Kenneth DeMason, June 11, 2020

2 Normed Linear Spaces

1. Suppose that X is a vector space.

   a) If $A, B \subset X$ are convex, show that $A + B$ and $A \cap B$ are convex. What about $A \cup B$ and $A \setminus B$?

   b) Show that $2A \subset A + A$. When is it true that $2A = A + A$?

Solution:

a) Let $A, B \subset X$ be convex. Let $x, y \in A + B$. Then there exist $a_x, a_y \in A$ and $b_x, b_y \in B$ such that $x = a_x + b_x$ and $y = a_y + b_y$. Let $t \in [0, 1]$. Then,

$$tx + (1-t)y = t(a_x + b_x) + (1-t)(a_y + b_y) = [ta_x + (1-t)a_y] + [tb_x + (1-t)b_y].$$

Since A, B are convex, $[ta_x + (1-t)a_y] \in A$ and $[tb_x + (1-t)b_y] \in B$. Thus $tx + (1-t)y$ can be written as the sum of an element of A and an element of B. By definition, $tx + (1-t)y \in A + B$, and hence $A + B$ is convex.

Let $x, y \in A \cap B$. Then $x, y \in A$. Since A is convex, for any $t \in [0, 1]$ we have $tx + (1-t)y \in A$. Similarly, $x, y \in B$ and B is convex. So $tx + (1-t)y \in B$. Thus $tx + (1-t)y \in A \cap B$ and hence $A \cap B$ is convex.

$A \cup B$ need not be convex. For example, let $A = [-2, -1]$ and $B = [1, 2]$. Then for $x = -1$, $y = 1$, $t = 1/2$ we have

$$tx + (1-t)y = \frac{1}{2}(1-1) = 0 \notin A \cup B.$$

Similarly, the difference $A \setminus B$ need not be convex. For example, take $A = [-1, 1]$ and $B = \{0\}$. Then $A \setminus B = [-1, 0) \cup (0, 1]$, and for $x = -1, y = 1, t = 1/2$,

$$tx + (1-t)y = \frac{1}{2}(1-1) = 0 \notin A \setminus B.$$

b) Let $x \in 2A$. Then there exists a $y \in A$ such that $x = 2y$. On the other hand,

$$x = 2y = y + y,$$

so that x is written as the sum of two elements of A. Thus $x \in A + A$, and $2A \subset A + A$.

If A is convex, then $2A = A + A$. To see this, let $z \in A + A$. Then $z = x + y$ where $x, y \in A$. In order for z to be in $2A$, it must be that $(x + y)/2 \in A$. However, if A is convex, then $tx + (1-t)y \in A$ for any $x, y \in A$ and $t \in [0, 1]$.

Taking $t = 1/2$ shows that, for the chosen x, y, we have $(x+y)/2 \in A$ as desired.

From the above, it is clear that $2A = A + A$ iff for all $x, y \in A$, $(x + y)/2 \in A$. This convexity-type property is called *midpoint convexity*. It is equivalent to the following: Let D denote the set of dyadic rationals. That is,

$$D = \left\{ \sum_{n=1}^{\infty} \frac{x_n}{2^n} \;\middle|\; x_n = 0 \text{ or } 1 \right\}.$$

Say A has the property *dyadic convexity* if for all $x, y \in A$ and $d \in D$,

$$dx + (1 - d)y \in A.$$

Clearly, $1/2 \in D$. So, if A is dyadic convex, it is midpoint convex. We now show that midpoint convex implies dyadic convex. Let $x, y \in A$ and $d \in D$. Then

$$d = \sum_{n=1}^{\infty} \frac{x_n}{2^n}$$

Note that $d, 1 - d$ are complementary dyadic rationals in the following sense. If $\overline{x_n}$ denotes swapping 0 with 1 (that is, $\overline{x_n} = 1$ if $x_n = 0$ and $\overline{x_n} = 0$ if $x_n = 1$), then

$$d = \sum_{n=1}^{\infty} \frac{x_n}{2^n}, \quad (1 - d) = \sum_{n=1}^{\infty} \frac{\overline{x_n}}{2^n}$$

which follows since $1 = 1/2 + 1/4 + 1/8 + ...$
We show that $dx + (1 - d)y \in A$ by stages. Fix an $N > 1 \in \mathbb{N}$. Define $z = (x + y)/2 \in A$ since A is midpoint convex. Set z_0 as

$$z_0 = \frac{z + x_N x + \overline{x_N} y}{2},$$

which is in A by midpoint convexity (either $x_N = 1$ and $\overline{x_N} = 0$ or vice-verse). Set z_1 as

$$z_1 = \frac{z_0 + x_{N-1} x + \overline{x_{N-1}} y}{2},$$

which is similarly in A. Continuing in this fashion, define z_k by

$$z_k = \frac{z_{k-1} + x_{N-k} x + \overline{x_{N-k}} y}{2},$$

until $k = N - 1$. All the z_k, in particular z_{N-1}, are in A by midpoint convexity. Expanding z_{N-1} gives

$$\begin{aligned}
z_{N-1} &= \frac{z_{N-2}}{2} + \frac{x_1}{2} x + \frac{\overline{x_{N-k}}}{2} y = \frac{1}{2}\left[\frac{z_{N-3} + x_2 x + \overline{x_2} y}{2}\right] + \frac{x_1}{2} x + \frac{\overline{x_1}}{2} y \\
&= \frac{z_{N-3} + x_2 x + \overline{x_2} y}{2^2} + \frac{x_1}{2} x + \frac{\overline{x_1}}{2} y \\
&= \frac{1}{2^2}\left[\frac{z_{N-4} + x_3 x + \overline{x_3} y}{2}\right] + \left[\frac{x_1}{2} + \frac{x_2}{2^2}\right] x + \left[\frac{\overline{x_1}}{2} + \frac{\overline{x_2}}{2^2}\right] y = ... \\
&= \frac{z}{2^{N-1}} + \left[\sum_{n=1}^{N-1} \frac{x_n}{2^n}\right] x + \left[\sum_{n=1}^{N-1} \frac{\overline{x_n}}{2^n}\right] y
\end{aligned}$$

We may repeat this process for any N. Crucially, z_{N-1} is always in A. So, taking the limit as $N \to \infty$, we get

$$z_\infty = \left[\sum_{n=1}^{\infty} \frac{x_n}{2^n}\right] x + \left[\sum_{n=1}^{\infty} \frac{\overline{x_n}}{2^n}\right] y = dx + (1-d)y \in A.$$

2. Let (X, d) be a metric space.

   a) Show that
   $$\rho(x, y) = \min(1, d(x, y))$$
   is also a metric.

   b) Show that $U \subset X$ is open in (X, d) if and only if U is open in (X, ρ).

   c) Repeat the above for
   $$\sigma(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Solution:

a)
- Since $d(x, y)$ is a metric, $d(x, y) \geq 0$ for all x, y. Thus $\rho(x, y) \geq 0$ since it is either 1 or $d(x, y)$.
- Suppose $x = y$. Then $d(x, y) = 0$, and since $0 < 1$, $\rho(x, y) = d(x, y) = 0$. Suppose now that $\rho(x, y) = 0$. Then it must be that $d(x, y) = 0$ (since $\rho(x, y)$ is either 1 or $d(x, y)$), and hence $x = y$ because d is a metric.
- Let $x, y \in X$. Then
  $$\rho(x, y) = \min(1, d(x, y)) = \min(1, d(y, x)) = \rho(y, x)$$
  so ρ is symmetric.
- Let $x, y \in X$. Let $z \in X$. By definition of ρ,
  $$\rho(x, y) \leq 1$$
  If either $d(x, z)$ or $d(y, z)$ exceeds 1, then $\rho(x, z)$ or $\rho(y, z) = 1$. Thus
  $$\rho(x, z) + \rho(y, z) > 1 \geq \rho(x, y).$$
  Now assume both $d(x, z), d(y, z) \leq 1$. So $\rho(x, z) = d(x, z)$ and $\rho(y, z) = d(y, z)$. Then by the triangle inequality of d,
  $$\rho(x, y) \leq d(x, y) \leq d(x, z) + d(y, z) = \rho(x, z) + \rho(y, z).$$

Thus ρ is a metric.

b) First observe if $R \leq 1$, we have $B_R^d(x) = B_R^\rho(x)$. Let $y \in B_R^d(x)$. Then, $d(x,y) < R \leq 1$. Thus, $\rho(x,y) = d(x,y) < R$. Hence, $y \in B_R^\rho(x)$ and $B_R^d(x) \subset B_R^\rho(x)$. Now let $y \in B_R^\rho(x)$. Then $d(x,y) \leq \rho(x,y) < R$, and $y \in B_R^d(x)$. So $B_R^\rho(x) \subset B_R^d(x)$ and $B_R^\rho(x) = B_R^d(x)$.
Let U be open in (X,d). Fix an $x \in U$. Then there exists an $R > 0$ such that $B_R^d(x) \subset U$. We may take R sufficiently small, for example $R \leq 1$. The above then shows $B_R^\rho(x) = B_R^d(x) \subset U$, and U is open in (X,ρ). Essentially the same proof shows U open in (X,ρ) implies U is open in (X,d).

c) We first show σ is a metric on X.

- Since $d(x,y)$ is a metric, $d(x,y) \geq 0$ for all x, y. Thus, $0 \geq \sigma(x,y) \leq 1$.
- Suppose $x = y$. Then $d(x,y) = 0$ and $\sigma(x,y) = 0/1 = 0$. Suppose now that $\sigma(x,y) = 0$. Then

$$\sigma(x,y) = \frac{d(x,y)}{1+d(x,y)} = 0 \quad \Leftrightarrow \quad d(x,y) = 0(1+d(x,y)) = 0.$$

- Let $x, y \in X$. Then

$$\sigma(x,y) = \frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)} = \sigma(y,x),$$

  so σ is symmetric.

- Observe that the function $t/(1+t)$ is monotone increasing for nonnegative t. (Write $t/(1+t)$ as $1 - 1/(1+t)$, and note that $1/(1+t)$ is monotone decreasing). Thus, since $d(x,y) \leq d(x,z) + d(z,y)$,

$$\begin{aligned}
\sigma(x,y) &= \frac{d(x,y)}{1+d(x,y)} \leq \frac{[d(x,z)+d(z,y)]}{1+[d(x,z)+d(y,z)]} \\
&= \frac{d(x,z)}{1+d(x,z)+d(y,z)} + \frac{d(z,y)}{1+d(x,z)+d(y,z)} \\
&\leq \frac{d(x,z)}{1+d(x,z)} + \frac{d(z,y)}{1+d(z,y)} = \sigma(x,z) + \sigma(z,y)
\end{aligned}$$

Now we show that σ generates the same topology as d. Observe that, since $1 + d(x,y) \geq 1$, we have

$$\sigma(x,y) = \frac{d(x,y)}{1+d(x,y)} \leq d(x,y)$$

So that for any $R > 0$, $B_R^d(x) \subset B_R^\sigma(x)$. Now let $y \in B_R^\sigma(x)$. Then,

$$\frac{d(x,y)}{1+d(x,y)} < R \quad \Leftrightarrow \quad d(x,y) < R + Rd(x,y) \quad \Leftrightarrow \quad d(x,y) < \frac{R}{1-R}$$

so, if $R < 1$, we have $B_{R/(1-R)}^d(x) \subset B_R^\sigma(x)$. Since open balls in (X,σ) are open in (X,d) and vice verse, σ and d generate the same topology.

3. Let X be a NLS, x_0 be a fixed vector in X, and $\alpha \neq 0$ a fixed scalar. Show that the mappings $x \mapsto x + x_0$ and $x \mapsto \alpha x$ are homeomorphisms of X onto itself.

Solution: The topology on X is induced by the metric $d(x, y) = \|x - y\|$, where $\|\cdot\|$ is the norm on X.

i) $T(x) = x + x_0$ is a homeomorphism. Let $x, y \in X$ be such that $T(x) = T(y)$. Then $x + x_0 = y + y_0$, and $x = y$. So, T is injective.
Let $y \in X$. Then $x := y - x_0 \in X$ (since X is a vector space), and $T(x) = x + x_0 = y - x_0 + x_0 = y$. So T is surjective.
Let $U \subset X$ be open. It is easy to see that $T^{-1}(x) = x - x_0$. Let $x \in T^{-1}(U) = U - x_0$. Then $x + x_0 \in U$ and there exists an $r > 0$ such that $B_r(x + x_0) \subset U$. We show $B_r(x) \subset T^{-1}(U)$. Let $y \in B_r(x)$. Since $y \in B_r(x)$, we have $\|x - y\| < r$. Then, $\|x + x_0 - (y + x_0)\| < r$. It follows that $y + x_0 \in B_r(x + x_0) \subset U$. By definition of T^{-1}, we have $T^{-1}(y + x_0) = y + x_0 - x_0 = y$. So, $y \in T^{-1}(U)$. A similar argument (replacing x_0 with $-x_0$) shows that T^{-1} is continuous.

ii) $S(x) = \alpha x$ is a homeomorphism. Let $x, y \in X$ be such that $S(x) = S(y)$. Then $\alpha x = \alpha y$. Since $\alpha \neq 0$, we may divide by it and conclude $x = y$. So, S is injective.
Let $y \in X$. Since $\alpha \neq 0$ and X is a vector space, $x := y/\alpha \in X$. Thus $S(x) = \alpha(y/\alpha) = y$, and S is surjective.
Let $U \subset X$ be open. We see that $S^{-1}(x) = x/\alpha$. Then, $S^{-1}(U) = 1/\alpha U$. Let $x \in S^{-1}(U)$. Then $\alpha x \in U$, and there exists an $r > 0$ such that $B_r(\alpha x) \subset U$. We show $B_{r/\alpha}(x) \subset S^{-1}(U)$. Let $y \in B_{r/|\alpha|}(x)$. Then,

$$\|x - y\| < \frac{r}{|\alpha|} \quad \Leftrightarrow \quad |\alpha|\|x - y\| < r \quad \Leftrightarrow \quad \|\alpha x - \alpha y\| < r$$

and $\alpha y \in B_r(\alpha x) \subset U$. Thus, $\alpha y \in U$. It follows that $S^{-1}(\alpha y) = y \in S^{-1}(U)$.

4. Show that if X is a NLS, then X is homeomorphic to $B_r(0)$ for fixed r. [Hint: consider the mapping $x \mapsto xr/(1 + \|x\|)$].

Solution: Fix an $r > 0$. Define $T(x) = xr/(1 + \|x\|)$. Let $x, y \in X$ be such that $T(x) = T(y)$. Then,

$$\frac{xr}{1 + \|x\|} = \frac{yr}{1 + \|y\|} \quad \Leftrightarrow \quad x = \left(\frac{1 + \|x\|}{1 + \|y\|}\right) y.$$

Geometrically, this means that x, y are parallel ($x = \lambda y$ for some $\lambda \in \mathbb{F}$). Taking norms of both sides shows that

$$\|x\| = \left(\frac{1 + \|x\|}{1 + \|y\|}\right)(\|y\|) \quad \Leftrightarrow \quad \|x\| + \|x\|\|y\| = \|y\| + \|x\|\|y\| \quad \Leftrightarrow \quad \|x\| = \|y\|.$$

So, x, y are parallel and have the same norm. This implies that $x = \lambda y$ where $|\lambda| = 1$. Applying T shows that

$$\frac{yr}{1 + \|y\|} = T(y) = T(x) = T(\lambda y) = \frac{\lambda(yr)}{1 + |\lambda|\|y\|},$$

and so,

$$y(1+|\lambda|\|y\|) = y(\lambda + \lambda\|y\|).$$

Since $1+|\lambda|\|y\|$ is real and positive, it follows that λ is real and positive. Hence $\lambda = 1$. Thus $x = y$ and T is injective.
Let $y \in B_r(0)$. Then $\|y\| < r$ so that $r - \|y\| > 0$. Now, $x := \alpha y \in X$ where $\alpha = 1/(r-\|y\|) > 0$. It follows that

$$T(x) = T(\alpha y) = \frac{\alpha(yr)}{1+|\alpha|\|y\|} = \frac{\alpha(yr)}{1+\alpha\|y\|} = \frac{yr}{1/\alpha + \|y\|} = \frac{yr}{r-\|y\|+\|y\|} = y,$$

and T is surjective.
Since d is a metric, $d(x,0) = \|x\|$ is continuous. Thus $T(x) = r\mathrm{id}_X(x)/(1+\|x\|)$ is continuous, as the quotient of two continuous functions. Note that $1+\|x\| \neq 0$, so that $T(x)$ is continuous everywhere. Now, $T^{-1}(x) = \mathrm{id}_X(x)/(r-\|x\|)$, which is the quotient of two continuous functions and hence continuous. Since $x \in B_r(0)$ in this case, $\|x\| > r$ and $r - \|x\| \neq 0$. So T^{-1} is continuous on all of $B_r(0)$.

5. In $\mathbb{R}^d$ show that any two norms are equivalent. Hint: Consider the unit sphere, which is compact.

Solution: We need only check that $\|x\|$ is equivalent to

$$\|x\|_1 := \sum_{n=1}^{d} |x_n|,$$

i.e. the l^1 norm on $\mathbb{R}^d$. To see this, note that norm equivalence is transitive. That is, let $\|\cdot\|_a$ and $\|\cdot\|_b$ be equivalent to $\|\cdot\|_1$. Then there exist c_a, C_a and c_b, C_b (all positive) such that

$$c_a\|\cdot\|_1 \leq \|\cdot\|_a \leq C_a\|\cdot\|_1,$$
$$c_b\|\cdot\|_1 \leq \|\cdot\|_b \leq C_b\|\cdot\|_1.$$

Since $\|\cdot\|_b \leq C_b\|\cdot\|_1$,

$$\|\cdot\|_a \geq c_a\|\cdot\|_1 \geq c_a\left(\frac{\|\cdot\|_b}{C_b}\right) = \left(\frac{c_a}{C_b}\right)\|\cdot\|_b.$$

Similarly, since $c_b\|\cdot\|_1 \leq \|\cdot\|_b$,

$$\|\cdot\|_a \leq C_a\|\cdot\|_1 \leq C_a\left(\frac{\|\cdot\|_b}{c_b}\right) = \left(\frac{C_a}{c_b}\right)\|\cdot\|_b,$$

and thus $\|\cdot\|_a$ is equivalent to $\|\cdot\|_b$ (with $c = c_a/C_b > 0$ and $C = C_a/c_b > 0$). Let $\{e_n\}_{n=1}^d$ be a basis for $\mathbb{R}^d$. Then by the triangle inequality,

$$\|x\| = \|x_1e_1+...+x_de_d\| \leq \|x_1e_1\|+...+\|x_de_d\| = |x_1|\|e_1\|+...+|x_d|\|e_d\| = \sum_{n=1}^{d}|x_n|\|e_n\|.$$

Let $M = \max \|e_n\|$. Then,

$$\|x\| \leq \sum_{n=1}^{d} |x_n| \|e_n\| \leq \sum_{n=1}^{d} |x_n| M = M\|x\|_1.$$

This inequality shows that balls in $(X, \|\cdot\|)$ are open in $(X, \|\cdot\|_1)$. Then if U is open in $\mathbb{R}$, $\|\cdot\|^{-1}(U)$ is open in $(X, \|\cdot\|)$ (by continuity of the norm in its generated topology), and by the above, $\|\cdot\|^{-1}(U)$ is open in $(X, \|\cdot\|_1)$. So $\|\cdot\|$ is continuous on $(X, \|\cdot\|_1)$. Let $m = \inf_{x \in S_1^1} \|x\|$. Since $\|\cdot\|$ is a continuous function on the compact set S_1^1 (because it's continuous in $(X, \|\cdot\|_1)$), it attains its minimum. Thus there exists an $x_0 \in S_1^1$ such that $\|x_0\| = m$. This shows that $m > 0$ (since $0 \notin S_1^1$). Now let $x \in X$ be nonzero. Then $x/\|x\|_1 \in S_1^1$ and by definition of the infimum, $\|(x/\|x\|_1)\| \geq m$. Hence

$$\|(x/\|x\|_1)\| \geq m \quad \Leftrightarrow \quad \|x\| \geq m\|x\|_1$$

If $x = 0$, we trivially have $m\|x\|_1 \leq \|x\|$. Together, for all $x \in X$,

$$m\|x\|_1 \leq \|x\| \leq M\|x\|_1$$

so that $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent.

6. Let X and Y be NLS over the same field, both having the same finite dimension n. Then prove that X and Y are topologically isomorphic, where a topological isomorphism is defined to be a mapping that is simultaneously an isomorphism and a homeomorphism.

Solution: It suffices to show that $X, Y \simeq \mathbb{F}^n$, where $\simeq$ denotes topologically isomorphic. This follows since $\simeq$ is an equivalence relation.
Let $\{e_i\}_{i=1}^n$ be a basis for X and define $T : X \to \mathbb{F}^n$ by

$$T(x) = (x_1, ..., x_n) \quad \text{where} \quad x = \sum_{i=1}^{n} x_i e_i$$

is the unique representation of x in the basis $\{e_i\}$. Now define $\|\cdot\|_1 := \|T(\cdot)\|_{l^1}$. Then $\|\cdot\|_1$ is a norm on X. Since X is finite dimensional, all norms on it are equivalent (Prop 2.10 in notes). Then there exist $c, C > 0$ such that

$$c\|\cdot\|_1 \leq \|\cdot\| \leq C\|\cdot\|_1.$$

We first show that T is an isomorphism of vector spaces (that is, T is an invertible linear map). Let $x, y \in X$ and $\alpha \in \mathbb{F}$. Then

$$x = \sum_{i=1}^{n} x_i e_i, \quad y = \sum_{i=1}^{n} y_i e_i.$$

Now we have

$$\begin{aligned} T(x+\alpha y) &= T\left(\sum_{i=1}^{n} x_i e_i + \alpha \sum_{i=1}^{n} y_i e_i\right) = T\left(\sum_{i=1}^{n} (x_i + \alpha y_i) e_i\right) \\ &= (x_1 + \alpha y_1, ..., x_n + \alpha y_n) = (x_1, ..., x_n) + (\alpha y_1, ..., \alpha y_n) \\ &= (x_1, ..., x_n) + \alpha(y_1, ..., y_n) = T(x) + \alpha T(y) \end{aligned}$$

where the last few equalities follow from the vector space structure on $\mathbb{F}^n$. So, T is linear. To show T is invertible, note that for $x = (x_1, ..., x_n) \in \mathbb{F}^n$, the map $S(x) = \sum_{i=1}^{n} x_i e_i$ is such that $ST = \mathrm{id}_X$ and $TS = \mathrm{id}_{F^n}$. So, $S = T^{-1}$ and T is invertible. Otherwise, we show that T is bijective as follows:
Suppose that $x \in X$ is such that $T(x) = 0$. Then $x_i = 0$ for all $1 \leq i \leq n$. So, $x = \sum_{i=1}^{n} x_i e_i = 0$. Thus $\ker(T) = \{0\}$ and T is injective. By the rank-nulllity theorem, it follows that range(T) has dimension n, and hence must coincide with $\mathbb{F}^n$. Then, T is bijective.
So, T is an invertible linear map, and is hence an isomorphism. We must show now that T is a homeomorphism. It remains to prove that T and T^{-1} are continuous. But for $x \in X$,

$$\|T(x)\|_{l^1} = \|x\|_1 \leq \frac{1}{c}\|x\|$$

by definition of $\|\cdot\|_1$ and the equivalence of norms. Similarly for $x \in \mathbb{F}^n$,

$$\|T^{-1}(x)\| \leq C\|T^{-1}(x)\|_1 = C\|TT^{-1}(x)\|_{l^1} = C\|x\|_{l^1}$$

so T and T^{-1} are bounded linear maps, and hence are continuous.

7. Show that $(C[a,b], \|\cdot\|_\infty)$, the set of real-valued continuous functions in the interval $[a,b]$ with the sup-norm (L^∞-norm), is a Banach space.

Solution: First recall that the sup-norm is given by

$$\|f\|_\infty = \sup_{x \in [a,b]} |f(x)| < \infty$$

where finiteness comes from the fact that $|f|$ is a continuous function on a compact set. Let $\{f_n\}_{n=1}^{\infty} \subset C[a,b]$ be a Cauchy sequence. Let $x \in [a,b]$. Then,

$$|f_n(x) - f_m(x)| \leq \sup_{x \in [a,b]} |f_n - f_m| = \|f_n - f_m\|_\infty$$

and it follows that $\{f_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{F}$. Since $\mathbb{F}$ is complete, the sequence converges to some point – define f by

$$f(x) = \lim_{n \to \infty} f_n(x).$$

Let $\epsilon > 0$ and N such that for $m, n \geq N$,

$$\|f_n - f_m\|_\infty < \epsilon.$$

It follows that for all $x \in [a, b]$,

$$|f_n(x) - f_m(x)| < \epsilon.$$

Taking the limit as $m \to \infty$ in the above shows that

$$|f_n(x) - f(x)| \leq \epsilon,$$

and therefore

$$\|f_n - f\|_\infty = \sup_{x\in[a,b]} |f_n(x) - f(x)| \leq \sup_{x\in[a,b]} \epsilon = \epsilon$$

So $f_n \to f$ in norm. This means that $f_n \to f$ uniformly. Since f_n are continuous, and the uniform limit of continuous functions is continuous, f is continuous. Thus $f \in C[a, b]$, and $C[a, b]$ is complete.

8. If $f \in L^p(\Omega)$ show that

$$\|f\|_p = \sup \left| \int_\Omega fg \, dx \right| = \sup \int_\Omega |fg| \, dx$$

where the supremum is taken over all $g \in L^q(\Omega)$ such that $\|g\|_q \leq 1$ and $1/p+1/q = 1$, where $1 \leq p, q \leq \infty$.

Solution: First note that the above equalities are trivially true if $f = 0$ a.e. So, throughout, we will assume that $f \neq 0$. By Hölder's inequality,

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

where $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ with $1/p + 1/q = 1$ and $1 \leq p, q \leq \infty$. Assuming that $\|g\|_q \leq 1$,

$$\int_\Omega |fg| \, dx \leq \|f\|_p \quad \Rightarrow \quad \sup \int_\Omega |fg| \, dx \leq \|f\|_p.$$

By the trivial inequality

$$\left| \int f \, dx \right| \leq \int |f| \, dx$$

it is easily seen that

$$\sup \left| \int_\Omega fg \, dx \right| \leq \sup \int_\Omega |fg| \, dx \leq \|f\|_p.$$

To prove equality, we must appeal to several cases.

- $[p = 1, q = \infty]$. Let $g(x) = \operatorname{sgn} f(x)$, where $\operatorname{sgn}(x)$ is defined as

$$\operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

It is clear that $\|g\|_\infty = 1$ so that $g \in L^q(\Omega) = L^\infty(\Omega)$, and satisfies the condition $\|g\|_q \leq 1$. Moreover,

$$\int_\Omega fg \; dx = \int_\Omega |f| \; dx = \|f\|_1$$

so that the sup is attained.

- $[1 < p, q < \infty]$. Define $g(x)$ by

$$g(x) = \frac{|f(x)|^{p-1} \operatorname{sgn} f(x)}{\|f\|_p^{p-1}}$$

Observe that $q(p-1) = pq(1-1/p) = pq(1/q) = 1$. Then,

$$\|g\|_q^q = \frac{1}{\|f\|_p^p} \int_\Omega |f|^p |\operatorname{sgn}(f)|^q \; dx = \frac{1}{\|f\|_p^p} \int_\Omega |f|^p \; dx = \frac{\|f\|_p^p}{\|f\|_p^p} = 1$$

by noting trivial properties of the sgn function. Then $g \in L^q(\Omega)$ and $\|g\|_q \leq 1$. Finally,

$$\int_\Omega fg \; dx = \int_\Omega \frac{|f|^{p-1} f \operatorname{sgn} f}{\|f\|_p^{p-1}} \; dx = \frac{1}{\|f\|_p^{p-1}} \int_\Omega |f|^p = \frac{\|f\|_p^p}{\|f\|_p^{p-1}} = \|f\|_p,$$

and the sup is attained.

- $[p = \infty, q = 1]$. Let $\epsilon > 0$ and let E be a set of positive measure such that $|f(x)| > \|f\|_\infty - \epsilon > 0$. Such a set exists, since $\|f\|_\infty$ is the smallest number such that

$$\mu\{|f| > \|f\|_\infty\} = 0.$$

That E can be chosen to have finite measure follows from the σ-finiteness of μ. Now set $g = \chi_E \operatorname{sgn} f / \mu(E)$. Then,

$$\|g\|_1 = \frac{1}{\mu(E)} \int_\Omega \chi_E |\operatorname{sgn} f| \; dx = \frac{1}{\mu(E)} \int_\Omega \chi_E \; dx = \frac{\mu(E)}{\mu(E)} = 1$$

where the second to last equality follows since, if x is such that $f(x) = 0$, then $x \notin E$. Thus $g \in L^1(\Omega) = L^q(\Omega)$ and $\|g\|_q \leq 1$ as required. Then,

$$\begin{aligned} \left| \int_\Omega fg \; dx \right| &= \frac{1}{\mu(E)} \left| \int_\Omega \chi_E |f| \; dx \right| = \frac{1}{\mu(E)} \int_\Omega \chi_E |f| \; dx \\ &= \frac{1}{\mu(E)} \int_E |f| \; dx \geq \int_E (\|f\|_\infty - \epsilon) \; dx = \|f\|_\infty - \epsilon \end{aligned}$$

Taking $\epsilon \to 0$ shows that

$$\|f\|_\infty \leq \left| \int_\Omega fg \; dx \right|$$

So, in all the above cases, there exists a $g \in L^q(\Omega)$ with $\|g\|_q \leq 1$ such that

$$\|f\|_p \leq \left| \int_\Omega fg \, dx \right|.$$

9. Suppose that $\Omega \subset \mathbb{R}^d$ has finite measure and $1 \leq p \leq q \leq \infty$.

   a) Prove that if $f \in L^q(\Omega)$, then $f \in L^p(\Omega)$ and
   $$\|f\|_p \leq (\mu(\Omega))^{1/p-1/q} \|f\|_q.$$

   b) Prove that if $f \in L^\infty(\Omega)$, then
   $$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty.$$

   c) Prove that if $f \in L^p(\Omega)$ for all $1 \leq p < \infty$, and there is $K > 0$ such that $\|f\|_p \leq K$, then $f \in L^\infty(\Omega)$ and $\|f\|_\infty \leq K$.

Solution: Throughout, we assume $\mu(\Omega) \neq 0$. Otherwise, all equalities are trivial since every function f is a.e. equal to 0 on Ω.

a) First, we may assume that $p < q$, otherwise (in the case $p = q$), the conclusion and inequality are trivially true. Now, observe that for any $r \in [1, \infty]$, $\chi_\Omega \in L^r(\Omega)$. This is easy to see since, for r finite,

$$\|\chi_\Omega\|_r^r = \int_\Omega |\chi_\Omega|^r \, dx = \int_\Omega \chi_\Omega \, dx = \mu(\Omega) < \infty$$

For $r = \infty$, it is easy to see that $\|\chi_\Omega\|_\infty = 1$. Now,

$$\|f\|_p^p = \int_\Omega |f|^p \, dx = \int_\Omega |f|^p |\chi_\Omega| \, dx.$$

Observe that q/p and $1/(1 - p/q)$ are conjugate exponents since

$$\frac{1}{q/p} + \frac{1}{1/(1-p/q)} = \frac{p}{q} + 1 - \frac{p}{q} = 1$$

Furthermore, since $p < q$, $p/q < 1$ and $1 > 1 - p/q > 0$. It follows that $1/(1 - p/q) \in (1, \infty)$. Next, $f^p \in L^{q/p}(\Omega)$, since

$$\|f\|_{q/p} = \left(\int_\Omega |f^p|^{q/p} \, dx \right)^{p/q} = \left(\int_\Omega |f|^q \, dx \right)^{p/q} = \|f\|_q^p < \infty$$

Thus, $f^p \in L^{q/p}(\Omega)$ and $\chi_\Omega \in L^{1/(1-p/q)}(\Omega)$. By Hölder's inequality,

$$\int_\Omega |f|^p \chi_\Omega \, dx = \|f^p \chi_\Omega\|_1 \leq \|f\|_{q/p} \|\chi_\Omega\|_{1/(1-p/q)} = \mu(\Omega)^{1-p/q} \|f\|_q^p.$$

In total,

$$\|f\|_p^p \leq \mu(\Omega)^{1-p/q} \|f\|_q^p$$

from the conclusion follows, since $x \mapsto x^{1/p}$ is monotone increasing and hence preserves inequalities.

b) By a), since $f \in L^\infty(\Omega)$, $f \in L^p(\Omega)$ for all $1 \leq p \leq \Omega$, and

$$\|f\|_p \leq \mu(\Omega)^{1/p-1/\infty}\|f\|_\infty = \mu(\Omega)^{1/p}\|f\|_\infty.$$

Since $\mu(\Omega) \neq 0$, taking $p \to \infty$ gives

$$\limsup_{p\to\infty} \|f\|_p = \|f\|_\infty.$$

Let $E_t = \{x \in \Omega \mid |f(x)| > t\}$. Then,

$$\mu(E_t) = \int_{E_t} 1 \, dx \leq \int_{E_t} \frac{|f|^p}{t^p} \, dx \leq \frac{1}{t^p}\int_X |f|^p \, dx = \frac{\|f\|_p^p}{t^p}$$

since $|f(x)|/t > 1$ on E_t. This is known as Chebyshev's inequality. Now let $t \in (0, \|f\|_\infty)$. Then $\mu(E_t) > 0$ (otherwise, $t < \|f\|_\infty$ would be such that $|f| \leq t$ a.e., a contradiction of the definition of $\|f\|_\infty$). Then by Chebyshev's inequality,

$$\|f\|_p \geq t\mu(E_t)^{1/p}.$$

Since Ω has finite measure, all subsets do. It follows that $\mu(E_t)^{1/p} \to 0$ as $p \to \infty$. Hence,

$$\liminf_{p\to\infty} \|f\|_p \geq t,$$

and taking $t \to \|f\|_\infty$ gives

$$\liminf_{p\to\infty} \|f\|_p \geq \|f\|_\infty.$$

Thus,

$$\|f\|_\infty \leq \liminf_{p\to\infty} \|f\|_p \leq \lim_{p\to\infty} \|f\|_p \leq \limsup_{p\to\infty} \|f\|_p \leq \|f\|_\infty$$

and $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.

c) The above proof showed that

$$\liminf_{p\to\infty} \|f\|_p \geq \|f\|_\infty$$

regardless whether $f \in L^\infty(\Omega)$ or not. But, $\|f\|_p \leq K$, so that

$$\|f\|_\infty \leq \liminf_{p\to\infty} \|f\|_p \leq \liminf_{p\to\infty} K = K.$$

So, $f \in L^\infty(\Omega)$ and $\|f\|_\infty \leq K$.

10. Finite dimensional matrices.

a) Let $M^{n\times m}$ be the set of matrices with real valued coefficients a_{ij}, for $1 \leq i \leq n$ and $1 \leq j \leq m$. For every $A \in M^{n\times m}$, define

$$\|A\| = \max_{x\neq 0 \in \mathbb{R}^m} \frac{|Ax|_{\mathbb{R}_n}}{|x|_{\mathbb{R}^m}}.$$

Show that $(M^{n\times m}, \|\cdot\|)$ is a NLS.

b) Each $A \in M^{n\times n}$ defines a linear map of $\mathbb{R}^n$ into itself. Show that

$$\|A\| = \max_{|x|=|y|=1} y^T A x,$$

where y^T is the transpose of y.

c) Show that each $A \in M^{n\times n}$ is continuous.

d) Prove that the convex hull is convex, and that it is the intersection of all convex subsets of X containing A.

e) If X is a normed linear space, prove that the convex hull of an open set is open.

f) If X is a normed linear space, is the convex hull of a closed set always closed?

g) Prove that if X is a normed linear space, then the convex hull of a bounded set is bounded.

Solution:

a)

- Since each of $|\cdot|_{\mathbb{R}^n}, |\cdot|_{\mathbb{R}^m}$ are nonnegative, we have that the quotient is always nonnegative. Therefore $\|A\| \geq 0$.
- Suppose $A = 0$, the zero matrix. Then by simple matrix multiplication, it is seen for any $x \in \mathbb{R}^m$ that $Ax = 0$. Thus $|Ax|_{\mathbb{R}^n} = 0$ for all x, and $\|A\| = 0$. Suppose now that $\|A\| = 0$. Then for all $x \neq 0 \in \mathbb{R}^m$,

$$\frac{|Ax|_{\mathbb{R}^n}}{|x|_{\mathbb{R}^m}} \leq 0 \quad \Leftrightarrow \quad |Ax|_{\mathbb{R}^n} \leq 0|x|_{\mathbb{R}^m} = 0$$

  On the other hand, as just discussed, $|Ax|_{\mathbb{R}^n}$ is always nonnegative. Thus, $Ax = 0$ for all $x \in \mathbb{R}^m$, which by definition means that $A = 0$.
- Let $\alpha \in \mathbb{R}$. Then

$$\|\alpha A\| = \max_{x\neq 0\in\mathbb{R}^m} \frac{|\alpha Ax|_{\mathbb{R}_n}}{|x|_{\mathbb{R}^m}} = \max_{x\neq 0\in\mathbb{R}^m} |\alpha| \frac{|Ax|_{\mathbb{R}_n}}{|x|_{\mathbb{R}^m}} = |\alpha| \max_{x\neq 0\in\mathbb{R}^m} \frac{|Ax|_{\mathbb{R}_n}}{|x|_{\mathbb{R}^m}} = |\alpha|\|A\|.$$

- Appealing to the triangle inequality of $|\cdot|_{\mathbb{R}^n}$,

$$\begin{aligned} \|A+B\| &= \max_{x\neq 0\in\mathbb{R}^m} \frac{|Ax+Bx|_{\mathbb{R}_n}}{|x|_{\mathbb{R}^m}} \leq \max_{x\neq 0\in\mathbb{R}^m} \frac{|Ax|_{\mathbb{R}_n}}{|x|_{\mathbb{R}^m}} + \frac{|Bx|_{\mathbb{R}_n}}{|x|_{\mathbb{R}^m}} \\ &\leq \max_{x\neq 0\in\mathbb{R}^m} \frac{|Ax|_{\mathbb{R}_n}}{|x|_{\mathbb{R}^m}} + \max_{x\neq 0\in\mathbb{R}^m} \frac{|Bx|_{\mathbb{R}_n}}{|x|_{\mathbb{R}^m}} = \|A\| + \|B\| \end{aligned}$$

  where the second inequality comes from the fact that, for any U, V, $\max(U+V) \leq \max(U) + \max(V)$.

b) As a remark, recall that $\langle y, x\rangle = y^T I_n x$, where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$ and I_n is the $n \times n$ identity matrix. Therefore, $y^T Ax = y^T I_n AX = \langle y, Ax\rangle$. Now, by Cauchy-Schwarz, and assuming x, y have unit norm,

$$\langle y, Ax\rangle \leq |y||Ax| = |Ax|$$

Taking the maximum of both sides shows that

$$\max_{|x|=|y|=1} y^T Ax \leq \max_{|x|=1} |Ax| = \|A\|.$$

Now for the reverse inequality,

$$\max_{|x|=|y|=1} y^T Ax \geq \max_{|x|=1, Ax\neq 0} \left\langle \frac{Ax}{|Ax|}, Ax \right\rangle = \max_{|x|=1, Ax\neq 0} \frac{|Ax|^2}{|Ax|} = \max_{|x|=1, Ax\neq 0} |Ax|$$

where the inequality comes from taking the maximum over smaller sets. Now, if $Ax = 0$ then $|Ax| = 0$, and hence does not contribute to the max (unless $A = 0$, in which case there is nothing to prove). Thus,

$$\max_{|x|=|y|=1} y^T Ax \geq \max_{|x|=1, Ax\neq 0} |Ax| = \max_{|x|=1} |Ax| = \|A\|.$$

c) Note that $M^{n\times n}$ is a finite dimensional vector space, so all norms on it are equivalent. Define

$$\|A\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}|.$$

It can easily be checked that $\|\cdot\|_1$ is a norm on $M^{n\times n}$. Now, clearly, $\|A\|_1 < \infty$ for all matrices A. So A is bounded, and hence continuous. Since all norms generate the same topology on a finite dimensional vector space, we see that A is continuous in $\|\cdot\|$.

d) Note that $M^{n\times n} \simeq \mathbb{R}^{n^2}$, so continuing forward we work with subsets of $\mathbb{R}^d$. Define the convex hull of a set $\Omega \subset \mathbb{R}^d$ by

$$\mathrm{co}(\Omega) = \left\{ \sum_{i=1}^{n} \lambda_i x_i \mid n \in \mathbb{N}, x_i \in \Omega, \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1 \right\}$$

This is clearly a generalization of the condition $tx + (1-t)y \in \Omega$ whenever $t \in [0,1]$ and $x, y \in \Omega$ (take $n = 2$, $x_1 = x, x_2 = y$, $\lambda_1 = t, \lambda_2 = 1 - t$). We can then prove the following proposition:

Proposition: A set $C \subset \mathbb{R}^d$ is convex iff $\sum_{i=1}^n \lambda_i x_i \in C$ for all $n \in \mathbb{N}$, $x_1, ..., x_n \in C$, and $\lambda_1, ..., \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$.

Proof: ($\Leftarrow$) Take the case $n = 2$, then the discussion above shows that C is convex.

($\Rightarrow$) Let C be convex. Let $\{x_i\}, \{\lambda_i\}$ satisfy the required properties. We show that $\sum_{i=1}^n \lambda_i x_i \in C$. We prove this inductively. Suppose it holds for $n-1$; we show it holds for n. Note that we may take $n \geq 2$, since the case $n = 1$ is trivial. Further, we may take $\lambda_i > 0$ (since, otherwise, we may omit the point x_i from the sum and decrease n). Define $\lambda = 1 - \lambda_n = \sum_{i=1}^{n-1} \lambda_i$. By the assumption $\lambda_i > 0$, we see that $\lambda \neq 0$. Define $\mu_i = \lambda_i/\lambda$ for $1 \leq i \leq n-1$. Then $\mu_i \geq 0$ and

$$\sum_{i=1}^{n-1} \mu_i = \frac{1}{\lambda} \sum_{i=1}^{n-1} \lambda_i = \frac{\lambda}{\lambda} = 1$$

By the inductive hypothesis, $y := \sum_{i=1}^{n-1} \mu_i x_i \in C$. By convexity of C,

$$\lambda y + (1-\lambda) x_n \in C$$

But,

$$\lambda y + (1-\lambda) x_n = \frac{\lambda}{\lambda} \sum_{i=1}^{n-1} \lambda_i x_i + \lambda_n x_n = \sum_{i=1}^{n} \lambda_i x_i.$$

This completes the proof of the proposition. Note that, rephrased, this says that C is convex iff $\text{co}(C) = C$.

Now, we show that $\text{co}(\Omega)$ is convex. Let $p = \sum_{i=1}^n \lambda_i p_i$ and $q = \sum_{i=1}^n \mu_i p_i$ be in $\text{co}(\Omega)$. Note that we can take p_i in both by adding terms with coefficients of zero to each sum. Now, for $\lambda \in [0,1]$, we have

$$\lambda p + (1-\lambda) q = \sum_{i=1}^{n} (\lambda \lambda_i + (1-\lambda)\mu_i) p_i$$

Observe that $\lambda\lambda_i + (1-\lambda)\mu_i \geq 0$ for all $1 \leq i \leq n$ since $\lambda, \lambda_i, \mu_i \geq 0$. Next,

$$\sum_{i=1}^{n} (\lambda \lambda_i + (1-\lambda)\mu_i) = \lambda \sum_{i=1}^{n} \lambda_i + (1-\lambda) \sum_{i=1}^{n} \mu_i = \lambda + (1-\lambda) = 1$$

since $p, q \in \text{co}(\Omega)$. It follows that $\lambda p + (1-\lambda) q \in \text{co}(\Omega)$, and $\text{co}(\Omega)$ is convex.

We now show that $\text{co}(\Omega) = \cap_{\alpha \in A} C_\alpha$ where $\{C_\alpha\}_{\alpha \in A}$ is the set of convex sets C_α containing Ω. Clearly, $\Omega \subset \text{co}(\Omega)$ (by taking $n = 1$ and $\lambda = 1$), and $\text{co}(\Omega)$ is convex. Hence, $\text{co}(\Omega) \in \{C_\alpha\}$, and $\cap_{\alpha \in A} C_\alpha \subset \text{co}(\Omega)$. Let $C \in \{C_\alpha\}$. Then C is convex, and by the proposition $C = \text{co}(C)$. But, by definition of C, $\Omega \subset C$, and hence $\text{co}(\Omega) \subset \text{co}(C) = C$. Since C was arbitrary, $\text{co}(\Omega) \subset C$ for all $C \in \{C_\alpha\}$, and hence $\text{co}(\Omega) \subset \cap_{\alpha \in A} C_\alpha$.

So, $\text{co}(\Omega)$ is convex, and is equal to the intersection of all convex sets containing Ω.

e) Let $U \subset X$ be open. Let $x \in \text{co}(U)$. Then,

$$x = \sum_{i=1}^{n} \lambda_i x_i, \quad x_i \in U,\ \lambda_i \geq 0,\ \sum_{i=1}^{n} \lambda_i = 1$$

By the sum condition of $\{\lambda_i\}$, there exists a $\lambda_k \neq 0$. WLOG, we may choose this to be λ_1. Now define $f : X \to X$ by

$$f(z) = \frac{1}{\lambda_i}\left(z - \sum_{i=2}^{n} \lambda_i x_i\right)$$

It is clear that $f(z)$ is continuous (it's a polynomial – actually, it is a homothety fixing $(x - \lambda_1 x_1)/(1 - \lambda_1)$). Thus, $f^{-1}(U)$ is open. But, $f^{-1}(U) = \lambda_1 U + \sum_{i=2}^{n} \lambda_i x_i$. In particular, since $x_1 \in U$, we have that $\lambda_1 x_1 + \sum_{i=2}^{n} \lambda_i x_i = x \in f^{-1}(U)$. Furthermore, it is clear that $f^{-1}(U) \subset \text{co}(U)$. Thus $\text{co}(U)$ is open.

f) No. For example, take $\Omega = \{(x, y) \mid y \geq 1/(1 + x^2)\}$. Then Ω is closed, but $\text{co}(\Omega) = \mathbb{R}^2_{y>0}$, which is open.

g) A set B is bounded if there exists $B_R(0)$ containing B. Since X is a normed linear space, $B_R(0)$ is convex. Hence,

$$\text{co}(B) \subset \text{co}(B_R(0)) = B_R(0)$$

and $\text{co}(B)$ is bounded.

11. Prove that if X is a normed linear space and $B = B_1(0)$ is the unit ball, then X is infinite dimensional if and only if B contains an infinite collection of non-overlapping balls of diameter $1/2$.

Solution: We need the following lemma due to Riesz:

Lemma [Riesz' Lemma]: Let M be a closed, proper subspace of X and $1 > r > 0$. Then there exists an $x \in X$ such that $\|x\| = 1$ and $\inf_{m \in M} \|x - m\| \geq r$.

Proof: Let $x_1 \notin M$ and set $R = \inf_{m \in M} \|x_1 - m\|$. Then $R > 0$ (if not, there exists a sequence $m_n \in M$ such that $\|x_1 - m_n\| \to 0$, which means that $m_n \to x_1$ in norm. But M is closed so that $x_1 \in M$, a contradiction). Let $\epsilon > 0$. Then there exists an $m_1 \in M$ such that $\|x_1 - m_1\| < R + \epsilon$. Set $x = (x_1 - m_1)/\|x_1 - m_1\|$, which clearly has unit norm. Then,

$$\inf_{m \in M} \|x - m\| = \inf_{m \in M} \left\| m + \frac{m_1}{\|x_1 - m_1\|} - \frac{x_1}{\|x_1 - m_1\|} \right\| = \frac{1}{\|x_1 - m_1\|} \inf_{m \in M} \|m\|x_1 - m_1\| + m_1 - x_1\|$$

But M is a proper subspace so that $m\|x_1 - m_1\| + m_1 \in M$. Thus,

$$\inf_{m \in M} \|x - m\| = \frac{1}{\|x_1 - m_1\|} \inf_{m \in M} \|m - x_1\| > \frac{R}{R + \epsilon}$$

For a prescribed $r < 1$, there exists an $\epsilon > 0$ such that $R/(R+\epsilon) = r$, completing the proof.

Next, observe that if X is infinite dimensional, then for any $x_1, ..., x_n \in X$, the set $M = \text{Span}(x_1, ..., x_n)$ is a closed, proper subspace of X. Clearly M is finite dimensional, and hence (by Exercise 6) is topologically isomorphic to $\mathbb{F}^n$. But $\mathbb{F}^n$ is closed, and so M is closed too. Further, since M is finite dimensional, it is a proper subspace of X.

So, let $x_1 \in X$ have unit norm. Then $M_1 = \text{Span}(x_1)$ is a closed, proper subspace of X. Hence, for $r = 2/3$, there exists by Riesz' Lemma an $x_2 \notin M_1$ such that $d(x_2, M_1) \geq 2/3$ and $\|x_2\| = 1$. Let $M_2 = \text{Span}(x_1, x_2)$. By the same logic, there exists an $x_3 \notin M_2$ such that $d(x_3, M_2) \geq 2/3$ and $\|x_3\| = 1$. Continuing inductively to obtain $x_1, x_2, ...$ all with unit norm such that for any m, n, $d(x_m, x_n) \geq 1/2$. Now let B^i denote the ball centered at $3/4x_i$ with radius $1/4$. Then $B^i \cap B^j$ are disjoint for $i \neq j$, since if $x \in B^i \cap B^j$,

$$\left\|\frac{3}{4}x_i - \frac{3}{4}x_j\right\| \leq \left\|\frac{3}{4}x_i - x + x - \frac{3}{4}x_j\right\| \leq \left\|\frac{3}{4}x_i - x\right\| + \left\|x - \frac{3}{4}x_j\right\| < \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

But, by construction,

$$\|x_i - x_j\| \geq 2/3 \quad \Leftrightarrow \quad \left\|\frac{3}{4}x_i - \frac{3}{4}x_j\right\| \geq \frac{1}{2}$$

Next, $B^i \subset B$ for all i. To see this, let $x \in B^i$. Then,

$$\|x\| \leq \left\|x - \frac{3}{4}x_i + \frac{3}{4}x_i\right\| \leq \left\|x - \frac{3}{4}x_i\right\| + \left\|\frac{3}{4}x_i\right\| \leq \frac{1}{4} + \frac{3}{4} = 1$$

by using the fact that x_i was chosen so that $\|x_i\| = 1$. Hence, there exists an infinite collection of non-overlapping balls with diameter $1/2$ (equivalently, radius $1/4$).

For the other direction, suppose X is finite dimensional. Let $\{B^i\}$ be a non-overlapping collection of balls with diameter $1/2$. We show that such a collection cannot exist. Since X is finite dimensional, the closed unit ball is compact. Consider the sequence $\{x_i\}$ where each x_i is the center of B^i. Then there exists a convergent subsequence, which we will still denote by $\{x_i\}$. Then for $\epsilon = 1/4$, there exists an N such that for $m, n \geq N$, $\|x_m - x_n\| < 1/4$. It follows that $x_m \in B^n$, so that $x_m \in B^n \cap B^m$ (similarly for x_n). Thus the balls are overlapping.

12. Prove that a subset A of a metric space (X, d) is bounded if and only if every countable subset of A is bounded.

Solution: Clearly if A is bounded, then every subset of A (countable or not) is bounded. Hence we need only prove the reverse direction. Suppose that every countable subset of A is bounded. To achieve a contradiction, suppose that A itself is not

bounded. Then, for every $R > 0$, $A \subsetneq B_R(0)$. In particular, this holds for $R = n \in \mathbb{N}$. Now let $x_n \in A \setminus B_n(0)$. Then $\tilde{A} = \{x_n\}$ is a countable subset of A, and hence is bounded. On the other hand, for any $R > 0$, there exists an $N > R$, and thus $x_N \notin B_R(0)$ (by definition of x_N). So, $\tilde{A}$ is not bounded, a contradiction.

13. Consider $(l^p, \|\cdot\|_p)$.

a) Prove that l^p is a Banach space for $1 \leq p \leq \infty$. Hint: Use that $\mathbb{R}$ is complete.

b) Show that $\|\cdot\|_p$ is not a norm for $0 < p < 1$. Hint: First show the result on $\mathbb{R}^2$.

Solution:

a) Throughout, we will use the fact that $x \mapsto x^p$ and $x \mapsto x^{1/p}$ are monotone nondecreasing for all p. Hence, they preserve order.
First suppose that $p \in [1, \infty)$. Then $\|\cdot\|_p$ is defined by

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}},$$

where $x = (x_1, x_2, ...)$. Let $\{x^k\}_{k=1}^{\infty}$ be a Cauchy sequence in l^p. Then, for $\epsilon > 0$, there exists a $K > 0$ such that for $m, n \geq K$,

$$\sum_{i=1}^{\infty} |x_i^m - x_i^n|^p = \|x^m - x^n\|_p^p < \epsilon^p.$$

Since all the $|x_i^m - x_i^k|$ are nonnegative, this implies that, for fixed i,

$$|x_i^m - x_i^n|^p < \epsilon^p,$$

and thus, for fixed i, there exists a $K > 0$ such that for all $m, n \geq K$,

$$|x_i^m - x_i^n| < \epsilon.$$

This means that for fixed i, $\{x_i^k\}$ is Cauchy. But, this is a sequence in $\mathbb{F}$, which is complete. So, it converges to some x_i^∞. Define x^∞ by

$$x^\infty = (x_1^\infty, x_2^\infty, ...).$$

We must show that $x^k \to x^\infty$ in norm, and that $x^\infty \in l^p$. Since $\{x^k\}$ is Cauchy, for any $\epsilon > 0$, there exists K such that for $m, n \geq K$, and for any I,

$$\left(\sum_{i=1}^{I} |x_i^m - x_i^n|^p \right)^{\frac{1}{p}} \leq \|x^m - x^n\|_p < \epsilon.$$

Letting $m \to \infty$ gives

$$\left(\sum_{i=1}^{I} |x_i^\infty - x_i^n|^p \right)^{\frac{1}{p}} < \epsilon.$$

Finally, letting $I \to \infty$ gives

$$\|x^\infty - x^n\| < \epsilon,$$

and $x^k \to x$ in norm.
Again since $\{x^k\}$ is Cauchy, it is bounded (say, by $C > 0$). Then for any I,

$$\left(\sum_{i=1}^{I} |x_i^k|^p\right)^{\frac{1}{p}} \leq \|x^k\|_p \leq C.$$

Taking the limit as $k \to \infty$ gives

$$\left(\sum_{i=1}^{I} |x_i^\infty|^p\right)^{\frac{1}{p}} \leq \|x^k\|_p \leq C,$$

and then taking the limit as $I \to \infty$ gives

$$\|x^\infty\|_p \leq C$$

so that $x^\infty \in l^p$.

For $p = \infty$, let $\{x^k\}$ be Cauchy in l^∞. Let $\epsilon > 0$. Then there exists a $K > 0$ such that for $m, n \geq K$,

$$\sup_i |x_i^n - x_i^m| < \epsilon$$

which implies, for any fixed i, that

$$|x_i^n - x_i^m| \leq \sup_i |x_i^n - x_i^m| < \epsilon$$

and hence $\{x_i^k\}$ is Cauchy in $\mathbb{F}$. Since $\mathbb{F}$ is complete, this sequence converges, and define x^∞ as above. Since $\{x^k\}$ is Cauchy, it is bounded, and therefore

$$\sup_i |x_i^k| \leq C$$

Then for fixed i,

$$|x_i^k| \leq C,$$

and taking the limit as $k \to \infty$ gives

$$|x_i^\infty| \leq C.$$

Hence, taking the sup over i of both sides,

$$\|x^\infty\|_\infty = \sup_i |x_i^\infty| \leq C$$

so that $x^\infty \in l^\infty$. Now, as before, for $m, n \geq K$ we have

$$\|x^m - x^n\|_\infty = \sup_i |x_i^m - x_i^n| < \epsilon.$$

Thus, for fixed i,

$$|x_i^m - x_i^n| < \epsilon$$

Taking the limit as $m \to \infty$ and then suping over i gives

$$\|x^\infty - x^n\|_\infty < \epsilon$$

so that $x^k \to x^\infty$ in norm.

b) We show that the triangle inequality fails. Consider $(1,0)$ and $(0,1)$, which both have norm 1 for any p. But,

$$\|(1,0)+(0,1)\|_p = (1^p + 1^p)^{\frac{1}{p}} = 2^{\frac{1}{p}}$$

Since $0 < p < 1$, we have that $1/p > 1$, and thus

$$\|(1,0)+(0,1)\|_p = 2^{\frac{1}{p}} \geq 2 = 1+1 = \|(1,0)\|_p + \|(0,1)\|_p$$

The same result holds in l^p for $(1,0,0,0...)$ and $(0,1,0,0,...)$.

14. If an infinite dimensional vector space X is also a NLS and contains a sequence $\{e_n\}_{n=1}^\infty$ with the property that for every $x \in X$ there is a unique sequence of scalars $\{\alpha_n\}_{n=1}^\infty$ such that

$$\|x - (\alpha_1 e_1 + ... + \alpha_n e_n)\| \to 0 \quad \text{as} \quad n \to \infty,$$

then $\{e_n\}_{n=1}^\infty$ is called a *Schauder basis* for X, and we have the *expansion* of x

$$x = \sum_{n=1}^{\infty} \alpha_n e_n.$$

a) Find a Schauder basis for $l^p, 1 \leq p < \infty$.

b) Show that if a NLS has a Schauder basis, then it is separable. [Remark: The converse is not true.]

Solution:

a) Define e^i by $(e^i)_n = \delta_{in}$. We easily get an expansion for $x \in X$ by

$$x = (x_1, x_2, ...) = \sum_{n=1}^{\infty} x_n e_n.$$

Then,

$$\left\| x - \sum_{n=1}^{N} x_n e_n \right\| = \left\| \sum_{N-1}^{\infty} x_n e_n \right\| \to 0$$

as $N \to \infty$, since the sum becomes empty.

b) Note that a Schauder basis $\{e_n\}$ is countable, so its span is countable. Then, it suffices to show that Span$\{e_n\}$ dense in X. Let $x \in X$. Then by definition of a Schauder basis, there exists an expansion of x,

$$x = \sum_{n=1}^{\infty} \alpha_n e_n.$$

Let x_N be defined by

$$x_N = \sum_{n=1}^{N} \alpha_n e_n,$$

which by definition of a Schauder basis satisfies

$$\|x - x_N\| \to 0 \quad \text{as } N \to \infty.$$

Therefore, $\{x_N\}$ is a sequence in Span$\{e_n\}$ such that $x_N \to x$ in norm. Hence, Span$\{e_n\}$ is dense in X.

15. Let Y be a subspace of a vector space X. The *coset* of an element $x \in X$ with respect to Y is denoted by $x + Y$ and is defined to be the set

$$x + Y = \{z \in X \mid z = x + y \text{ for some } y \in Y\}.$$

Show that the distinct cosets form a partition of X. Show that under algebraic operations defined by

$$(x_1 + Y) + (x_2 + Y) = (x_1 + x_2) + Y \quad \text{and} \quad \lambda(x + Y) = \lambda x + Y,$$

for any $x_1, x_2, x \in X$ and λ in the field, these cosets form a vector space. This space is called the *quotient space* of X by (or *modulo* Y, and it is denoted X/Y).

Solution: The cosets form a partition since $\simeq$ defined by $x+Y \simeq z+Y$ iff $x-z \in Y$ is an equivalence relation. Alternatively, since Y is a subspace, $0 \in Y$. Thus, $x \in x+Y$, and each coset is nonempty.
The above also easily implies that

$$\bigcup_{x \in X} x + Y = X,$$

since each coset is a subset of X, and each point $x \in X$ is in the coset $x + Y$.
Suppose now that $x_1 + Y$ and $x_2 + Y$ are distinct cosets. We show that they are disjoint. Suppose there exists an $x \in (x_1 + Y) \cap (x_2 + Y)$. Then $x = x_1 + y_1$ and $x = x_2 + y_2$ for some $y_1, y_2 \in Y$. Now let $z \in x_1 + Y$. Then $z = x_1 + y$ for some $y \in Y$. By the above, $x_1 = x_2 + y_2 - y_1$ so that

$$z = x_1 + y = (x_2 + y_2 - y_1) + y = x_2 + (y_2 - y_1 + y).$$

Since Y is a subspace, $y_2 - y_1 + y \in Y$, and $z \in x_2 + Y$. The reverse inclusion is similar, and hence $x_1 + Y = x_2 + Y$, a contradiction.

Thus, the cosets form a partition.

We now show that the set of cosets forms a vector space. To this end, we must show eight properties.

i) (Associativity of addition). Let $x_1, x_2, x_3 \in X$. Then,

$$\begin{aligned}[(x_1+Y)+(x_2+Y)]+(x_3+Y) &= [(x_1+x_2)+Y]+(x_3+Y) = ((x_1+x_2)+x_3)+Y \\ &= (x_1+(x_2+x_3))+Y = (x_1+Y)+[(x_2+x_3)+Y] \\ &= (x_1+Y)+[(x_2+Y)+(x_3+Y)].\end{aligned}$$

ii) (Commutativity of addition). Let $x_1, x_2 \in X$. Then,

$$(x_1+Y)+(x_2+Y) = (x_1+x_2)+Y = (x_2+x_1)+Y = (x_2+Y)+(x_1+Y).$$

iii) (Identity under addition) Let 0 denote the 0 element of X. Then for any $x \in X$,

$$(x+Y)+(0+Y) = (x+0)+Y = x+Y.$$

iv) (Inverse under addition). Let $x \in X$. Since X is a vector space, $-x \in X$. Then,

$$(x+Y)+(-x+Y) = (x-x)+Y = 0+Y.$$

v) (Compatibility of scalar multiplication with field multiplication). Let $a, b \in \mathbb{F}$. Then, for any $x \in X$,

$$a(b(x+Y)) = a(bx+Y) = abx+Y = (ab)(x+Y).$$

vi) (Identity under multiplication). Let 1 be the multiplicative identity of $\mathbb{F}$. Then for any $x \in X$,

$$1(x+Y) = (1x)+Y = x+Y$$

vii) (First distributivity law). Let $a \in \mathbb{F}$ and $x_1, x_2 \in X$. Then,

$$\begin{aligned}a((x_1+Y)+(x_2+Y)) &= a((x_1+x_2)+Y) = (a(x_1+x_2)+Y) \\ &= (ax_1+ax_2)+Y = (ax_1+Y)+(ax_2+Y) \\ &= a(x_1+Y)+a(x_2+Y).\end{aligned}$$

viii) (Second distributivity law). Let $a, b \in \mathbb{F}$ and $x \in X$. Then,

$$\begin{aligned}(a+b)(x+Y) &= (a+b)x+Y = (ax+bx)+Y = (ax+Y)+(bx+Y) \\ &= a(x+Y)+b(x+Y).\end{aligned}$$

Thus, the set of cosets of X (under Y) is a vector space.

16. Let Y be a closed subspace of a NLS X. Show that a norm on the quotient space X/Y is given for $\hat{x} \in X/Y$ by

$$\|\hat{x}\|_{X/Y} = \inf_{x \in \hat{x}} \|x\|_X.$$

Solution:

- Evidently for any $x \in \hat{x}$, since $\|\cdot\|_X$ is a norm, $\inf_{x\in\hat{x}} \geq 0$.
- Suppose $\hat{x}$ is the zero coset, $\hat{x} = 0 + Y$. Since Y is a subspace, $0 \in Y$, and therefore $0 \in \hat{x}$. But $\|0\|_X = 0$, so that the inf is zero. Thus, $\|\hat{x}\|_{X/Y} = 0$. Now suppose that $\|\hat{x}\|_{X/Y} = 0$. Then there exists a sequence $\{x_n\} \subset \hat{x}$ such that $\|x_n\|_X \to 0$. Since $x_n \in \hat{x}$ for all n, we may write them as $x_n = x + y_n$ (where $\hat{x} = x + Y$) for some $y_n \in Y$. Since Y is closed,

  $$x_n - x = y_n \to y \in Y.$$

  Now, the x_n converge in X to 0 so that, on the other hand,

  $$y_n = x_n - x \to 0 - x = -x.$$

  Since limits are unique, $-x \in Y$. It follows that $x \in Y$, and so $x + Y = 0 + Y$.
- Observe that the definition given is equivalent to the following. Let $\hat{x} = x + Y$. Then,

  $$\|\hat{x}\|_{X/Y} = \inf_{y \in Y} \|x - y\|_X = d(x, Y).$$

  This follows since every $z \in \hat{x}$ is of the form $x - y$ for some $y \in Y$, so

  $$\inf_{z\in\hat{x}} \|z\|_X = \inf_{y\in Y} \|x - y\|_X.$$

  Now let $\alpha \neq 0 \in \mathbb{F}$. Then,

  $$\|\alpha\hat{x}\|_{X/Y} = \inf_{y\in Y} \|\alpha x - y\|_X = \inf_{y\in Y} \|\alpha x - \alpha y\|_X = |\alpha| \inf_{y\in Y} \|x - y\|_X = |\alpha|\|\hat{x}\|_{X/Y},$$

  since Y is a subspace. If $\alpha = 0$, then $\alpha\hat{x} = (\alpha x) + Y = 0 + Y$ is the zero coset, and

  $$\|\alpha\hat{x}\|_{X/Y} = 0 = |\alpha|\|\hat{x}\|_{X/Y}.$$
- The triangle inequality follows from the triangle inequality of $\|\cdot\|_X$ and of inf. Let $\hat{x_1} = x_1 + Y$ and $\hat{x_2} = x_2 + Y$. Then,

  $$\begin{aligned} \|\hat{x_1} + \hat{x_2}\|_{X/Y} &= \inf_{y\in Y} \|x_1 + x_2 - y\|_X = \inf_{y_1,y_2\in Y} \|x_1 + x_2 - y_1 - y_2\|_X \\ &\leq \inf_{y_1,y_2\in Y} \|x_1 - y_1\|_X + \|x_2 - y_2\|_X \\ &\leq \inf_{y_1\in Y} \|x_1 - y_1\|_X + \inf_{y_2\in Y} \|x_2 - y_2\|_X = \|\hat{x_1}\|_{X/Y} + \|\hat{x_2}\|_{X/Y}. \end{aligned}$$

17. If X and Y are NLS, then the product space $X \times Y$ is also a NLS with any of the norms

$$\|(x,y)\|_{X\times Y} = \max(\|x\|_X, \|y\|_Y)$$

or, for any $1 \le p < \infty$,

$$\|(x,y)\|_{X\times Y} = (\|x\|_X^p + \|y\|_Y^p)^{\frac{1}{p}}.$$

Why are these norms equivalent?

Solution: Denote the max norm by $\|\cdot\|$ and the p-norms by $\|\cdot\|_p$. We will show three inequalities. Fix a p. We have that, for any $(x,y) \in X \times Y$,

$$(\|x\|_X + \|y\|_Y)^p \le (2\max\{\|x\|_X, \|y\|_Y)^p \le 2^p(\|(x,y)\|^p).$$

Next, $\|(x,y)\|^p$ is one of $\|x\|_X^p$ or $\|y\|_Y^p$. Thus, we may add the other to it. Since both are nonnegative, this either increases or does not change the value. Hence using the previous inequality,

$$\|(x,y)\|_1^p \le 2^p(\|(x,y)\|^p) \le 2^p(\|x\|_X^p + \|y\|_Y^p) = 2^p\|(x,y)\|_p^p.$$

It follows that

$$\|(x,y)\|_1 \le 2\|(x,y)\|_p.$$

By employing the same trick,

$$\|(x,y)\| \le \|x\|_X + \|y\|_Y = \|(x,y)\|_1.$$

Finally,

$$\|(x,y)\|_p = (\|x\|_X^p + \|y\|_Y^p)^{\frac{1}{p}} \le (2\|(x,y)\|^p)^{\frac{1}{p}} = 2^{\frac{1}{p}}\|(x,y)\|.$$

Together, we conclude that

$$\frac{1}{2^{1+1/p}}\|(x,y)\|_1 \le \frac{1}{2^{1/p}}\|(x,y)\|_p \le \|(x,y)\| \le \|(x,y)\|_1$$

This shows that $\|\cdot\|_1$ is equivalent to $\|\cdot\|_p$ for any p and $\|\cdot\|$. Thus they are all equivalent (by transitivity).

Another way to see this is that they all define norms on $\mathbb{R}^2$, which is finite dimensional, and so all norms are equivalent.

18. If X and Y are Banach spaces, prove that $X \times Y$ is a Banach space.

Solution: Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. Equip $X \times Y$ with the norm $\|(x,y)\|_1 = \|x\|_X + \|y\|_Y$. Let $\{(x_n, y_n)\}$ be Cauchy in $(X \times Y, \|\cdot\|_1)$. Then for any $\epsilon > 0$ there exists N such that for $m, n \ge N$,

$$\|x_n - x_m\|_X + \|y_n - y_m\|_Y = \|(x_n - x_m, y_n - y_m)\|_1 = \|(x_n, y_n) - (x_m, y_m)\|_1 < \epsilon.$$

Then, the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy in $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, respectively. Thus, they converge, say to x, y. In the above inequality, take $m \to \infty$. Then,

$$\|x_n - x\|_X + \|y_n - y\|_Y = \|(x_n, y_n) - (x, y)\|_1 < \epsilon$$

so that $(x_n, y_n) \to (x, y)$ in norm. Thus, $(X \times Y, \|\cdot\|_1)$ is complete.

19. Let $T : C[0,1] \to C[0,1]$ be defined by

$$(Tx)(t) = \int_0^t x(\tau)\ d\tau.$$

Find the range $R(T)$ of T, and show that T is invertible on its range, $T^{-1} : R(T) \to C[0,1]$. Is T^{-1} linear and bounded?

Solution: The range of T is the set of all functions f on $[0,1]$ such that $f(0) = 0$, f' exists, and f' is continuous. That is, $R(T) = C^1[0,1]$. To see this, note that $(Tx)(t)$ gives a C^1 function since

$$\frac{d}{dt}(Tx)(t) = \left(\frac{d}{dt}\ t\right) x(t) - \left(\frac{d}{dt}\ 0\right) x(0) = x(t)$$

and by definition, $x(t)$ is continuous on $[0,1]$. Moreover,

$$(Tx)(0) = \int_0^0 x(\tau)\ d\tau = 0.$$

So that Tx is a C^1 function on $[0,1]$ such that $(Tx)(0) = 0$. Now, if $f \in C^1[0,1]$ with $f(0) = 0$, then f' is continuous on $[0,1]$ and

$$(Tf')(t) = \int_0^t f'(\tau)\ d\tau = f(t) - f(0) = f(t).$$

So, T surjects onto the set of $C^1[0,1]$ functions which fix 0. The inverse has already been found – it is $(T^{-1}x)(t) = d/dtx(t)$. Since the differential operator is linear, T^{-1} is linear. However, T^{-1} is not bounded. For example, define f_n by

$$f_n(x) = \sin(nx)$$

Then,

$$f_n'(x) = n\cos(nx).$$

It follows that

$$\|f_n\|_{C[0,1]} = 1, \quad \|T^{-1}f_n\|_{C[0,1]} = \|f_n'\|_{C[0,1]} = n.$$

Therefore, there cannot be a $C > 0$ such that

$$\|T^{-1}f\|_{C[0,1]} \leq C\|f\|_{C[0,1]}$$

for all $f \in C^1[0,1]$ with $f(0) = 0$.

20. Show that on $C[a,b]$, for any $y \in C[a,b]$ and scalars α and β, the functionals

$$f_1(x) = \int_a^b x(t)y(t)\ dt \quad \text{and} \quad f_2(x) = \alpha x(a) + \beta x(b)$$

are linear and bounded.

Solution:

- We show that f_1 is linear and bounded. Let $x_1, x_2 \in C[a,b]$ and $\lambda \in \mathbb{F}$. Then,

$$\begin{aligned} f_1(x_1 + \lambda x_2) &= \int_a^b (x_1(t) + \lambda x_2(t))y(t)\ dt = \int_a^b x_1(t)y(t) + \lambda x_2(t)y(t)\ dt \\ &= \int_a^b x_1(t)y(t)\ dt + \int_a^b \lambda x_2(t)y(t)\ dt = \int_a^b x_1(t)y(t)\ dt + \lambda \int_a^b x_2(t)y(t)\ dt \\ &= f_1(x_1) + \lambda f_1(x_2) \end{aligned}$$

  following linearity of the integral. To show that f_1 is bounded,

$$|f_1(x)| \leq \left| \int_a^b x(t)y(t)\ dt \right| \leq \int_a^b |x(t)||y(t)|\ dt \leq \int_a^b \|x\|\|y\|\ dt = [(b-a)\|y\|]\|x\|$$

  where $\|\cdot\| = \|\cdot\|_{C[a,b]}$ is the sup norm. It follows that $\|f_1\|_{C[a,b]^*} \leq (b-a)\|y\|$, and f_1 is bounded.

- We show that f_2 is linear and bounded. Let $x_1, x_2 \in C[a,b]$ and $\lambda \in \mathbb{F}$. Then,

$$\begin{aligned} f_2(x_1 + \lambda x_2) &= \alpha(x_1(a) + \lambda x_2(a)) + \beta(x_1(b) + \lambda x_2(b)) \\ &= \alpha x_1(a) + \lambda\alpha x_2(a) + \beta x_1(b) + \lambda\beta x_2(b) \\ &= [\alpha x_1(a) + \beta x_1(b)] + \lambda[\alpha x_2(a) + \beta x_2(b)] \\ &= f_2(x_1) + \lambda f_2(x_2), \end{aligned}$$

  and f_2 is linear. Now to show that f_2 is bounded,

$$|f_2(x)| \leq |\alpha x(a) + \beta x(b)| \leq |\alpha||x(a)| + |\beta||x(b)| \leq (|\alpha| + |\beta|)\|x\|.$$

  So, $\|f_2\|_{C[a,b]^*} \leq |\alpha| + |\beta|$, and f_2 is bounded.

21. Find the norm of the linear functional f defined on $C[-1,1]$ by

$$f(x) = \int_{-1}^0 x(t)\ dt - \int_0^1 x(t)\ dt.$$

Solution: By an easy inequality,

$$\begin{aligned} |f(x)| &\leq \left| \int_{-1}^0 x(t)\ dt - \int_0^1 x(t)\ dt \right| \leq \left| \int_{-1}^0 x(t)\ dt \right| + \left| \int_0^1 x(t)\ dt \right| \\ &\leq \int_{-1}^0 |x(t)|\ dt + \int_0^1 |x(t)|\ dt = \int_{-1}^1 |x(t)|\ dt \\ &\leq (1 - (-1))\|x\| = 2\|x\|_{C[-1,1]} \end{aligned}$$

Thus, $\|f\|_{C[-1,1]^*} \leq 2$. To show $\|f\|_{C[-1,1]^*} = 2$, we approximate by functions. Note that the simple function $x(t) = \chi_{[-1,0]} - \chi_{[0,1]}$ is such that

$$\int_{-1}^{0} x(t)\ dt - \int_{0}^{1} x(t)\ dt = 2 = 2\|x\|_{C[-1,1]}.$$

However, $x(t)$ is not continuous. Instead, define $x_n(t)$ on $[-1,1]$ by

$$x_n(t) = \begin{cases} 1 & -1 \leq x \leq -1/n \\ -nx & |x| \leq 1/n \\ -1 & 1/n \leq x \leq 1 \end{cases}$$

Then,

$$f(x_n) = 2\left(1 - \frac{1}{n}\right) + 2\left(\frac{1}{2n}\right) = 2 - \frac{1}{n}$$

(geometrically, we compute the area of two triangles with the same base length $1/n$ and height 1, and the area of two rectangles with the same base length $(1 - 1/n)$ and height 1). It follows that

$$|f(x_n)| = \left(2 - \frac{1}{n}\right)\|f_n\|_{C[-1,1]}$$

Thus, $\|f\|_{C[-1,1]^*} \geq 2 - 1/n$ for all n, and it follows that $\|f\|_{C[-1,1]^*} = 2$. (If it were less than 2, we could find an N such that $2 - 1/N$ exceeds the norm, and apply the above to x_N).

23. The space $C^1[a,b]$ is the NLS of all continuously differentiable functions defined on $[a,b]$ with the norm

$$\|x\| = \sup_{t\in[a,b]} |x(t)| + \sup_{t\in[a,b]} |x'(t)|.$$

a) Show that $\|\cdot\|$ is indeed a norm.

b) Show that $f(x) = x'((a+b)/2)$ defines a continuous linear functional on $C^1[a,b]$.

c) Show that f defined above is not bounded on the subspace of $C[a,b]$ consisting of all continuously differentiable functions with the norm inherited from $C[a,b]$.

Solution: Note that differentiation is a linear mapping. Most properties follow from this fact.

a) We show that $\|\cdot\|$ is a norm.

- First, since $|x(t)|, |x'(t)| \geq 0$ for all $t \in [a,b]$, we have that the supremae are nonnegative and hence $\|\cdot\| \geq 0$. Furthermore, since x, x' are continuous on a compact set, they are bounded, and $\|\cdot\| < \infty$.

- Suppose that $x(t) = 0$, the constant zero function on $[a, b]$. Then $x'(t) = 0$, and
$$\|x\| = \sup_{t\in[a,b]} 0 + \sup_{t\in[a,b]} 0 = 0.$$
Suppose now that $\|x\| = 0$. Since both supremae are nonnegative, for their sum to be zero, both must be zero. In particular, for any $t \in [a, b]$,
$$0 \geq |x(t)| \leq \sup_{t\in[a,b]} |x(t)| = 0,$$
and thus $x(t) = 0$ on $[a, b]$. So, x is the constant zero function. Note that we could have not reached the same conclusion if $\|x\|$ was defined to be $\sup_{t\in[a,b]} |x'(t)|$, since all constant functions have a derivative of 0.
- Let $\alpha \in \mathbb{R}$. Then for any $x \in C^1[a, b]$,
$$\begin{aligned}\|\alpha x\| &= \sup_{t\in[a,b]} |(\alpha x)(t)| + \sup_{t\in[a,b]} |(\alpha x)'(t)| = \sup_{t\in[a,b]} |\alpha x(t)| + \sup_{t\in[a,b]} |\alpha x'(t)| \\ &= |\alpha| \sup_{t\in[a,b]} |x(t)| + |\alpha| \sup_{t\in[a,b]} |x'(t)| = |\alpha|\|x\|.\end{aligned}$$
- Let $x, y \in C^1[a, b]$. Then,
$$\begin{aligned}\|x + y\| &= \sup_{t\in[a,b]} |(x + y)(t)| + \sup_{t\in[a,b]} |(x + y)'(t)| = \sup_{t\in[a,b]} |x(t) + y(t)| + \sup_{t\in[a,b]} |x(t) + y'(t)| \\ &\leq \sup_{t\in[a,b]} [|x(t)| + |y(t)|] + \sup_{t\in[a,b]} [|x(t)| + |y(t)|] \\ &\leq \sup_{t\in[a,b]} |x(t)| + \sup_{t\in[a,b]} |y(t)| + \sup_{t\in[a,b]} |x'(t)| + \sup_{t\in[a,b]} |y'(t)| = \|x\| + \|y\|\end{aligned}$$
which follows from the triangle inequality of $|\cdot|$ and of sets, $\sup(A+B) \leq \sup(A) + \sup(B)$.

Thus, $\|\cdot\|$ is a norm.

b) We first show f is a linear functional. Clearly $f : C^1[a, b] \to \mathbb{R}$ so that it is a functional. To show it is linear, let $x, y \in C^1[a, b]$ and $\alpha \in \mathbb{R}$. Then,
$$\begin{aligned}f(x + \alpha y) &= (x + \alpha y)'\left(\frac{a+b}{2}\right) = (x' + \alpha y')\left(\frac{a+b}{2}\right) = x'\left(\frac{a+b}{2}\right) + (\alpha y')\left(\frac{a+b}{2}\right) \\ &= x'\left(\frac{a+b}{2}\right) + \alpha y'\left(\frac{a+b}{2}\right) = f(x) + \alpha f(y).\end{aligned}$$
So, f is linear. To show that f is continuous, we show it is bounded. Let $x \in C^1[a, b]$. Then,
$$|f(x)| = \left|x'\left(\frac{a+b}{2}\right)\right| \leq \sup_{t\in[a,b]} |x'(t)| \leq \sup_{t\in[a,b]} |x(t)| + \sup_{t\in[a,b]} |x'(t)| = \|x\|$$
where the first inequality follows by definition of the supremum, and the second inequality follows since we simply add something nonnegative. Thus, $\|f\|_{C^1[a,b]^*} \leq 1$, and f is bounded, hence continuous.

c) Consider $x_n(t) = \sin(n(t-(a+b)/2))$. Then, $x_n'(t) = n\cos(n(t-(a+b)/2))$ and

$$x_n'((a+b)/2) = n\cos\left(n\left(\frac{a+b}{2} - \frac{a+b}{2}\right)\right) = n\cos(0) = n$$

On the other hand, $\|x_n\|_{C[a,b]} = 1$ for all n. So, f cannot possibly be bounded.

24. Suppose X is a vector space. The algebraic dual of X is the set of all linear functionals on X, and is also a vector space. Suppose also that X is a NLS. Show that X has finite dimension if and only if the algebraic dual and the dual space X^* coincide.

Solution: Suppose that X has finite dimension. We need to show that the algebraic dual and X^* coincide. Clearly, X^* (the set of all bounded linear functionals) is a subset of the algebraic dual. So, we need to show that every linear functional on X is bounded. To this end, let T be a linear functional and define a norm on X by

$$\|x\|_T := |T(x)|.$$

This is easily seen to be a norm by noting that T is linear. Since all norms are equivalent on a finite dimensional space, there exists a $C > 0$ such that $\|x\|_T \leq C\|x\|$. Hence,

$$|T(x)| = \|x\|_T \leq C\|x\|,$$

and T is bounded. Now suppose X is infinite dimensional. We construct a discontinuous (not bounded) linear functional. First note that every vector space has a Hamel basis. This is defined to be a set B such that B is linearly independent, and every vector $x \in X$ can be written as a finite linear combination of vectors in B. By Zorn's lemma, every linearly independent set E can be extended to a Hamel basis. So, take $\{e_n\}_{n=1}^{\infty}$ a linearly independent set (such a set can be chosen inductively). We may further take $\|e_n\| = 1$. Then extend this to a Hamel basis B. Define $T : X \to \mathbb{R}$ by

$$T(e_n) = n\|e_n\| = n,$$

and zero for all other vectors in B. Since any vector x can be represented as a linear combination of vectors, we can extend T to a linear functional on X by

$$\tilde{T}(x) = \tilde{T}\left(\sum_{n=1}^{N} x_n b_n\right) = \sum_{n=1}^{N} x_n T(b_n)$$

where $b_n \in B$. Some care must be taken to show this is linear (in particular, we must write x, y as sums over the same basis vectors, adding terms of zero when necessary). It is clear that

$$\tilde{T}(e_n) = T(e_n) = n$$

so that $\tilde{T}$ is unbounded. So, the algebraic dual and X^* do not coincide when X is infinite dimensional.

25. Let X be a NLS and M a nonempty subset. The annihilator M^a of M is

defined to be the set of all bounded linear functionals $f \in X^*$ such that f restricted to M is zero. Show that M^a is a closed subspace of X^*. What are X^a and $\{0\}^a$?

Solution: Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in M^a such that $f_n \to f \in X^*$. We show that $f \in M^a$. For any fixed $x \in M$, we have

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} 0 = 0,$$

and therefore f vanishes on M. Another way to see M^a is closed is as follows. Define for $x \in X$ the element of $\hat{x}$ of X^{**} by

$$\hat{x}(\tau) = \tau(x),$$

where $\tau \in X^*$. Then τ vanishes on M if $\tau \in \ker \hat{x}$ (where $x \in M$), and $M^a \subset \ker \hat{x}$. Thus,

$$M^a = \bigcap_{x\in M} \ker \hat{x}$$

Each kernel is closed since $\ker(\hat{x}) = \hat{x}^{-1}(0)$, and each $\hat{x}$ is continuous (they are all in the double dual, hence bounded). Thus M^a is closed.

The annihilator of X is the set of bounded linear functionals which are zero on X. Thus, $X^a = \{0\}$, where 0 is the zero function. On the other hand, $\{0\}^a$ is the set of functionals which are zero at zero. Hence, $\{0\}^a$ is the set of bounded linear functionals such that $f(0) = 0$. But, all linear functionals fix zero (by linearity), and so $\{0\}^a = X^*$.

26. Define the operator T by the formula

$$(Tf)(x) = \int_a^b K(x,y)f(y)\ dy.$$

Suppose that $K \in L^q([a,b] \times [a,b])$, where q lies in the range $1 \le q \le \infty$. Determine the values of p for which T is necessarily a bounded linear operator from $L^p(a,b)$ to $L^q(a,b)$. In particular, if a and b are both finite, show that $K \in L^\infty([a,b] \times [a,b])$ implies T to be bounded on all the L^p-spaces.

Solution: An important result, proved previously, is if $f \in L^p$, then $f \in L^r$ for all $1 \le r \le p$. Consider the case $T : L^q(a,b) \to L^q(a,b)$.
If $q = \infty$, then

$$\|(Tf)(x)\|_q = \sup_{x\in[a,b]} \left| \int_a^b K(x,y)f(y)\ dy \right| \le \sup_{x\in[a,b]} \int_a^b |K(x,y)||f(y)|\ dy$$

NEED TO FINISH.

27. Let $U = B_r(0) = \{x \in X \mid \|x\| < r\}$ be an open ball about 0 in a real normed

linear space, and let $y \notin \mathrm{cl}(U)$. Show that there is a bounded linear functional f that separates U from y. (That is, U and y lie in opposite half spaces determined by f, which is to say there is an α such that U lies in $\{x \mid f(x) < \alpha\}$ and $f(y) > \alpha$.)

Solution: Since $y \notin \mathrm{cl}(U)$, $y \in \mathrm{cl}(U)^c$. Hence, y lies in some open set, and there exists an $R > 0$ such that $B_R(y) \subset \mathrm{cl}(U)^c$. This exactly means that $\mathrm{cl}(U) \cap B_R(y) = \emptyset$. Hence, $U \cap B_R(y) = \emptyset$. By the separating hyperplane theorem, there exists an $f \in X^*$ and $\gamma \in \mathbb{R}$ such that $f(x) < \gamma$ for all $x \in U$, and $f(x) > \gamma$ for all $x \in B_R(y)$. In particular, this implies that $U \subset \{f(x) < \gamma\}$ and $f(y) > \gamma$.

28. Prove that $L^2[0,1]$ is of the first category in $L^1[0,1]$. (Recall that a set is of first category if it is a countable union of nowhere dense sets, and that a set is nowhere dense if its closure has an empty interior.) Hint: Show that $A_k = \{f \mid \|f\|_2 \leq k\}$ is closed in L^1 but has empty interior.

Solution: Defining A_k as in the hint, let $\{f_n\}_{n=1}^{\infty}$ be a sequence in A_k such that $f_n \to f \in L^1$. By Fatou's Lemma:

$$\int_0^1 |f|^2 \, dx = \int_0^1 \liminf_{n\to\infty} |f_n|^2 \, dx \leq \liminf_{n\to\infty} \int_0^1 |f_n|^2 \, dx \leq k.$$

Hence, $f \in A_k$. where $f_n \to f$ is a subsequence that converges a.e. to f. Let $f \in A_k$ and $g \in L^1[0,1] \setminus L^2[0,1]$. Then $f + 1/ng \to f$ in the L^1 norm, since

$$\left\| f + \frac{1}{n}g - f \right\|_1 = \frac{1}{n}\|g\|_1 \to 0$$

where $\|g\|_1$ is some fixed, finite number. Note, however, that $f + 1/ng \notin A_k$ for any n. To see this, observe that

$$\left\| f + \frac{1}{n}g \right\|_2 = \|f\|_2 + \|g\|_2 + \frac{2}{n}\int_0^1 |fg| \, dx$$

but $\|g\|_2 = \infty$ since $g \notin L^2[0,1]$, and every other term is nonnegative. Hence, $f + 1/ng \notin A_k$, and since every (L^1) neighborhood of f will contain $f + 1/ng$, it must be that the neighborhood is not entirely contained in A_k. Thus A_k has empty interior. Clearly, $L^2[0,1]$ is the union of all A_k.

Alternatively, one may note that $L^2[0,1] \subset L^1[0,1]$. Then, if any A_k has an interior point f in L^1, we may (via Exercises 3 and 4) find a homeomorphism to $B_r(f) \subset A_k$. Then, by taking the inverse of this homeomorphism, one may show that the inclusion operator $j : L^2[0,1] \to L^1[0,1]$ is surjective. However, this is a contradiction.

29. If a Banach space X is reflexive, show that X^* is also reflexive. ($*$) Is the converse true? Give a proof or a counterexample.

Solution: We prove the following lemma first.

Lemma: If $Y \subset X$ is closed, and X is reflexive, than Y is reflexive.

Proof: Let $\varphi \in Y^{**}$ and define $\tilde{\varphi} \in X^{**}$ by $\tilde{\varphi}(f) = \varphi(f|_Y)$. Since X is reflexive, $\tilde{\varphi} = E_x$ for some $x \in X$. We show that $x \in Y$, and that φ is an evaluation map at x. Suppose not; then by the Mazur separation lemma, there exists an $f : Y \to \mathbb{F}$ such that $f(x) \neq 0$ and $f|_Y = 0$. But,

$$f(x) = \tilde{\varphi}(f) = \varphi(f|_Y) = \varphi(0) = 0$$

a contradiction. So $x \in Y$. By Hahn-Banach, every bounded linear functional $f : Y \to \mathbb{F}$ can be written as $\tilde{f} : X \to \mathbb{F}$, where $\tilde{f}|_Y = f$. Then,

$$\varphi(f) = \tilde{\varphi}(\tilde{f}) = \tilde{f}(x) = f(x)$$

since $x \in Y$. Thus, φ is an evaluation map

We now prove the following

Prop: X is reflexive iff X^* is reflexive.

Solution: Suppose first that X^* is reflexive. Let $F : X \to X^{**}$ denote the canonical inclusion. Consider $X^{***} = (X^{**})^*$, and let $F^* : X^* \to X^{***}$ be the embedding of X^* into its double dual. Let $\eta \in X^{***}$, and define $f_\eta \in X^*$ by

$$f_\eta(x) = \eta(F(x)).$$

Since X is reflexive, for each $\varphi \in X^{**}$, there exists an $x \in X$ such that $\varphi = F(x)$. Then,

$$(F^*(f_\eta))(\varphi) = \varphi(f_\eta) = f_\eta(x).$$

By definition of f_η,

$$f_\eta(x) = \eta(F(x)) = \eta(\phi)$$

so that $F^*(f_\eta) = \eta$. It follows that F^* is surjective.
Suppose now that X^* is reflexive. Then X^{***} is reflexive. By the above, X^{**} is reflexive. But, $X \simeq F(X) \subset_{cl} X^{**}$. Applying the lemma, we see that X is reflexive.

30. Let $y = (y_1, y_2, y_3, ...) \in \mathbb{C}^\infty$ be a vector of complex numbers such that $\sum_{i=1}^\infty y_i x_i$ converges for every $x = (x_1, x_2, x_3, ...) \in \mathbb{C}_0$, where $\mathbb{C}_0 = \{x\mathbb{C}^\infty \mid x_i \to 0 \text{ as } i \to \infty\}$. Prove that

$$\sum_{i=1}^{\infty} |y_i| < \infty.$$

Solution: NEED TO FINISH

31. Let X and Y be normed linear spaces, $T \in B(X, Y)$, and $\{x_n\}_{n=1}^\infty$. If $x_n \rightharpoonup x$,

prove that $Tx_n \xrightarrow{w^*} Tx$ in Y. Thus a bounded linear operator is weakly sequentially continuous. Is a weakly sequentially continuous linear operator necessarily bounded?

Solution: That $x_n \xrightarrow{w} x$ means, for any $f \in X^*$, $f(x_n) \to f(x)$. So, let $g \in Y^*$. Consider the operator $T_g := g \circ T : X \to \mathbb{F}$. Then, $T_g \in X^*$ since it is the composition of bounded linear maps. Thus, applying weak convergence to T_g, we see that

$$g(Tx_n) = g \circ T(x_n) = T_g(x_n) \to T_g(x) = g \circ T(x) = g(Tx).$$

In total, for any $g \in Y^*$, we have that $g(Tx_n) \to g(Tx)$. Hence, $Tx_n \xrightarrow{w} Tx$ in Y.

Suppose, for any weakly convergent sequence x_n, we have that Tx_n is weakly convergent. Then, by Proposition 2.91 (as a consequence of PUB), it follows that $\{\|Tx_n\|\}_{n=1}^{\infty}$ is bounded. Suppose that T is unbounded. Then there is a sequence $x_n \to 0$ in norm (hence, weakly to 0, since strong implies weak convergence) such that $\|Tx_n\| \to \infty$. Then, Tx_n does not converge weakly (otherwise, it would be bounded), a contradiction.

32. Suppose that X is a Banach space, M and N are linear subspaces, and that $X = M \oplus N$, which means that

$$X = M + N = \{m + n \mid m \in M, n \in N\}$$

and $M \cap N$ is the trivial linear subspace consisting only of the zero element. Let P denote the projection of X onto M. That is, if $x = m + n$, then

$$P(x) = m.$$

Show that P is well defined and linear. Prove that P is bounded if and only if both M and N are closed.

Solution: Let $m_1 + n_1$ and $m_2 + n_2$ are representations of x. We show that the representation is actually unique. We have that $m_1 + n_1 = m_2 + n_2$. After rearranging, we get $m_1 - m_2 = n_2 - n_1$. Since M, N are linear subspaces, the LHS is an element of M, while the RHS is an element of N. The only element shared in both is the zero element, hence $m_1 - m_2 = 0 = n_2 - n_1$, and $m_1 = m_2$, $n_1 = n_2$. Hence the representation is unique, and P is well defined. Let $x = m_x + n_x$ and $y = m_y + n_y$ be two elements of X and $\lambda \in \mathbb{F}$. Then,

$$P(x + \lambda y) = P(m_x + \lambda m_y + n_x + \lambda n_y) = m_x + \lambda m_y = P(x) + \lambda P(y).$$

Suppose P is bounded (hence continuous). There are several ways to show that both M and N are closed. First, observe that $M = \text{range}(P)$ and $N = \ker(P)$. By definition, $\text{range}(T) \subset M$; on the other hand, every $m \in M$ can be written as $m + 0 \in X$, so that $P(m) = m$. Every $n \in N$ can be written as $0 + n \in X$, so that $P(n) = P(0 + n) = 0$. Hence $N \subset \ker(P)$. On the other hand, if $x \in \ker(P)$, then $P(x) = 0$. This exactly means that if $x = m + n$, then $m = 0$. Hence

$x = 0 + n = n \in X$.
So, if P is continuous, the kernel is clearly closed, for $\ker(P) = P^{-1}\{0\}$. To show that M is closed, note that $Q = I_X - P$ is a projection onto N. Then $\ker(Q) = M = \text{range}(P)$ and $\text{range}(Q) = N = \ker(P)$. Since P is continuous, Q is continuous, and therefore its kernel M is closed.

Suppose now that both M and N are closed. Since X is a Banach space, and M is closed in X, M is a Banach space. By the Closed Graph Theorem, it suffices to prove that T is closed. So, let $x_k \to x$ and $Px_k \to y$. We must show that $y = Px$. Note that $x_k - Px_k \in N$ for all k (since Px_k strips off the "M" part of x_k). Since N is closed, $x_k - Px_k \to x - y \in N$. Hence, $P(x - y) = 0$. On the other hand, $P(x-y) = Px - Py$ so that $Px = Py$. Now, M is closed so that $Px_k \to y \in M$, and therefore $Py = y$. It follows that $y = Px$.

33. Let X be a Banach space, Y a NLS, and $T_n \in B(X, Y)$ such that $\{T_n x\}_{n=1}^{\infty}$ is a Cauchy sequence in Y. Show that $\{\|T_n\|\}_{n=1}^{\infty}$ is bounded. If, in addition, Y is a Banach space, show that if we define T by $T_n x \to Tx$, then $T \in B(X, Y)$.

Solution: Since each sequence $\{T_n x\}$ is Cauchy in Y, they are bounded. Hence, none of them are unbounded. By PUB, it follows that $\{\|T_n\|\}$ is uniformly bounded; say, by M. Then,

$$\|Tx\| = \lim_{n\to\infty} \|T_n x\| \leq \lim_{n\to\infty} \|T_n\|\|x\| \leq \lim_{n\to\infty} M\|x\| = M\|x\|$$

So, T is bounded.

34. Let X be the normed space of sequences of complex numbers $x = \{x_i\}_{i=1}^{\infty}$ with only finitely many nonzero terms and norm defined by $\|x\| = \sup_i |x_i|$. Let $T : X \to X$ be defined by

$$y = Tx = \left\{ x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, ... \right\}.$$

Show that T is a bounded linear map, but that T^{-1} is unbounded. Why does this not contradict the Open Mapping Theorem?

Solution: If $x = 0$, then clearly $\|Tx\| \leq M\|x\|$ for any M. Suppose $x \neq 0$. Since only finitely many terms are nonzero, it follows that $\|x\| = \max_i |x_i|$, and is actually attained by one of the x_i; say, x_n. Then, $|x_i| \leq |x_n|$ for all i. It follows that $1/i|x_i| \leq |x_n|$, so that $\|Tx\| \leq \|x\|$. Hence T has norm at most one. This is actually the best we can do, for example by taking the sequence $x = (1, 0, 0...)$, for which $Tx = x$.

The inverse map T^{-1} is defined by

$$T^{-1}x = \{x_1, 2x_2, 3x_3, ...\}.$$

Consider the sequence $(x^k)_i = \delta_i^k$. Then, $\|T^{-1}x^k\| = k$ whereas $\|x^k\| = 1$ for all k.

This does not contradict the Open Mapping Theorem since X is not a Banach space. E.g., take $x_n = (1, 1/2, 1/3, ..., 1/n, 0, 0, ...)$. Then $\{x_n\}$ is a Cauchy sequence in X, but does not converge in X; the limit is $x = (1, 1/2, 1/3, ...)$, all of whose entries are nonzero.

35. Give an example of a function that is closed but *not* continuous.

Solution: Consider $f(x) = 1/(x^2 + 1)$. Then $f(\mathbb{R}) = (0, 1]$, which is neither open nor closed.

36. For each $\alpha \in \mathbb{R}$, let E_α be the set of all continuous functions f on $[-1, 1]$ such that $f(0) = \alpha$. Show that the E_α are convex, and that each is dense in $L^2([-1, 1])$.

Solution: Let $f, g \in E_\alpha$. Let $t \in [0, 1]$. Then,

$$(tf + (1-t)g)(0) = tf(0) + (1-t)g(0) = t\alpha + (1-t)\alpha = \alpha,$$

so that $tf + (1-t)g \in E_\alpha$. Thus, the E_α are convex.

To prove density, we use the fact that a subset of a Hilbert space is dense iff its orthogonal complement is $\{0\}$. The inner product on L^2 is precisely

$$\langle f, g\rangle = \int_{-1}^{1} fg \; dx.$$

The orthogonal complement of E_α is defined by

$$E_\alpha^\perp = \left\{ f \in L^2[-1, 1] \mid \langle f, g\rangle = 0 \;\text{ for all } g \in E_\alpha \right\}.$$

Note that the polynomials $g_n = x^n + \alpha$ are all in E_α. Let $f \in E_\alpha^\perp$. Then,

$$\langle f, g_n\rangle = \int_{-1}^{1} x^n f \; dx + \alpha \int_{-1}^{1} f \; dx = 0.$$

Note that, in particular $\langle f, g_0\rangle = 0$ so that

$$\int_{-1}^{1} x^n f \; dx = 0$$

for all n. By Stone-Weierstrass, this implies that $f = 0$.

37. Suppose that X, Y, and Z are Banach spaces and that $T : X \times Y \to Z$ is bilinear and continuous. Prove that there is a constant $M < \infty$ such that

$$\|T(x, y)\| \le M\|x\|\|y\| \;\text{ for all } x \in X, y \in Y.$$

Is completeness needed here?

Solution: Fix a $y \in Y$ and define $T_y : X \to Z$ by $T_y(x) = T(x, y)$. Then, as a restriction of a continuous map, T_y is continuous. It follows that

$$\|T(x, y)\| = \|T_y(x)\| \leq \|T_y\|\|x\|$$

Next, consider the set $\{T_y \mid y \in Y\}$. Then, this set is pointwise bounded by continuity of T in the first variable. That is, defining $T_x(y) := T(x, y)$, we have

$$\sup_{y \in B_1^Y(0)} \|T_y(x)\| = \sup_{y \in B_1^Y(0)} \|T(x, y)\| = \sup_{y \in B_1^Y(0)} \|T_x(y)\| \leq \|T_x\|.$$

So, by PUB, the T_y are uniformly bounded; say, by M. Since the above was for y with norm 1, it follows that $\|T_y\| \leq M\|y\|$. Hence,

$$\|T(x, y)\| \leq \|T_y\|\|x\| \leq M\|x\|\|y\|.$$

Completeness is not necessary here. It is only necessary if T is instead continuous in each variable separately (in the product topology, this implies that T is continuous on the product space, but the topology on a normed product space need not be the product topology).

38. Prove that a bilinear map is continuous if it is continuous at the origin $(0, 0)$.

Solution: We show that continuity at the origin implies boundedness, and that boundedness implies continuity everywhere.
Suppose T is a bilinear map continuous at the origin. Then there exist $\delta_1, \delta_2 > 0$ such that if $\|x\| < \delta_1$ and $\|y\| < \delta_2$,

$$\|T(x, y)\| = \|T(x, y) - T(0, 0)\| < 1.$$

Now let $x \in X$ and $y \in Y$. Set $\tilde{x} = x/\|x\|\delta_1$ and $\tilde{y} = y/\|y\|\delta_2$. Then,

$$\|T(\tilde{x}, \tilde{y}\| \leq 1 \;\Rightarrow\; \|T(x, y)\| \leq \frac{1}{\delta_1\delta_2}\|x\|\|y\|.$$

Then, T is bounded.

Suppose now that T is bounded by $C > 0$. Let $\epsilon > 0$ and fix $(x_0, y_0) \in X \times Y$. Set $\delta = \min(1, \epsilon/(C(1 + \|x_0\| + \|y_0\|)))$. Then, if $\|x - x_0\|, \|y - y_0\| < \delta$, we have

$$\begin{aligned} \|T(x, y) - T(x_0, y_0)\| &= \|T(x - x_0, y_0) + T(x - x_0, y - y_0) + T(x_0, y - y_0)\| \\ &\leq C\|x - x_0\|\|y_0\| + C\|x - x_0\|\|y - y_0\| + C\|x_0\|\|y - y_0\| \\ &< C\delta(\delta + \|x_0\| + \|y_0\|) \leq C\delta(1 + \|x_0\| + \|y_0\|) \leq \epsilon \end{aligned}$$

So, T is continuous at (x_0, y_0).

39. Consider $X = C[a,b]$, the continuous functions defined on $[a,b]$ with the maximum norm. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in X and suppose that $f_n \xrightarrow{w} f$. Prove that $\{f_n\}_{n=1}^{\infty}$ is pointwise convergent. That is,

$$f_n(x) \to f(x) \text{ for all } x \in [a,b].$$

Prove that a weakly convergent sequence in $C^1[a,b]$ is convergent in $C[a,b]$. $(*)$ Is this still true when $[a,b]$ is replaced by $\mathbb{R}$?

Solution: By definition, $f_n \xrightarrow{w} f$ weakly if, for all $\Lambda \in C[a,b]^*$ we have

$$\Lambda(f_n) \to \Lambda(f).$$

By Proposition 2.91, $\sup_n \|f_n\| < \infty$. For $x \in [a,b]$, let δ_x be the Dirac measure. Observe that $\delta_x \in C[a,b]^*$ via

$$\delta_x(f) = \int_a^b f \ \delta_x = f(x)$$

Hence, $\delta_x(f_n) \to \delta_x(f)$, which precisely means $f_n(x) \to f(x)$.

Let f_n be a weakly convergent sequence in $C^1[a,b]$. By Banach Saks, we may upgrade this to a strongly convergent sequence.
NEED TO FINISH THIS

40. Let X be a normed linear space and Y a closed subspace. Show that Y is weakly sequentially closed.

Solution: Since Y is convex (it's a subspace), Corollary 2.104 applies and we see that the weak and strong closures of Y are equal. But, Y is closed, so the weak closure of Y is Y. Hence Y is weakly closed. Now let $y_n \xrightarrow{w} y$. Let U be an open neighborhood of y. Then some tail lies in U, and hence $U \cap Y \setminus \{y\} \neq \emptyset$. Thus $y \in \mathrm{cl}(Y)^w$. (It is actually true for any topological space that closed implies sequentially closed).

41. Let X be a normed linear space. We say that a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ is weakly Cauchy if $\{Tx_n\}_{n=1}^{\infty}$ is Cauchy for all $T \in X^*$, and we say that X is weakly complete if each weak Cauchy sequence converges weakly. If X is reflexive, prove that X is weakly complete.

Solution: Consider the sequence $\{E_{x_n}\}$. Since $\{Tx_n\}$ is Cauchy in $\mathbb{F}$, it is bounded. However, $Tx_n = E_{x_n}(T)$, so that $\{E_{x_n}T\}$ is bounded over x_n. It follows by PUB that $\{E_{x_n}\}$ are uniformly bounded. But, $\|E_{x_n}\| = \|x_n\|$, so that the x_n are uniformly bounded, say by $M < \infty$. Since each sequence $\{Tx_n\}$ is Cauchy in a complete space, it converges. Define $\varphi : X^* \to \mathbb{F}$ by

$$\varphi(T) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} E_{x_n}(T).$$

The map φ is linear, and bounded since

$$\|\varphi(T)\| \leq \lim_{n\to\infty} \|x_n\|\|T\| = \lim_{n\to\infty} \sup_n \|x_n\|\|T\| \leq \lim_{n\to\infty} M\|T\| = M\|T\|$$

So, $\varphi \in X^{**}$. Since X is reflexive, there exists an $x \in X$ such that $E_x = \varphi$. But, then

$$T(x) = \varphi(T) = \lim_{n\to\infty} T(x_n)$$

so that $x_n \xrightarrow{w} x$.

42. Show that every finite dimensional vector space is reflexive.

Solution: Let X be a d dimensional normed space with $d < \infty$. Given a basis $\{e_i\}_{i=1}^d$, we may define a basis on X^* by

$$e^i(e_j) = \delta^i_j$$

and extending e^i linearly to all of X. Now, we can repeat the same thing for X^* and deduce that X^{**} has dimension d. Thus, F is a linear, injective map between two d dimensional spaces. It follows that F is surjective, hence X is reflexive.

43. Show that $C[0,1]$ is not reflexive.

Solution: Suppose $C[0,1]$ is reflexive. By Banach-Alaoglu, we see that the unit ball B_1^{**} is weak-$*$ (hence weakly) compact. Since $C[0,1]$ is reflexive, we see that B_1 is also weakly compact. Hence, for any $\varphi \in C[0,1]^*$, there exists an $f \in C[0,1]$ such that φ attains its norm at f (that is, $\varphi(f) = \|\varphi\|$). Now consider the functional

$$\varphi(f) = \int_0^{1/2} f\ dx - \int_{1/2}^1 f\ dx$$

Let us limit our attention to functions with $\|f\| = 1$; that is, f is essentially bounded between -1 and 1. The way to optimize φ is to choose the function $f = \chi_{[0,1/2]} - \chi_{[1/2,1]}$, for which $\varphi(f) = 1 = \|\varphi\|$. Any other function decreases φ in one way or another (cancellation of area, smaller function, etc.) But, the optimal f is not in $C[0,1]$. See Exercise 21 for further discussion.

44. If X and Y are Banach spaces, show that $E \subset B(X,Y)$ is equicontinuous if, and only if, there is an $M < \infty$ such that $\|T\| \leq M$ for all $T \in E$.

Solution: Suppose first E is equicontinuous. Then, in particular, it is equicontinuous at 0. Let $\epsilon = 1$. Then there exists a $\delta > 0$ such that if $\|x\| < \delta$ then $\|Tx\| < \epsilon = 1$ for all $T \in E$. Now let $x \in X$. Then,

$$\|Tx\| = \frac{\|x\|}{\delta} \left\| T\left(\frac{x}{\|x\|}\delta\right)\right\| \leq \frac{\|x\|}{\delta}$$

Then, $\|T\| \leq 1/\delta$.
Now suppose the T are uniformly bounded by M. Let $\epsilon > 0$. Set $\delta = \epsilon/M$. Fix an $x_0 \in X$. Then, if $x \in X$ is such that $\|x - x_0\| < \delta$,

$$\|Tx - Tx_0\| = \|T(x - x_0)\| \leq M\|x - x_0\| < M\delta < \epsilon$$

and hence E is equicontinuous at x_0.

45. Let X be a Banach space and $T \in X^* = B(X, \mathbb{F})$. Identify the range of $T^* \in B(\mathbb{F}, X^*)$.

Solution: The adjoint is defined by

$$T^*\alpha(x) = \alpha(Tx) = (\alpha T)(x)$$

for $\alpha \in \mathbb{F}$. So, the action of T^* is precisely to multiply T by some scalar. Thus, range$(T) = \mathbb{F}T \subset X^*$. It should be noted that the dual of $\mathbb{F}$ is the set of continuous functions from $\mathbb{F} \to \mathbb{F}$. So, really T^* should multiply T by such a function. In which case, the range is actually $C[\mathbb{F}]T$.

46. Let X be a Banach space, $S, T \in B(X, X)$, and I be the identity map.

a) Show by example that $ST = I$ does not imply $TS = I$.

b) If T is compact, show that $S(I - T) = I$ if, and only if, $(I - T)S = I$.

c) If $S = (I - T)^{-1}$ exists for some T compact, show that $I - S$ is compact.

Solution:

a) Let $X = l^1$. Consider the linear operator T defined by

$$T(x) = T(x_1, x_2, x_3, ...) = (0, x_1, x_2, x_3, ...)$$

and the linear operator S defined by

$$S(x) = S(x_1, x_2, x_3, ...) = (x_2, x_3, ...).$$

Then, $ST(x) = x$, but $TS(x) = (0, x_2, x_3, ...) = x - x_1e_1$. Note that this occurs since T is injective but not surjective, while S is surjective but not injective. Such an event can only happen if X is infinite dimensional.

b) An operator T is compact if, for every bounded subset B, the image $T(B)$ has compact closure. Alternatively, T is compact iff for every sequence $\{x_n\}_{n=1}^{\infty} \subset X$, $\{Tx_n\}$ has a convergent subsequence.
It takes a little work to set up, but there exists a theorem called the Fredholm alternative. It is stated here:

Proposition: Let T be a compact operator on a Banach space X. Then exactly one of the following holds

i) $I - T$ is invertible with bounded inverse

ii) There exists a nonzero $x \in X$ such that $Tx = x$ (i.e., 1 is an eigenvalue)

We show that $I - T$ is invertible. Suppose that $S(I - T) = I$. Suppose by way of contradiction that there exists a nonzero x such that $Tx = x$. Then, $S(I - T)x = S(x - x) = S(0) = 0$. On the other hand, $S(I - T)x = Ix = x$, which implies $x = 0$. This is a contradiction. The converse is similar (may need to use adjoints?)

So, in either case $I - T$ is invertible with bounded inverse (by applying the open mapping principle). It follows in either case that S is the inverse of $I - T$, and hence the other equality holds.

c) Let $A = I - S = I - (I - T)^{-1}$. Then, $A(I - T) = (I - T) - I = -T$. Hence, $A = -(I - T)^{-1}T$, which is the product of a compact and bounded operator (hence compact). To see this, suppose B is the closed unit ball. Let T compact and S bounded; we show ST is compact. Since S is bounded, it is continuous. Since T is compact, $\mathrm{cl}(TB)$ is compact. Hence, the image under S is compact, and ST is a compact operator.

For more facts on compact operators see: Appendix

47. Let $1 \leq p < \infty$ and define, for each $r \in \mathbb{R}^d$, $T_r : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ by

$$T_r(f)(x) = f(x + r).$$

a) Verify that $T_r(f) \in L^p(\mathbb{R}^d)$ and that T_r is bounded and linear. What is the norm of T_r?

b) Show that as $r \to s, \|T_r f - T_s f\|_p \to 0$. Hint: Use that the set of continuous functions with compact support are dense in $L^p(\mathbb{R}^d)$ for $p < \infty$.

Solution:

a) Linearity follows easily by the vector space structure of $L^p(\mathbb{R}^d)$. Let $f \in L^p(\mathbb{R}^d)$. Then, by the translation invariance of Lebesgue measure,

$$\|T_r(f)\|_p = \left(\int_{\mathbb{R}^d} |f(r + x)|^p \, dx \right)^{1/p} = \left(\int_{\mathbb{R}^d} |f(x)|^p \, dx \right)^{1/p} = \|f\|_p.$$

It follows from this that T_r is bounded with norm 1.

b) Show that as $r \to s$, $\|T_r f - T_s f\|_p \to 0$. Hint: Use that the set of continuous functions with compact support are dense in $L^p(\mathbb{R}^d)$ for $p < \infty$.
Let $f \in L^p(\mathbb{R}^d)$. Then there exists a sequence of continuous functions with compact support f_n such that $f_n \to f$. Then,

$$\lim_{r \to s} T_r f = \lim_{r \to s} \lim_{n \to \infty} T_r f_n = \lim_{n \to \infty} \lim_{r \to s} T_r f_n = \lim_{n \to \infty} T_s f_n = T_s f,$$

which follows by continuity of f_n and the fact that $f_n \to f$. By continuity of the norm, the conclusion follows.