Non-Euclidean Geometry

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Outline

i) Axioms of Euclidean Geometry.
ii) Explore the parallel postulate.
iii) Generalize the idea of straight lines to other spaces.
iv) Spherical Geometry.
v) Hyperbolic Geometry.

What is Euclidean Geometry?

This is the geometry we are all familiar with, and study in our grade school geometry courses! Like all things in math, it is built from *axioms* which are built in "truths". These axioms were developed by Euclid in *Elements*, around 300 B.C.E.

i) Any two points can be joined by a straight line.
ii) Any straight line segment can be extended indefinitely in a straight line.
iii) Given any straight line segment, a circle can be drawn having the segment as its radius and one endpoint as its center.
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The Parallel Postulate:

There are many ways to reformulate the final axiom:

a) Through a point not on a given straight line, exactly one straight line can be drawn that never meets the given line.

b) If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough. (This is the version Euclid used)

c) The sum of the angles in a triangle is exactly 180 degrees.

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The Parallel Postulate:

The first four axioms can actually be used to show existence of parallel lines: Given a straight line and a point not on it, we can always find a parallel line passing through that point.

Meta-question: Can there be more than one such parallel line?

That is, if we assume the parallel postulate is false, can we derive a contradiction in the first four axioms? If we can, then there is only one geometry: Euclidean geometry. If not, then multiple *non-Euclidean* geometries exist.

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Some History:

In 1733, the Jesuit priest Giovanni Saccheri, believing in Euclidean geometry, tried to establish that only one parallel line could be drawn (the parallel postulate follows from the first four axioms). He failed, and at the end of his work went on a rant about the absurdity of everything he wrote.

Gauss believed that non-Euclidean geometries. Much like the existence of complex numbers, such an idea was considered heretic. Gauss never published his thoughts, fearing his reputation would suffer.

Somewhat unrelated: Euclidean geometry was also restricted to at most three dimensions. Higher dimensional objects were not believed to exist. Gauss also privately refuted this idea, and tasked his student Riemann to develop a theory of high dimensional surfaces. This led to *Riemannian Geometry*.

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Euclid's Definitions:

Euclid was really quite pedantic, often times at the consequence of being obscure, with his definitions.

a) A *point* is that which has no part.
b) A *line* is breadthless length.
c) A *straight line* is a line which lies evenly with the points on itself. (???)

On the other hand, it's important to make clear some of his definitions.

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Straight Lines:

Some alternatives:

a) A *straight line* is the shortest path connecting two points.

b) A *straight line* is a path having no acceleration/constant velocity.

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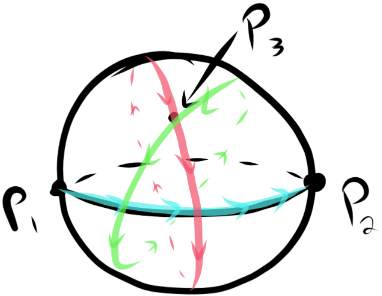
Geodesics on a Sphere:

Imagine you're on a (smallish) sphere, and you start walking forward. What kind of path do you trace out? Do you end up back where you started?

Geodesics on a Sphere:



The geodesics can be described in the following way: Take any plane through the origin and trace its intersection along the sphere. This path is a geodesic.

Geodesics on a Sphere:

NO parallel lines exist! Any two “straight lines” (geodesics) intersect at least at two points.

Spherical Geometry:

This is the geometry on the surface of the sphere, and is given by the following five axioms.

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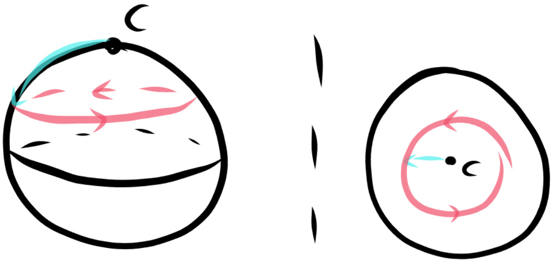
The Third Axiom of Spherical Geometry:

Given any straight line segment, a circle can be drawn having the segment as its radius and one endpoint as its center.



The Parallel Postulate and Spherical Geometry:

Spherical geometry gives an example of a geometry where the parallel postulate is false. Hence, it is a non-Euclidean geometry. The fifth axiom, like the parallel postulate, can be reformulated.

a) Through a point not on a given straight line, no straight lines can be drawn that never meet the given line.

b) Any two geodesics will intersect in exactly two points.

c) On the sphere the sum of the angles in a triangle is always strictly greater than 180 degrees.

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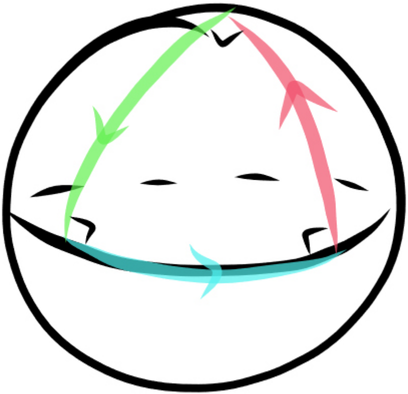
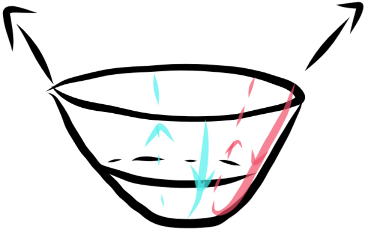


Geodesics on a Hyperboloid:

Here we consider the following hyperboloid.



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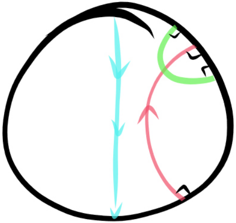
Geodesics on a Hyperboloid:

It's a little hard to visualize these geodesics, so we search for a planar description. To do this, we place a disc below the hyperboloid and project it onto it.

The Poincaré Disc Model:

What we've just done is construct the *Poincaré disc*. The geodesics on the hyperboloid become curved arcs on the disc, which intersect the boundary at right angles.



Hyperbolic Geometry:

This is the geometry on the surface of the hyperboloid, visualized using the Poincaré disc model, and is given by the following five axioms.

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The Parallel Postulate and Hyperbolic Geometry:

Since the parallel postulate is also false in Hyperbolic geometry, it is another example of non-Euclidean geometry. The fifth axiom, as before, can be reformulated.

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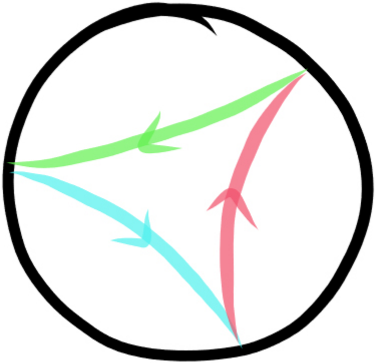

Comparing the Fifth Axioms:

Writing out the fifth axioms together, we have:

a) On the plane the sum of the angles in a triangle is exactly 180 degrees.

b) On the sphere the sum of the angles in a triangle is always strictly greater than 180 degrees.

c) On the hyperboloid the sum of the angles in a triangle is always strictly less than 180 degrees.

Comparing the Fifth Axioms:

We can combine these in the following theorem:

Theorem (Gauss-Bonnet)

If α, β, γ *denote the angles of a triangle* Δ *(in Euclidean, Spherical, or Hyperbolic geometry) then*

$$\alpha + \beta + \gamma = \pi + K\,\text{Area}(\Delta)$$

where $K = 0$ *in Euclidean geometry,* $K = 1$ *in Spherical, and* $K = -1$ *in Hyperbolic.*

The number K is the *curvature* of the corresponding geometry.