PROBLEM SET 9

DUE DATE: APRIL 3 ON GRADESCOPE

To numerically compute determinants or row reductions, feel free to use any relevant linear algebra software or online mathematical tools (such as WolframAlpha), and just cite which tools you used to compute the result.

(1) (3 points) Using the method from class, or otherwise, prove that the following sets form a basis for $\mathbb{R}^4$:
  (a) $\{[1,5,6,-7],[1,3,9,-8],[-6,2,6,4],[5,2,1,0]\}$
  (b) $\{[5,1,3,-4],[9,2,-1,3],[4,-1,0,4],[2,8,1,7]\}$
  (c) $\{[1,1,1,1],[1,1,1,-2],[1,1,-2,-3],[1,-2,-3,-4]\}$

(2) (4 points) Let $v_1,\dots,v_n \in \mathbb{R}^m$ be any vectors, and define the $m \times n$ matrix

$$A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}.$$

Prove that $\mathcal{B} = \{v_j\}_{j=1}^n$ spans $\mathbb{R}^m$, if and only if the system $AX = B$ has a solution $X \in \mathbb{R}^n$ for any $B \in \mathbb{R}^m$. Prove that $\mathcal{B}$ is linearly independent if and only if the system $AX = 0$ admits only the trivial solution $X = 0 \in \mathbb{R}^n$.

(3) (4 points) For any vector spaces $\mathcal{V}$ and $\mathcal{W}$, define $\mathcal{L}(\mathcal{V},\mathcal{W})$ as the set of linear functions from $\mathcal{V}$ to $\mathcal{W}$. For any $f,g \in \mathcal{L}(\mathcal{V},\mathcal{W})$ and any scalar $c \in \mathbb{R}$, define the functions $f+g$ and cf by

$$(f+g)(v) = f(v) + g(v) \text{ for all } v \in V,$$
$$(cf)(v) = cf(v) \text{ for all } v \in V.$$

Prove that under these operations, $\mathcal{L}(\mathcal{V},\mathcal{W})$ is a vector space. If $\mathcal{V}$ and $\mathcal{W}$ are finite dimensional, with dimensions n and m respectively, prove that $\dim \mathcal{L}(\mathcal{V},\mathcal{W}) = nm$. (Hint: fix bases for $\mathcal{V}$ and $\mathcal{W}$, and construct a basis for $\mathcal{L}(\mathcal{V},\mathcal{W})$, by analogy with the canonical basis for $\mathcal{M}_{m,n}$.)

(4) (3 points) Show that the set

$$\left\{ \begin{bmatrix} 4 & 3 \\ -7 & 9 \end{bmatrix}, \begin{bmatrix} 8 & -1 \\ 7 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 5 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 5 \\ 4 & -3 \end{bmatrix} \right\}$$

forms a basis for $\mathcal{M}_{2,2}$. (Hint: use a canonical basis for $\mathcal{M}_{2,2}$, use Theorem 1, and show that a certain matrix is invertible.)

(5) (3 points) Prove that for any degree n polynomial f, the set $\{f, f^{(1)}, \dots, f^{(n)}\}$ forms a basis for $\mathcal{P}_n$. (Hint: use the determinant method, and observe that the resulting matrix has a simple form.)

(6) (3 points) Let A be an $m \times n$ matrix, and let $v_1,\dots,v_k \in \mathbb{R}^n$. Prove that if $Av_1,\dots,Av_k$ are linearly independent, then so are $v_1,\dots,v_k$. Prove that the converse is not always true. (Hint: pick a matrix with a nontrivial kernel.)

(7) (bonus 10 points) Let $\mathcal{W} \subseteq \mathbb{R}^n$ be a subspace, and let $k = \dim \mathcal{W}$. Define the *orthogonal complement* of $\mathcal{W}$ as the set

$$\mathcal{W}^{\perp} = \{v \in \mathbb{R}^n \ : \ v \cdot w = 0 \text{ for all } w \in \mathcal{W}\}\,.$$

Show that $\mathcal{W}^{\perp}$ is a vector subspace, and prove that $\dim \mathcal{W}^{\perp} = n - k$.

Hint: By picking a basis for $\mathcal{W}$, express $\mathcal{W}^{\perp}$ as the solution set of a $k \times n$ matrix, reduce the system using the simplified span method, and use Theorem 2 to show that the resulting matrix has no row of zeros. Conclude the proof by constructing a basis for $\mathcal{W}^{\perp}$ in terms of fundamental solutions to the homogeneous problem.

Cheat Sheet

Determinant Test for Bases. Suppose we are given a set $\mathcal{B} \subset \mathbb{R}^n$. The following test tells us whether $\mathcal{B}$ is a basis for $\mathbb{R}^n$.

Step 1. Check that $\#(\mathcal{B}) = n$ (we have the right number of vectors).

Step 2. If $\mathcal{B} = \{v_1, \dots, v_n\}$, check that the matrix

$$A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$$

has nonzero determinant (which is equivalent to checking that it is invertible).

Finding a basis for a homogeneous solution set. Let A be an $m \times n$ matrix, and define its kernel

$$\ker A = \{X \in \mathbb{R}^n \ : \ AX = 0\},$$

that is, the solution set to the homogeneous problem $AX = 0$. The following procedure gives a basis for $\ker A$.

(1) Reduce A to its reduced row echelon form $\tilde{A}$.
(2) Determine which columns don't have pivot entries, and list them as the j_1'th, j_2'th, ..., j_k'th columns, in increasing order. (These correspond to the independent/free variables of the solution set.)
(3) For any $s \in \{1, 2, \dots, k\}$, solve for the unique vector $\mathbf{v}_s \in \ker A$ such that v_s has entry 1 in the j_s coordinate, and has entry 0 for the $j_1, j_2 \dots, j_{s-1}, j_{s+1}, \dots, j_n$ coordinates.

The vectors $v_1, \dots, v_k$ form a basis for $\ker A$, and furthermore $\ker A$ has dimension k.

Key Results from the Lectures. The following two theorems are often useful when proving results about bases.

Theorem 1. *Let $\mathcal{V}$ be a vector space, with a finite basis $\mathcal{B} = \{v_1, \dots, v_n\}$. Suppose*

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

is an invertible $n \times n$ matrix, and define $w_j = \sum_{i=1}^n a_{ij} v_i$ for all $1 \le j \le n$. Then the set $\{w_1, \dots, w_n\}$ is also a basis for $\mathcal{V}$.

Proof. This will be proved on Tuesday, April 1 in class. □

Theorem 2. *Let S be a finite set which spans the vector space $\mathcal{V}$, and suppose $T \subseteq \mathcal{V}$ is a subset such that $\#(T) > \#(S)$. Then T is not linearly independent.*

Proof. It suffices to assume that T is finite. (If T is infinite, take any finite subset $T_0 \subset T$ such that $\#(T_0) > \#(S)$. If T_0 is linearly dependent, so is T.)

Write $S = \{v_1, \dots, v_m\}$, and express $T = \{w_1, \dots, w_n\}$, where $m < n$. Since S spans $\mathcal{V}$, we can write

$$w_j = \sum_{i=1}^m c_{ij} v_i$$

for any $1 \leq j \leq n$, and we define the $m \times n$ matrix

$$C = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix},$$

then since the homogeneous system $CX = 0$ has more variables than equations, it admits a nonzero solution

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},$$

such that

$$\sum_{j=1}^{m} c_{ij} x_j = 0, \qquad i = 1, \ldots, n.$$

But this implies that

$$\sum_{j=1}^{n} x_j w_j = \sum_{j=1}^{n} x_j \sum_{i=1}^{m} c_{ij} v_i = \sum_{i=1}^{m} \sum_{j=1}^{n} x_j c_{ij} v_i = \sum_{i=1}^{m} \left(\sum_{j=1}^{m} c_{ij} x_j \right) v_i = 0,$$

and since the values $\{x_j\}_{j=1}^{n}$ are not all zero, this shows that the set $T = \{w_1, \ldots, w_n\}$ is linearly dependent. $\square$