The Gauss-Bonnet Theorem Revisited

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Abstract

In this work, we develop the Gauss-Bonnet theorem from first principles using calculus on surfaces.

We begin by defining some important quantities on manifolds, and an essential coordinate system used to ease computations. Thereafter, we perform these computations, calculating variations of Christoffel symbols, components of the Riemann curvature tensor, and the volume element. Following this, we specialize to dimension $n = 2$ and explicitly write out the formulae – this helps in interpreting the variations. We then provide a reformulation of the variation of the Gauss curvature in terms of laplacians and divergences, which becomes important in computing a key integral. Finally, we put all of the above together and reprove the Gauss-Bonnet theorem.

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1 Acknowledgments

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2 Introduction

The Gauss-Bonnet theorem is the single most important theorem about compact, orientable 2-manifolds. It provides a beautiful and remarkable connection between the geometry and topology of such manifolds. We first provide some history, and higher dimensional analogs. Then, we detail the structure of the thesis.

In the 1600s, Albert Girard proved the following result relating the interior angles of a spherical triangle to its area.

Theorem 2.1 (Girard)**.** *Let Δ geodesic triangle on S^2 with interior angles α, β, γ. Then the area of Δ is given by*

$$A(\Delta) = \alpha + \beta + \gamma - \pi.$$

See [Nic03]. A historical remark: this theorem is normally attributed to Legendre, but he proved a different, yet related theorem approximating spherical excess.
In modern language, this can be restated as

$$\int_{\Delta} 1 \; dA = \alpha + \beta + \gamma - \pi,$$

where dA is the area element on S^2. While rudimentary in form, this equation actually provides the first insight into the Gauss-Bonnet theorem.

In 1827, Gauss published his fundamental treatise "*Disquisitiones generales circa superficies curvas*". Here, he lays the foundation of classical surface theory. Among these is Gauss' Theorem Egregium:

Theorem 2.2 (Theorem Egregium, Gauss 1827)**.** *Let M be an orientable surface embedded in $\mathbb{R}^3$. Then its Gauss curvature depends only on the coefficients of its first fundamental form.*

A translation can be found in Chapter 3 of [Spi99].
Yet there is another landmark theorem provided in his book. It is as follows:

Theorem 2.3 (Gauss 1827)**.** *Let $M \subset \mathbb{R}^3$ be an orientable surface and Δ be a geodesic triangle on M with angles α, β, γ. Then,*

$$\int_{\Delta} K \; dA = \alpha + \beta + \gamma - \pi.$$

In proving this, Gauss essentially converts the integral into one using geodesic polar coordinates. These are coordinates obtained as follows: Suppose $\exp_p v$ is defined. Then there exists $\epsilon > 0$ such that $\exp_p u$ is defined for

$$u = tv(s) \quad 0 \le t \le 1, \quad -\epsilon < s < \epsilon.$$

Using the fact that $\exp_p v$ is a local diffeomorphism, we obtain geodesic polar coordinates (the geodesic portion comes from the fact that $t \mapsto \exp_p tv(s_0)$ is a geodesic for fixed s_0). Here, t plays the role of a radial parameter while s plays the role of an angular parameter.

This theorem provides the beginning of the Gauss-Bonnet theorem, a remarkable result in differential geometry. Even Gauss held the simplistic version above in high regards, calling it "...among the most elegant in the theory of curved surfaces." In fact, historically, it seems that Gauss proved Theorem 2.3 *first*, then saw as a corollary the Gauss curvature ought to only depend on the metric.

In general we can consider what is called the defect of a geodesic polygon. Let P be a polygon in a surface $M \subset \mathbb{R}^3$ with geodesic sides. Denote by $\alpha_1, ..., \alpha_n$ the angles of P. Let P' be a polygon with the same number of edges (here, n edges) in Euclidean space. Then the defect $\delta(P)$ is defined as the difference in the total angle measures,

$$\delta(P) = (\alpha_1 + \alpha_2 + ... + \alpha_n) - (n-2)\pi.$$

Observe that the defect may be rewritten as

$$\delta(P) = 2\pi - \sum_j (\pi - \alpha_j).$$

In "*Mémoire sur la théorie générale des surfaces*", Bonnet proves a result relating the angle defect of a simply connected region. It reads

Theorem 2.4 (Bonnet, 1848)**.** *On a surface M in $\mathbb{R}^3$, let $\mathcal{R}$ be a simply connected region of M with the boundary $\partial\mathcal{R}$ consisting of a finite number of smooth curves. Then*

$$\int_{\partial R} k_g \, ds + \sum_j (\pi - \alpha_j) + \int_{\mathcal{R}} K \, dA = 2\pi,$$

where k_g is the geodesic curvature of the boundary curve. Each α_j is the interior angle at a vertex of the boundary.

Note that this is only a local version of the theorem – it is restricted to a simply connected region $\mathcal{R}$. By using Gauss' theorem for geodesic triangles, one may take a regular region of M and triangulate it. This results in the following global version

Theorem 2.5. *Let $\mathcal{R}$ be a regular region of an orientable surface M embedded in $\mathbb{R}^3$. Then*

$$\int_{\partial R} k_g \, ds + \sum_j (\pi - \alpha_j) + \int_{\mathcal{R}} K \, dA = 2\pi\chi(M).$$

See section 4.5 of [dC76] for a proof.

Though the Euler characteristic was defined relatively early in the history of mathematics, its importance had not yet been revealed. Indeed, this would not occur until the rise of algebraic topology. During the 19th century, the Gauss-Bonnet theorem often took the form with $\chi(M) = 1$ – that is, they proved the formula for disk like surfaces. Walther von Dyck, in 1888, first realized this should hold true in general and proved the following:

Theorem 2.6 (von Dyck 1888)**.** *Let M be a compact (orientable) surface without boundary embedded in $\mathbb{R}^3$. Then*

$$\int_M K\ dA = 2\pi\chi(M).$$

Note: compact hypersurfaces of $\mathbb{R}^n$ are automatically orientable, hence the parenthesis. See [Sam69] for a proof.

Observe that the boundary terms from the classical Gauss-Bonnet theorem – those involving the geodesic curvature and interior angles – disappear.

It still remains that these theorems hold for surfaces embedded in $\mathbb{R}^3$. The next natural generalization would be to prove it for hypersurfaces embedded in $\mathbb{R}^n$. In the early 1920s, Heinz Hopf began to work on such a generalization and proved:

Theorem 2.7 (Hopf 1925, 26)**.** *Let M be an even dimensional compact (orientable) hypersurface without boundary embedded in $\mathbb{R}^{2n+1}$. Define the map $G : M \to S^{2n}$ as the Gauss map. Let ω_{2n} be the volume form on S^{2n}. It follows that the pullback $G^*\omega_{2n}$ is a $2n$-form on M. Then*

$$\int_M G^*\omega_{2n} = \frac{\mathrm{Vol}(S^{2n})}{2}\chi(M).$$

A summary of his work can be found in [Wu08], and is loosely presented here. This is the appropriate higher dimensional analog of the Gauss-Bonnet theorem for hypersurfaces. As an example, when $n = 1$ it is well known that $G^*\omega_2 = KdA$. In the case of surfaces in $\mathbb{R}^3$, we see that

$$\int_M K\ dA = \int_M G^*\omega_2 = \deg(G)\int_{S^2}\omega_2,$$

where $\deg(G)$ is the topological degree of G. What Hopf further proved is that, for an even dimensional compact hypersurface,

$$\deg(G) = \frac{1}{2}\chi(M).$$

This proof critically relies on the Hopf Index Theorem. The above equality immediately proves Theorem 2.6 in a more geometric fashion. Further, the same computation holds in arbitrary even dimension, thus proving Theorem 2.7.

Though it is not necessary in the above proof, Hopf also showed that

$$G^*\omega_{2n} = \frac{\mathrm{Vol}(S^{2n})}{2}\ \Omega,$$

where Ω is called the *Gauss-Bonnet integrand of M*. Observe that, restated, Hopf proves that $\int_M \Omega = \chi(M)$.

Although the Gauss map cannot be classically defined when a submanifold has arbitrary codimension, the $2n$-form on the right hand side *can*. This fact is essential in generalizing Gauss-Bonnet to arbitrary codimension. Indeed, 1940, Allendoerfer and Frenchel independently proved the following

Theorem 2.8 (Allendoerfer, Frenchel 1940). *Let M be an even dimensional compact orientable submanifold without boundary embedded in $\mathbb{R}^{2n+k}$ with $k > 1$. Then*

$$\int_M \Omega = \chi(M).$$

They proved this by using a tubular neighborhood, whose boundary turns out to be a hypersurface in $\mathbb{R}^{2n+k}$, and applying Theorem 2.7 to it. Through a nontrivial calculation involving integrating along fibers, one arrives at Theorem 2.8.

Up until now, all Gauss-Bonnet type theorems have involved submanifolds of $\mathbb{R}^n$. It was an open problem to show a Gauss-Bonnet theorem for an arbitrary Riemannian manifold. Given the Nash Embedding Theorem, this could easily be solved, but that had not yet been proven. The Nash Embedding Theorem, proven in 1956, roughly reads

Theorem 2.9 (Nash 1956). *Every smooth Riemannian manifold of dimension n can be isometrically embedded into some Euclidean space $\mathbb{R}^N$, where $N \leq n(n+1)(3m+11)/2$.*

See [Nas56] for details. In 1943 though, Allendoerfer and Weil pushed Theorem 2.8 to its limits and obtained a Gauss-Bonnet theorem for arbitrary Riemannian manifolds. To do so, they appealed to the following

1. The Whitney Embedding Theorem, which allows them to embed M into Euclidean space locally.

2. Theorem 2.8, or the Gauss-Bonnet theorem for submanifolds.

3. A combinatorial argument to patch everything up, and obtain a global theorem.

Once again, a combinatorial argument appears, and geometric intuition is lacking. In August 1943, Chern visited the Institute for Advanced Study. There, Weil pointed out the lack of a geometric argument – Chern solved this issue within weeks. In 1945, he published his results, and proved the remarkable Chern-Gauss-Bonnet theorem.

Theorem 2.10 (Chern 1945). *Let (M, g) be a compact, orientable $2n$-dimensional Riemannian manifold without boundary. Let Ω be the curvature form associated to the Levi-Civita connection. Denote by* Pf *the Pfaffian. Then*

$$\int_M \mathrm{Pf}(\Omega)\ d\mu_g = (2\pi)^n \chi(M).$$

See [Che45]. Another issue Chern solved is what the analog of $G^*\omega_{2n}$ is when the Gauss map does not exist (as is the case for arbitrary codimension submanifolds). Thus, he found the most generalized form of KdA.

While Theorem 2.6 has a geometric proof, it is also combinatorial in flavor. The proof most texts present, for example as in [dC76], involves triangulating the manifold. Then, local Gauss-Bonnet is used to prove a global Gauss-Bonnet. Although it is geometric in nature, it seems to reduce Gauss-Bonnet to mere combinatorial happenstance.

The aim of this paper is to provide a self contained proof of the following altered version of Theorem 2.6 using only calculus techniques.

Theorem 2.11 (Gauss-Bonnet)**.** *Let (M, g) be a compact orientable 2-dimensional Riemannian manifold without boundary. Then,*

$$\int_M K d\mu_g = 2\pi\chi(M),$$

where K is the Gauss curvature.

Why should such proof relying only on calculus of surfaces have any hope of working? Consider a Riemannian manifold (M, g_0) as in Theorem 2.11. Let $\mathcal{M}$ denote the space of metrics on M. We then appeal to the following proposition:

Proposition 2.12. *The space of metrics on a manifold is path connected.*

Now consider a path γ in $\mathcal{M}$ emanating from g_0. This induces a one-parameter family of metrics on M, denoted $(M, g(t))$. We may instead investigate the quantity

$$I(t) = \int_M K \ d\mu_{g(t)}.$$

Since the Euler characteristic is a topological quantity, it does not depend on the metric. Thus it satisfies to show

$$\frac{d}{dt}\int_M K \ d\mu_{g(t)} = 0,$$

which implies that $I(t)$ is locally constant. Since $\mathcal{M}$ is path connected, $I(t)$ is constant. Then we look at the individual genus cases and show $I(t) = 2\pi\chi(M)$. Finally, by the classification of compact oriented surfaces in $\mathbb{R}^3$, we obtain Theorem 2.11.

To this end, we must compute a variational formula for the Gauss curvature. First we define important quantities, prove Prop 2.12, and establish some crucial results in Section 3. Included in theses is the existence of Geodesic (Normal) coordinates. Without these, the computations are too cumbersome to carry out. A combinatorial proof sketched in Section 6 shows that there would be at least 200 terms, barring cancellation, to handle.

We then compute variational formulas for the inverse metric, Christoffel symbols, Riemann curvature tensor. These computations are carried out in full generality in Section 4.

In Section 5, we explicitly write out these formulae in dimension 2 in terms of the metric and its time derivative. We break up the computation quantity by quantity for organizational purposes, eventually culminating in explicit formula for the variation of the Riemann curvature tensor. Having this in readily leads to a variational formula for the Gauss curvature. Indeed, in dimension 2 we have $\text{Ric}_g = Kg$. Therefore, one only needs to compute the variation of the components of the Ricci tensor, which may be obtained from the Riemann curvature tensor.

Section 6 reformulates the result in Section 5. The primary focus here is to restate everything in terms of geometric operators on M. This provides for a purely geometric interpretation of the variation in Gauss curvature.

Finally, Section 7 shows that $d/dtI(t)$ is indeed zero and that $I(t) = 2\pi\chi(M)$, by following the instructions set forth above. To achieve this, we employ a clever trick involving the connected sum. This allows us to inductively prove the theorem using only information from the genus 0 and 1 cases.

Henceforth, we apply the following conventions. Partial derivatives will be written as $\partial_i = \partial/\partial x^i$. Riemannian manifolds will always be connected and of arbitrary dimension unless otherwise stated. Denote by $\mathfrak{X}(M), \mathcal{D}(M)$ the set of vector fields and smooth functions, respectively, on M.

3 Preliminaries

We present several definitions and fundamental results. Sections 3.2 and 3.3 follow [dC92] whereas Sections 3.4 and 3.5 follow [CK04] and [LeB04] respectively.

3.1 Path Connectedness of $\mathcal{M}$

Recall that $\mathcal{M}$ denotes the space of metrics on a Riemannian manifold (M, g). An essential step in proving relies on the aforementioned proposition:

Proposition 2.12. *The space of metrics on a manifold is path connected.*

Proof. Let g, h be two Riemannian metrics on M. We show that there exists a path $\gamma : [0, 1] \to \mathcal{M}$ such that $\gamma(0) = g$ and $\gamma(1) = h$. Tentatively define γ as

$$\gamma(t) = (1 - t)g + th,$$

we show that $\gamma(t) \in \mathcal{M}$ for all $t \in (0, 1)$ (really, for $t \in [0, 1]$, but the $t = 0, 1$ cases are already handled since $\gamma(0) = g$ and $\gamma(1) = h$). That is, we must show $\gamma(t)$ is a symmetric bilinear positive-definite form which varies smoothly. First, g, h are symmetric forms, so that $\gamma(t)$ is too. Evidently for $u, v \in T_pM$,

$$\gamma(t)(u, v) = (1 - t)g(u, v) + th(u, v) = (1 - t)g(v, u) + th(v, u) = \gamma(t)(v, u).$$

Next, g, h are bilinear forms on T_pM for all $p \in M$. Thus for $u, v, w \in T_pM$ and $\lambda \in \mathbb{R}$ we have

$$\begin{aligned} \gamma(t)(\lambda u + v, w) &= (1 - t)g(\lambda u + v, w) + th(\lambda + v, w) \\ &= (1 - t)(\lambda g(u, w) + g(v, w)) + t(\lambda h(u, w) + h(v, w)) \\ &= \lambda[(1 - t)g(u, w) + th(u, w)] + [(1 - t)g(v, w) + th(v, w)] \\ &= \lambda\gamma(t)(u, w) + \gamma(t)(v, w). \end{aligned}$$

Similarly,

$$\gamma(t)(u, \lambda w + v) = \lambda\gamma(t)(u, w) + \gamma(t)(u, v),$$

by applying the symmetry of $\gamma(t)$.

We must now show that γ is a positive definite form. Indeed, since g, h are positive definite, $g(v, v) = \|v\|_g > 0$ and $h(v, v) = \|v\|_h > 0$ for nonzero $v \in T_pM$. Thus

$$\gamma(t)(v, v) = (1 - t)\|v\|_g + t\|v\|_h > 0,$$

since $\gamma(t)(v, v)$ is a convex combination of positive numbers. Now let $v = 0$. Then $g(v, v) = h(v, v) = 0$ so that $\gamma(t)(v, v) = 0$. Finally suppose $\gamma(t)(v, v) = 0$. Then $(1 - t)\|v\|_g = -t\|v\|_h$. If $\|v\|_h > 0$ we would obtain $\|v\|_g < 0$, a contradiction. So $\|v\|_h = 0$, which implies $v = 0$.

It is also a well known fact that the set of positive definite matrices is convex.

We now show $\gamma(t)$ varies smoothly as p varies. Since g, h are metrics, their components g_{ij} and h_{ij} are smooth. Then the components of $\gamma(t)$ are given by $\gamma_{ij}(t) = (1 - t)g_{ij} + th_{ij}$, which is smooth (as the sum of smooth functions). $\square$

3.2 Geodesic Coordinates

We now define the aforementioned Geodesic coordinates. To do so, we will need to define some geometric quantities on a Riemannian manifold.

Definition 3.1. The *Christoffel symbols* on M are defined by

$$\Gamma_{ij}^k = \frac{1}{2} \{\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\} g^{lk}.$$

The Christoffel symbols play a role in measuring how curved a space is. They often provide a defect factor for curved spaces. Some examples of this are seen later with the Levi-Civita connection and with the Laplacian. It is important to recall here the definition of the Levi-Civita connection.

Definition 3.2. The *Levi-Civita connection* ∇ is the unique symmetric affine connection compatible with the metric. In coordinates, it is given by

$$\nabla_V W = \left(v^i \partial_i w^k + v^i w^j \Gamma_{ij}^k\right) \partial_k,$$

where $V = v^i \partial_i$, $W = w^i \partial_i$ are vector fields on M. It is a fundamental theorem of Riemannian geometry that such a connection exists. Notice for flat spaces ($\Gamma_{ij}^k = 0$) the covariant derivative coincides with the standard covariant derivative in Euclidean space.

We now turn to another fundamental concept in Riemannian geometry, that of the exponential map. It relates vectors in the tangent space at a point to geodesics emanating from that point.

Definition 3.3. Let (M^n, g) be an n-dimensional Riemannian manifold equipped with the Levi-Civita connection ∇. Let $p \in M$. We define the *exponential map at* p, denoted $\exp_p$, by

$$\exp_p(v) = \gamma(1, p, v)$$

where $\gamma(1, p, v)$ denotes the geodesic γ such that $\gamma(0) = p$, $\gamma'(0) = v$ evaluated at $t = 1$.

Remark 3.4. The exponential map is well defined since the geodesic γ described above is unique. This follows from the uniqueness of ODEs.

Geodesic coordinates arise from the following well-known theorem:

Theorem 3.5. *There exist neighborhoods $U \subset M$ and $V \subset T_pM$ such that $\exp_p$ is a diffeomorphism of V onto U.*

See chapter 3 of [dC92] for a proof. The heuristic is as follows: We have a standard orthonormal coordinate system on $\mathbb{R}^n$. Since $T_pM \simeq \mathbb{R}^n$, this naturally carries over to T_pM. Since a diffeomorphism exists between a neigborhood of $p \in M$ and $0 \in T_pM$, we can carry these coordinates over to M. We call them geodesic coordinates. Some useful properties of geodesic coordinates are given below.

Theorem 3.6 (Properties of geodesic coordinates)**.** *Let (M, g) be a manifold with geodesic coordinates at $p \in M$. Then,*

1. $g_{ij}(p) = \delta_{ij}$.
2. $\partial g_{ij}/\partial x^k(p) = 0$.
3. $\Gamma^k_{ij}(p) = 0$.

Property (1) follows from the heuristic given. (2) is easily implied by (1), and (3) follows from (2) using the formula for the Christoffel symbols.

Remark 3.7. In general, quantities like curvature and the Christoffel symbols are viewed as functions on the manifold, with the background metric fixed. Instead, we will be fixing a point and varying the metric. Thus, at $t = 0$, when we have the original metric in place, we will be able to apply geodesic coordinates.

3.3 Tensors and the Covariant Derivative of Tensors

Here we review the concept of a tensor and covariant differentiation of them.

Definition 3.8. Let (M, g) be a Riemannian manifold. A *rank r tensor* is a multilinear mapping

$$T : \mathfrak{X}(M) \times ... \times \mathfrak{X}(M) \to \mathcal{D}(M)$$

(where there are r copies of $\mathfrak{X}(M)$). We refer to T as an r-tensor.

Like with many objects in Riemannian geometry, we may consider the action of T in a coordinate system and define, accordingly, the components of T.

Definition 3.9. The *components* of T are defined as

$$T_{ij...k} = T(\partial x^i, ..., \partial x^k)$$

where there are r indices.

Remark 3.10. Tensors are pointwise objects in the following sense. Let $V_1, ...V_n$ be given by

$$V_k = \sum_{i_k} v^{i_k} \partial x_{i_k}.$$

Then by linearity,

$$T(V_1, ..., V_r) = \sum_{i_1,...,i_r} v^{i_1} ... v^{i_r} T(\partial x_{i_1}, ..., \partial x_{i_r}) = \sum_{i_1,...,i_r} v^{i_1} ... v^{i_r} T_{i_1...i_r}.$$

So, the value of $T(V_1, ..., V_r)$ at a specific point $p \in M$ depends only on the value of the components of T at p and the values of $V_1, ..., V_r$ at p.

We may also consider the covariant derivative of an r-tensor.

Definition 3.11. The *covariant derivative of an r-tensor*, denoted ∇T, is given by

$$\nabla T(V_1, ..., V_r, Z) = Z(T(V_1, ..., V_r)) - T(\nabla_Z V_1, ..., V_r) - ... - T(V_1, ..., \nabla_Z V_r).$$

It is an $(r+1)$-tensor. We sometimes fix a particular Z and write $\nabla_Z T(...) = \nabla T(..., Z)$.

Remark 3.12. Recall that the covariant derivative of a vector field $V = v^j \partial_j$ is given by

$$\nabla_i V = \nabla_{\partial_i} V = \left\{ \partial_i v^k - v^j \Gamma_{ij}^k \right\} \partial_k.$$

Then, at p in geodesic coordinates we get

$$\nabla_i V|_p = \partial_i v^k \partial_k|_p.$$

From this, in geodesic coordinates we conclude that

$$\nabla_i T(V_1, ..., V_r) \quad = \quad \partial_i T(V_1, ..., V_r).$$

3.4 The Laplacian, Divergence of Tensors, and the Divergence Theorem

Here we introduce some geometric operators on manifolds. This will be relevant later on in reformulating an abstract result into a more geometric one. We also discuss the Divergence Theorem, which plays an essential role in computing the integral of the variation of the Gauss curvature. We begin with some terminology concerning function spaces on manifolds.

Definition 3.13. A function $f : M \to \mathbb{R}$ is said to be of *class* C^k if for every combination of nonnegative integers $\alpha_1, ..., \alpha_n$ such that $\alpha_1 + ... + \alpha_n \leq k$ the partial derivatives

$$\partial_1^{\alpha_1} \partial_2^{\alpha_2} ... \partial_n^{\alpha_n} f$$

exist and are continuous. We will abuse notion and write $f \in C^k$ to say f is a function of class C^k. Smooth functions are said to be of class C^∞.

In other words, a function is of class C^k if its partial derivatives up to (and including) order k exist and are continuous. We now turn to defining an important operator on manifolds.

Definition 3.14. The *Laplacian* with respect to g on a Riemannian manifold (M, g), denoted Δ_g, is an operator $\Delta_g : \mathcal{D}(M) \to \mathcal{D}(M)$ defined by

$$\Delta_g = g^{mn} \left\{ \partial_m \partial_n - \Gamma_{mn}^l \partial_l \right\}.$$

(realistically, we only need to require that $f \in C^2$).

Observe then, in geodesic coordinates at p, we have

$$\Delta_g|_p = \partial_1^2|_p + \partial_2^2|_p$$

since $\Gamma_{ij}^k(p) = 0$.

An important operator on tensors is that of the divergence. In general, it takes an r-tensor and results in an $(r-1)$-tensor (whereas the covariant derivative resulted in an $(r+1)$-tensor).

Definition 3.15. The *divergence* of a 2-tensor A is given by

$$(\operatorname{div} A)_k = g^{ij}\nabla_i A_{jk},$$

which results in a 1-tensor.

In geodesic coordinates at p, we obtain the following particularly simple form of the divergence.

$$(\operatorname{div} A)_k|_p = g^{ij}\nabla_i A_{jk}|_p = \sum_{i=1}^n \partial_i A_{ik}|_p.$$

Definition 3.16. The divergence of a 1-tensor α with components α_k is given by

$$\operatorname{div} \alpha = g^{lk}\nabla_l \alpha_k,$$

which results in a smooth function.

Likewise, at p, we see that

$$\operatorname{div}(\operatorname{div} A)|_p = \sum_{k=1}^n \partial_k \sum_{i=1}^n \partial_i A_{ik}|_p.$$

Remark 3.17. When $n = 2$ we obtain

$$\begin{aligned} \operatorname{div}(\operatorname{div} A)|_p &= \sum_{k=1}^n \partial_k(\partial_1 A_{1k} + \partial_2 A_{2k})|_p \\ &= \partial_1(\partial_1 A_{11} + \partial_2 A_{21}) + \partial_2(\partial_1 A_{12} + \partial_2 A_{22}) \\ &= \partial_1^2 A_{11} + 2\partial_1\partial_2 A_{12} + \partial_2^2 A_{22}, \end{aligned}$$

where the RHS is evaluated at p.

One last important theorem we need is the so-called divergence theorem.

Theorem 3.18. *Let (M, g) be a compact, orientable Riemannian manifold without boundary. Let X be a differentiable vector field on M. Then*

$$\int_M \operatorname{div} X \; d\mu_g = 0.$$

See [Cha06] for details.

Remark 3.19. This also implies for $f \in C^2(M)$ that

$$\int_M \Delta f \; d\mu_g = 0.$$

This follows easily via a direct application of the divergence theorem with $X = \nabla f$.

3.5 The Connected Sum

A key concept used in the latter half of the proof is that of the connected sum. We define it here for surfaces.

Definition 3.20. Let M_1 and M_2 be two closed, oriented Riemannian 2-manifold. The *connected sum* of M_1 and M_2, denoted $M_1 \# M_2$ is the resulting manifold obtained via the following procedure:

Step 1: Remove a (small) ball from each of M_1 and M_2.

Step 2: Identify the resulting S^1 boundaries (via a reflection). That is, glue a cylinder connecting the two S^1 boundaries.

Step 3: Smooth any corners.

The connected sum results in a closed, oriented Riemannian 2-manifold, unique up to diffeomorphism. An example is shown below.

Example 3.21. Here we demonstrate the process in obtaining the connected sum of two tori.

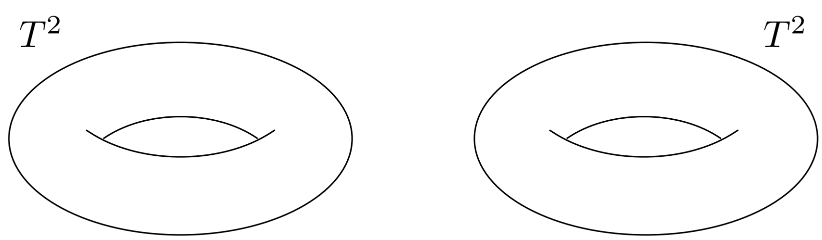

Figure 3.22. *Two tori embedded in* $\mathbb{R}^3$*.*

Step 1: We remove a small ball from each torus. This results in S^1 boundaries, shown below in Figure 3.23.

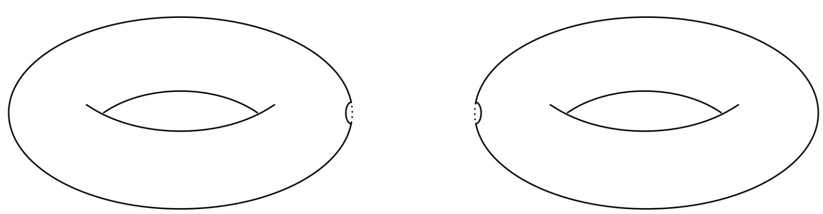

Figure 3.23. *Two tori with a small ball removed.*

Step 2: We identify the resulting S^1 boundaries via a reflection.

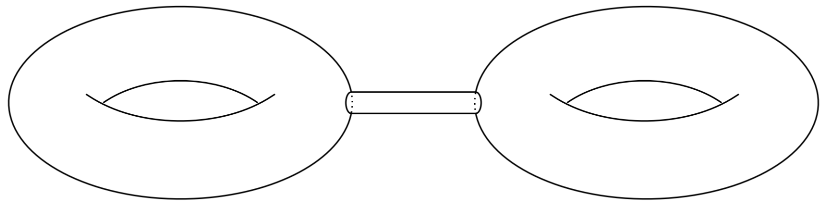

Figure 3.24. *Two tori with identified S^1 boundaries.*

Step 3: We smooth out the manifold.

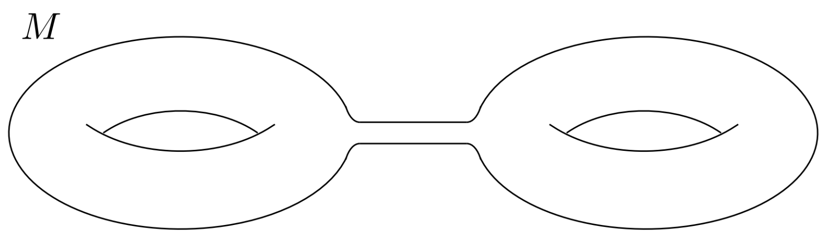

Figure 3.25. *The connected sum $M = T^2 \# T^2$, once smoothed out.*

The connected sum is only unique up to diffeomorphism! The following genus 2 surface is diffeomorphic to it.

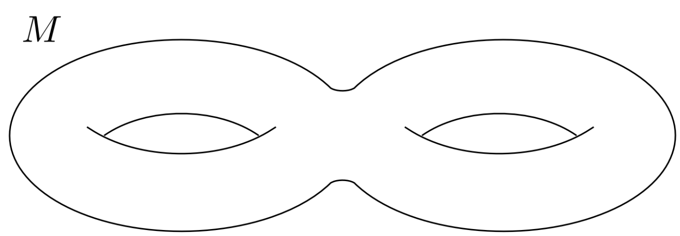

Figure 3.26. *A diffeomorphic genus* 2 *surface.*

We could have also done this process *anywhere* along each T^2.

Remark 3.27. Note the following property of integrals on a connected sum: Let $M = M_1 \# M_2$ with metric g and let $f : M \to \mathbb{R}$ be smooth. Let S^1 be in the cylindrical portion of M. Then M is separated into two halves – denote by N_1 one half and N_2 the other. Then,

$$\int_M f \; d\mu_g = \int_{N_1} f \; d\mu_g + \int_{N_2} f \; d\mu_g.$$

The above works since S^1 is a measure zero set in M. In a measure theoretic sense, we are integrating $\tilde{f}$ which is f on N_1, N_2 and zero on S^1. Then, appeal to the fact that $f = \tilde{f}$ a.e.

It should be made clear here that when speaking of the connected sum, we mean the manifold given in Figure 3.25. Although it is diffeomorphic to the manifold in Figure 3.26, integration on the two are very different.

Connected sums allow for a fundamental classification of compact, orientable Riemannian 2-manifolds.

Theorem 3.28. *Let M be a compact, orientable Riemannian 2-manifold. Then M is diffeomorphic to either a sphere or the connected sum of n tori for $n \geq 1$.*

Thus, it suffices to perform all integrals on a sphere or a connected sum of n tori. The Euler characteristic distinguishes all of these, and is given by

$$\chi(M) = 2(1 - n),$$

where n is the number of tori in the connected sum (equivalently, the number of holes in M, usually called the genus).

4 Variational Formulas

Here we prove variational formulas for several geometric quantities, including the Christoffel symbols, volume form, and Riemann curvature tensor.

Let $(M, g(t))$ be a compact, orientable Riemannian manifold equipped with the one parameter family of metrics $g(t)$. Set $h_{ij} = \partial_t g_{ij}$. This set of computations follows the one given in [CK04].
In what follows, we will make use of Remark 3.12. That is, a tensor evaluated at p depends only on the value of its components at p. We can then perform the following:

Step 1: Approach the problem at one particular point using geodesic coordinates.

Step 2: By Remark 3.12, that formula will hold in any coordinate system

Step 3: Since the same process can be repeated for any point, it holds over M.

We first compute the variation of the inverse metric.

Proposition 4.1. *The inverse metric evolves by*

$$\partial_t g^{ij} = -g^{ik} g^{jl} h_{kl}.$$

Proof. The inverse metric is defined by

$$g^{ik} g_{kj} = \delta^i_j.$$

Here,

$$\delta_{ij} = \delta^i_j = \delta^{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

is the standard Kronecker delta. The indices are raised or lowered to follow the summation convention. Taking time derivatives we get

$$g_{kj} \partial_t g^{ik} + g^{ik} \partial_t g_{kj} = 0.$$

By definition, $\partial_t g_{ij} = h_{ij}$ so that

$$g_{kj} \partial_t g^{ik} = -g^{ik} h_{kj}.$$

It follows, after changing some notation on the indices, that

$$\partial_t g^{ij} = -g^{ik} g^{jl} h_{kl}.$$

□

We can also easily compute the variation in $d\mu_{g(t)}$.

Proposition 4.2. *The volume form $d\mu_{g(t)}$ evolves by*

$$\frac{\partial}{\partial t} d\mu_{g(t)} = \frac{1}{2}\left(g^{ij}\frac{\partial}{\partial t} g_{ij}\right) d\mu_{g(t)}.$$

Proof. In local coordinates,

$$d\mu_{g(t)} = \sqrt{\det g}\ dx^1 dx^2 ... dx^n.$$

For invertible matrices $A(t)$ we have

$$\frac{d}{dt}\det A(t) = \det A(t) \operatorname{tr}\left(A^{-1}(t)\frac{d}{dt}A(t)\right).$$

Then,

$$\begin{aligned}
\frac{\partial}{\partial t}\sqrt{\det g} &= \frac{1}{2\sqrt{\det g}}\frac{\partial}{\partial t}\det g \\
&= \frac{1}{2\sqrt{\det g}}\det g \operatorname{tr}\left(g^{-1}\frac{\partial}{\partial t}g\right) \\
&= \frac{1}{2}\operatorname{tr}\left(g^{-1}\frac{\partial}{\partial t}g\right)\sqrt{\det g}.
\end{aligned}$$

Substituting this gives

$$\begin{aligned}
\frac{\partial}{\partial t} d\mu_{g(t)} &= \frac{\partial}{\partial t}\sqrt{\det g}\ dx^1 dx^2 ... dx^n \\
&= \frac{1}{2}\operatorname{tr}\left(g^{-1}\frac{\partial}{\partial t}g\right)\sqrt{\det g}\ dx^1 dx^2 ... dx^n \\
&= \frac{1}{2}\operatorname{tr}\left(g^{-1}\frac{\partial}{\partial t}g\right) d\mu_{g(t)},
\end{aligned}$$

as desired. □

Using the variation of the inverse metric, we can find the variation of the Christoffel symbols.

Proposition 4.3. *The Christoffel symbols evolve by*

$$\partial_t \Gamma^l_{ij} = \frac{1}{2}\left\{\nabla_i h_{jk} + \nabla_j h_{ik} - \nabla_k h_{ij}\right\} g^{kl}$$

where $\nabla_i = \nabla_{\partial_i}$.

Proof. First recall that the Christoffel symbols are given by

$$\Gamma^l_{ij} = \frac{1}{2}\left\{\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right\} g^{kl}.$$

Recall that the covariant derivative differs from the regular derivative by a factor involving Christoffel symbols. It follows that, in geodesic coordinates centered at p, $\nabla_i(p) = \partial_i(p)$. Taking the time derivative at p gives,

$$\begin{aligned}
\frac{\partial}{\partial t}\Gamma^l_{ij}\bigg|_p &= \frac{1}{2}\left\{\frac{\partial}{\partial t}(\partial_i g_{jk}) + \frac{\partial}{\partial t}(\partial_j g_{ik}) - \frac{\partial}{\partial t}(\partial_k g_{ij})\right\} g^{kl}\bigg|_p \\
&+ \frac{1}{2}\{\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\}\frac{\partial}{\partial t} g^{kl}\bigg|_p \\
&= \frac{1}{2}\left\{\partial_i\left(\frac{\partial}{\partial t} g_{jk}\right) + \partial_j\left(\frac{\partial}{\partial t} g_{ik}\right) - \partial_k\left(\frac{\partial}{\partial t} g_{ij}\right)\right\} g^{kl}\bigg|_p \\
&= \frac{1}{2}\{\partial_i h_{jk} + \partial_j h_{ik} - \partial_k h_{ij}\} g^{kl}\bigg|_p \\
&= \frac{1}{2}\{\nabla_i h_{jk} + \nabla_j h_{ik} - \nabla_k h_{ij}\} g^{kl}\bigg|_p,
\end{aligned}$$

since $\partial_i g_{jk}(p) = 0$, and by commuting partial derivatives. Since both sides are components of tensors, this holds at any point in any coordinate system. □

From here we find the variation of the components of the Riemann curvature tensor.

Proposition 4.4. *The components of the Riemann curvature tensor evolve by*

$$\frac{\partial}{\partial t} R^l_{ijk} = \nabla_i\left(\frac{\partial}{\partial t}\Gamma^l_{jk}\right) - \nabla_j\left(\frac{\partial}{\partial t}\Gamma^l_{ik}\right).$$

Proof. First recall that the components are given by

$$R^l_{ijk} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^s_{jk}\Gamma^l_{is} - \Gamma^s_{ik}\Gamma^l_{js}.$$

A derivation of this formula can be found in [dC92]. At p, the time derivatives of all Christoffel symbols vanish. Thus

$$\frac{\partial}{\partial t}\Gamma^s_{jk}\Gamma^l_{is}\bigg|_p = 0.$$

We are then left with

$$\frac{\partial}{\partial t} R^l_{ijk}\bigg|_p = \partial_i\left(\frac{\partial}{\partial t}\Gamma^l_{jk}\right)\bigg|_p - \partial_j\left(\frac{\partial}{\partial t}\Gamma^l_{ik}\right)\bigg|_p = \nabla_i\left(\frac{\partial}{\partial t}\Gamma^l_{jk}\right)\bigg|_p - \nabla_j\left(\frac{\partial}{\partial t}\Gamma^l_{ik}\right)\bigg|_p$$

after commuting partial derivatives and again applying $\partial_i(p) = \nabla_i(p)$. This computation holds everywhere. □

The computations performed in this section were done in full generality using geodesic coordinates. In theory, one can specialize to $n = 2$ and attempt to use isothermal coordinates. The advantage is that the Gauss curvature becomes a Laplacian, which is easy to work with. The disadvantage is that the variation must stay conformal to Euclidean space – hence, the matrix h must be diagonal. Furthermore, the computations are more cumbersome. Early work on this thesis was done in this setting. A special case of Gauss-Bonnet can be proven using this method. Interestingly, the techniques employed were available to Gauss, since he showed existence of isothermal coordinates on a surface with real analytic metric [Gau73].

5 Explicit Formulae For Variational Quantities

In this section, we explicitly compute the variations in Section 4 in dimension 2. All computations are done in geodesic coordinates at p.

5.1 The Inverse Metric

Recall from Prop 4.1 that the variation of the inverse metric is given by

$$\frac{\partial}{\partial t} g^{ij} = -g^{ik} g^{jl} h_{kl}.$$

Therefore,

$$\begin{aligned}
\left.\frac{\partial}{\partial t} g^{11}\right|_p &= -g^{1k} g^{1l} h_{kl}|_p \\
&= -g^{1k}(g^{11} h_{k1} + g^{12} h_{k2})|_p = -g^{1k}(g^{11} h_{k1})|_p \\
&= -g^{11}(g^{11} h_{11} + g^{12} h_{12})|_p = -(g^{11})^2 h_{11}|_p = -h_{11}|_p.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\left.\frac{\partial}{\partial t} g^{12}\right|_p &= -h_{12}|_p, \\
\left.\frac{\partial}{\partial t} g^{22}\right|_p &= -h_{22}|_p.
\end{aligned}$$

5.2 The Christoffel Symbols

Recall from Prop 4.3 that the variation of the Christoffel symbols is given by

$$\frac{\partial}{\partial t} \Gamma^k_{ij} = \frac{1}{2} g^{kl} \left\{ \nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij} \right\},$$

so that at p,

$$\left.\frac{\partial}{\partial t} \Gamma^k_{ij}\right|_p = \frac{1}{2} g^{kl} \left\{ \partial_i h_{jl} + \partial_j h_{il} - \partial_l h_{ij} \right\} |_p.$$

Then,

$$\begin{aligned}
\left.\frac{\partial}{\partial t} \Gamma^1_{11}\right|_p &= \frac{1}{2} g^{11} \left\{ \partial_1 h_{11} + \partial_1 h_{11} - \partial_1 h_{11} \right\} |_p \\
&+ \frac{1}{2} g^{12} \left\{ \partial_1 h_{12} + \partial_1 h_{12} - \partial_2 h_{11} \right\} |_p \\
&= \frac{1}{2} \left\{ \partial_1 h_{11} + \partial_1 h_{11} - \partial_1 h_{11} \right\} |_p = \frac{1}{2} \partial_1 h_{11}|_p,
\end{aligned}$$

$$
\begin{aligned}
\left.\frac{\partial}{\partial t}\Gamma^2_{11}\right|_p &= \frac{1}{2}g^{21}\left\{\partial_1 h_{11} + \partial_1 h_{11} - \partial_1 h_{11}\right\}|_p \\
&+ \frac{1}{2}g^{22}\left\{\partial_1 h_{12} + \partial_1 h_{12} - \partial_2 h_{11}\right\}|_p \\
&= \frac{1}{2}\left\{\partial_1 h_{12} + \partial_1 h_{12} - \partial_2 h_{12}\right\}|_p = \frac{1}{2}\left\{2\partial_1 h_{12} - \partial_2 h_{11}\right\}|_p, \\
\left.\frac{\partial}{\partial t}\Gamma^1_{12}\right|_p &= \frac{1}{2}g^{11}\left\{\partial_1 h_{21} + \partial_2 h_{11} - \partial_1 h_{12}\right\}|_p \\
&+ \frac{1}{2}g^{12}\left\{\partial_1 h_{22} + \partial_2 h_{12} - \partial_2 h_{12}\right\}|_p \\
&= \frac{1}{2}\left\{\partial_1 h_{21} + \partial_2 h_{11} - \partial_1 h_{12}\right\}|_p = \frac{1}{2}\partial_2 h_{11}|_p.
\end{aligned}
$$

In each, expand the summation on l first, then evaluate the inverse metric at p, then simplify. Observe that the remaining Christoffel symbols may be computed by interchanging $(1 \leftrightarrow 2)$. In summary,

$$
\begin{aligned}
\left.\frac{\partial}{\partial t}\Gamma^1_{11}\right|_p &= \frac{1}{2}\partial_1 h_{11}|_p & \left.\frac{\partial}{\partial t}\Gamma^2_{11}\right|_p &= \frac{1}{2}\left\{2\partial_1 h_{12} - \partial_2 h_{11}\right\}|_p \\
\left.\frac{\partial}{\partial t}\Gamma^1_{12}\right|_p &= \frac{1}{2}\partial_2 h_{11}|_p & \left.\frac{\partial}{\partial t}\Gamma^2_{12}\right|_p &= \frac{1}{2}\partial_1 h_{22}|_p \\
\left.\frac{\partial}{\partial t}\Gamma^1_{22}\right|_p &= \frac{1}{2}\left\{2\partial_2 h_{12} - \partial_1 h_{22}\right\}|_p & \left.\frac{\partial}{\partial t}\Gamma^2_{22}\right|_p &= \frac{1}{2}\partial_2 h_{22}|_p
\end{aligned}
$$

5.3 Components of the Riemann Curvature Tensor

Recall from Prop 4.4 that the variation of the components of the Riemann curvature tensor is given by

$$
\left.\frac{\partial}{\partial t}R^l_{ijk}\right|_p = \partial_i\left(\frac{\partial}{\partial t}\Gamma^l_{jk}\right)\Bigg|_p - \partial_j\left(\frac{\partial}{\partial t}\Gamma^l_{ik}\right)\Bigg|_p
$$

Since this formula is antisymmetric in i, j we see that

$$
\left.\frac{\partial}{\partial t}R^l_{iik}\right|_p = 0,
$$

so that

$$
\begin{aligned}
\left.\frac{\partial}{\partial t}R^1_{111}\right|_p = \left.\frac{\partial}{\partial t}R^2_{111}\right|_p = \left.\frac{\partial}{\partial t}R^1_{112}\right|_p = \left.\frac{\partial}{\partial t}R^2_{112}\right|_p = \\
\left.\frac{\partial}{\partial t}R^1_{221}\right|_p = \left.\frac{\partial}{\partial t}R^2_{221}\right|_p = \left.\frac{\partial}{\partial t}R^1_{222}\right|_p = \left.\frac{\partial}{\partial t}R^2_{222}\right|_p = 0.
\end{aligned}
$$

This takes care of 8 of the 16 components. Now,

$$\begin{aligned} \frac{\partial}{\partial t} R^1_{121}\Big|_p &= \partial_1 \left(\frac{\partial}{\partial t} \Gamma^1_{21} \right) \Big|_p - \partial_2 \left(\frac{\partial}{\partial t} \Gamma^1_{11} \right) \Big|_p \\ &= \partial_1 \left(\frac{1}{2} \partial_2 h_{11} \right) \Big|_p - \partial_2 \left(\partial_1 h_{11} \right) \Big|_p = 0. \end{aligned}$$

Note that this cancellation occurs since $\partial_i \partial_j = \partial_j \partial_i$, and

$$\frac{\partial}{\partial t} \Gamma^k_{ik}\Big|_p = \frac{1}{2} \partial_i h_{kk}|_p.$$

Thus,

$$\frac{\partial}{\partial t} R^1_{121}\Big|_p = \frac{\partial}{\partial t} R^1_{211}\Big|_p = \frac{\partial}{\partial t} R^2_{122}\Big|_p = \frac{\partial}{\partial t} R^2_{212}\Big|_p = 0.$$

(you can also use antisymmetry of R^l_{ijk} in i, j). Finally,

$$\begin{aligned} \frac{\partial}{\partial t} R^2_{121}\Big|_p &= \partial_1 \left(\frac{\partial}{\partial t} \Gamma^2_{21} \right) \Big|_p - \partial_2 \left(\frac{\partial}{\partial t} \Gamma^2_{11} \right) \Big|_p \\ &= \partial_1 \left(\frac{1}{2} \partial_1 h_{22} \right) \Big|_p - \partial_2 \left(\frac{1}{2} (2\partial_1 h_{12} - \partial_2 h_{11}) \right) \Big|_p \\ &= \frac{1}{2} \left[\partial_1^2 h_{22} + \partial_2^2 h_{11} - 2\partial_1 \partial_2 h_{12} \right] |_p. \end{aligned}$$

Note that this is symmetric with respect to interchanging $(1 \leftrightarrow 2)$. Thus,

$$\frac{\partial}{\partial t} R^1_{212}\Big|_p = \frac{\partial}{\partial t} R^2_{121}\Big|_p = \frac{1}{2} \left[\partial_1^2 h_{22} + \partial_2^2 h_{11} - 2\partial_1 \partial_2 h_{12} \right] |_p.$$

Moreover, by applying antisymmetry of R^l_{ijk} in i, j we obtain

$$\frac{\partial}{\partial t} R^2_{211}\Big|_p = \frac{\partial}{\partial t} R^1_{122}\Big|_p = -\frac{\partial}{\partial t} R^2_{121}\Big|_p = -\frac{1}{2} \left[\partial_1^2 h_{22} + \partial_2^2 h_{11} - 2\partial_1 \partial_2 h_{12} \right] |_p.$$

5.4 Components of the Ricci Tensor

Recall that the components of the Ricci tensor are given by

$$R_{ik} = R^j_{ijk} = \sum_{j=1}^{n} R^j_{ijk}.$$

See [dC92] for a definition. It follows that

$$\frac{\partial}{\partial t} R_{ik} = \sum_{j=1}^{n} \frac{\partial}{\partial t} R^j_{ijk}.$$

Thus,

$$\begin{aligned}
\frac{\partial}{\partial t} R_{11}\bigg|_p &= \frac{\partial}{\partial t} R^1_{111}\bigg|_p + \frac{\partial}{\partial t} R^2_{121}\bigg|_p \\
&= \frac{1}{2}\left[\partial_1^2 h_{22} + \partial_2^2 h_{11} - 2\partial_1\partial_2 h_{12}\right]|_p, \\
\frac{\partial}{\partial t} R_{12}\bigg|_p = \frac{\partial}{\partial t} R_{21}\bigg|_p &= \frac{\partial}{\partial t} R^1_{112}\bigg|_p + \frac{\partial}{\partial t} R^2_{122}\bigg|_p = 0, \\
\frac{\partial}{\partial t} R_{22}\bigg|_p &= \frac{\partial}{\partial t} R^1_{212}\bigg|_p + \frac{\partial}{\partial t} R^2_{222}\bigg|_p \\
&= \frac{1}{2}\left[\partial_1^2 h_{22} + \partial_2^2 h_{11} - 2\partial_1\partial_2 h_{12}\right]|_p.
\end{aligned}$$

5.5 The Gauss Curvature

Recall (from [dC92]) that the scalar curvature is given by

$$R = g^{ij} R_{ij},$$

so that

$$\frac{\partial R}{\partial t} = \left(\frac{\partial}{\partial t} g^{ij}\right) R_{ij} + g^{ij}\left(\frac{\partial}{\partial t} R_{ij}\right).$$

At p, this reduces to

$$\begin{aligned}
\frac{\partial R}{\partial t}\bigg|_p &= -\sum_{i=1}^{2}\sum_{j=1}^{2} h_{ij} R_{ij}|_p \\
&+ \left[g^{11}\left(\frac{\partial}{\partial t} R_{11}\right) + 2g^{12}\left(\frac{\partial}{\partial t} R_{12}\right) + g^{22}\left(\frac{\partial}{\partial t} R_{22}\right)\right]_p \\
&= -\sum_{i=1}^{2}\sum_{j=1}^{2} h_{ij} R_{ij}|_p + \left(\frac{\partial}{\partial t} R_{11}\right)_p + \left(\frac{\partial}{\partial t} R_{22}\right)_p \\
&= -\sum_{i=1}^{2}\sum_{j=1}^{2} h_{ij} R_{ij}|_p + \left[\partial_1^2 h_{22} + \partial_2^2 h_{11} - 2\partial_1\partial_2 h_{12}\right]|_p.
\end{aligned}$$

We therefore conclude that

$$\frac{\partial R}{\partial t} = -\sum_{i=1}^{2}\sum_{j=1}^{2} h_{ij} R_{ij} + \left[\partial_1^2 h_{22} + \partial_2^2 h_{11} - 2\partial_1\partial_2 h_{12}\right].$$

For $n = 2$, recall that $R_{ij} = K g_{ij}$ since $\mathrm{Ric}_g = Kg$. It follows that

$$R = g^{ij} K g_{ij} = K \operatorname{tr} g = 2K.$$

Thus,

$$\frac{\partial K}{\partial t} = \frac{1}{2}\frac{\partial R}{\partial t} = -\frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} h_{ij} R_{ij} + \frac{1}{2}\left[\partial_1^2 h_{22} + \partial_2^2 h_{11} - 2\partial_1\partial_2 h_{12}\right]. \tag{1}$$

6 Reformulation of the Variation of Gauss Curvature

In this section, we reformulate the quantity in the right hand side of (1). This will make it easier to geometrically interpret. These will naturally come from the previous computations. We will also begin to omit the evaluation at p when it becomes cumbersome to show.

Definition 6.1. let g, h be matrices. Then the trace of h with respect to g, denoted $\mathrm{tr}_g(h)$, is defined as

$$\mathrm{tr}_g(h) = g^{ij} h_{ij}.$$

Next, define by H the quantity

$$H := \mathrm{tr}_g \left(\frac{\partial}{\partial t} g \right)$$

so that in geodesic coordinates at p,

$$H|_p = h_{11}|_p + h_{22}|_p.$$

Now observe that

$$\begin{aligned}
\langle h, \mathrm{Ric}_g \rangle &= g^{ik} g^{jl} h_{ij} R_{kl} \\
\langle h, \mathrm{Ric}_g \rangle|_p &= g^{ik} g^{jl} h_{ij} R_{kl}|_p \\
&= g^{ik} (g^{11} h_{i1} R_{k1} + g^{12} h_{i1} R_{k2} + g^{21} h_{i2} R_{k1} + g^{22} h_{i2} R_{k2})_p \\
&= g^{ik} (h_{i1} R_{k1} + h_{i2} R_{k2})_p \\
&= (g^{11} h_{11} R_{11} + g^{12} h_{11} R_{21} + g^{21} h_{21} R_{11} + g^{22} h_{21} R_{21})_p \\
&+ (g^{11} h_{12} R_{12} + g^{12} h_{12} R_{22} + g^{21} h_{22} R_{12} + g^{22} h_{22} R_{22})_p \\
&= (h_{11} R_{11} + h_{21} R_{21})_p + (h_{12} R_{12} + h_{22} R_{22})_p = \sum_{i=1}^{2} \sum_{j=1}^{2} h_{ij} R_{ij}|_p.
\end{aligned}$$

By Definition 3.14, we obtain

$$\Delta_g H = \partial_1^2 h_{11} + \partial_2^2 h_{11} + \partial_1^2 h_{22} + \partial_2^2 h_{22},$$

where each side is evaluated at p. Then by applying Remark 3.17 to $A = \partial g / \partial t|_{t=0} = h$,

$$\begin{aligned}
\Delta_g H - \mathrm{div}(\mathrm{div}\, h) &= \partial_1^2 h_{11} + \partial_2^2 h_{11} + \partial_1^2 h_{22} + \partial_2^2 h_{22} \\
&\quad - \partial_1^2 h_{11} - 2\partial_1 \partial_2 h_{12} - \partial_2^2 h_{22} \\
&= \partial_1^2 h_{22} + \partial_2^2 h_{11} - 2\partial_1 \partial_2 h_{12}.
\end{aligned}$$

Combining all of this, it follows that

$$\frac{\partial K}{\partial t} = -\frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} h_{ij}R_{ij} + \frac{1}{2}\left[\partial_1^2 h_{22} + \partial_2^2 h_{11} - 2\partial_1\partial_2 h_{12}\right] = -\frac{1}{2}\langle h, \mathrm{Ric}_g\rangle + \frac{1}{2}\Delta_g H - \frac{1}{2}\operatorname{div}(\operatorname{div} h). \tag{2}$$

Note also that

$$\frac{\partial}{\partial t} d\mu_{g(t)} = \frac{H}{2} d\mu_{g(t)}.$$

Remark 6.2. We briefly demonstrate that it is imperative to use geodesic coordinates to vastly simplify the computations. Without geodesic coordinates, we sketch a combinatorial argument that there are more than 200 terms in the expression for $\partial K/\partial t$ (assuming there is little to no cancellation). To do so, we estimate the number of terms in $\Delta_g H$.

For simplicity, consider the 2 dimensional case. Observe that, without geodesic coordinates, $H = g^{11}h_{11} + 2g^{12}h_{12} + g^{22}h_{22}$. Recall that the Laplacian is

$$\Delta_g = g^{mn}\left\{\partial_m\partial_n - \Gamma^l_{mn}\partial_l\right\}.$$

Let us look at $g^{mn}\left\{\partial_m\partial_n - \Gamma^l_{mn}\partial_l\right\}$ for fixed m, n. The index l can take values $l = 1, 2$, so there are three terms here. Since $m, n = 1, 1; 1, 2; 2, 2$, there are three terms like this, giving a total of 9 terms in the Laplacian. Six of these involve a first derivative while three involve a second derivative.
Those six which involve a first derivative also involve a Christoffel symbol. Recall that

$$\Gamma^l_{mn} = \frac{1}{2}\left\{\partial_m g_{nk} + \partial_n g_{mk} - \partial_k g_{mn}\right\} g^{kl}.$$

So, for each k there are three terms. Once more, because we are in dimension 2, k can range from 1 to 2, and hence each Christoffel symbol has six terms. In total, there are 36 terms involving a first derivative (by expanding each Christoffel symbol) and three that involve a second derivative.

Fixing i, j and applying the product rule on $g^{ij}h_{ij}$ shows that, for each k, $\partial_k g^{ij}h_{ij}$ is the sum of two terms, each being a product of two terms. Applying product rule once more shows that, for each l, k, $\partial_l\partial_k g^{ij}h_{ij}$ is the sum of four terms. Thus, for fixed i, j, m, n, the quantity $g^{mn}\left\{\partial_m\partial_n - \Gamma^l_{mn}\partial_l\right\}(g^{ij}h_{ij})$ has $(1)(4) + (6)(2) + (6)(2) = 28$ terms. Again, the contribution of 1 comes from the second partial derivative term, which results in four terms. The contributions of 6 come from each Christoffel symbol, which is multiplied by a first partial.

Finally, allowing m, n to range, we get three copies of this. So there are approximately 84 terms in $\Delta_g g^{ij}h_{ij}$. But this is for fixed i, j. We are interested in H, which is the sum of three terms like $g^{ij}h_{ij}$. Thus $\Delta_g H$ has approximately 252 terms. An attempt by hand shows that there is little to no cancellation. And this is only one of the quantities involved in computing $\partial K/\partial t$.

7 Gauss-Bonnet Revisited

We now have everything necessary to prove Theorem 2.11 stated in the introduction.

Theorem 2.11 (Gauss-Bonnet)**.** *Let (M, g) be a compact, orientable 2-dimensional Riemannian manifold without boundary. Then,*

$$\int_M K d\mu_g = 2\pi\chi(M)$$

where K is the Gauss curvature.

The proof is broken into two parts. First, we show that the time derivative $\partial/\partial t \int_M K d\mu_{g(t)}$ is zero. Hence, $\int_M K d\mu_{g(t)}$ is locally constant. By the path connectedness of the space of metrics, it is actually constant. Then, we must establish which constant. To this end, we consider a general compact, orientable 2-dimensional Riemannian manifold. By the classification result of surfaces, it follows that this manifold is diffeomorphic to a sphere, torus, or the connected sum of tori. It suffices then to compute $\partial/\partial t \int_M K d\mu_g$ for these special cases.

7.1 Step 1: Showing the time derivative is zero

Here we aim to prove the following lemma:

Lemma 7.1. *Let (M, g_0) be a compact, orientable 2-dimensional Riemannian manifold without boundary. Consider a one-parameter family of metrics $g(t)$ on M such that $g(0) = g_0$. Then,*

$$\frac{\partial}{\partial t}\int_M K d\mu_{g(t)} = 0$$

Proof. First, observe that in the case $n = 2$ at p we have

$$\begin{aligned} \langle h, \mathrm{Ric}_g\rangle &= \langle h, Kg\rangle = K\langle h, g\rangle \\ &= Kg^{ik}g^{jl}h_{ij}g_{kl} \\ &= Kg^{ik}h_{ij}\delta^j_k = Kg^{ij}h_{ij} = KH \end{aligned}$$

(since $\delta^j_k = 1$ iff $j = k$).
Notice that, due to Prop 4.2 and (2), the time derivative is

$$\begin{aligned} \frac{\partial}{\partial t}\int_M K d\mu_{g(t)} &= \int_M \frac{\partial}{\partial t} K d\mu_{g(t)} + \int_M K\frac{\partial}{\partial t} d\mu_{g(t)} \\ &= \int_M \left[-\frac{HK}{2} + \frac{1}{2}\Delta_g H - \frac{1}{2}\operatorname{div}(\operatorname{div} h)\right] d\mu_{g(t)} + \int_M \frac{HK}{2} d\mu_{g(t)} \\ &= \int_M \left[\frac{1}{2}\Delta_g H - \frac{1}{2}\operatorname{div}(\operatorname{div} h)\right] d\mu_{g(t)} \end{aligned}$$

Now, by application of Theorem 3.18, we see that

$$\frac{\partial}{\partial t}\int_M K d\mu_{g(t)} = \int_M \left[\frac{1}{2}\Delta_g H - \frac{1}{2}\operatorname{div}(\operatorname{div} h)\right] d\mu_{g(t)} = 0.$$

□

7.2 Step 2: Computation of the Integral

We now turn to which constant $\int_M K d\mu_g$ is equal to when M is a genus $n = 0, 1, ...$ surface. We first compute this in the genus 0 and 1 cases. Then, we use an inductive proof to show the general case. Henceforth, I write $d\mu_g$ as dS.

Since surfaces of revolution are relatively simple, we may compute the integral for surfaces of revolution in general. Afterwards, we may specialize to S^2 and T^2 to obtain the genus 0 and 1 cases.

Let $(g(t), f(t))$ be a simple curve in $\mathbb{R}^2$ parameterized by $t \in [a, b]$. We may graph this as a curve in $\mathbb{R}^3$ as $\gamma = (g(t), 0, f(t))$. Then, the surface of revolution S generated by rotating γ around the z axis is given by

$$X(u, v) = (g(u)\cos(v), g(u)\sin(v), f(u))$$

where $u \in [a, b]$ and $v \in [0, 2\pi)$. The first partial derivatives are

$$\begin{aligned} X_u &= (g'(u)\cos(v), g'(u)\sin(v), f'(u)), \\ X_v &= (-g(u)\sin(v), g(u)\cos(v), 0). \end{aligned}$$

The coefficients of the first fundamental form are given by

$$\begin{aligned} E &= \langle X_u, X_u \rangle = (g'(u)\cos(v))^2 + (g'(u)\sin(v))^2 + f'(u)^2 = g'(u)^2 + f'(u)^2, \\ F &= \langle X_u, X_v \rangle = -g(u)g'(u)\cos(v)\sin(v) + g(u)g'(u)\sin(v)\cos(v) + f'(u)(0) = 0, \\ G &= \langle X_v, X_v \rangle = (-g(u)\sin(v))^2 + (g(u)\cos(v))^2 + 0^2 = g(u)^2. \end{aligned}$$

The unit normal is given by

$$\begin{aligned} X_u \times X_v &= (-f'(u)g(u)\cos(v), -f'(u)g(u)\sin(v), g(u)g'(u)) \\ &= g(u)(-f'(u)\cos(v), -f'(u)\sin(v), g'(u)), \\ \|X_u \times X_v\| &= |g(u)|\sqrt{f'(u)^2 + g'(u)^2}, \\ N(u, v) &= \frac{g(u)}{|g(u)|\sqrt{f'(u)^2 + g'(u)^2}}(-f'(u)\cos(v), -f'(u)\sin(v), g'(u)). \end{aligned}$$

The second partial derivatives are

$$\begin{aligned} X_{uu} &= (g''(u)\cos(v), g''(u)\sin(v), f''(u)), \\ X_{uv} &= (-g'(u)\sin(v), g'(u)\cos(v), 0), \\ X_{vv} &= (-g(u)\cos(v), -g(u)\sin(v), 0). \end{aligned}$$

The coefficients of the second fundamental form are given by

$$\begin{aligned} e &= \langle X_{uu}, N \rangle = \frac{g(u)(g'(u)f''(u) - f'(u)g''(u))}{|g(u)|\sqrt{f'(u)^2 + g'(u)^2}}, \\ f &= \langle X_{uv}, N \rangle = \frac{g(u)(f'(u)g'(u)\cos(v)\sin(v) - f'(u)g'(u)\sin(v)\cos(v)}{|g(u)|\sqrt{f'(u)^2 + g'(u)^2}} = 0, \\ g &= \langle X_{vv}, N \rangle = \frac{g(u)^2 f'(u)}{|g(u)|\sqrt{f'(u)^2 + g'(u)^2}}. \end{aligned}$$

The Gauss curvature may be computed by

$$K = \frac{eg - f^2}{EG - F^2} = \frac{eg}{EG}$$

since the parameterization of S is orthogonal ($F = f = 0$). Then

$$eg = \frac{g(u)(g'(u)f''(u) - f'(u)g''(u))}{|g(u)|\sqrt{f'(u)^2 + g'(u)^2}} \frac{g(u)^2 f'(u)}{|g(u)|\sqrt{f'(u)^2 + g'(u)^2}} = \frac{g(u)f'(u)(g'(u)f''(u) - f'(u)g''(u))}{(f'(u)^2 + g'(u)^2)}$$

and

$$EG = g(u)^2(f'(u)^2 + g'(u)^2).$$

Thus,

$$K = \frac{f'(u)(g'(u)f''(u) - f'(u)g''(u))}{g(u)(f'(u)^2 + g'(u)^2)^2}.$$

We are now ready to compute the integral of the Gaussian curvature. Recall that the surface and area elements are related by

$$dS = \sqrt{EG - F^2}dA = |g(u)|\sqrt{f'(u)^2 + g'(u)^2}dA$$

so that

$$\int_S KdS = \int_A K\sqrt{EG - F^2}dA = \int_0^{2\pi}\int_a^b \frac{|g(u)|f'(u)(g'(u)f''(u) - f'(u)g''(u))}{g(u)(f'(u)^2 + g'(u)^2)^{3/2}}dudv$$

(recall that u ranges over $[a, b]$ and v ranges over $[0, 2\pi)$). By Fubini's theorem, we conclude that

$$\int_S KdS = 2\pi \int_a^b \Lambda du$$

where

$$\Lambda = \frac{|g(u)|f'(u)(g'(u)f''(u) - f'(u)g''(u))}{g(u)(f'(u)^2 + g'(u)^2)^{3/2}}.$$

We can then prove the following specialize to S^2 and T^2 to obtain the following lemmas:

Lemma 7.2 (Genus 0 case)**.** *For $M = S^2$ embedded in $\mathbb{R}^3$, we have*

$$\int_M KdS = 2\pi\chi(M).$$

Proof. Consider the curve

$$(\sin t, \cos t)$$

where $t \in [0, \pi]$. Following the procedure above we set $g(u) = \sin u$ and $f(u) = \cos u$. Note that $g(u) \geq 0$ on $[0, \pi]$ and $g'(u) = f(u)$. Thus

$$\Lambda = \frac{f'(u)(f(u)f''(u) - f'(u)^2)}{(f'(u)^2 + f(u)^2)^{3/2}}.$$

Now $f''(u) = -f(u)$ so that

$$\Lambda = \frac{f'(u)(-f(u)^2 - f'(u)^2)}{(f'(u)^2 + f(u)^2)^{3/2}} = -\frac{f'(u)}{(f'(u)^2 + f(u)^2)^{1/2}}.$$

Finally we obtain

$$\Lambda = \sin(u).$$

Thus,

$$\int_{S^2} K dS = 2\pi \int_0^\pi \sin u \; du = -2\pi \cos u|_0^\pi = 4\pi = 2\pi\chi(S^2).$$

□

Lemma 7.3 (Genus 1 case)**.** *For $M = T^2$ embedded in $\mathbb{R}^3$, we have*

$$\int_M K dS = 2\pi\chi(M).$$

Proof. Consider the curve $r = \cos\theta$. In Cartesian coordinates this is given by

$$(\cos^2 t, \sin t \cos t)$$

where $t \in [0, \pi]$. Following the procedure above we set $g(u) = \cos^2 u$ and $f(u) = \sin u \cos u$. Note that $g(u) \geq 0$ on $[0, \pi]$ and $g'(u) = -2f(u)$. Thus,

$$\Lambda = \frac{f'(u)(-2f(u)f''(u) + 2f'(u)^2)}{(f'(u)^2 + 4f(u)^2)^{3/2}}.$$

Now $f(u) = 1/2 \sin(2u)$ so that $f''(u) = -4f(u) = -2\sin(2u)$. Then

$$\Lambda = \frac{\cos(2u)(-\sin(2u)(-2\sin(2u)) + 2\cos(2u)^2)}{(\cos(2u)^2 + \sin(2u)^2)^{3/2}} = 2\cos(2u).$$

It follows that

$$\int_{T^2} K dS = 2\pi \int_0^\pi 2\cos(2u) du = -2\pi \sin(2u)|_0^\pi = 0 = 2\pi\chi(T^2).$$

□

We now turn to prove the general genus case:

Lemma 7.4 (Genus $g \geq 2$ case)**.** *For M with genus $g \geq 2$ embedded in $\mathbb{R}^3$, we have*

$$\int_M K dS = 2\pi\chi(M).$$

We cannot simply apply the earlier result for surfaces of revolution – there is no clear way to realize, for example, a genus 2 surface in $\mathbb{R}^3$ as a surface of revolution. However, we can apply the fact that any surface with genus $g \geq 2$ is a connected sum of tori, and apply Lemmas 7.2 and 7.3.

Proof. We proceed by induction. Suppose Lemma 7.4 holds for genus $g = n \geq 2$; We show it holds for genus $g = n + 1$.

Let M be a compact, orientable Riemannian 2-manifold with genus $g = n + 1$. Then M is diffeomorphic to a genus $g = n$ compact surface connected sum a torus via a cylindrical portion with unit radius.

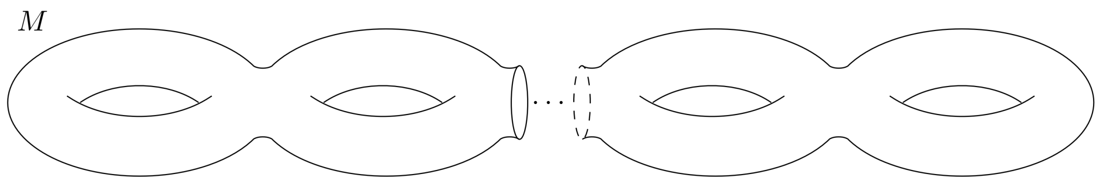

Figure 7.5. *The genus $n + 1$ manifold M defined above. Note that not all $n + 1$ handles are depicted.*

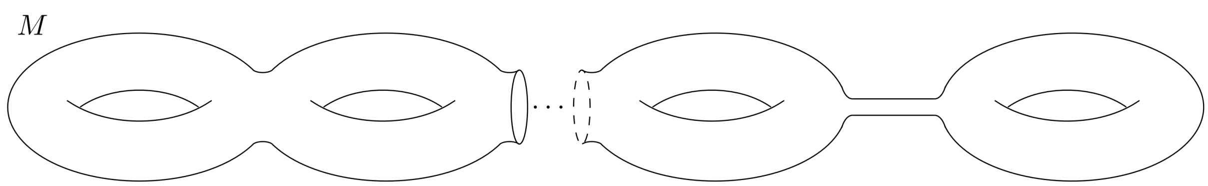

Figure 7.6. *The diffeomorphic manifold described above*

We now perform the following series of cuts and gluings:

Step 1: Cut M (as shown in Figure 7.6) orthogonally through the cylindrical portion. This results in two new closed manifolds. Call these N_1 and N_2, as depicted below.

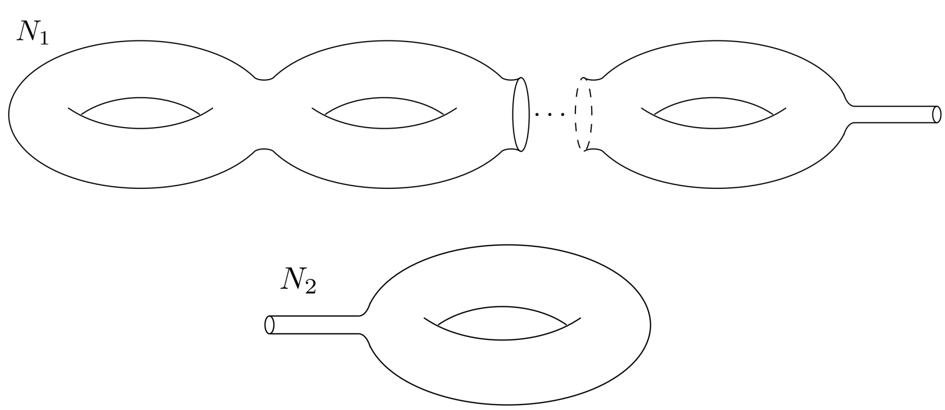

Figure 7.7. *Manifolds N_1 (above) and N_2 (below)*

Observe that $M = N_1 \# N_2$, with identification along the obvious S^1 boundary.

Step 2: Let H be a (closed) unit hemisphere. We may attach a copy of H to each of N_1 and N_2 (along the obvious S^1 boundaries). Call these M' and T', as depicted below. Note that $S^2 = H \# H$.

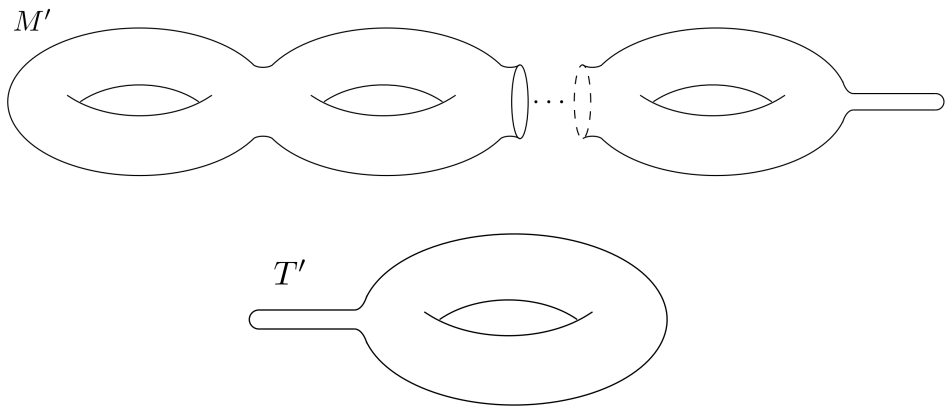

Figure 7.8. *Manifolds M' (above) and T' (below)*

The resulting surfaces M' and T' are diffeomorphic to general $g = n$ and $g = 1$ surfaces. Therefore, we may apply Lemmas 7.4 and 7.3. Doing so, in conjunction with Remark 3.27, results in the following:

$$\begin{aligned}\int_{N_1} K\ dS + \int_H K\ dS &= \int_{M'} K\ dS = 2\pi\chi(M'),\\ \int_{N_2} K\ dS + \int_H K\ dS &= \int_{T'} K\ dS = 2\pi\chi(T') = 2\pi\chi(T^2).\end{aligned}$$

By observation that $S^2 = H \# H$, application of Remark 3.27 with Lemma 7.2 also gives

$$\int_H K\ dS + \int_H K\ dS = \int_{S^2} K\ dS = 2\pi\chi(S^2),$$

and it follows that

$$\int_{N_1} K\ dS + \int_{N_2} K\ dS = 2\pi\chi(M') + 2\pi\chi(T^2) - 2\pi\chi(S^2).$$

Finally, splitting the integral across the connected sum and substituting the above gives

$$\begin{aligned}\int_M K\ dS &= \int_{N_1} K\ dS + \int_{N_2} K\ dS\\ &= 2\pi\chi(M') + 2\pi\chi(T^2) - 2\pi\chi(S^2)\\ &= 2\pi(\chi(M') - \chi(S^2))\\ &= 4\pi(1 - n - 1) = 4\pi(1 - (n+1)) = 2\pi\chi(M),\end{aligned}$$

where we have applied the fact that $\chi(M) = 2(1 - g)$. □

7.3 Completing the Proof

We are now ready to prove Theorem 2.11.

Proof of Thm 2.11. Lemma 7.1 asserts that

$$\frac{\partial}{\partial t}\int_M K \; d\mu_{g(t)} = 0.$$

Thus, $\int_M K \; d\mu_g$ is a constant. Lemmas 7.2, 7.3, and 7.4 together assert that if M is a sphere, torus, or connected sum of tori then

$$\int_M K \; d\mu_g = 2\pi\chi(M).$$

By the classification of compact orientable surfaces, it follows that any prescribed M as in Theorem 2.11 is diffeomorphic to one of these surfaces – call it M'. By applying Lemma 7.1, we see that $\int_M K \; d\mu_g$ coincides with $\int_{M'} K' \; d\mu_{g'}$.
Thus,

$$\int_M K \; d\mu_g = \int_{M'} K' \; d\mu_{g'} = 2\pi\chi(M') = 2\pi\chi(M),$$

where the last equality follows since M and M' are diffeomorphic and $\chi(M)$ is a topological quantity. This completes the proof. □

8 Conclusion and Further Work

We presented an overview of the history of Gauss-Bonnet and provided an alternative, self-contained proof of it for compact, orientable 2-manifolds. To this end, we computed variations of geometric quantities. We reformulated the variation of Gauss curvature in terms of the Laplacian on M and divergence of tensors. Finally, we made use of the classification of compact, orientable surfaces to show that the integrals evaluate to the $2\pi\chi(M)$. The advantage of this proof is that it relies only on calculus of surfaces.

One can ask whether such an approach would work for higher dimensional cases. The first obstacle to overcome is to find the correct higher dimensional analog of KdA. For hypersurfaces of $\mathbb{R}^{2n+1}$, this was answered by Hopf and is $G^*\omega_{2n}$. In full generality, it was answered by Chern and is $\mathrm{Pf}(\Omega)d\mu_g$ where Pf is the Pfaffian, Ω is the so-called curvature form of the Levi-Civita connection. Note the presence of the familiar volume form, which we have already computed the variation of. In theory, one could compute the variation of $\mathrm{Pf}(\Omega)$ as well.

The second part of the proof, though, relied on the classification of surfaces. Such a result exists in dimension 3 due to the work of Perelman, but does not exist for higher dimensions. Thus, the second part of the proof does not immediately generalize.

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