1 Introduction: In $\mathbb{R}^3$ we can locally describe surfaces via a parameterization $X(u, v) : \mathbb{R}^2 \to \mathbb{R}^3$. Such surfaces are easily visualized, and include many common ones such as planes, spheres, cones, and quadric surfaces. For these surfaces, it is easy to imagine an intuitive notion of how curved the surface is at a particular point – formally, the Gauss curvature, denoted by $K(p)$. By considering a geometric map defined for orientable surfaces, the Gauss map N, Gauss proved that $K(p) = \det((dN)_p)$. But there is another invariant associated to $(dN)_p$ – its trace – and one can ask whether any geometric information is stored in it. This turns out to be true, and we define the so-called mean curvature H to be $-1/2\,\mathrm{tr}((dN)_p)$.

The study of surfaces with certain conditions imposed on the mean curvature is deeper than that of the Gauss curvature. For example, suppose we wanted to analyze the surfaces with constant Gauss or mean curvature. It can easily be shown that *any* surface with constant Gauss curvature must be a sphere or a plane. However, there are many (non-diffeomorphic) surfaces with constant mean curvature; for example, the sphere, catenoid, and helicoid. In fact, there are so many that those with a constant mean curvature of 0 are put into their own class, and are called minimal surfaces. Interestingly, minimal surfaces themselves have not been fully classified; Minimal surface theory is an extremely active research area.

2 Future Directions: Constant Mean Curvature (CMC) surfaces are of profound interest. In particular, examine the following theorem regarding them:

Theorem 2.1. *(Geometric Maximum Principle, see* [Men13]*). Let Σ_1 and Σ_2 be two constant mean curvature surfaces in $\mathbb{R}^3$. Suppose there exists $p \in \Sigma_1 \cap \Sigma_2$ such that Σ_1 and Σ_2 are tangent at p, and Σ_2 lies in the mean convex side of Σ_1 in a neighborhood of p. Then $H_2 \geq H_1$, and the equality holds if, and only if, $\Sigma_1 = \Sigma_2$.*

The Geometric Maximum Principle (GMP) is an important theorem in the study of CMC surfaces. Importantly, it can be extended to CMC surfaces embedded in other manifolds, such as $\mathbb{R}^n, \mathbb{H}^n, \mathbb{S}^n, \mathbb{H}^2 \times \mathbb{R}$, etc. Its versatility is demonstrated in proving half-space type theorems. The concept is simple: suppose a manifold M is divided into two half-spaces by a hypersurface Σ. Then, if $\tilde{\Sigma}$ is a CMC surface living in one of the half-spaces formed by Σ, what does it look like? In 1990, the first of these theorems, given below, was proven by Hoffman-Meeks using GMP.

Theorem 2.2. *(Hoffman-Meeks, 1990) If Σ is a properly immersed minimal surface lying on one side of a plane P in $\mathbb{R}^3$, then Σ is a plane parallel to P.* [HM90]

Finally, GMP can be used to attack asymptotic Plateau problems. The Plateau problem was originally posed in the 1770s. It was not solved, however, until the 1930s, by Douglas and Radó. Douglas would later be awarded the first Fields medal for his contribution.

Theorem 2.3. *(Radó 1930, Douglas 1931) Given $\Gamma \subset \mathbb{R}^3$ a simple, closed curve, there exists $\Sigma \subset \mathbb{R}^3$ a compact minimal surface with $\partial\Sigma = \Gamma$.* [Rad30] [Dou31]

(Note: The surface Σ is unique under a convexity restriction on Γ and is proven using GMP, see [Sch83]). This question can be extended into other spaces, such as $\mathbb{H}^2 \times \mathbb{R}$. Here, we may consider curves Γ with boundary "at infinity" (asymptotic boundary). Does such a result still hold? This turns out to be true in $\mathbb{H}^2 \times \mathbb{R}$ [NR02] under certain conditions, but the theory is still incomplete. The proof relies on the construction of barrier CMC surfaces. GMP then forces a prescribed minimal surface Σ to have asymptotic boundary Γ.

The above theorems are extremely rigid – all of them follow the form "if the hypotheses are met, then *only one possible thing* can occur". GMP therefore provides for rich study of CMC surfaces, and I aim to explore half-space type theorems, asymptotic Plateau problems in $\mathbb{H}^2 \times \mathbb{R}$, and more using it.

3 Intellectual Merit: To prepare for my graduate studies in surfaces of constant mean curvature, I have read several books on differential geometry – Montiel & Ros, O'Neill, and the widely used *Riemannian Geometry* by do Carmo. I have also read some more specialized books, including *A Course in Minimal Surfaces* by Colding and Minicozzi and *Eigenvalues in Riemannian Geometry* by Chavel. Furthermore, I am working on my senior thesis with Dr. Luca Di Cerbo, where I am examining Geometric Flows on Delaunay surfaces. These are the only six surfaces of revolution in $\mathbb{R}^3$ with constant mean curvature. Moreover, under the direction of Dr. Neves at UChicago, I proved the following compactness theorem regarding stable minimal hypersurfaces:

Theorem 3.1. *Let $U \subset \mathbb{R}^n$ be open, and $K \subset U$ compact with $3 \leq n \leq 6$. Suppose $\Sigma_k \subset U$ is a sequence of stable minimal hypersurfaces with $\mathrm{Vol}(\Sigma_k) \leq a$. Then there exists a subsequence converging to a stable minimal hypersurface Σ in U.*

Though the theorem is well known among experts, a formal proof is not given in the literature. Therefore, my proof, which uses GMP, will be a valuable resource for beginning researchers studying minimial surfaces. I plan to disseminate it in article form, submit it to a good geometric analysis journal (e.g., *J. Geom. Anal.*), and present my work at conferences (e.g., JMM). I am well equipped to do so, since I have given several math talks.

Furthermore, it may be possible to extend this in some meaningful way to CMC surfaces. I conjecture that this is the case, and aim to work on it under Drs. Marques and Menezes at Princeton. Alongside this, I plan to complete the theory for the asymptotic Plateau problem in $\mathbb{H}^2 \times \mathbb{R}$, including providing necessary and sufficient conditions for existence.

4 Broader Impacts: Geometric Analysis has many applications in physics. For example, the Plateau problem originally arose by trying to categorize the surfaces formed by soap films bounded by a wire frame. It was recognized early on by Plateau that these surfaces all minimize surface area.

I plan to continue the mathematical outreach that I conducted in college throughout my life. First, I will work on additional solution manuals, share lecture notes, and improve math courses so as to transform math pedagogy. I intend to help review local International Baccalaureate students' Internal Assesments, as I have done for several years. Finally, I will showcase my research to my students and local high school students, specifically geared towards underrepresented minorities. Geometry provides a beautiful way to achieve this. Due to its visual nature, many of the arguments can be understood at a fundamental level by simply drawing pictures. For example, a surface which has small curvature does not vary much in its height. A precise version of this theorem exists for minimal surfaces, but the underlying idea is easily understood. From this, I plan to present my research to high school and collegiate students alike, inspire them, and motivate them to appreciate mathematics.

5 References: [Dou31] Douglas *Trans. of the AMS*, 33(1):263-321, 1931; [HM90] Hoffman & Meeks *Invent. math*, 101:373-377, 1990; [Men13] Menezes, Ph.D. thesis, Instituto de Matemática Pura e Aplicada, 2013; [NR02] Nelli & Rosenberg *Bulletin of the Brazilian Mathematical Society*, 33(2):263-292, 2002; [Rad30] Radó *Ann. Math. Second Series*, 31(3):457-469, 1930; [Sch83] Schoen *J. Diff. Geom.*, 18(4):791-809, 1983.