Non-Euclidean Geometry:

i) Axioms of Euclidean Geometry
  ↳ Explore parallel postulate
ii) Generalize idea of straight line
iii) Spherical & Hyperbolic Geometry

Euclidean Geometry

Describes our world, geometry we learn in grade school
Built from five axioms developed in Euclid's <u>Elements</u>, 300 BCE.

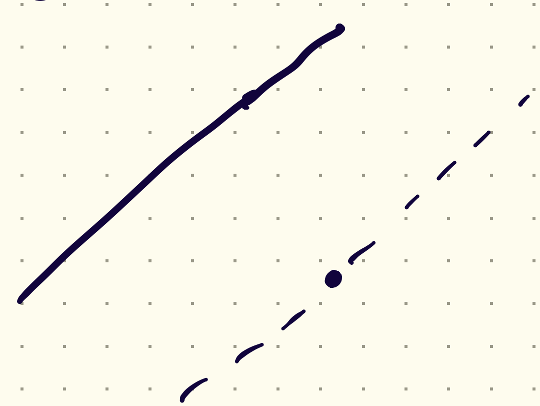
i) Any two points can be joined by a straight line

ii) Any straight line segment can be extended indefinitely in a straight line

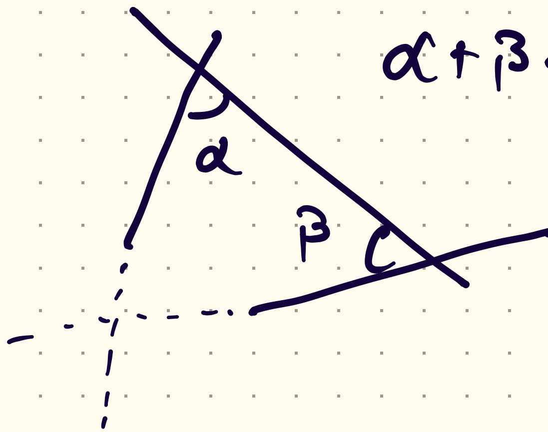
iii) Given any straight line segment, a circle can be drawn having the segment as its radius and one endpoint as its center.

iv) All right angles are congruent

v) Through a point not on a given straight line, exactly one straight line can be drawn that never meets the given line:



Equivalent to: If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough



$\alpha + \beta < \pi$

This is the version Euclid used

Also equivalent: The sum of the angles in a triangle is exactly 180° (more on this later)

Def: Two straight lines which do not intersect are called parallel.

First four axioms show existence of parallel lines

↳ ie. Given a straight line and a point not on it we can always find a parallel line passing through it

Axiom v), the parallel postulate, asserts uniqueness.

Question: Can we derive v) from i) – iv)?

History:

In 1773 Giovanni Saccheri tried to show i) - iv) => v)

↪ Failed, at the end ranted about the absurdity of his work

Gauss believed in non-Euclidean Geometries, which satisfy i) - iv) but not v).
Feared his reputation would suffer. This idea, like the existence of complex numbers, was considered heretic.

Related: Same thing with dim > 3, pushed it onto Riemann

Euclid's definitions:

We intuitively know what point, line, etc. mean.
What formal definition did Euclid use?
↳ very pedantic often at clarity's expense.

Def: A point is that which has no part

A line is breadthless length

A straight line is a line which lies evenly with the points on itself
↳ trans. invariance?

Let's define straight lines more formally

Physics:

Imagine a ball on a frictionless table, which you apply a force to. How does it move?

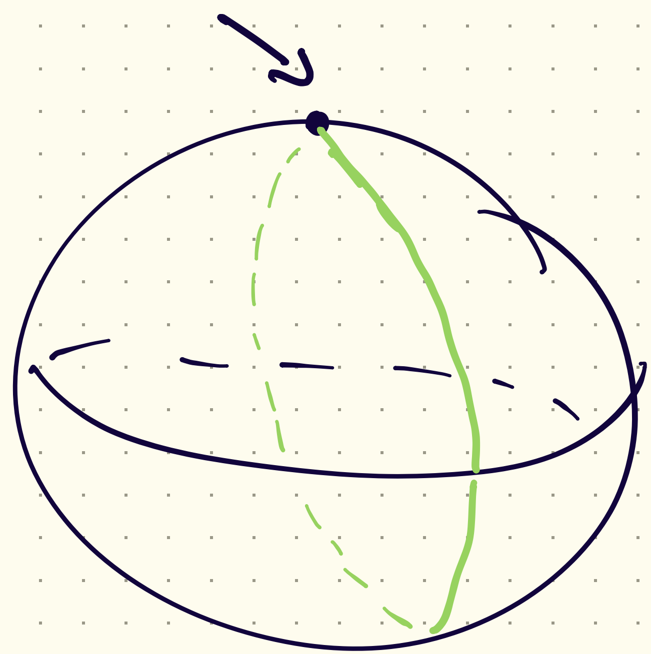
So in some sense straight lines are those paths which have no acceleration.

Def: Let $\gamma : (-\varepsilon, \varepsilon) \to M$. Then γ is a <u>geodesic</u> ("straight line") if $\frac{d^2}{dt^2} \gamma(t) \equiv 0$.

↳ Hiding a lot of technical machinery... what do derivatives in M mean?

With this definition, consider the following thought experiment:

Place our ball on a frictionless sphere & apply a force. What path does it trace?



Geodesics on the sphere are equators / great circles.

Question: Are there any parallel lines on S^2?

<u>No!</u> Segments must be extended indefinitely, will eventually close up

Any two parallel lines intersect at exactly two (antipodal) points

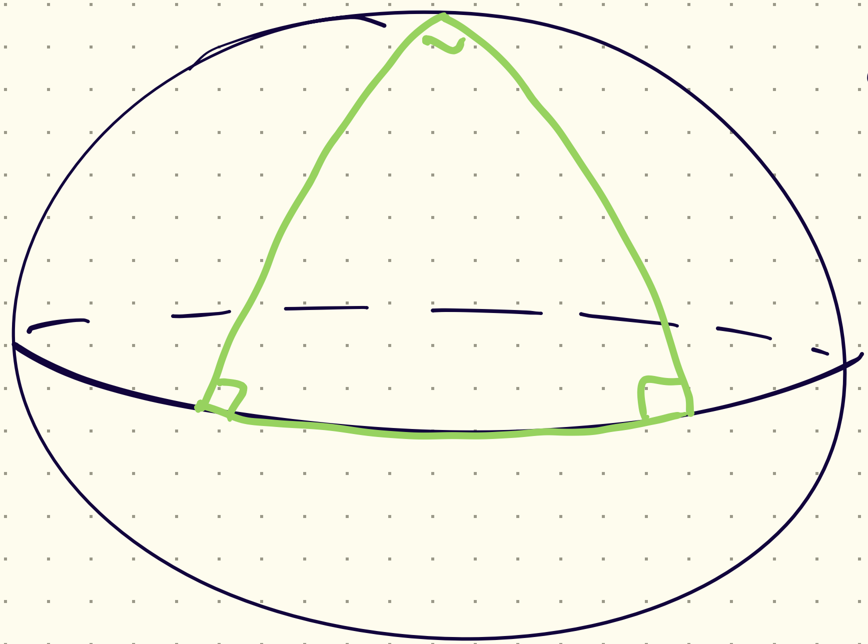
Can still model spherical geometry by Euclid's axioms.

i) – iv) are unchanged, v) $\rightsquigarrow$ "... no straight lines can be drawn ..."

What about the equivalent formulations?

Equivalent to:

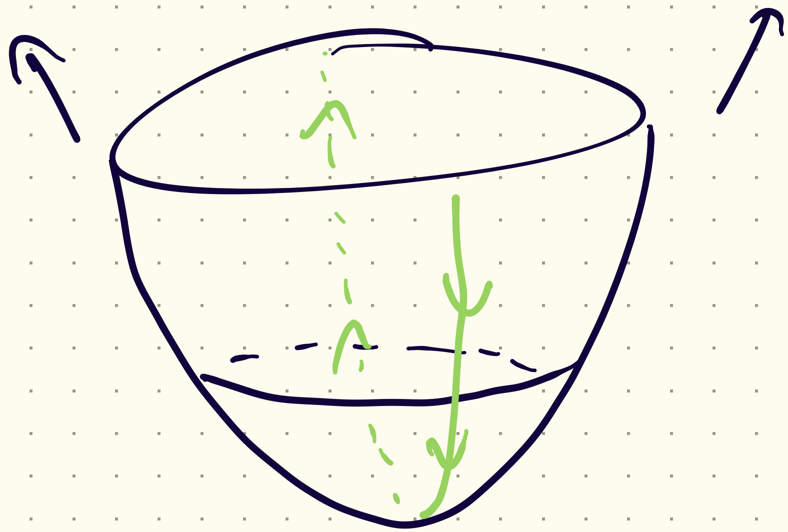
- Sum of angles in a triangle is always strictly greater than 180°



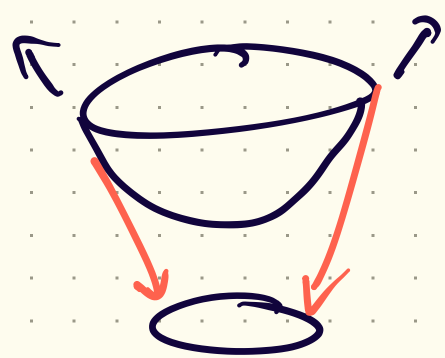
Sum is $\frac{3\pi}{2}$!!

Hyperbolic geometry:

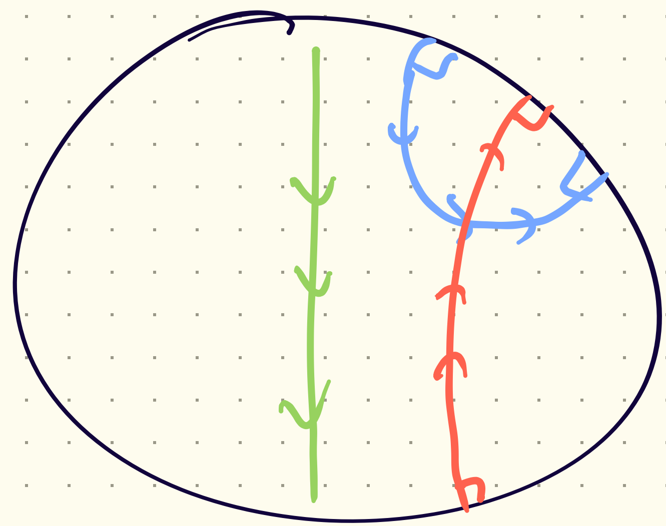
How do we visualize? With a hyperboloid!



These are hard to understand. Can project onto a disc



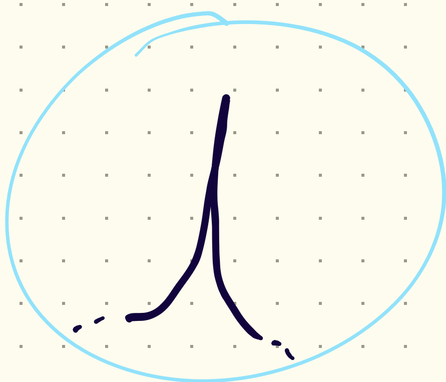
This is called the Poincaré disc.
Geodesics become curved arcs:

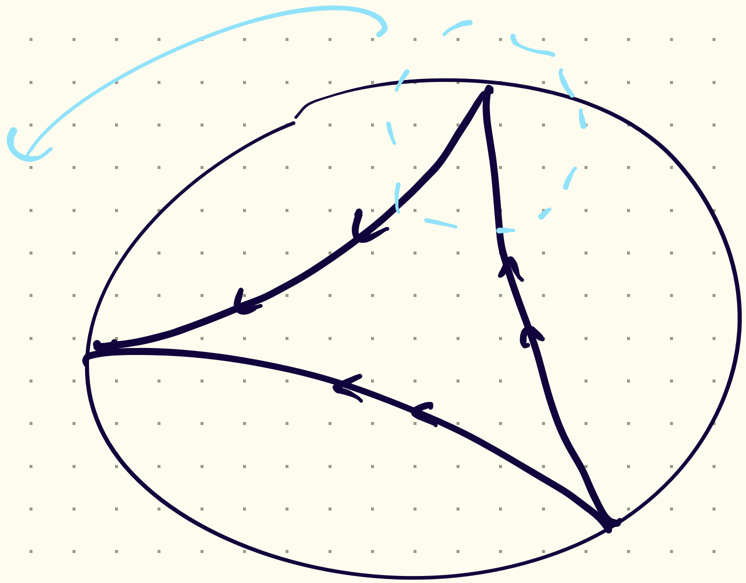


$\mathcal{H}^2$

Can also realize w/ Euclid's axioms.
i) – iv) are the same, v) $\rightsquigarrow$ "... infinitely many..."

Equivalent to sum of angles in a triangle $< 180°$





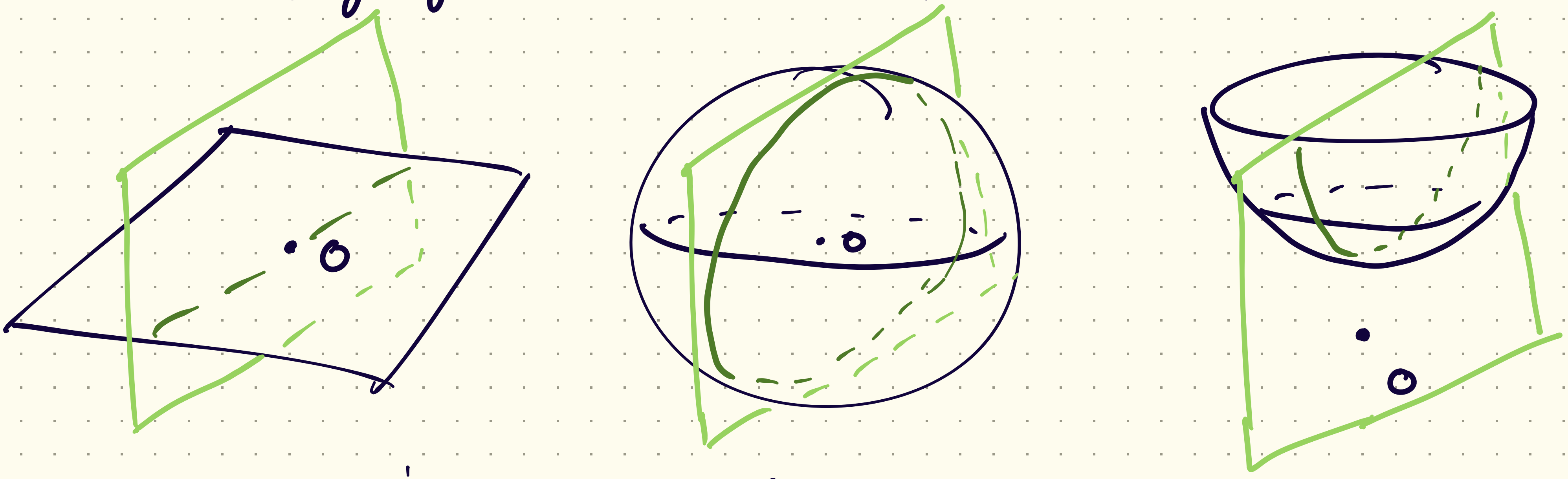
Angle sum is ....
zero degrees ??

THM [Gauss-Bonnet] If α, β, γ denote the angles of a Δ (in Euclidean, spherical, hyperbolic geometry) then

$$\alpha + \beta + \gamma = \pi + K \, \mathrm{Area}(\Delta)$$

where $K = -1, 0, 1$ in hyp., euc., sph. resp.

Characterizing geodesics on $\mathbb{R}^2, S^2, H^2$:



Geodesics can also be related to curvature.

Curvature <=> acceleration, imagine turning a car.
(feel a force, centripetal acceleration)

The acceleration felt along a curve $\gamma : (-\varepsilon, \varepsilon) \to M$ is called the geodesic curvature K_g.

So if γ is a geodesic then $K_g \equiv 0$.

"Non-geodesics" try to pull you off curved surfaces.

THM [Gauss-Bonnet] If M^2 has constant K then:

$$\int_M K + \int_{\partial M} \kappa_g = 2\pi \chi(M)$$

Eg with $M = \Delta$ in $\mathbb{R}^2$, S^2, H^2, get above.