cell. So $\mathbb{Z}/\operatorname{Im}(\partial_1) = \mathbb{Z}$. This only occurs if $\operatorname{Im}(\partial_1) = 0$, and therefore $\partial_1 = 0$.

Since all the boundary maps are zero, the chain complex directly gives the homology groups. Hence,

$$H_i(S^1 \times (S^1 \vee S^1)) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}^3 & i = 1 \\ \mathbb{Z}^2 & i = 2 \\ 0 & \text{else} \end{cases}$$

c) We've seen this space in a previous homework as well. We showed that the abelianization of its fundamental group is isomorphic to $\mathbb{Z}^2$. A CW structure is shown below.

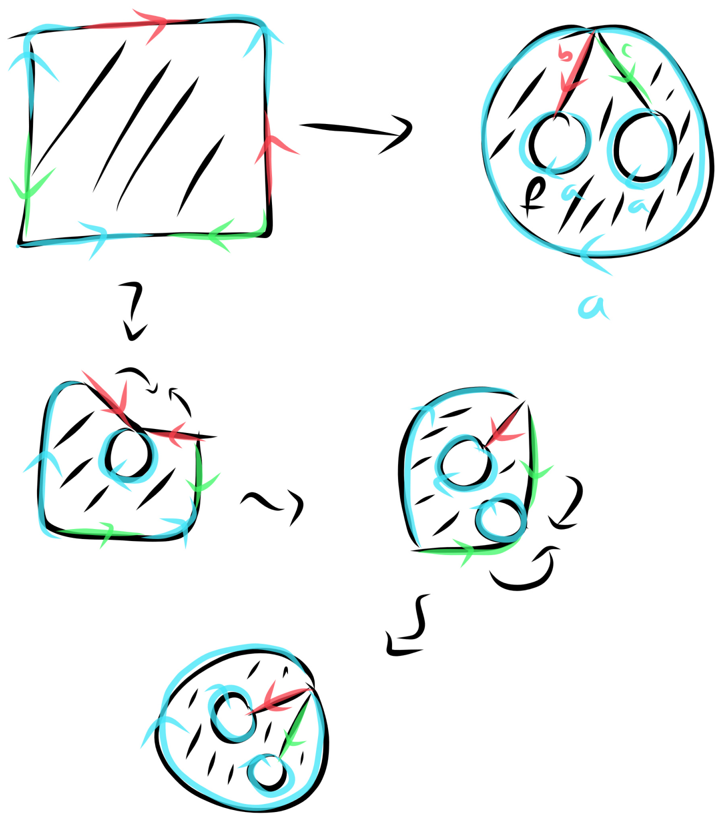
Hence, the chain complex is

$$\ldots \to \mathbb{Z} \to \mathbb{Z}^3 \to \mathbb{Z} \to 0$$

As shown in b), since there is only one 0-cell and the space is path-connected, this implies that $\partial_1 = 0$. To compute ∂_2, we need only check its action on the 2-cell f. We have that

$$\partial_2[f] = k_1[a] + k_2[b] + k_2[c].$$

Let's first compute the coefficient k_1. To do so, let's first find the attaching map. It is shown below.



To find k_1, we should do the whole "look at the image of the boundary in the crushed 1-skeleton, where we crush everything but a" and look at the degree. What we essentially do there is travel around the boundary of the attaching square and count (with sign!) all the appearances of a. So, in this case, it is one; all but one of the arrows point in the same direction, and there are three of them (there may be something here to say about why the degree is 1 and not -1, but I think everything works out fine). Similarly, $k_2 = k_3 = 0$ since there are two red and two green arrows, and they are oppositely oriented. It follows that

$$\partial_2[f] = [a].$$

Hence, $\text{Ker}(\partial_2) = 0$ while $\text{Im}(\partial_2) = \mathbb{Z}$. Computing the first and second homology gives

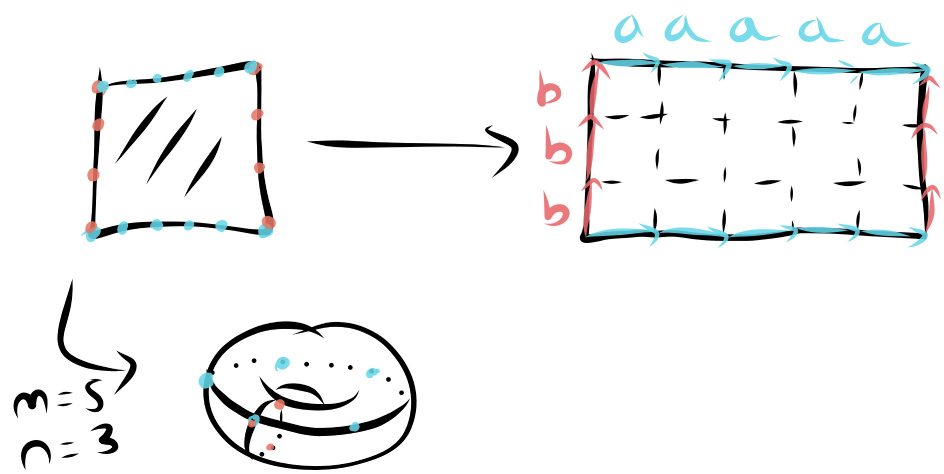
$$H_1(X) = \text{Ker}(\partial_1)/\text{Im}(\partial_2) = \mathbb{Z}^3/\mathbb{Z} = \mathbb{Z}^2, \quad H_2(X) = \text{Ker}(\partial_2)/\text{Im}(\partial_3) = 0/0 = 0.$$

In total,

$$H_i(X) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}^2 & i = 1 \\ 0 & \text{else} \end{cases}$$

Note that the abelianization of the first fundamental group and the first homology coincide.

d) The quotient effectively partitions the usual CW structure on T^2 into mn "regions". It looks like the following



The above left shows the identification of the wedge point via the $2\pi/m$ and $2\pi/n$ rotations. This happens to each point along $S^1 \times \{x_0\}$ and $\{x_0\} \times S^1$ though, hence why all the vertical and horizontal 1-cells are identified, respectively. Thus there is one 0-cell, two 1-cells, and one 2-cell. The chain complex is

$$... \to 0 \to \mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z} \to 0.$$

The space is path-connected with one 0-cell, so $\partial_1 = 0$. To compute ∂_2, we look at the attaching map of the 2-cell. It has degree zero since it attaches to a m times in one direction, then m times in the opposite direction, and similarly for b and n. Hence, all the boundary maps are zero, and the homology is just the chain complex. In total,

$$H_i(X) = \begin{cases} \mathbb{Z} & i = 0, 2 \\ \mathbb{Z}^2 & i = 1 \\ 0 & \text{else} \end{cases}$$

2.2.10. Let X be the quotient space of S^2 under the identifications $x \sim -x$ for x in the equator S^1. Compute the homology groups $\tilde{H}_i(X)$. Do the same for S^3 with antipodal points of the equatorial $S^2 \subset S^3$ identified.

Solution: The identification on the equator results in $\mathbb{R}\mathrm{P}^1 \simeq S^1$ embedded inside the quotient space. We can put a CW structure on X using this fact. First, a CW structure on S^2 is as follows: we have one 0-cell, one 1-cell, and two 2-cells. The 0- and 1- cells are used to make the equator while the two 2-cells are glued by attaching the boundary of each onto the equator identically. So, each is a hemisphere.

Upon taking the quotient, we still have the same number cells. What differs though is the attaching map of the 2-cells. We can regard going around $\mathbb{R}\mathrm{P}^1$ once as going around S^1 twice. Hence, to get a CW structure we attach each 2-cell onto S^1 with the usual degree 2 maps ($t \mapsto (\cos(2t), \sin(2t))$). Giving the 2-cells names, f_1, f_2, and calling the 1-cell e, we have $\partial_2[f_i] = 2[e]$. It follows that $(a, b) \mapsto 2(a + b)$. Since there is only one zero cell and the space is path-connected, $\partial_1 = 0$. The chain complex is then given by

$$... \to 0 \to \mathbb{Z}^2 \xrightarrow{(a,b)\mapsto 2(a+b)} \mathbb{Z} \xrightarrow{0} 0.$$

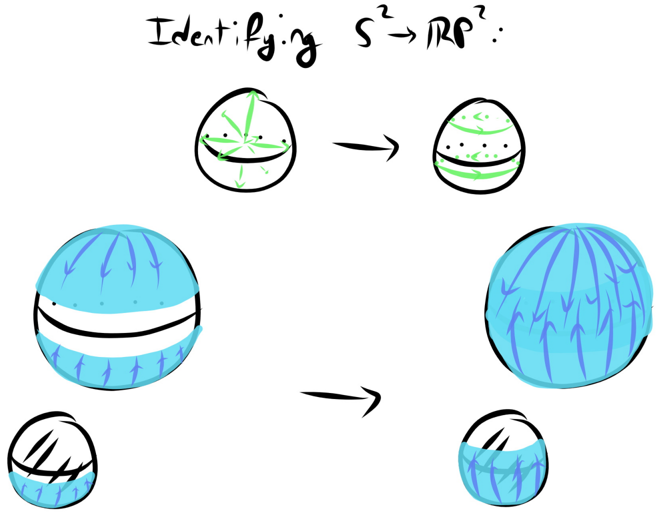
We can explicitly compute $\mathrm{Ker}(\partial_2)$ and $\mathrm{Im}(\partial_2)$. If $2(a+b) = 0$ then $a = -b$. So $\mathrm{Ker}(\partial_2) = \langle (1, -1) \rangle$. This is isomorphic to $\mathbb{Z}$ by the isomorphism $(a, -a) \mapsto a$. So, $\mathrm{Ker}(\partial_2) = \mathbb{Z}$. As for the image, every element in the image is an even integer. To show that all the even integers are in the image, consider the images of $(a, 0)$. Hence, $\mathrm{Im}(\partial_2) = 2\mathbb{Z}$. It follows that the first and second homology are given as

$$H_2(X) = \mathrm{Ker}(\partial_2)/\mathrm{Im}(\partial_3) = \mathbb{Z}/0 = \mathbb{Z}, \quad H_1(X) = \mathrm{Ker}(\partial_1)/\mathrm{Im}(\partial_2) = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2.$$

In total, the homology is

$$H_i(X) = \begin{cases} \mathbb{Z} & i = 0, 2 \\ \mathbb{Z}_2 & i = 1 \\ 0 & \text{else} \end{cases}$$

We can use similar reasoning for the case S^3. We build S^3 from S^2 by attaching two 3-cells by the identity attaching map. The quotient space consists of $\mathbb{R}\mathrm{P}^2$ with two 3-cells attached to it by modifying the above attaching maps. We obtain $\mathbb{R}\mathrm{P}^2$ by identifying the two 2-cells in the CW structure of S^2 via the antipodal map. So, the two become one, and we modify the attaching maps to be degree zero maps! These are degree zero because the degree of the antipodal map changed. In the previous case, the degree was 2 since the attaching map wrapped around twice in the same way. However, the degree here is zero since the attaching map wraps around twice in opposite directions. I depict this below. In the depiction, I use the CW structure on S^3 consisting of attaching two balls identically to S^2. For clarity, I only show attachment of one, but the same argument works for attaching the other.



At the top, I show how the antipodal map identifies points of S^2. To the left, I show how one ball is starting to be attached to this quotient of S^2. To the right, I show the same attachment further progressed – it is clear that the attachment starts to overlap in the opposite direction. We have the following chain complex

$$\ldots \to 0 \to \mathbb{Z}^2 \xrightarrow{0} \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{0} 0.$$

Now, ∂_2 can be found solely by looking at $\mathbb{R}\mathrm{P}^2$. Its 2-cell f is also attached to its 1-cell by a degree 2 map. Hence, $\partial_2[f] = 2[e]$. From this, it follows that $\mathrm{Ker}(\partial_2) = 0$ and $\mathrm{Im}(\partial_2) = 2\mathbb{Z}$. Thus,

$$\begin{aligned} H_1(X) &= \mathrm{Ker}(\partial_1)/\mathrm{Im}(\partial_2) = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2 \\ H_2(X) &= \mathrm{Ker}(\partial_2)/\mathrm{Im}(\partial_3) = 0/0 = 0 \\ H_3(X) &= \mathrm{Ker}(\partial_3)/0 = \mathbb{Z}^2 \end{aligned}$$

In total,

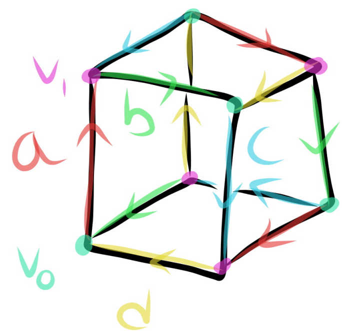
$$H_i(X) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}_2 & i = 1 \\ \mathbb{Z}^2 & i = 3 \\ 0 & \text{else} \end{cases}$$

2.2.11. In an exercise for §1.2 we described a 3-dimensional CW complex obtained from the cube I^3 by identifying opposite faces via a one-quarter twist. Compute the homology groups of this complex.

Solution: A CW structure can be given with two 0-cells, four 1-cells, three 2-cells, and one 3-cell. The chain complex is

$$\dots \to 0 \xrightarrow{0} \mathbb{Z} \xrightarrow{\partial_3} \mathbb{Z}^3 \xrightarrow{\partial_2} \mathbb{Z}^4 \xrightarrow{\partial_1} \mathbb{Z}^2 \xrightarrow{0} 0.$$

The following is a CW structure.



Call the front/back face f_1, the left/right face f_2, and the top/bottom face f_2. Call the 3-cell U.

We will use Smith normal form to compute the homology, see the appendix at the end. First, we need to find each ∂_i. Unlike the previous problems, we cannot immediately conclude $\partial_1 = 0$. However, it is clear from the depiction what the attaching maps are. It follows that

$$\partial_1[a] = [v_1] - [v_0] \quad \partial_1[b] = [v_0] - [v_1]$$
$$\partial_1[c] = [v_1] - [v_0] \quad \partial_1[d] = [v_0] - [v_1]$$

Hence,

$$\partial_1[k_1 a + k_2 b + k_3 c + k_4 d] = -(k_1 - k_2 + k_3 - k_4)[v_0] + (k_1 - k_2 + k_3 - k_4)[v_1]$$

We can represent this by the matrix C as follows:

$$C = \begin{pmatrix} -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

Transforming this into Smith normal form yields

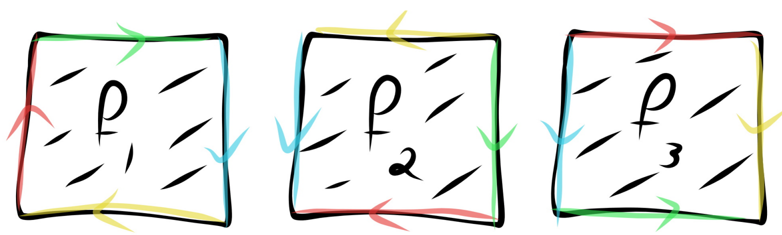
$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

It follows that

$$H_0(X) = \mathbb{Z}_1 \oplus \mathbb{Z}^{2-1-0} = \mathbb{Z}$$

since C has rank 1 with elementary divisor 1 and $\partial_0 = 0$ has rank 0.

Now the attaching maps for the 2-cells are given as follows:



Hence,

$$\begin{aligned}\partial_2[f_1] &= [a]+[b]+[c]+[d]\\ \partial_2[f_2] &= [a]+[b]-[c]-[d]\\ \partial_2[f_3] &= [a]-[b]-[c]+[d]\end{aligned}$$

Together, we conclude

$$\begin{aligned}\partial_2[k_1f_1+k_2f_2+k_3f_3] &= (k_1+k_2+k_3)[a]+(k_1+k_2-k_3)[b]\\ &+ (k_1-k_2-k_3)[c]+(k_1-k_2+k_3)[d]\end{aligned}$$

A matrix representing this is

$$B=\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & -1\\ 1 & -1 & -1\\ 1 & -1 & 1\end{pmatrix}$$

We can also put this into Smith normal form as follows

$$\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & -1\\ 1 & -1 & -1\\ 1 & -1 & 1\end{pmatrix}\to\begin{pmatrix}1 & 1 & 1\\ 0 & 0 & -2\\ 0 & -2 & -2\\ 0 & -2 & 0\end{pmatrix}\to\begin{pmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\\ 0 & 2 & 2\end{pmatrix}\to\begin{pmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\\ 0 & 0 & 0\end{pmatrix}$$

Hence, the rank of B is 3 with elementary divisors $1,2,2$. It follows that

$$H_1(X)=\left(\bigoplus_{i=1}^{3}\mathbb{Z}_{b_i}\right)\oplus\mathbb{Z}^{4-3-1}=\mathbb{Z}_2\oplus\mathbb{Z}_2$$

Notice this is the abelianization of the quaternion group, $\pi_1(X)$, as expected. Now we must investigate the attaching map of the 3-cell. To do so, we will use local homology. Fix a point y in the interior of a face. Then there are two preimages x_1,x_2 under the attaching map in ∂D^3. Now, whatever the local degrees are at these points, they must have opposite sign; this is because the faces are identified by a reflection and a quarter twist. Hence, adding up the local degrees gives 0. The same holds for each face so that $\partial_3=0$, which has rank 0. Hence,

$$H_2(X)=\mathbb{Z}^{3-3}=0 \quad H_3(X)=\mathrm{Ker}(\partial_3)=\mathbb{Z}.$$

In total

$$H_i(X)=\begin{cases}\mathbb{Z} & i=0,3\\ \mathbb{Z}_2\oplus\mathbb{Z}_2 & i=1\\ 0 & \text{else}\end{cases}$$

2.2.33. Suppose the space X is the union of open sets $A_1,...,A_n$ such that each intersection $A_{i_1}\cap ...\cap A_{i_k}$ is either empty or has trivial reduced homology groups. Show that $\tilde{H}_i(X)=0$ for $i\geq n-1$, and give an example showing this inequality is best possible, for each n.

Solution: We prove this by induction on n. The base case is $n=1$, in which case $\tilde{H}_i(X)=\tilde{H}_i(A_1)$. This is either trivial for all i, or if $A_1=\emptyset$ then it is trivial for all $i\geq 0$ (the empty set has nontrivial reduced homology at $i=-1$). Now assume it holds for $n-1$. Given $A_1,...,A_n$, let $Y=A_1\cup...\cup A_{n-1}$ and apply Mayer-Vietoris with Y and A_n. Then we get

$$...\to\tilde{H}_i(Y\cap A_n)\to\tilde{H}_i(Y)\oplus\tilde{H}_i(A_n)\to\tilde{H}_i(X)\to\tilde{H}_{i-1}(Y\cap A_n)\to...$$

Since $A_k\cap A_n$ is an open set for all k, it follows that $Y\cap A_n$ is the union of at most $n-1$ many open sets $A_k\cap A_n$. The intersection of a subcollection of these is either empty or has trivial homology since we assume $A_{i_1}\cap...\cap A_{i_k}$ does (for $\{i_1,...,i_k\}\subset\{1,...,n\}$). Hence, by the inductive hypothesis $\tilde{H}_i(Y\cap A_n)\simeq 0$ for $i\geq n-2$ and

$$...\to 0\to\tilde{H}_i(Y)\oplus\tilde{H}_i(A_n)\to\tilde{H}_i(X)\to 0\to...$$

Thus $\tilde{H}_i(Y) \oplus \tilde{H}_i(A_n) \simeq \tilde{H}_i(X)$ for $i \geq n-1$. But, by the inductive hypothesis,

$$\tilde{H}_i(Y) = \tilde{H}_i(A_1 \cup ... \cup A_{n-1}) \simeq 0$$

for $i \geq n-2$. Furthermore, $\tilde{H}_i(A_n) \simeq 0$ for $i \geq 0$ from the base case. It follows that $\tilde{H}_i(X) \simeq 0$ for $i \geq n-1$.

To show that this is the best we can do, we need a space X with nontrivial homology at $n-2$. The easiest such space is S^{n-2} which is such that $\tilde{H}_{n-2}(S^{n-2}) \simeq \mathbb{Z}$. Consider $X = \partial\Delta^{n-1} \simeq S^{n-2}$. Then $\tilde{H}_{n-2}(X) \simeq \mathbb{Z}$. We now wish to find n open sets satisfying the required criteria. Notice that $\partial\Delta^{n-1}$ has n faces. We take A_i to be a small open neighborhood of each face. By taking these sufficiently small, we can guarantee that $A_{i_1} \cap ... \cap A_{i_k}$ is either empty, or has trivial reduced homology. The latter case occurs since the intersection is some neighborhood of lower dimension faces, which are contractible. For example, in $\partial\Delta^3$, the intersection of two such A_i is an open neighborhood of an edge, which deformation retracts to the edge, which is contractible.

2.2.42. Let X be a finite connected graph having no vertex that is the endpoint of just one edge, and suppose that $H_1(X;\mathbb{Z})$ is free abelian of rank $n > 1$, so the group of automorphisms of $H_1(X,\mathbb{Z})$ is $GL_n(\mathbb{Z})$, the group of invertible $n \times n$ matrices with integer entries whose inverse matrix also has integer entries. Show that if G is a finite group of homeomorphisms of X, then the homomorphism $G \to GL_n(\mathbb{Z})$ assigning to $g : X \to X$ the induced homomorphism $g_* : H_1(X;\mathbb{Z}) \to H_1(X;\mathbb{Z})$ is injective. Show the same result holds if the coefficient group $\mathbb{Z}$ is replaced by $\mathbb{Z}_m$ with $m > 2$. What goes wrong when $m = 2$?

Solution: I do not have a fully-fleshed proof, but I can perhaps provide some insight into my thoughts. First, by removing valence 2 vertices (simply convert the two edges connected to the vertex into a single, continuous vertex) we see that any homeomorphism sends edges to edges and vertices to vertices (this follows from computing local homology, as in a previous homework problem). Hence, any homeomorphism is also a graph automorphism. Since there are no 2-cells, there are no boundaries. Hence, $H_1(X;\mathbb{Z})$ is just $\text{Ker}(\partial_1)$ consisting of cycles – not equivalence classes. Now suppose we have a homeomorphism $g : X \to X$ such that $g \mapsto I_n$. Since $H_1(X;\mathbb{Z}) = \text{Ker}(\partial_1)$, for a cycle $c = k_1e_1 + k_2e_2 + ... + k_me_m$ with $k_i \neq 0$ (I think actually all the k_i are ± 1, but I digress), it follows that

$$g_*(c) = \sum_{i=1}^{m} k_i g(e_i)$$

where g keeps track of the orientation on e_l (if you imagine directing e_l, then g just pushes forward the orientation). OTOH, since g is represented by I_n, we have that $g_*(c) = c$. Hence,

$$\sum_{i=1}^{m} k_i g(e_i) = \sum_{i=1}^{m} k_i e_i$$

Both of these are some combination of edges. So, the only way for the above statement to make sense is if g permutes the e_i while also preserving orientation and the coefficients match (e.g., if $g(e_i) = e_j$, then $k_j = k_i$). I think, since this must hold for each cycle, this implies g is the identity. The idea is that permuting the edges in one cycle will change the edges in other cycles. So, g cannot permute the edges in these cycles since a new edge was introduced.
Maybe something goes wrong if you have a graph like a bowtie – two triangles connected at a common vertex.

For coefficients in $\mathbb{Z}_m$ with $m > 2$, I imagine that everything you do in the $\mathbb{Z}$ case carries over since orientations are preserved. However, in $\mathbb{Z}_2$, since $1 = -1$ orientations are not preserved. For example, if you consider S^1 as a graph with two edges and two vertices, then a reflection permuting the vertices will be represented by the identity matrix.

Appendix: It is useful to develop some theory relating the Smith normal form and computing homology. First, the Smith normal form of a matrix $A \in \text{Mat}_{m\times n}(\mathbb{Z})$ is a unique (up to sign on D) factorization $A = UDV$ where

- $D \in \text{Mat}_{m\times n}(\mathbb{Z})$ is such that $(D)_{ij} = 0$ whenever $i \neq j$. So, for $m = n$, D is a diagonal matrix.
- The diagonal entries of D satisfy $(D)_{ii} \mid (D)_{i+1,i+1}$; that is, $(D)_{ii}$ divides $(D)_{i+1,i+1}$. These are called the elementary divisors of A. The number of nonzero elementary divisors is the rank r of A.
- $U \in \text{Mat}_{m\times m}(\mathbb{Z})$ and $V \in \text{Mat}_{n\times n}(\mathbb{Z})$ with $\det(U), \det(V) = \pm 1$.

We have the following proposition relating the cokernels of A and D.

Proposition: Let $a_1, ..., a_r$ be the nonzero elementary divisors of $A \in \text{Mat}_{m\times n}(\mathbb{Z})$. Then, $\text{Coker}(A) \simeq \text{Coker}(D)$ via the isomorphism $x + \text{Im}(A) \mapsto U^{-1}x + \text{Im}(D)$.

Proof. First recall that the cokernel is defined by $\text{Coker}(A) = \mathbb{Z}^m/\text{Im}(A)$. So, any element $[x] \in \text{Coker}(A)$ can be represented as $x + \text{Im}(A)$ where $x \in \mathbb{Z}^m$. Technically here, anything in $\text{Im}(A)$ is an $m \times 1$ integer matrix, but there is a canonical isomorphism between these with the $1 \times m$ integer matrices in $\mathbb{Z}^m$. We use this identification freely.

Let us show this map is well defined. Suppose that y represents the same equivalence class as x in the cokernel. Then $y - x \in \text{Im}(A)$. Let $z \in \mathbb{Z}^n$ be such that $y - x = Az$. Then,

$$U^{-1}(y - x) = U^{-1}Az = U^{-1}UDVz = D(Vz)$$

where $Vz \in \mathbb{Z}^n$. It follows that $U^{-1}(y - x) \in \text{Im}(D)$. Hence, $U^{-1}y$ and $U^{-1}x$ represent the same equivalence class in the cokernel of D.

Now let us show this is a homomorphism. Consider two equivalence classes $[x]$ and $[y]$ in $\text{Coker}(A)$. We wish to show that $\varphi[x] + \varphi[y] = \varphi[x + y]$ where φ is the defined mapping. But this is obvious since U^{-1} is a linear transformation. That is,

$$\varphi[x] + \varphi[y] = [U^{-1}x] + [U^{-1}y] = [U^{-1}x + U^{-1}y] = [U^{-1}(x + y)] = \varphi[x + y].$$

Suppose now that $[x]$ and $[y]$ in $\text{Coker}(A)$ are such that $\varphi[x] = \varphi[y]$. It follows that $U^{-1}x = U^{-1}y$. But U is invertible so that $x = y$. Hence, φ is injective. Now let $z + \text{Im}(D) \in \text{Coker}(D)$. Then $Uz + \text{Im}(A) \to U^{-1}Uz + \text{Im}(D) = z + \text{Im}(D)$ so that φ is surjective.

Hence, $x + \text{Im}(A) \mapsto U^{-1}x + \text{Im}(D)$ is a well defined isomorphism. □

As a corollary to this, we have

Corollary: For $A \in \text{Mat}_{m\times n}(\mathbb{Z})$, $\text{Coker}(A) \simeq (\bigoplus_{i=1}^r \mathbb{Z}_{a_i}) \oplus \mathbb{Z}^{m-r}$.

Proof. Let $\{e_i\}_{i=1}^m$ be the standard basis of $\mathbb{Z}^m$. Then clearly the image of D is spanned by $\{a_ie_i\}_{i=1}^r$. It follows that

$$\text{Coker}(D) = \mathbb{Z}^m/\text{Im}(D) = \mathbb{Z}^m/\bigoplus_{i=1}^r a_i\mathbb{Z} = \left(\bigoplus_{i=1}^r \mathbb{Z}_{a_i}\right) \oplus \mathbb{Z}^{m-r}.$$

□

From this, we can compute homology of CW complexes rather easily once we know the boundary maps. We have the following theorem.

Theorem: Suppose we have a CW complex X whose chain complex at the i-th position is $... \to \mathbb{Z}^n \xrightarrow{\partial_{i+1}} \mathbb{Z}^m \xrightarrow{\partial_i} \mathbb{Z}^k \to ...$ Let $B \in \text{Mat}_{k\times m}(\mathbb{Z})$ be a matrix representation of ∂_i and $A \in \text{Mat}_{m\times n}(\mathbb{Z})$

of ∂_{i+1}. Then,

$$H_i(X) = \operatorname{Ker}(\partial_i)/\operatorname{Im}(\partial_{i+1}) \simeq \left(\bigoplus_{i=1}^{r} \mathbb{Z}_{a_i}\right) \oplus \mathbb{Z}^{m-r-s}$$

where r is the rank of A and s is the rank of B and the a_i are the nonzero elementary divisors of A.

Proof. Since we have a chain complex, we know that $\operatorname{Im}(\partial_{i+1}) \subset \operatorname{Ker}(\partial_i)$ so that ∂_i is zero on $\operatorname{Im}(\partial_{i+1})$. In particular, this implies we can descend to a homomorphism $\tilde{\partial}_i : \mathbb{Z}^m/\operatorname{Im}(\partial_{i+1}) \to \mathbb{Z}^k$ by $\tilde{\partial}_i[x] = \partial_i(x)$. This does not depend on the choice of representative, since if y is another representative then $y - x \in \operatorname{Im}(\partial_{i+1})$. Hence, $y - x \in \operatorname{Ker}(\partial_i)$. It follows that

$$\tilde{\partial}_i[y] = \tilde{\partial}_i[x + y - x] = \partial_i(x + y - x) = \partial_i(x) + \partial_i(y - x) = \partial_i[x].$$

It is clear from this that $\operatorname{Ker}(\tilde{\partial}_i) \simeq \operatorname{Ker}(\partial_i)/\operatorname{Im}(\partial_{i+1}) = H_i(X)$ via the isomorphism $[x] \mapsto x + \operatorname{Im}(\partial_{i+1})$. Let $\tilde{B}$ be a matrix representation of $\tilde{\partial}_i$.

By the previous proposition, $\mathbb{Z}^m/\operatorname{Im}(\partial_{i+1}) \simeq (\bigoplus_{i=1}^{r} \mathbb{Z}_{a_i}) \oplus \mathbb{Z}^{m-r}$. Note that $\tilde{B}$ maps into a free group, namely $\mathbb{Z}^k$. Since all nonidentity elements of a free group H have infinite order, and the torsion subgroup G_T of a group G consists of all finite order elements, it follows that any homomorphism $\varphi : G \to H$ must satisfy $G_T \subset \operatorname{Ker}(\varphi)$. In our case, $G = (\bigoplus_{i=1}^{r} \mathbb{Z}_{a_i}) \oplus \mathbb{Z}^{m-r}$ so that $G_T = \bigoplus_{i=1}^{r} \mathbb{Z}_{a_i}$. Hence,

$$\operatorname{Ker}(\tilde{B}) = \left(\bigoplus_{i=1}^{r} \mathbb{Z}_{a_i}\right) \oplus \operatorname{Ker}(\tilde{B}|_{\mathbb{Z}^{m-r}}).$$

In the above, we've split the kernel of $\tilde{B}$ into the torsion part and the remaining part, which is just the kernel of what's leftover. By assumption, B has rank s. By construction, $\tilde{B}$ has the same image (since we only quotiented out by $\operatorname{Im}(A)$ which is in the kernel) so that $\operatorname{Ker}(\tilde{B}|_{\mathbb{Z}^{m-r}})$ has rank $m - r - s$. Finally, the subgroup of any free group is free, so $\operatorname{Ker}(\tilde{B}|_{\mathbb{Z}^{m-r}}) = \mathbb{Z}^{m-r-s}$. □

HW12

Hatcher Chapter 2.3, problems 3, 4:

2.3.3. Show that if $\tilde{h}$ is a reduced homology theory, then $\tilde{h}_n(point) = 0$ for all n. Deduce that there are suspension isomorphisms $\tilde{h}_n(X) \simeq \tilde{h}_{n+1}(SX)$ for all n.

Solution: We have for any CW pair (X, A) that there exists boundary homomorphisms $\partial : \tilde{h}_n(X/A) \to \tilde{h}_{n-1}(A)$ such that the following sequence is exact

$$... \to \tilde{h}_{n+1}(X/A) \xrightarrow{\partial} \tilde{h}_n(A) \xrightarrow{i_*} \tilde{h}_n(X) \xrightarrow{q_*} \tilde{h}_n(X/A) \to ...$$

Now consider the case when $A = X$. Then,

$$... \to \tilde{h}_{n+1}(point) \xrightarrow{\partial} \tilde{h}_n(X) \xrightarrow{i_*} \tilde{h}_n(X) \xrightarrow{q_*} \tilde{h}_n(point) \to ...$$

is exact. But the inclusion homomorphism i_* is actually an isomorphism so that $\text{Ker}(i_*) = 0$ and $\text{Im}(i_*) = \tilde{h}_n(X)$. Hence $\text{Im}(\partial) = \text{Ker}(i_*) = 0$ and $\text{Ker}(q_*) = \text{Im}(i_*) = \tilde{h}_n(X)$. It follows that both q_* and ∂ are the zero homomorphism. Now exactness at $\tilde{h}_n(point)$ implies that $\text{Ker}(\partial) = \text{Im}(q_*) = 0$. But ∂ is the zero homomorphism, which is injective iff $\tilde{h}_n(point) = 0$. This same logic holds for all n; in particular, it still holds at $n = 0$ since the last morphism ($\tilde{h}_0(X/A) \to 0$) is the zero map by definition.

Now recall that $SX = (X \times I)/\sim$ where $\sim$ crushes $X \times \{0\}$ and $X \times \{1\}$ to distinct points. We can just as easily view SX as $CX \cup_f CX$ where f glues the two bases of CX together identically. With this in mind, there are two natural subcomplexes of SX; namely, the upper and lower cones $A = C_+X$ and $B = C_-X$. Then $A \cap B = X$ while $A \cup B = SX$. Applying Mayer-Vietoris gives an exact sequence

$$... \to \tilde{h}_{n+1}(C_+X) \oplus \tilde{h}_{n+1}(C_-X) \to \tilde{h}_{n+1}(SX) \to \tilde{h}_n(X) \to \tilde{h}_n(C_+X) \oplus \tilde{h}_n(C_-X) \to ...$$

But C_+X and C_-X are contractible so that $\tilde{h}_n(C_+X) \simeq \tilde{h}_n(C_-X) \simeq \tilde{h}_n(point) = 0$ for all n. Thus we obtain an isomorphism $\tilde{h}_{n+1}(SX) \simeq \tilde{h}_n(X)$.

2.3.4. Show that the wedge axiom for homology theories follows from the other axioms in the case of finite wedge sums.

Solution: We proceed by induction. It trivially holds when $n = 1$; that is when X is just itself. Now in $n = 2$, with $X = X_1 \vee X_2$, consider the CW pair (X, X_1). Then we obtain a long exact sequence

$$... \to \tilde{h}_{n+1}(X_2) \xrightarrow{\partial} \tilde{h}_n(X_1) \xrightarrow{i_*} \tilde{h}_n(X) \xrightarrow{q_*} \tilde{h}_n(X_2) \to ...$$

Note that the quotient map $p : X \to X_1$ which crushes X_2 to a point is a retraction of X to X_1. It follows by functorality that $i_* : \tilde{h}_n(X_1) \to \tilde{h}_n(X)$ is injective (this was a previous hw exercise that depended only on the fact that H_n is a functor). It follows that $\text{Im}(\partial) = \text{Ker}(i_*) = 0$ so that ∂ is the zero map. Consequently, $\text{Im}(q_*) = \text{Ker}(\partial) = \tilde{h}_n(X_2)$. Hence, for each n we have a short exact sequence

$$0 \to \tilde{h}_n(X_1) \xrightarrow{i_*} \tilde{h}_n(X) \xrightarrow{q_*} \tilde{h}_n(X_2) \to 0.$$

Finally observe that $q : X \to X_2$ is a retraction of X to X_2. It follows that $q \circ j = \text{Id}_{X_2}$ where j is the inclusion of X_2 into X. By functorality, $q_* \circ j_* = \text{Id}_{\tilde{h}_n(X_2)}$. The splitting lemma holds, so that

$$\tilde{h}_n(X) \simeq \tilde{h}_n(X_1) \oplus \tilde{h}_n(X_2).$$

Now suppose $X = X_1 \vee ... \vee X_{n+1}$ and assume inductively that

$$\tilde{h}_i(Y) \simeq \bigoplus_{k=1}^{n} \tilde{h}_i(Y_k)$$

whenever $Y = Y_1 \vee ... \vee Y_n$. Consider the pair (X, X_{n+1}). Then by the same logic there is an exact sequence

$$... \to \tilde{h}_{i+1}(Y) \to \tilde{h}_i(X_{n+1}) \to \tilde{h}_i(X) \to \tilde{h}_i(Y) \to ...$$

where $Y = X_1 \vee ... \vee X_n$. By the same reasoning we can extract a short exact sequence for each i

$$0 \to \tilde{h}_i(X_{n+1}) \to \tilde{h}_i(X) \to \tilde{h}_i(Y) \to 0$$

which splits so that

$$\tilde{h}_i(X) \simeq \tilde{h}_i(Y) \oplus \tilde{h}_i(X_{n+1}) = \bigoplus_{k=1}^{n+1} \tilde{h}_i(X_k).$$

Hatcher Chapter 2.C, problems 2, 3, 4, 9:

2.C.2. Use the Lefschetz fixed point theorem to show that a map $S^n \to S^n$ has a fixed point unless its degree is equal to the degree of the antipodal map $x \mapsto -x$.

Solution: First we can show that for a path-connected simplicial complex X any simplicial map $f : X \to X$ is such that $f_* : H_0(X) \to H_0(X)$ is the identity. The reasoning is as follows: $H_0(X)$ is defined to be $C_0(X)/\operatorname{Im}(\partial_1)$. I claim that all the vertices v of X lie in the same homology class. Let v_1 and v_2 be two vertices of X. Then there exists a path from v_1 to v_2. This is a 1-simplex $[v_1, v_2]$. Its boundary is $[v_2] - [v_1]$, and it follows that $[v_1]$ and $[v_2]$ are the same homology class. From this, one also deduces that $[v]$ generates $H_0(X)$ for any vertex v. Hence, since f is simplicial

$$f_*[v] = [f(v)] = [v]$$

and f_* is the identity. By simplicial approximation, it follows that any continuous map $f : X \to X$ is homotopic to a simplicial map provided X is a finite simplicial complex (using a barycentric subdivision if necessary). Hence, any map $f : X \to X$ in this situation is such that $\operatorname{Tr}(f_*) = 1$. In particular, this holds for $X = S^n$.

Now, the Lefschetz number for a map $f : S^n \to S^n$ is

$$\begin{aligned} \tau(f) &= \sum_i (-1)^i \operatorname{Tr}(f_* : H_i(S^n) \to H_i(S^n)) \\ &= \operatorname{Tr}(f_* : H_0(S^n) \to H_0(S^n)) + (-1)^n \operatorname{Tr}(f_* : H_n(S^n) \to H_n(S^n)) \\ &= 1 + (-1)^n \operatorname{Tr}(f_* : H_n(S^n) \to H_n(S^n)) \end{aligned}$$

since $H_i(S^n) = 0$ except when $i = 0, n$. By the Lefschetz fixed point theorem, there exists a fixed point when $\tau(f) \neq 0$. We see that $\tau(f) = 0$ iff $\deg(f) = (-1)^{n+1}$; that is if f has the same degree as the antipodal map.

2.C.3. Verify that the formula $f(z_1, ..., z_{2k}) = (\overline{z}_2, -\overline{z}_1, \overline{z}_4, -\overline{z}_3, ..., \overline{z}_{2k}, -\overline{z}_{2k-1})$ defines a map $f : \mathbb{C}^{2k} \to \mathbb{C}^{2k}$ inducing a quotient map $\mathbb{C}\mathrm{P}^{2k-1} \to \mathbb{C}\mathrm{P}^{2k-1}$ without fixed points.

Solution: Observe that f is a bijection. For any $z = (z_1, ..., z_{2k}) \in \mathbb{C}^{2k}$ we have

$$f(-\overline{z}_2, \overline{z}_1, -\overline{z}_4, \overline{z}_3, ..., -\overline{z}_{2k}, \overline{z}_{2k-1}) = (\overline{\overline{z}_1}, -(\overline{-\overline{z}_2}), \overline{\overline{z}_3}, -(\overline{-\overline{z}_4}), ..., \overline{\overline{z}_{2k-1}}, -(\overline{-\overline{z}_{2k}})) = (z_1, ..., z_{2k})$$

so that f is surjective. Clearly f is injective; e.g. if $f(z) = f(z')$ then for all j we have $(-1)^j \overline{z}_j = (-1)^j \overline{z}'_j$ and $z_j = z'_j$. So f defines a bijective map $f : \mathbb{C}^{2k} \to \mathbb{C}^{2k}$. We can show that f takes lines through the origin to lines through the origin. A line through the origin in $\mathbb{C}^{2k}$ looks like

$$\sum_{j=1}^{2k} c_j z_j = 0$$

for real c_j possibly zero. Under the change of variables given by f, this becomes

$$\sum_{j=1}^{2k-1} c_j \overline{z}_{j+1} - \sum_{j=1}^{2k-1} c_{j+1} \overline{z}_j = 0$$

which a line through the origin since conjugating both sides gives a line through the origin. Hence, f induces a well defined map on $\mathbb{C}\mathrm{P}^{2k-1}$ to itself.

Recall in the real case $\mathbb{R}^n \to \mathbb{R}^n$ that every linear transformation takes lines through the origin to lines through the origin and hence defines a map $\mathbb{R}\mathrm{P}^{n-1} \to \mathbb{R}\mathrm{P}^{n-1}$. The nonzero eigenvectors of the linear transformation correspond to fixed points of this quotient map. Indeed, if we have a nonzero eigenvector, then the 1-dimensional span of it is a line through the origin kept fixed under the linear transformation. Hence, this is a fixed point in $\mathbb{R}\mathrm{P}^{n-1}$. The converse is also true.

Here we have something slightly different. The above map is not a linear transformation, but it is conjugate linear. We can still talk about eigenvectors, and the same reasoning as above will show that if f does not have a nonzero eigenvector then the quotient map has no fixed points. So, suppose we have an eigenvector $f(z) = \lambda z$ with $\lambda > 0$. Then,

$$(\lambda z_1, ..., \lambda z_{2k}) = (\overline{z}_2, -\overline{z}_1, ..., \overline{z}_{2k}, -\overline{z}_{2k-1}).$$

Hence for each odd j we have $\lambda z_j = \overline{z}_{j+1}$ and for each even j we have $\lambda z_j = -\overline{z}_{j-1}$. Now fix an odd j. Then,

$$\lambda z_j = \overline{z}_{j+1} = \frac{1}{\lambda}(\overline{-\overline{z}_j}) = -\frac{1}{\lambda} z_j.$$

Rearranging this gives

$$\lambda\overline{\lambda} z_j = -z_j \quad \Rightarrow \quad (|\lambda|^2 + 1) z_j = 0$$

Thus $z_j = 0$. Using the relation $\lambda z_j = \overline{z}_{j+1}$ we see that $z_{j+1} = 0$. This holds for all j, so the only eigenvector is $z = 0$. It follows that there are no fixed points in the quotient map.

2.C.4. If X is a finite simplicial complex and $f : X \to X$ is a simplicial homeomorphism, show that the Lefschetz number $\tau(f)$ equals the Euler characteristic of the set of fixed points of f. In particular, $\tau(f)$ is the number of fixed points if the fixed points are isolated. [Hint: Barycentrically subdivide X to make the fixed point set a subcomplex.]

Solution: Consider the action of f on a simplex. Since f is a simplicial homeomorphism, it sends k-dimensional simplices to k-dimensional simplices. Hence, we may view $f_* : C_k(X) \to C_k(X)$ as a permutation of the k-dimensional simplices. Now, suppose we may barycentrically subdivide X so that the fixed point set X_f is a subcomplex. Then $C_k(X_f)$ has basis all the fixed k-dimensional simplices under f. It follows that $\dim(C_k(X_f))$ is the number of fixed k-dimensional simplices. Since $f_* : C_k(X) \to C_k(X)$ is a permutation of the k-simplices, it follows that $\mathrm{Tr}(f_* : C_k(X) \to C_k(X))$ is precisely the number of fixed k-simplices. Hence, $\mathrm{Tr}(f_* : C_k(X) \to C_k(X)) = \dim(C_k(X_f))$ and the result follows.

We now show why the hint works. First let us set up some notation and definitions. For a simplex $S = [v_0, ..., v_n]$ in a simplicial complex, its barycenter $\beta(S)$ is given by

$$\beta(S) = \frac{1}{n+1} \sum_{i=0}^{n} v_i.$$

For a simplicial complex K we denote the first barycentric subdivision by $\mathrm{sd}\, K$. A general simplex takes the form

$$[\beta(S_1), ..., \beta(S_m)], \quad S_i \in K, \quad S_1 \subset S_2 \subset ... \subset S_m$$

E.g., for a simplex $S = [v_0, ..., v_n]$ we can construct a simplex in the barycentric subdivision by setting $S_i = [v_0, ..., v_{i-1}]$ and letting i range from 1 to $n + 1$.
Recall that simplicial maps are uniquely determined by their action on vertices of simplices. If $f : K \to K$ is a simplicial map then we may define a simplicial map $\mathrm{sd}\, f : \mathrm{sd}\, K \to \mathrm{sd}\, K$ as folllows: For $\mathrm{sd}\, f$ to be simplicial, we need only define it on vertices of simplices in $\mathrm{sd}\, K$, and moreover it must send vertices in $\mathrm{sd}\, K$ to vertices in $\mathrm{sd}\, K$. But vertices in $\mathrm{sd}\, K$ are precisely barycenters in K,

which are defined in terms of linear combinations of the vertices in K. Since f sends vertices to vertices, we can use it to define $\operatorname{sd} f$ (sending barycenters to barycenters). This is given below

$$(\operatorname{sd} f)(\beta(S)) = \beta(f(S)).$$

Clearly $\operatorname{sd} f$ sends barycenters to barycenters (hence vertices to vertices in $\operatorname{sd} K$) and therefore extends to a simplicial map.

What's the point of this? Well now we have a simplicial map on $\operatorname{sd} K$ rather than K. Now let's check the action of $\operatorname{sd} f$ on vertices of K, which are barycenters of 0-simplices. If $S = [v_0]$ then its barycenter is $\beta(S) = v_0$. Hence,

$$(\operatorname{sd} f)(v_0) = (\operatorname{sd} f)(\beta(S)) = \beta(f(S)) = \beta([f(v_0)]) = f(v_0)$$

Thus $\operatorname{sd} f = f$, and they clearly have the same fixed point set. So, instead of talking about the fixed points on $f : K \to K$ we can talk about the fixed points on $\operatorname{sd} f : \operatorname{sd} K \to \operatorname{sd} K$. This turns out to be advantageous for the following reason: for a general $x \in K$ we of course have that x is an interior point of some n-simplex $S = [v_0, ..., v_n]$. Hence,

$$x = \sum_{i=0}^{n} t_i v_i,$$

where $t_i > 0$ for all i and $\sum_{i=0}^{n} t_i = 1$. Now, f sends vertices to vertices so that we may view f as simply permuting the vertices. That is, define $\sigma(i)$ by the index of $f(v_i)$. Then,

$$f(x) = \sum_{i=0}^{n} t_i v_{\sigma(i)} = \sum_{i=0}^{n} t_{\sigma^{-1}(i)} v_i.$$

If x is a fixed point we have

$$\sum_{i=0}^{n} t_{\sigma^{-1}(i)} v_i = \sum_{i=0}^{n} t_i v_i.$$

We want to be able to say that the entire simplex S is fixed under f (indeed, this is what it means to say that the fixed point set is a subcomplex). We can do this if we know that the vertices of S are fixed – i.e. $\sigma^{-1}(i) = i$ for all i. Herein lies our issue: we cannot immediately conclude this, for it may be that multiple t_i are equal. For example, if $t_1 = t_2$ and we know that $t_{\sigma^{-1}(1)} = t_1$, it could be that $\sigma^{-1}(1) = 1$ or 2. Passing to the barycentric subdivision allows us to circumvent this. Let us see how.

Let $x \in K$ be a fixed point of f. Since $x \in K$, it is an interior point of some unique k-dimensional simplex $[w_0, ..., w_k]$ in $\operatorname{sd} K$ and an n-dimensional simplex $S = [v_0, ..., v_n]$ in K. We have that $w_i = \beta(S_i)$ where $S_i \subset S$ is a sub-simplex. Let $d_i = \dim(S_i)$ so that $S_i = [v_0, ..., v_{d_i}]$ (up to an initial relabelling of the vertices). Since $S_0 \subset S_1 \subset ...$ we can always keep the first however many vertices the same. The only thing that differs between S_i and S_{i+1} is that $d_{i+1} - d_i$ many vertices are added. It is important that we have the same vertices being used! We can write w_i as

$$w_i = \beta(S_i) = \frac{1}{d_i + 1} \sum_{l=0}^{d_i} v_l.$$

Now, since x is in the interior of $[w_0, ..., w_k]$, we can write

$$x = \sum_{i=0}^{k} t_i w_i$$

where $t_i > 0$ and $\sum_i t_i = 1$ as before. But we can now substitute our definition of w_i and write

$$x = \sum_{i=0}^{n} \tilde{t}_i v_i$$

where $\tilde{t}_i \geq \tilde{t}_{i+1}$ for all i. The reasoning is as follows: As i increases, w_i "turns on" more and more of the vertices v, but does not turn off any of the ones it already turned on. So, the coefficients on $v_0, ..., v_{d_0}$ is

$$\frac{t_0}{d_0+1} + \frac{t_1}{d_1+1} + ... + \frac{t_k}{d_k+1}$$

whereas the coefficient on v_{d_0+1} to v_{d_1} misses the coefficient from w_0, and therefore is

$$\frac{t_1}{d_1+1} + ... + \frac{t_k}{d_k+1}$$

and it continues in this fashion until $\tilde{t}_i$ possibly becomes zero. Since all the $t_i > 0$ it follows that $\tilde{t}_i \geq \tilde{t}_{i+1}$.

Now remember the entire issue is that it's a priori possible to write $x = \sum s_i v_i$ where $s_i = s_j$ for $i \neq j$. If we could achieve a strict inequality in $\tilde{t}_i \geq \tilde{t}_{i+1}$, we would be done, but this is not possible. What we can do is group together vertices based on their coefficients, as done above:

$$x = \sum_{i=0}^{k} s_i \left(\sum_{l=d_{i-1}+1}^{d_i} v_l \right)$$

where

$$s_i = \sum_{j=i}^{k} \frac{t_j}{d_j+1}$$

and we set $d_{-1} = -1$. Hence $\tilde{t}_0 = ... = \tilde{t}_{d_0} = s_0$, and so on. In expanded form, this is just

$$\begin{aligned} x &= \left(\frac{t_0}{d_0+1} + ... + \frac{t_k}{d_k+1} \right)(v_0 + ... + v_{d_0}) + \left(\frac{t_1}{d_1+1} + ... + \frac{t_k}{d_k+1} \right)(v_{d_0+1} + ... + v_{d_1}) \\ &\quad + ... + \left(\frac{t_k}{d_k+1} \right)(v_{d_{k-1}+1} + ... + v_{d_k}) \end{aligned}$$

which again comes from substituting

$$\begin{aligned} w_0 &= \frac{1}{d_0+1}(v_0 + ... + v_{d_0}) \\ w_1 &= \frac{1}{d_1+1}(v_0 + ... + v_{d_1}) = \frac{1}{d_1+1}(v_0 + ... + v_{d_0}) + \frac{1}{d_1+1}(v_{d_0+1} + ... + v_{d_1}) \\ &\ldots \\ w_k &= \frac{1}{d_k+1}(v_0 + ... + v_{d_k}) \\ &= \frac{1}{d_k+1}(v_0 + ... + v_{d_0}) + \frac{1}{d_k+1}(v_{d_0+1} + ... + v_{d_1}) + \frac{1}{d_k+1}(v_{d_{k-1}+1} + ... + v_{d_k}) \end{aligned}$$

into

$$x = t_0 w_0 + t_1 w_1 + ... + t_k w_k.$$

Now notice the following. In order for x to be a fixed point, it must be that

$$\sum_{i=0}^{k} \tilde{t}_i v_i = \sum_{i=0}^{k} \tilde{t}_i v_{\sigma(i)} = \sum_{i=0}^{k} \tilde{t}_{\sigma^{-1}(i)} v_i \quad \Rightarrow \quad \sum_{i=0}^{k} (\tilde{t}_i - \tilde{t}_{\sigma^{-1}(i)}) v_i = 0$$

This implies that, for each i, $\tilde{t}_i = \tilde{t}_{\sigma^{-1}(i)}$. So, f permutes vertices with the same coefficients. We've written x so that all vertices with the same coefficients are grouped together. It follows that $\{v_0, ..., v_{d_0}\}$ are permuted, $\{v_{d_0+1}, ..., v_{d_1}\}$ are permuted, and so on. This immediately says that w_i is a fixed point for each i! To see this, look at the groupings for each w_i above. I've grouped them based on the classes of permuted vertices. Each sum is preserved so that the total sum, w_i, is too. Hence barycenters are fixed under f, and therefore $\operatorname{sd} f$. But $\operatorname{sd} f$ is a simplicial map on $\operatorname{sd} K$ so that it is defined on the barycenters of K. It follows that $[w_0, ..., w_k]$ (the simplex x is in the interior of in sd K) is fixed by sd f, and therefore f. Thus, the fixed point set is a subcomplex of sd K.

As an example, consider the standard 2-simplex $S = [v_0, v_1, v_2]$. We can define a simplicial map by $f(v_0) = v_0, f(v_1) = v_2, f(v_2) = v_1$; i.e. a reflection. A simplex in the barycentric subdivision is $[w_0, w_1]$ where $w_0 = v_0$ and w_1 is the barycenter of $[v_0, v_1, v_2]$, so $w_1 = 1/3(v_0 + v_1 + v_2)$. We define $S_0 = [v_0]$ and $S_1 = S$ so that $d_0 = 0$ and $d_1 = 2$. This agrees with our above definitions of w_i, since

$$w_i = \frac{1}{d_i + 1} \sum_{l=0}^{d_i} v_l.$$

Consider $x = 1/4w_0 + 3/4w_1$, which is certainly fixed by f (it lies on the axis of reflection). Then, writing out x in terms of v_i gives

$$x = \frac{1}{4}(v_0) + \frac{3}{4}\left(\frac{1}{3}(v_1 + v_2 + v_3)\right) = \frac{1}{2}v_0 + \frac{1}{4}v_1 + \frac{1}{4}v_2.$$

We see here that the vertices with equal coefficients are permuted, and v_1 is a fixed point. Hence the barycenters w_0 and w_1 are fixed. From this, it follows that the simplex they span is fixed by f. This is a simplex in the barycentric subdivision by definition.

2.C.9. Show that there are only countably many homotopy types of finite CW complexes.

Solution: Every CW complex X is homotopy equivalent to a simplicial complex, which may be taken to be finite if X is finite. A simplicial complex can be regarded as a set V of vertices and subsets of V. These subsets give the simplices while the dimension can be extracted from the cardinality. E.g., if we have $V = \{v_0, v_1, v_2, v_3\}$ and we choose the subset $\{v_0, v_2, v_3\}$, this says that our simplicial complex contains the 2-simplex $[v_0, v_2, v_3]$. Clearly for a fixed vertex set V there are $2^{|V|}$ many subsets and $2^{|V|}!$ sets of subsets. Hence there are at most $2^{|V|}!$ simplicial complexes with $|V|$ vertices. Even if all these simplicial complexes have distinct homotopy types, it follows that if X has $|V|$ many 0-cells then there are finitely many homotopy types. Hence, among all finite CW complexes, there are countably many.

Note: I think the number of simplicial complexes with n vertices follows the sequence $1, 2, 9, 114, ...$ This is obviously much lower than the upper bound given of $2^n!$.

HW13

Hatcher Chapter 4.1, problems 2, 8, 11:

4.1.2. Show that if $\varphi : X \to Y$ is a homotopy equivalence, then the induced homomorphisms $\varphi_* : \pi_n(X, x_0) \to \pi_n(Y, \varphi(x_0))$ are isomorphisms for all n. [The case $n = 1$ is Proposition 1.18.]

Solution: Recall that π_n is a functor so that $(\varphi\psi)_* = \varphi_*\psi_*$ and $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_n(X,x_0)}$. Since $\varphi : X \to Y$ is a homotopy equivalence there exists a homotopy inverse $\psi : Y \to X$ such that $\varphi \circ \psi \simeq \mathrm{id}_Y$ and $\psi \circ \varphi \simeq \mathrm{id}_X$.

Next we need the following lemma: Let $h_t : X \to Y$ be a homotopy and γ the path $h_{1-t}(x_0)$. Then, $h_{0*} = \beta_\gamma h_{1*}$. [For $n \geq 2$ at least, the same idea holds for $n = 1$ but the notation is slightly different.]
To see this, let γ_t be the path along γ from $\gamma(1-t)$ to $\gamma(1)$ reparameterized with domain $[0, 1]$. Now let $[f] \in \pi_n(X, x_0)$ and consider $\gamma_t(h_t \circ f)$. Since $f : (I^n, \partial I^n) \to (X, x_0)$, and $h_t : X \to Y$, it follows that $(h_t \circ f) : (I^n, \partial I^n) \to (Y, h_t(x_0))$ and $[h_t \circ f] \in \pi_n(Y, h_t(x_0))$. The associated map $\gamma_t(h_t \circ f)$ then takes $(I^n, \partial I^n)$ to $(Y, h_0(x_0))$. Hence, $[\gamma_t(h_t \circ f)] \in \pi_n(Y, h_0(x_0))$ and moreover $\gamma_t(h_t \circ f)$ is a homotopy of maps $(I^n, \partial I^n) \to (Y, h_0(x_0))$. In particular, $[\gamma_0(h_0 \circ f)] = [\gamma_1(h_1 \circ f)]$. But γ_0 is a constant path and $\gamma_1 = \gamma$ so,

$$\beta_\gamma h_{1*}[f] = [\gamma_1(h_1 \circ f)] = [h_0 \circ f] = h_{0*}[f]$$

as desired.

Now, apply the lemma to $\psi \circ \varphi \simeq \mathrm{id}_X$. It follows that $\psi_* \circ \varphi_* \simeq \beta_\gamma$ for some γ. Since change of basepoints are isomorphisms, it follows that $\psi_* \circ \varphi_*$ is an isomorphism. Thus, φ_* is injective. We can do the same thing with $\varphi \circ \psi$ to conclude φ_* is surjective. Hence, φ_* is an isomorphism.

4.1.8. Show the sequence $\pi_1(X, x_0) \to \pi_1(X, A, x_0) \xrightarrow{\partial} \pi_0(A, x_0) \to \pi_0(X, x_0)$ is exact.

Solution: Recall that $\pi_0(A, x_0)$ and $\pi_0(X, x_0)$ are the (based) sets of path-components of A and X, respectively. Each path-component of A is contained in a path-component of X so that the last map – call it β – takes a path-component of A, looks at which path-component of X it resides in, and maps it there.

The set $\pi_1(X, A, x_0)$ is the (based) set of homotopy classes of maps $f : I \to X$ such that $f(0) = x_0$ and $f(1) \in A$. The boundary map $\partial : \pi_1(X, A, x_0) \to \pi_0(A, x_0)$ simply takes a representative f of $[f]$ and reads off the path-component of A which $f(1)$ lies in. This is well-defined since if two maps $f, g : I \to X$ are homotopic rel A via a basepoint preserving homotopy then there exists an $F : I \times I \to X$ such that $F(t, 0) = f(t)$, $F(t, 1) = g(t)$, $F(0, s) = x_0$, and $F(1, s) \in A$. By continuity, $F(1, s)$ is a curve in A with $F(1, 0) = f(1)$ and $F(1, 1) = g(1)$, so that $f(1)$ and $g(1)$ lie in the same path-component.

The first map – call it α – takes a representative of a homotopy class $[f] \in \pi_1(X, x_0)$ and allows the second endpoint to vary. That is, originally $f : I \to X$ with $f(0) = f(1) = x_0$, and in $\pi_1(X, x_0)$ we consider homotopy classes of f with both endpoints fixed. We can also consider homotopy classes of f with only the first endpoint fixed and with the second endpoint varying, but inside A; this gives $\pi_1(X, A, x_0)$.

Since we want to talk about kernels, we need to make sense of what a zero element for each of these sets is. The zero element of $\pi_1(X, x_0)$ is obvious, and should be the homotopy class of the constant map. In theory this zero element should be mapped to zero in $\pi_1(X, A, x_0)$, so our candidate for the zero element here is $[x_0]$. Then, the zero elements for $\pi_0(A, x_0)$ and $\pi_0(X, x_0)$ are the path-components (in A and X, respectively) containing x_0. Call these path components B and Y, respectively. So,

$$[x_0] \xrightarrow{\alpha} [x_0] \xrightarrow{\partial} B \xrightarrow{\beta} Y$$

The kernel of $\beta : \pi_0(A, x_0) \to \pi(X, x_0)$ is the set of path-components of A – call them A_1,...,A_k – such that $A_i \subset Y \subset X$. Now let f be a representative of a homotopy class $[f] \in \pi_1(X, A, x_0)$. Then $f : I \to X$ with $f(0) = x_0$ and $f(1) \in A$. But f is a path, so that x_0 and $f(1)$ must lie in the same path-component of X – namely, Y. It follows that $f(1)$ lies in one of the above A_i. Hence, $\beta \circ \partial = Y$ and $\text{Im}(\partial) \subset \text{Ker}(\beta)$. Now take any $A_i \in \text{Ker}(\beta)$. Since $A_i \subset Y$, and Y is the path-component of X containing x_0, there exists a curve $f : I \to X$ such that $f(0) = x_0$ and $f(1) \in A_i$. Then $\partial[f] = A_i$, so that $\text{Ker}(\beta) \subset \text{Im}(\partial)$. In total, $\text{Ker}(\beta) = \text{Im}(\partial)$.

Finally, the kernel of $\partial : \pi_1(X, A, x_0) \to \pi_0(A, x_0)$ is the set of homotopy classes of maps with endpoints lying in B (not just A). Given any representative f of a homotopy class $[f] \in \pi_1(X, x_0)$, the homotopy class $[f] \in \pi_1(X, A, x_0)$ allows us to move the second endpoint within A, as discussed. We also saw that all representatives of $[f] \in \pi_1(X, A, x_0)$ have their second endpoint in the same path-component. In particular, in the same path-component as the endpoint of f itself, which is x_0 since f is a loop. This path-component is of course B. Thus $\partial \circ \alpha = B$ and $\text{Im}(\alpha) \subset \text{Ker}(\partial)$. Now take any representative f of a homotopy class in $\text{Ker}(\partial)$. This is some path in X with one endpoint at x_0 and its other endpoint somewhere in B, the same path-component as x_0. Let γ be a path from $f(1)$ to x_0 contained entirely inside B; such a path exists since B is a path-component. Now consider a homotopy $f_s(t)$ which extends the endpoint of f along γ until $\gamma(s)$. So, for $s = 0$ we recover f, and for $s = 1$ we have $\gamma * f$, where $*$ is path concatenation. This homotopy preserves the basepoint and keeps the second endpoint $f_s(1) \in A$ for all s. Thus, this is a valid homotopy and $[f] = [\gamma * f]$ in $\pi_1(X, A, x_0)$. But $\gamma * f$ is a loop at x_0. It follows that $\text{Ker}(\partial) \subset \text{Im}(\alpha)$, so $\text{Ker}(\alpha) = \text{Im}(\partial)$.

4.1.11. Show that a CW complex is contractible if it is the union of an increasing sequence of subcomplexes $X_1 \subset X_2 \subset ...$ such that each inclusion $X_i \hookrightarrow X_{i+1}$ is nullhomotopic, a condition sometimes expressed by saying X_i is contractible in X_{i+1}. An example is S^∞, or more generally the infinite suspension $S^\infty X$ of any CW complex X, the union of the iterated suspensions $S^n X$.

Solution: We prove the following lemma:

Lemma: If X is a connected CW complex, x_0 a 0-cell, and $\pi_n(X, x_0) = 0$ for all n then X is contractible.

To see this, consider the inclusion $i : \{x_0\} \hookrightarrow X$. The induced map $i_* : \pi_n(\{x_0\}, x_0) \to \pi_n(X, x_0)$ is an isomorphism since all the $\pi_n(X, x_0) = \pi_n(\{x_0\}, x_0) = 0$. Hence, by Whitehead, X and $\{x_0\}$ are homotopy equivalent spaces, which implies that X is contractible.

To finish the solution, let x_0 be a 0-cell of X. Let $f : S^n \to X$. Since S^n is compact, the image $f(S^n)$ lies in some X_k. Thus, we regard f as a map $f : S^n \to X_k$. On the other hand, the inclusion $i : X_k \to X$ is nullhomotopic since $X_k \hookrightarrow X_{k+1}$ is nullhomotopic and $i = X_k \hookrightarrow X_{k+1} \hookrightarrow X$. It follows that $i \circ f = f$ is nullhomotopic, so that $\pi_n(X, x_0) = 0$. Now we need only show that X is connected. To do this, suppose $x_1, x_2 \in X$ lie in different components. Then, since X is the union of an increasing sequence of subcomplexes, it follows that $x_1, x_2 \in X_k$ for some k. Now if these were in different components (in X), we would have that X_k is not contractible in X_{k+1}. Intuitively, for X_k to be contractible in X_{k+1}, we need to be able to shrink X_k to a point solely by moving within X_{k+1}. If x_1, x_2 are in different components in X, they are in different components in X_{k+1}. There is no good way to shrink both of these components to a single point.

Hatcher Chapter 4.2, problems 1, 2, 14, 27:

4.2.1. Use homotopy groups to show there is no retraction $\mathbb{R}\text{P}^n \to \mathbb{R}\text{P}^k$ if $n > k > 0$.

Solution: Recall that $\mathbb{R}\text{P}^n$ is doubly covered by S^n. It follows by proposition 4.1 in Hatcher that

$\pi_k(S^n) \simeq \pi_k(\mathbb{R}\mathrm{P}^n)$ for all $k \geq 2$. We know that $\mathbb{R}\mathrm{P}^1 \simeq S^1$ so that $\pi_k(\mathbb{R}\mathrm{P}^1) \simeq \pi_k(S^1)$ for all k. By corollary 4.25 of Hatcher, $\pi_n(S^n) \simeq \mathbb{Z}$ for all $n \geq 1$. Hence, $\pi_n(\mathbb{R}\mathrm{P}^n) \simeq \mathbb{Z}$ for all $n \geq 1$. Now if $1 < k < n$ then $\pi_k(S^n) \simeq 0$ so that $\pi_k(\mathbb{R}\mathrm{P}^n) \simeq 0$. Suppose $\mathbb{R}\mathrm{P}^k$ is a retract of $\mathbb{R}\mathrm{P}^n$. Then the induced homomorphism $i_* : \pi_k(\mathbb{R}\mathrm{P}^k) \to \pi_k(\mathbb{R}\mathrm{P}^n)$ is injective. But, the first group is $\mathbb{Z}$ while the latter is 0. In the case $k = 1$, we have that $\pi_1(\mathbb{R}\mathrm{P}^1) \simeq \mathbb{Z}$ while $\pi_1(\mathbb{R}\mathrm{P}^n) \simeq \mathbb{Z}_2$ for $n > k = 1$. In this case, we also cannot have an injection $\mathbb{Z} \to \mathbb{Z}_2$.

4.2.2. Show the action of $\pi_1(\mathbb{R}\mathrm{P}^n)$ on $\pi_n(\mathbb{R}\mathrm{P}^n) \simeq \mathbb{Z}$ is trivial for n odd and nontrivial for n even.

Solution: One can show the action of $\pi_1(\mathbb{R}\mathrm{P}^n)$ on $\pi_n(\mathbb{R}\mathrm{P}^n)$ corresponds to the action of $\pi_1(\mathbb{R}\mathrm{P}^n)$ on $\pi_n(S^n)$ induced by the action of $\pi_1(\mathbb{R}\mathrm{P}^n)$ on S^n as deck transformations. Recall that in general the action of a group G on a space Y is a homomorphism $\rho : G \to \mathrm{Homeo}(Y)$, the group of all homeomorphisms from Y to itself. So, for each $g \in G$, ρ assigns to g a homeomorphism $\rho(g) : Y \to Y$. In this problem, we are looking at the action of $\pi_1(\mathbb{R}\mathrm{P}^n) \simeq \mathbb{Z}_2$ when $n > 1$. It follows that ρ sends the trivial class in $\pi_1(\mathbb{R}\mathrm{P}^n)$ to the identity while ρ sends the nontrivial class in $\pi_1(\mathbb{R}\mathrm{P}^n)$ to the antipodal map.

Now what is $\pi_n(S^n)$? These are homotopy classes of maps $S^n \to S^n$. Since homotopic maps have the same degree, there is a well defined map $\deg : \pi_n(S^n) \to \mathbb{Z}$ by $\deg[f] = \deg(f)$, where the latter is the usual degree. Corollary 4.25 tells us that this is an isomorphism, i.e. that two maps have the same degree iff they are homotopic.

The action of $\pi_1(\mathbb{R}\mathrm{P}^n)$ on $\pi_n(S^n)$ induced by deck transformations is as follows: For each element $g \in \pi_1(\mathbb{R}\mathrm{P}^n)$, we have by the above an associated deck transformation $\rho(g)$. Let $[\alpha] \in \pi_n(S^n)$. Then, $g \cdot [\alpha] := [\rho(g) \circ \alpha]$. We need only look at when g is the nontrivial class in $\mathbb{R}\mathrm{P}^n$; recall that in this case $\rho(g) = p$. If n is even, then the degree of the antipodal map is -1, and it follows that $[p \circ \alpha] \neq [\alpha]$. On the other hand, if n is odd, then the degree is 1 so that p is homotopic to the identity. Hence, $[p \circ \alpha] = [\alpha]$. I've been sloppy about basepoints here, for example the action should include a change of basepoint from $\rho(g)(s_0)$ to s_0. But I think everything works out since S^n is path-connected.

In the case $n = 1$, we look at the action of $\pi_1(\mathbb{R}\mathrm{P}^1) \simeq \mathbb{Z}$ on itself. In general, let $[\gamma] \in \pi_1(X)$. The action of $\pi_1(X)$ on itself is the homomorphism $\rho : \pi_1(X) \to \mathrm{Inn}(\pi_1(X))$ given by $\rho[\gamma] = \beta_\gamma$. We want to show that ρ is the trivial homomorphism. That is, we need to show $\beta_\gamma = \beta_{x_0}$, the constant change of basepoint. Suppose $\pi_1(X)$ is abelian. Then, for $[\alpha] \in \pi_1(X)$ we have $\beta_\gamma[\alpha] = [\gamma]^{-1}[\alpha][\gamma] = [\alpha] = \beta_{x_0}[\alpha]$. Thus whenever $\pi_1(X)$ is abelian, the action of $\pi_1(X)$ on itself is trivial. Since $\pi_1(\mathbb{R}\mathrm{P}^1) \simeq \mathbb{Z}$ is abelian, the result follows.

4.2.14. If an n dimensional CW complex X contains a subcomplex Y homotopy equivalent to S^n, show that the map $\pi_n(Y) \to \pi_n(X)$ induced by inclusion is injective. [Use the Hurewicz homomorphism.]

Solution: We showed on a previous homework that if X has dimension n then $H_i(X) = 0$ for $i > n$ and $H_n(X)$ is free. So $H_n(X)$ is a subgroup of the free abelian group generated by the n-cells of X. The same logic holds for Y, replacing X with Y in the above argument. Since Y is homotopy equivalent to S^n, they have the same homology. In particular, we know there is a canonical choice of generator, $[\sigma]$ for $H_n(Y)$. By definition, σ is a linear combination of n-cells in Y, which are n-cells in X, and hence $\sigma \in C_n(X)$. Since σ is a cycle in $C_n(Y)$, it is a cycle in $C_n(X)$. It follows that $[\sigma] \in H_n(X)$ and the subgroup it generates is $\mathbb{Z}$. Thus $H_n(Y) \to H_n(X)$ induced by inclusion is an injection.

We know that S^n is $n-1$-connected, and since $Y \simeq S^n$ it follows that Y is $n-1$ connected. Then by Hurewicz, $\pi_n(Y) \simeq H_n(Y)$ whenever $n \geq 2$. Hence, the composition $\pi_n(Y) \to H_n(Y) \to H_n(X)$ is injective. On the other hand, the composition $\pi_n(Y) \to \pi_n(X) \to H_n(X)$ is the same map. It follows that $\pi_n(Y) \to \pi_n(Y)$ is injective.

The cases when $n = 0, 1$ are obvious. For $n = 0$, X is just a discrete collection of points. The path-components of X are the points themselves. Then for any subcomplex Y (not even homotopy equivalent to S^0), we have an injection induced by inclusion – Y is some subcollection of path-components of X, each of which is mapped to itself by $\pi_0(Y) \to \pi_0(X)$. For $n = 1$, we have that X is a graph, possibly with loops attached and cycles within it. If Y is homotopy equivalent to S^1, then a loop in $\pi_1(Y)$ is also a loop in $\pi_1(X)$. Since X has dimension $n = 1$, there is no room to homotope the loop. E.g. if X was a disk, we can consider the boundary, which is S^1. A loop around the boundary, when regarded in X, can be pulled into the center and nullhomotoped. This cannot happen when X is a 1 dimensional CW complex.

4.2.27. Show that the image of the map $\pi_2(X, x_0) \to \pi_2(X, A, x_0)$ lies in the center of $\pi_2(X, A, x_0)$.

Solution: Lemma 4.39 states that if $a, b \in \pi_2(X, A, x_0)$ then

$$aba^{-1} = (\partial a)b$$

where $\partial : \pi_2(X, A, x_0) \to \pi_1(A, x_0)$ and $(\partial a)b$ is the action of ∂a on b. Now if a lies in the image of $\pi_2(X, x_0) \to \pi_2(X, A, x_0)$, then it lies in the kernel of ∂. Hence, $\partial a = 0$. It follows that the action is trivial, and thus

$$aba^{-1} = b$$

whenever $a \in \text{Im}(\pi_2(X, x_0) \to \pi_2(X, A, x_0))$ and $b \in \pi_2(X, A, x_0)$. Hence, $\text{Im}(\pi_2(X, x_0) \to \pi_2(X, A, x_0))$ lies in the center of $\pi_2(X, A, x_0)$.