PROBLEM SET 10

DUE DATE: APRIL 10 ON GRADESCOPE

To numerically compute determinants or row reductions, feel free to use any relevant linear algebra software or online mathematical tools (such as WolframAlpha), and just cite which tools you used to compute the result.

Definition 1. Let $\mathcal{V}$ be a vector space, and let $L : \mathcal{V} \to \mathcal{V}$ be a linear transformation. We say that a vector $X \in \mathcal{V}$ is an *eigenvector* for L if the following conditions hold:

(1) $X \neq \mathbf{0}$
(2) $L(X) = \lambda X$ for some $\lambda \in \mathbb{R}$.

If $L(X) = \lambda X$ for some nonzero $X \in \mathcal{V}$, then we say that λ is an *eigenvalue* for L. For any eigenvalue λ of L, we define the associated *eigenspace* as

$$E_\lambda = \{X \in \mathcal{V} \;:\; L(X) = \lambda X\},$$

that is, the set of eigenvectors associated to λ, combined with the 0 vector.

We recall that the notions of eigenvalues, eigenvectors, and eigenspaces were previously defined just for matrices.

Problems.

(1) (Bases of matrices) Define the linear operator $L : \mathcal{M}_{2,2} \to \mathcal{M}_{2,2}$ as $L(A) = A^\top$ for any 2×2 matrix A.
  (a) By constructing suitable eigenvectors, show that $\lambda = 1$ and $\lambda = -1$ are eigenvalues for L.
  (b) Prove that $\lambda = -1, 1$ are the only eigenvalues for L. (Hint: compute the eigenvalues of $L^2 = L \circ L$. Assume by contradiction that $L(X) = \lambda X$ for some nonzero $X \in \mathcal{M}_{2,2}$ and $\lambda \notin \{-1, 1\}$. What is $L^2(X)$?)
  (c) Characterize the eigenspaces E_{-1} and E_1 in terms of properties of matrices from earlier in the class. Construct bases for E_{-1} and E_1 (make a guess, and justify your reasoning – you do not need to prove that these are bases). What are the dimensions of E_{-1} and E_1?

(2) (Polynomial interpolation) Let $a_0, a_1, \ldots, a_n \in \mathbb{R}$ be n distinct real numbers. Define the polynomials

$$p_i(x) = \prod_{\substack{0 \leq j \leq n \\ j \neq i}} \frac{x - a_j}{a_i - a_j}, \qquad 0 \leq i \leq n.$$

  (a) Define the function $L : \mathcal{P}_n \to \mathbb{R}^{n+1}$ as

$$L(p) = \begin{bmatrix} p(a_0) \\ p(a_1) \\ \vdots \\ p(a_n) \end{bmatrix}.$$

  Prove that L is linear.

(b) By showing that the sequence $(L(p_i))_{i=0}^n \subset \mathbb{R}^{n+1}$ is linearly independent, show that the sequence $(p_i)_{i=0}^n \subset \mathcal{P}_n$ is linearly independent (you can quote results from previous problem sheets).

(c) Using Lemma 4.10 from the textbook, show that

$$\mathcal{B} = (p_i)_{i=0}^n$$

is a basis for $\mathcal{P}_n$.

(d) Let $b_0, \ldots, b_n \in \mathbb{R}$ be arbitrary numbers. Show that there exists a unique polynomial p of degree at most n (i.e. $p \in \mathcal{P}_n$), such that

$$p(a_i) = b_i \quad \text{for all } 0 \leq i \leq n. \tag{1}$$

Hint: construct p explicitly using the p_i. To prove uniquenss, show that $\text{range}(L) = \mathbb{R}^{n+1}$, and use the dimension theorem (Theorem 5.10 from the textbook) to show that $\ker(L) = \{\mathbf{0}\}$. If p and q both satisfy (1), show that $L(p - q) = \mathbf{0}$ and therefore $p - q = 0$.

(e) Let $\mathcal{C} = \{e_0, \ldots, e_n\}$ denote the standard basis on $\mathbb{R}^{n+1}$ (where for this question, we number coordinates starting from 0). Compute the matrix of the linear map $L : \mathcal{P}_n \to \mathbb{R}^{n+1}$ in $\mathcal{B} \to \mathcal{C}$ coordinates.

(3) (Determinants of operators) Let $\mathcal{V}$ be a finite, n-dimensional vector space, let $L : \mathcal{V} \to \mathcal{V}$ and let $\mathcal{B}, \mathcal{C}$ be any two bases for $\mathcal{V}$.

(a) Show that $\det[L]_{\mathcal{B},\mathcal{B}} = \det[L]_{\mathcal{C},\mathcal{C}}$. (Hint: use Theorems 5.6 and 4.19 from the textbook.) We call this value the *determinant* of L.

(b) Show that $\det(\lambda I_n - [L]_{\mathcal{B},\mathcal{B}}) = \det(\lambda I_n - [L]_{\mathcal{C},\mathcal{C}})$ for any $\lambda \in \mathbb{R}$. We call the polynomial

$$p_L(\lambda) = \det(\lambda I_n - [L]_{\mathcal{B},\mathcal{B}})$$

the *characteristic polynomial* of L.

(c) Show that L has λ as an eigenvalue if and only if $p_L(\lambda) = 0$. (Hint: coordinatize L with respect to a basis $\mathcal{B}$.)

(4) (Dimension theorem) Let $L : \mathbb{R}^3 \to \mathbb{R}^4$ be given by

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & -6 \\ -1 & 6 & 41 \\ 1 & -3 & -26 \\ 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Find bases for $\ker(L)$ and $\text{range}(L)$. Verify that

$$\dim\ker(L) + \dim\text{range}(L) = \dim(\mathbb{R}^3).$$

Hint: use the kernel method and range method in Section 5.3 of the textbook. You may use online solvers to compute the reduced row echelon form, as long as you cite which resource you used, and write down the resulting matrix (you do not need to write the row operations you used).