Name:	
Student Number:	

SOLUTIONS TO MIDTERM #1 – LINEAR ALGEBRA AND MATRIX THEORY

	Multiple Choice (10 max)	Problem 1 (25 max)	Problem 2 (15 max)	Problem 3 (20 max)	Problem 4 (15 max)	Problem 5 (15 max)	Total (100 max)
Score:							

1. Multiple Choice (10 points)

Let A, B be arbitrary $n \times n$ matrices, and let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ be arbitrary vectors, and let $c \in \mathbb{R}$ be a nonzero scalar. Which of the following are true in general? (2 points each.)

(i) $AB = BA$

a) TRUE   b) **FALSE**

(ii) Only square matrices can be invertible.

a) **TRUE**   b) FALSE

(iii) $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$

a) **TRUE**   b) FALSE

(iv) $(cA)^{-1} = cA^{-1}$

a) TRUE   b) **FALSE**

(v) If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ are nonzero vectors, and the angle between them is θ, then $\mathbf{x} \cdot \mathbf{y} = \sin(\theta) \|\mathbf{x}\| \|\mathbf{y}\|$.

a) TRUE   b) **FALSE**

2. Free Response

Problem 1. (25 points).

(i) (10 points.) Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ be any nonzero vectors. Using basic properties of the dot product (e.g. distributivity, non-negativity of the norm), prove the identity

$$-\|\mathbf{x}\| \|\mathbf{y}\| \leq \mathbf{x} \cdot \mathbf{y} \leq \|\mathbf{x}\| \|\mathbf{y}\| \qquad \text{(Cauchy-Schwartz inequality)}$$

(Hint: prove the identity first for the unit vectors $\mathbf{u} = \frac{\mathbf{x}}{\|\mathbf{x}\|}$ and $\mathbf{v} = \frac{\mathbf{y}}{\|\mathbf{y}\|}$ by distributivity for $\|\mathbf{u} \pm \mathbf{v}\|^2$, and factor out the scalar $\|\mathbf{x}\| \|\mathbf{y}\|$ to prove the general identity.)

Answer to 1(i). We prove the statement first for the unit vectors $\mathbf{u}$ and $\mathbf{v}$. We compute that

$$0 \leq \|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2 = 2 + 2\mathbf{u} \cdot \mathbf{v}$$

by distributivity, and therefore by dividing both sides by 2, we get $\mathbf{u} \cdot \mathbf{v} + 1 \geq 0$ and therefore $\mathbf{u} \cdot \mathbf{v} \geq -1$. Similarly, we compute that

$$0 \leq \|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 - 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2 = 2 - 2\mathbf{u} \cdot \mathbf{v}$$

by distributivity, and again dividing both sides by 2, we get $1 - \mathbf{u} \cdot \mathbf{v} \geq 0$ and therefore $\mathbf{u} \cdot \mathbf{v} \leq 1$. Combining these inequalities gives us that

$$-1 \leq \mathbf{u} \cdot \mathbf{v} = \frac{\mathbf{x}}{\|\mathbf{x}\|} \cdot \frac{\mathbf{y}}{\|\mathbf{y}\|} \leq 1,$$

and multiplying both inequalities by $\|\mathbf{x}\| \|\mathbf{y}\|$ gives the claim.

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(ii) (5 points.) Use the Cauchy-Schwartz inequality to prove Minkowski's inequality (also called the triangle inequality):

$$\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\| \qquad \text{(Minkowski's inequality)}$$

for any nonzero vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. (Hint: square both sides and use distributivity.)

Answer to 1(ii). We square the term on the left hand side and compute

$$\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2\mathbf{x}\cdot\mathbf{y} + \|\mathbf{y}\|^2 \leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\,\|\mathbf{y}\| + \|\mathbf{y}\|^2 = (\|\mathbf{x}\| + \|\mathbf{y}\|)^2,$$

where we have used distributivity of the dot product and the Cauchy-Schwartz inequality to bound the second equation. Taking square roots of both sides proved Minkowski's inequality (since both sides are non-negative).

(iii) (5 points.) Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n \in \mathbb{R}^n$ be pair-wise orthogonal unit vectors. (That is, assume $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ whenever $i \neq j$, and $\|\mathbf{u}_i\| = 1$ for all i.) Let $\mathbf{y} \in \mathbb{R}^n$ be a vector expressible in the form

$$\mathbf{y} = \sum_{i=1}^{n} \alpha_i \mathbf{u}_i$$

for some coefficients $\alpha_i \in \mathbb{R}$. Prove the identity

$$\|\mathbf{y}\|^2 = \sum_{i=1}^{n} (\mathbf{y}\cdot\mathbf{u}_i)^2. \qquad \text{(Parseval's identity)}$$

Answer to 1(iii). We expand the left hand side as

$$\|\mathbf{y}\|^2 = \left(\sum_{i=1}^{n} \alpha_i \mathbf{u}_i\right) \cdot \left(\sum_{j=1}^{n} \alpha_j \mathbf{u}_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j \mathbf{u}_i\cdot\mathbf{u}_j = \sum_{i=1}^{n} \alpha_i^2,$$

where we have used distributivity of the dot product, and where we have used the assumption that $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ when $i \neq j$ and $\mathbf{u}_i \cdot \mathbf{u}_i = 1$ for all i. But we can also compute the dot product

$$\mathbf{y}\cdot\mathbf{u}_i = \left(\sum_{j=1}^{n} \alpha_j \mathbf{u}_j\right) \cdot \mathbf{u}_i = \sum_{j=1}^{n} \alpha_j \mathbf{u}_j \cdot \mathbf{u}_i = \alpha_i$$

for any i, where we have again used the fact that $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ when $i \neq j$ and $\mathbf{u}_i \cdot \mathbf{u}_i = 1$ for all i. Plugging in this formula for α_i completes the proof.

(iv) (3 points.) Define the matrix $P = [\mathbf{u}_1\ \mathbf{u}_2\ \cdots\ \mathbf{u}_n]$, where we consider $\mathbf{u}_i$ as column vectors. Prove that $P^\top P = I_n$. Does this imply that P is invertible?

(Hint: use the definition of matrix multiplication in terms of dot products of columns and rows.)

Answer to 1(iv). Considering the vectors $\mathbf{u}_i$ as column vectors (i.e. $n \times 1$ matrices), we consider $P^\top = \begin{bmatrix} \mathbf{u}_1^\top \\ \vdots \\ \mathbf{u}_n^\top \end{bmatrix}$ as an $n \times n$ matrix. But then the (i, j) entry of $P^\top P$ is the dot product of the i'th row vector of $P^\top$ and the j'th column vector of P, but this is precisely $\mathbf{u}_i \cdot \mathbf{u}_j$, which is 1 exactly when $i = j$ and 0 otherwise. But these are precisely the entries of the identity matrix I_n, and therefore $P^\top P = I_n$. Since P is square and has a left inverse, it is therefore invertible.

(v) (2 points.) Give an example of nonsquare matrices A, B such that $AB = I_n$. This problem shows that nonsquare matrices can have left or right inverses.

(Hint: choose appropriate row/column vectors from the previous question.)

Answer to 1(v). If $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ is any column vector with unit norm, then $\mathbf{u}^\top \cdot \mathbf{u} = [1]$ is the 1×1 identity matrix, since $\mathbf{u}^\top \cdot \mathbf{u} = [u_1^2 + u_2^2 + \cdots + u_n^2] = \left[\ \|\mathbf{u}\|^2\ \right] = [1]$.

Problem 2. (15 points). Systems of equations depending on a parameter. In this problem, we consider the system $AX = B$, where A and B are given by the augmented matrix

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 0 & t & 0 \\ 4 & 1 & 2 & 4 \\ -2 & 4 & 8 & -2 \end{array}\right],$$

where $t \in \mathbb{R}$ is a parameter. In this question, you only need to give the answer, not the proof.

(i) (5 points.) For which $t \in \mathbb{R}$ does the system $AX = B$ have no solutions? When does it have a unique solution? When does it have infinitely many solutions? (Note that it's possible that only some of these possibilities will occur.)

Answer to 2(i). We reduce the augmented matrix $[A|B]$ to row echelon form, by computing

$$\left[\begin{array}{ccc|c} 1 & 0 & t & 0 \\ 4 & 1 & 2 & 4 \\ -2 & 4 & 8 & -2 \end{array}\right] \underset{\sim}{\langle 2\rangle \mapsto \langle 2\rangle - 4\langle 1\rangle} \left[\begin{array}{ccc|c} 1 & 0 & t & 0 \\ 0 & 1 & 2-4t & 4 \\ -2 & 4 & 8 & -2 \end{array}\right] \underset{\sim}{\langle 3\rangle \mapsto \langle 3\rangle + 2\langle 1\rangle} \left[\begin{array}{ccc|c} 1 & 0 & t & 0 \\ 0 & 1 & 2-4t & 4 \\ 0 & 4 & 8+2t & -2 \end{array}\right]$$

$$\underset{\sim}{\langle 3\rangle \mapsto \langle 3\rangle - 4\langle 2\rangle} \left[\begin{array}{ccc|c} 1 & 0 & t & 0 \\ 0 & 1 & 2-4t & 4 \\ 0 & 0 & 18t & -18 \end{array}\right]$$

When $t = 0$, we get the inconsistent system $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & -18 \end{array}\right]$ and we see that there are no solutions.

When $t \neq 0$, by the row operation $\langle 3\rangle \mapsto \frac{1}{18t}\langle 3\rangle$, we get the row echelon form $\left[\begin{array}{ccc|c} 1 & 0 & t & 0 \\ 0 & 1 & 2-4t & 4 \\ 0 & 0 & 1 & -1/t \end{array}\right]$ which we see has a unique solution.

(ii) (5 points.) Specify the solution set to $AX = B$ for each $t \in \mathbb{R}$.

Answer to 2(ii). When $t = 0$, we have just shown that there are no solutions, so the solution set is empty. When $t \neq 0$, we solve for the unique solution $X = \left[\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right]$. We see immediately that $x_3 = -1/t$, and we can solve for

$$x_2 + (2-4t)x_3 = x_2 - \frac{2-4t}{t} = x_2 - \frac{2}{t} + 4 = 4$$

to find that $x_2 = 2/t$. Similarly, we can solve for

$$x_1 + tx_3 = x_1 - \frac{t}{t} = x_1 - 1 = 0$$

to find that $x_1 = 1$, so that we have the unique solution $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2/t \\ -1/t \end{bmatrix}$ whenever $t \neq 0$.

(iii) (5 points.) Compute the rank of A for each $t \in \mathbb{R}$.

Answer to 2(iii). Using the matrices in row echelon form that we computed in 2(i), we see that A has two nonzero rows (or equivalently, two pivot entries) when $t = 0$, and has three nonzero rows (or equivalently, three pivot entries) whenever $t \neq 0$. Therefore A has rank 2 when $t = 0$, and has rank 3 whenever $t \neq 0$.

Hint: Reduce the augmented matrix $[A|B]$ to row echelon form, and separate into two cases, depending on $t \in \mathbb{R}$. In both cases, the solution set and rank can then be computed explicitly.

Problem 3. (20 points). In this problem, we define the matrices

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 3 & 2 \\ 6 & 4 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

You may use the fact that A is invertible, with inverse

$$A^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 3 & -2 \\ -2 & 2 & -1 \end{bmatrix}.$$

(i) (5 points.) Specify the unique solution for $AX = B$.

Answer to 3(i). We use that A is invertible, and left-multiplying the equation by A^{-1} gives us that $X = A^{-1}AX = A^{-1}B$, and so we can compute that

$$X = A^{-1}B = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 3 & -2 \\ -2 & 2 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -4 \\ 7 \\ -2 \end{bmatrix}.$$

(ii) (5 points.) Specify the unique solution for $A^\top X = B$.

Answer to 3(ii). Since A is invertible, we know that its transpose $A^\top$ is invertible with inverse $(A^{-1})^\top$, and therefore we can compute that

$$X = (A^{-1})^\top A^\top X = (A^{-1})^\top B = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 3 & 2 \\ 2 & -2 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 7 \\ -10 \\ 6 \end{bmatrix}$$

(iii) (3 points.) By row reduction or otherwise, compute the inverse matrix C^{-1}.

Answer to 3(iii). We do a row reduction on the augmented matrix $[C|I_3]$ to compute that

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \overset{\langle 1\rangle \mapsto \langle 1\rangle + 2\langle 2\rangle}{\rightsquigarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

so we have compute the inverse matrix

$$C^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(iv) (7 points.) Specify the unique solution for $ACX = B$.

Answer to 3(iv). We note that the matrix AC is a product of invertible $n \times n$ matrices, and therefore has inverse $C^{-1}A^{-1}$. Therefore, we can left-multiply by $C^{-1}A^{-1}$ to find that

$$X = C^{-1}A^{-1}ACX = C^{-1}A^{-1}B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 2 \\ 0 & 3 & -2 \\ -2 & 2 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ 7 \\ -2 \end{bmatrix} = \begin{bmatrix} 10 \\ 7 \\ -2 \end{bmatrix}.$$

Hint: while problems (i) and (ii) can be solved by row reduction, this would be time-consuming. It is much quicker to use the formula for the matrix A^{-1}. Problem (iv) is similar.

Problem 4 (15 points). Computations.

Let $\mathbf{a} = [-1, 1, -1]$ and $\mathbf{b} = [0, 1, 4]$ be vectors.

(i) (3 points.) Compute the dot product $\mathbf{a} \cdot \mathbf{b}$.

Answer to 4(i). We compute that

$$\mathbf{a} \cdot \mathbf{b} = [-1, 1, -1] \cdot [0, 1, 4] = (-1) \cdot 0 + 1 \cdot 1 + (-1) \cdot 4 = 1 - 4 = -3.$$

(ii) (3 points.) Compute the projection $\operatorname{proj}_{\mathbf{a}} \mathbf{b}$. (I.e., the projection of $\mathbf{b}$ onto $\mathbf{a}$)

Answer to 4(ii). We recall the general formula for a projection, that

$$\operatorname{proj}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2} \mathbf{a},$$

and computing the norm $\|\mathbf{a}\| = \sqrt{(-1)^2 + 1^2 + (-1)^2} = \sqrt{3}$, we find that

$$\operatorname{proj}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2} \mathbf{a} = \frac{-3}{\sqrt{3}^2}[-1, 1, -1] = (-1)[-1, 1, -1] = [1, -1, 1].$$

(iii) (9 points.) Specify the solution sets for each of the following augmented matrices. (No proof is required for this part.)

(a) (3 points.) Specify the solution set associated to:

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & -2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Answer to 4(iii)(a). Since the bottom row contains the equation $0 = 1$, the system is inconsistent, and therefore there are no solutions, and the solution set is empty.

(b) (3 points.) Specify the solution set associated to:

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Answer to 4(iii)(b). Since the matrix to the left of the augmentation bar is square with full rank, it is invertible, so we have a unique solution. This system is simple enough to compute by hand (though row reductions also work). We see immediately that $x_3 = 3$, and using the equation $x_2 + 2x_3 = 7$, we compute that $x_2 = 7 - 2x_3 = 7 - 2 \cdot 3 = 1$. Similarly, we can use the equation $x_1 + 3x_3 = 0$ to compute that $x_1 = -3x_3 = -3 \cdot 3 = -9$, which gives us the solution set

$$\left\{ \begin{bmatrix} -9 \\ 1 \\ 3 \end{bmatrix} \right\},$$

which is a set with a single vector, corresponding to the unique solution to this problem.

(c) (3 points.) Specify the solution set associated to:

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Answer to 4(iii)(c). This is a system in reduced row echelon form, and so we can read off the solution set immediately. Letting x_3 be the free variable, we solve for the variables x_1 and x_2, and we use the equations $x_1 + 3x_3 = 2$ and $x_2 + 2x_3 = 3$ to compute that $[x_1, x_2, x_3] = [2 - 3x_3, 3 - 2x_3, x_3]$ provides a solution for any x_3, and since we have solved for the dependent variables x_1 and x_2, these are the only possible solutions. Therefore, we have found the solution set to be

$$\left\{ \begin{bmatrix} 2 - 3x_3 \\ 3 - 2x_3 \\ x_3 \end{bmatrix} \ : \ x_3 \in \mathbb{R} \right\}.$$

Problem 5. (15 points). Let $\mathbf{a} \in \mathbb{R}^n$ be any nonzero vector.

(i) (7 points.) Let P be the $n \times n$ matrix such that $PX = \operatorname{proj}_{\mathbf{a}} X$ for any vector $X \in \mathbb{R}^n$. Specify the (i,j)-entry of P. Using this formula, or otherwise, prove that $P^2 = P$, and prove that P is symmetric.

Answer to 5(i). Recall from Lecture 6 that we derived the formula

$$P_{ij} = \frac{a_i a_j}{\|\mathbf{a}\|^2}$$

for the projection matrix P associated to the vector $\mathbf{a}$. We can therefore compute the matrix product as

$$[P^2]_{ik} = \sum_{j=1}^{n} P_{ij}P_{jk} = \sum_{j=1}^{n} \frac{a_i a_j a_j a_k}{\|\mathbf{a}\|^4} = \frac{a_i a_k}{\|\mathbf{a}\|^4} \sum_{j=1}^{n} a_j^2 = \frac{a_i a_k}{\|\mathbf{a}\|^4} \|\mathbf{a}\|^2 = \frac{a_i a_k}{\|\mathbf{a}\|^2} = [P]_{ik}$$

for any indices i and k, and therefore $P^2 = P$. To prove symmetry of P, we see that $P_{ij} = P_{ji}$ by the formula for the projection matrix, but by the definition of the matrix transpose, we have $[P^\top]_{ij} = P_{ji} = P_{ij}$ for any indices i and j, and therefore $P = P^\top$, so P is a symmetric matrix.

(ii) (3 points.) Define the reflection matrix in the $\mathbf{a}$ direction as $R = I_n - 2P$, where P is given in (i). Using the answer to (i), show that $R^2 = I_n$.

Answer to 5(ii). We compute that

$$R^2 = (I_n - 2P)^2 = I_n^2 - 4P + 4P^2 = I_n - 4P + 4P = I_n,$$

where we have used the property $P = P^2$ proved in 5(i), and this proves the claim.

(iii) (5 points.) Let $\mathbf{a} = [1, -1, 0]$. Compute the associated projection matrix $P \in \mathcal{M}_{3,3}$, and find its reduced row echelon form. What is the rank of P? Is P invertible?

Answer to 5(iii). We use the formula for the projection matrix in part (i), and compute that

$$P = \frac{1}{\|\mathbf{a}\|^2} \begin{bmatrix} a_1a_1 & a_1a_2 & a_1a_3 \\ a_2a_1 & a_2a_2 & a_2a_3 \\ a_3a_1 & a_3a_2 & a_3a_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

but we can compute its reduced row echelon form to be

$$\begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \overset{\langle 2\rangle \mapsto \langle 1\rangle + \langle 2\rangle}{\sim} \begin{bmatrix} 1/2 & -1/2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \overset{\langle 1\rangle \mapsto 2\langle 1\rangle}{\sim} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which has a single nonzero row, and therefore P has rank 1. Since P is a 3×3 matrix with non-maximum rank, it is singular, i.e. not invertible.