GMT Seminar: Introduction to Integral Varifolds

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1 Introduction and preliminary results

Definition 1.1. [Sim84]

Let $G(n,k)$ be the set of k-dimensional linear subspaces of $\mathbb{R}^n$. Let M be locally $\mathcal{H}^k$-rectifiable, and let $\theta : \mathbb{R}^n \to \mathbb{N}$ be in L^1_{loc}. Then, an **integral varifold** V of dimension k in U is a Radon measure on $U \times G(n,k)$ acting on functions $\varphi \in C^0_c(U \times G(n,k)$ by

$$V(\varphi) \;=\; \int_M \varphi(x, T_x M)\, \theta(x)\, d\mathcal{H}^k.$$

By "projecting" $U \times G(n,k)$ onto the first factor, we arrive at the following definition:

Definition 1.2. [Lel12]

Let $U \subset \mathbb{R}^n$ be an open set. An **integral varifold** V of dimension k in U is a pair $V = (\Gamma, f)$, where (1) $\Gamma \subset U$ is a $\mathcal{H}^k$-rectifiable set, and (2) $f : \Gamma \to \mathbb{N} \setminus \{0\}$ is an L^1_{loc} Borel function (called the **multiplicity function** of V).

We can naturally associate to V the following Radon measure:

$$\mu_V(A) \;=\; \int_{\Gamma \cap A} f\, d\mathcal{H}^k \qquad \text{for any Borel set } A.$$

We define the **mass** of V to be $\mathcal{M}(V) := \mu_V(U)$.

We define the **tangent space** $T_x V$ to be the approximate tangent space of the measure μ_V, whenever this exists. Thus, $T_x V = T_x \Gamma$ $\mathcal{H}^k$-a.e.

Definition 1.3. [Lel12]

If $\Phi : U \to W$ is a diffeomorphism and $V = (\Gamma, f)$ an integral varifold in U, then the **pushforward of** V is $\Phi_\# V = (\Phi(\Gamma), f \circ \Phi^{-1})$, which is itself an integral varifold in W.

Definition 1.4. [Lel12]

If V is a varifold in U and $X \in C^1_c(U; \mathbb{R}^n)$, then the **first variation of V along X** is defined by

$$\delta V(X) \;=\; \frac{d}{dt}\bigg|_{t=0} \mathcal{M}\big((\Phi_t)_\# V\big), \tag{1.1}$$

where Φ_t is the one-parameter family generated by X.

Proposition 1.5. [Lel12]

Let $V = (\Gamma, f)$ be an integral varifold in $U \subset \mathbb{R}^n$. Then the right hand side of (1.1) is well-defined and

$$\delta V(X) \;=\; \int_U \operatorname{div}_{T_x\Gamma} X \, d\mu_V \qquad \textit{for all } X \in C^1_c(U;\mathbb{R}^n). \tag{1.2}$$

Proof. By the standard simplifying arguments, we may assume $\Gamma = F(\mathbb{R}^k)$ for F Lipschitz. Then,

$$\begin{aligned}
\mathcal{M}((\Phi_t)_\# V) \;&=\; \int_{\Phi_t(\Gamma)} f(\Phi_t^{-1}(z))\, d\mathcal{H}^k(z) \\
&=\; \int_\Gamma f(z)\, J\Phi_t|_z \, d\mathcal{H}^k(z) \\
&=\; \int_{F(\mathbb{R}^k)} f(z)\, J\Phi_t|_z \, d\mathcal{H}^k(z) \\
&=\; \int_{\mathbb{R}^k} f(F(x)))\, J\Phi_t|_{F(x)}\, JF|_x \, d\mathcal{H}^k(x).
\end{aligned}$$

Hence,

$$\delta V(X) \;=\; \frac{d}{dt}\Big|_{t=0} \mathcal{M}((\Phi_t)_\# V) \;=\; \int_{\mathbb{R}^k} f(F(x))) \left(\frac{d}{dt}\Big|_{t=0} J\Phi_t|_{F(x)} \right) JF|_x \, d\mathcal{H}^k(x).$$

From computations before, we know that $\frac{d}{dt}\big|_{t=0} J\Phi_t|_y = \operatorname{div}_{T_x\Gamma} X(y)$, which concludes the proof. □

Definition 1.6. [Lel12]

We say that V has **bounded generalized mean curvature** if there exists a $C \geq 0$ such that

$$|\delta V(X)| \;\leq\; C \int_U |X| \, d\mu_V \qquad \text{for all } X \in C^1_c(U;\mathbb{R}^n). \tag{1.3}$$

Proposition 1.7. [Lel12]

If V is a varifold in U with bounded generalized mean curvature, then there is a bounded Borel map $H : U \to \mathbb{R}^n$ such that

$$\delta V(X) \;=\; -\int_U X \cdot \vec{H} \, d\mu_V \qquad \textit{for all } X \in C^1_c(U;\mathbb{R}^n). \tag{1.4}$$

$\vec{H}$ is called the **generalized mean curvature** *of V and is defined μ_V-a.e.*

Proof. First, (1.2) tells us that δV is continuous in $C^1_c(U;\mathbb{R}^n)$. Since inequality (1.3) holds for all $X \in C^1_c(U;\mathbb{R}^n)$ and doesn't involve derivatives, by density it extends to all $X \in C^0_c$. For $X \in C^0_c(U;\mathbb{R}^n)$, let $\operatorname{spt} X \subset B_R$. Thus, δV is a bounded linear functional on $C^0_c(U;\mathbb{R}^n)$, so by Riesz we can find a Radon measure $||\delta V||$, which is the **total variation measure of** δV, and a $||\delta V||$-measurable function $\vec{\nu}$ with $|\vec{\nu}| = 1$ $||\delta V||$-a.e. such that

$$\delta V(X) \;=\; \int_U X \cdot \vec{\nu} \, d||\delta V||.$$

Furthermore, for A open,

$$||\delta V||(A) \;=\; \sup\{\delta V(X) : X \in C^0_c(A;\mathbb{R}^n),\, ||X||_{C^0} \leq 1\}. \tag{1.5}$$

Looking back at (1.3), for each $X \in C_c^0(A;\mathbb{R}^n)$, we know

$$|\delta V(X)| \;\leq\; C\int_U |X|\,d\mu_V \;\leq\; \mu_V(A\cap U)\,||X||_{C^0}. \tag{1.6}$$

Hence, we combine (1.5) and (1.6) to get

$$||\delta V||(A) \;=\; \sup\{\delta V(X) : \ldots\} \;\leq\; \sup\{C\mu_V(U\cap A)||X||_{C^0} : \ldots\} \;\leq\; C\mu_V(U\cap A),$$

so $||\delta V||$ is absolutely continuous with respect to μ_V. Hence, if we label the Radon-Nikodym derivative of $||\delta V||$ with resepct to μ_V as $-H$, which exists μ_V-a.e., and we label $\vec{H} = H\vec{\nu}$ then $\vec{\nu}\,d||\delta V|| = -\vec{H}\,d\mu_V$ and

$$\delta V(X) \;=\; -\int_U X\cdot\vec{H}\,d\mu_V$$

as desired. □

Remark 1.8. If we didn't have the inequality

$$|\delta V(X)| \;\leq\; \mu_V(A\cap U)\,||X||_{C^0}$$

for $X \in C_c^0(A;\mathbb{R}^n)$, then we couldn't conclude that $||\delta V||$ is absolutely continuous with respect to μ_V. However, if we knew that V had **locally bounded first variation** given by

$$|\delta V(X)| \;\leq\; c||X||_{C^0},$$

we still could still use Riesz and apply a Lebesgue decomposition to $||\delta V||$ to write

$$\vec{\nu}\,d||\delta V|| \;=\; -\vec{H}\,d\mu_V + d\mu_{sing}.$$

Then, we use a polar decomposition on μ_{sing} to write

$$d\mu_{sing} \;=\; \vec{\nu}_{co}\,d\sigma,$$

where $\vec{\nu}_{co}$ is the **generalized co-normal** and σ is the **generalized boundary measure**. In this way, we recover the full tangential divergence theorem

$$\begin{aligned}\delta V(X) \;&=\; \int_U \mathrm{div}_{T_x\Gamma}X\,d\mu_V \\ &=\; -\int_U X\cdot\vec{H}\,d\mu_V + \int_U X\cdot\vec{\nu}_{co}\,d\sigma.\end{aligned}$$

See [Sim84] Ch.8 for more details.

Definition 1.9. [Lel12]
V is **stationary** if $\delta V(X) = 0$ for all $X \in C_c^1(U,\mathbb{R}^n)$.

Remark 1.10. If $V = (\Gamma, f)$ is stationary, then the proposition is telling us that $\vec{H} \equiv 0$, so V has zero *generalized* mean curvature. If we suppose $f \equiv 1$, we can see that $\mu_V = \mathcal{H}^k\big|_{\Gamma\cap U}$. Then, for all $X \in C_c^1(U;\mathbb{R}^n)$, we have

$$\delta V(X) \;=\; -\int_{\mathbb{R}^n} X\cdot\vec{H}\,d\mu_V \;=\; -\int_\Gamma X\cdot\vec{H}\,d\mathcal{H}^k \;=\; 0.$$

Hence, Γ has zero mean curvature in U, so Γ is a minimal surface in U.

2 Compactness

Theorem 2.1. [Sim84]

Suppose $\{V_j\}$ is a sequence of integral varifolds in U which are of locally bounded first variation in U,

$$\sup_j \Big(\mu_{V_j}(W) \;+\; ||\delta V_j||(W)\Big) \;<\; \infty \qquad \forall\, W \subset\subset U.$$

Then, there exists a subsequence $\{V_{j'}\} \subset \{V_j\}$ and an integral varifold V of locally bounded first variation in U such that $V_{j'} \to V$ in the sense of Radon measures on $U \times G(n,k)$, and $||\delta V||(W) \leq \liminf_j ||\delta V_{j'}||(W)$ for all $W \subset\subset U$.

Corollary 2.2. If in addition the $\{V_j\}$ are stationary, then, by the LSC property of the total variation measure $||\delta V||$, the limit varifold V is also stationary.

Remark 2.3. First, we note that convergence in the sense of Radon measures on $U \times G(n,k)$ is called **varifold convergence**. Second, for fixed $X \in C_c^1(U;\mathbb{R}^n)$, we note that the first variation functional is continuous with respect to varifold convergence. By definition, $V_j \to V$ as varifolds if $V_j(\varphi) \to V(\varphi)$ for all $\varphi \in C_c^0(U \times G(n,k);\mathbb{R}^n)$. For $X \in C_c^1(U;\mathbb{R}^n)$,

$$\delta V(X) \;=\; \int_{U\times G(n,k)} \operatorname{div}_S X(x)\, dV(x,S) \;=\; \int_{U\times G(n,k)} \varphi(x,S)\, dV(x,S),$$

for $\varphi(x,S) = \operatorname{div}_S X(x) \in C_c^0(U \times G(n,k);\mathbb{R}^n)$. Hence,

$$\delta V_j(X) \to \delta V(X).$$

3 Monotonicity formula

For a differentiable function $g : U \to \mathbb{R}$ and a varifold $V = (\Gamma, f)$ in U, we denote by $\nabla^\perp g(x)$ the orthogonal projection of ∇g onto $(T_x\Gamma)^\perp$, e.g. the normal part of the gradient. For fixed $\xi \in U$, define $r(x) := |x - \xi|$.

Theorem 3.1. [Lel12]

Let V be an integral varifold of dimension k in U with bounded generalized mean curvature $\vec{H}$. Fix $\xi \in U$. For every $0 < \sigma < \rho < \operatorname{dist}(\xi, U)$ we have the **Monotonicity Formula**

$$\frac{\mu_V(B_\rho(\xi))}{\rho^k} - \frac{\mu_V(B_\sigma(\xi))}{\sigma^k} \;=\; \int_{B_\rho(\xi)} \frac{\vec{H}}{k}\cdot(x-\xi)\left(\frac{1}{m(r)^k} - \frac{1}{\rho^k}\right) d\mu_V \;+\; \int_{B_\rho(\xi)\setminus B_\sigma(\xi)} \frac{|\nabla^\perp r|^2}{r^k}\, d\mu_V,$$

where $m(r) = \max\{r, \sigma\}$. Hence, the map $\rho \mapsto e^{\rho||H||_\infty}\rho^{-k}\mu_V(B_\rho(\xi))$ is monotone increasing.

Proof. Without loss of generality, we assume $\xi = 0$. We fix a function $\gamma \in C_c^1([0,1])$ such that $\gamma \equiv 1$ in some neighborhood of 0. For $s \in [0, \operatorname{dist}(0, \partial U)]$, we define the vector field $X_s(x) := \gamma\left(\frac{|x|}{s}\right) x$. Then, $X_s \in C_c^1(U)$, so we can combine (1.2) and (1.4) to conclude

$$\int \operatorname{div}_{T_x\Gamma} X_s \, d\mu_V \;=\; -\int X_s \cdot \vec{H}\, d\mu_V. \tag{3.1}$$

Our goal now is compute both sides of (3.1), rearrange it in a smart way, use a dominated convergence argument to replace γ by the indicator function $\mathbb{1}_{[0,1]}$, and conclude the identity.

We fix a point x, and let $\pi = T_x\Gamma$. Let $e_1, \ldots, e_k$ be an orthonormal basis for π, and complete it to an orthonormal basis for $\mathbb{R}^n$. Now, recalling that $r := |x - \xi| = |x|$, we compute

$$\begin{aligned} \operatorname{div}_\pi X_s &= k\,\gamma\left(\frac{r}{s}\right) + \sum_{j=1}^{k} e_j \cdot x\,\gamma'\left(\frac{r}{s}\right)\,\frac{x \cdot e_j}{|x|\,s} \\ &= k\,\gamma\left(\frac{r}{s}\right) + \frac{r}{s}\gamma'\left(\frac{r}{s}\right)\,\sum_{j=1}^{k}\left(\frac{x\cdot e_j}{|x|}\right)^2 \\ &= k\,\gamma\left(\frac{r}{s}\right) + \frac{r}{s}\gamma'\left(\frac{r}{s}\right)\left[1 - \sum_{j=k+1}^{n}\left(\frac{x\cdot e_j}{|x|}\right)^2\right] \\ &= k\,\gamma\left(\frac{r}{s}\right) + \frac{r}{s}\gamma'\left(\frac{r}{s}\right)\left(1 - |\nabla^\perp r|^2\right). \end{aligned} \tag{3.2}$$

Now, we insert (3.2) into (3.1), divide both sides by s^{k+1}, and integrate s between σ and ρ:

$$\begin{aligned} &\int_\sigma^\rho \int_{\mathbb{R}^n} \frac{k}{s^{k+1}}\gamma\left(\frac{|x|}{s}\right)\,d\mu_V(x)\,ds + \int_\sigma^\rho\int_{\mathbb{R}^n}\frac{r}{s^{k+2}}\,\gamma'\left(\frac{|x|}{s}\right)\left(1 - |\nabla^\perp r|^2\right)\,d\mu_V(x)\,ds \\ &= -\int_\sigma^\rho\int_{\mathbb{R}^n}\frac{\vec{H}\cdot x}{s^{k+1}}\,\gamma\left(\frac{|x|}{s}\right)\,d\mu_V(x)\,ds. \end{aligned}$$

Then, we use Fubini's theorem to change the order of integration, we distribute the integrand in the term $(1 - |\nabla^\perp r|^2)$, and we move the $|\nabla^\perp r|^2$ term to the right hand side to find:

$$\int_{\mathbb{R}^n}\int_\sigma^\rho \left[\frac{k}{s^{k+1}}\gamma\left(\frac{|x|}{s}\right) + \frac{|x|}{s^{k+2}}\,\gamma'\left(\frac{|x|}{s}\right)\right]\,ds\,d\mu_V(x) \tag{3.3}$$

$$= \int_{\mathbb{R}^n}|\nabla^\perp r|^2\int_\sigma^\rho \frac{|x|}{s^{k+2}}\,\gamma'\left(\frac{|x|}{s}\right)\,ds\,d\mu_V(x) - \int_{\mathbb{R}^n}\vec{H}\cdot x\int_\sigma^\rho\frac{1}{s^{k+1}}\,\gamma\left(\frac{|x|}{s}\right)\,d\mu_V(x)\,ds. \tag{3.4}$$

Looking at (3.3), we note that

$$\frac{d}{ds}\left[\frac{1}{s^k}\,\gamma\left(\frac{|x|}{s}\right)\right] = -\frac{k}{s^{k+1}}\,\gamma\left(\frac{|x|}{s}\right) + \frac{1}{s^k}\,\gamma'\left(\frac{|x|}{s}\right)\,\frac{-|x|}{s^2},$$

so that

$$\begin{aligned} &-\int_\sigma^\rho\left[\frac{k}{s^{k+1}}\gamma\left(\frac{|x|}{s}\right) + \frac{|x|}{s^{k+2}}\,\gamma'\left(\frac{|x|}{s}\right)\right]\,ds \\ &= \int_\sigma^\rho \frac{d}{ds}\left[\frac{1}{s^k}\,\gamma\left(\frac{|x|}{s}\right)\right]\,ds \\ &= \frac{1}{\rho^k}\,\gamma\left(\frac{|x|}{\rho}\right) - \frac{1}{\sigma^k}\,\gamma\left(\frac{|x|}{\sigma}\right). \end{aligned}$$

Now, we subsititute this indentity and the resulting identities

$$\begin{aligned} -\int_\sigma^\rho \frac{k}{s^{k+1}}\gamma\left(\frac{|x|}{s}\right)\,ds &= \int_\sigma^\rho\frac{|x|}{s^{k+2}}\,\gamma'\left(\frac{|x|}{s}\right)\,ds + \frac{1}{\rho^k}\,\gamma\left(\frac{|x|}{\rho}\right) - \frac{1}{\sigma^k}\,\gamma\left(\frac{|x|}{\sigma}\right) \\ -\int_\sigma^\rho\frac{|x|}{s^{k+2}}\,\gamma'\left(\frac{|x|}{s}\right)\,ds &= \int_\sigma^\rho\frac{k}{s^{k+1}}\gamma\left(\frac{|x|}{s}\right)\,ds + \frac{1}{\rho^k}\,\gamma\left(\frac{|x|}{\rho}\right) - \frac{1}{\sigma^k}\,\gamma\left(\frac{|x|}{\sigma}\right) \end{aligned}$$

into (3.3) to find

$$\rho^{-k}\int_{\mathbb{R}^n}\gamma\left(\frac{|x|}{\rho}\right)\,d\mu_V(x)-\sigma^{-k}\int_{\mathbb{R}^n}\gamma\left(\frac{|x|}{\sigma}\right)\,d\mu_V(x)-\int_{\mathbb{R}^n}\vec{H}\cdot x\int_\sigma^\rho\frac{1}{s^{k+1}}\gamma\left(\frac{|x|}{s}\right)\,ds\,d\mu_V(x)$$
$$=\int_{\mathbb{R}^n}|\nabla^\perp r|^2\left[\rho^{-k}\gamma\left(\frac{|x|}{\rho}\right)-\sigma^{-k}\gamma\left(\frac{|x|}{\sigma}\right)+\int_\sigma^\rho\frac{k}{s^{k+1}}\gamma\left(\frac{|x|}{s}\right)\,ds\right]\,d\mu_V(x). \tag{3.5}$$

Our initial choice of $\gamma\in C_c^1([0,1])$ was arbitrary, and we arrived at (3.6). This expression doesn't involve any derivatives of γ, and, since $0<\sigma<\rho<\operatorname{dist}(\xi,\partial U)$, the integrands are products of nice bounded functions. Hence, we can use a dominated convergence theorem argument to pass from a sequence of nonnegative $C_c^1([0,1])$ functions γ_n converging from below to $\mathbb{1}_{[0,1]}$ and directly insert $\mathbb{1}_{[0,1]}$ into (3.6) to find:

$$\rho^{-k}\mu_V(B_\rho(0))-\sigma^{-k}\mu_V(B_\sigma(0))-\int_{\mathbb{R}^n}\vec{H}\cdot x\int_\sigma^\rho\frac{1}{s^{k+1}}\mathbb{1}_{[0,1]}\left(\frac{|x|}{s}\right)\,ds\,d\mu_V(x)$$
$$=\int_{\mathbb{R}^n}|\nabla^\perp r|^2\left[\rho^{-k}\mathbb{1}_{B_\rho(0)}(x)-\sigma^{-k}\mathbb{1}_{B_\sigma(0)}(x)+\int_\sigma^\rho\frac{k}{s^{k+1}}\mathbb{1}_{[0,1]}\left(\frac{|x|}{s}\right)\,ds\right]\,d\mu_V(x). \tag{3.6}$$

Finally, we compute the integral

$$\int_\sigma^\rho\frac{k}{s^{k+1}}\mathbb{1}_{[0,1]}\left(\frac{|x|}{s}\right)\,ds. \tag{3.7}$$

Observe that (3.7) is equal to 0 if $|x|>\rho$, since then $\frac{|x|}{s}>1$ for all $\sigma<s<\rho$. If $|x|\le\rho$, then $\frac{|x|}{s}\le 1$ for $s\in[|x|,\rho]$. In this case, we can change our limits of integration to $[\max\{\sigma,|x|\},\rho]$, where we have to take the max of $\sigma,|x|$ in case $|x|<\sigma$. Thus, (3.7) becomes

$$\int_\sigma^\rho\frac{k}{s^{k+1}}\mathbb{1}_{[0,1]}\left(\frac{|x|}{s}\right)\,ds\;=\;\mathbb{1}_{B\rho(0)}(x)\int_{\max\{|x|,\sigma\}}^\rho\frac{k}{s^{k+1}}\,ds\;=\;\left(\frac{1}{\max\{|x|,\sigma\}^k}-\frac{1}{\rho^k}\right)\mathbb{1}_{B_\rho(0)}(x). \tag{3.8}$$

We can insert this identity directly into the right-hand-side of (3.6) and $\frac{1}{k}$ times this identity into the left-hand-side of (3.6) to find

$$\rho^{-k}\mu_V(B_\rho(0))-\sigma^{-k}\mu_V(B_\sigma(0))-\int_{B_\rho(0)}\frac{\vec{H}\cdot x}{k}\left(\frac{1}{\max\{|x|,\sigma\}^k}-\frac{1}{\rho^k}\right)\,d\mu_V(x)$$
$$=\int_{\mathbb{R}^n}|\nabla^\perp r|^2\left[\rho^{-k}\mathbb{1}_{B_\rho(0)}(x)-\sigma^{-k}\mathbb{1}_{B_\sigma(0)}(x)+\left(\frac{1}{\max\{|x|,\sigma\}^k}-\frac{1}{\rho^k}\right)\mathbb{1}_{B_\rho(0)}(x)\right]\,d\mu_V(x)$$
$$=\int_{\mathbb{R}^n}|\nabla^\perp r|^2\left[\frac{1}{\max\{|x|,\sigma\}^k}\mathbb{1}_{B_\rho(0)}(x)-\frac{1}{\sigma^k}\mathbb{1}_{B_\sigma(0)}(x)\right]\,d\mu_V(x) \tag{3.9}$$

Finally, we note that

$$\frac{1}{\max\{|x|,\sigma\}^k}\mathbb{1}_{B_\rho(0)}(x)-\frac{1}{\sigma^k}\mathbb{1}_{B_\sigma(0)}(x)\;=\;\begin{cases}0 & |x|\le\sigma\\ \frac{1}{|x|^k} & \sigma<|x|<\rho\\ 0 & \rho\le|x|\end{cases}$$

so that (3.9) becomes

$$\int_{\mathbb{R}^n}|\nabla^\perp r|^2\left[\frac{1}{\max\{|x|,\sigma\}^k}\mathbb{1}_{B_\rho(0)}(x)-\frac{1}{\sigma^k}\mathbb{1}_{B_\sigma(0)}(x)\right]\,d\mu_V(x)\;=\;\int_{B\rho(0)\setminus B_\sigma(0)}\frac{|\nabla^\perp|^2}{r^k}\,d\mu_V,$$

which conludes the Monotonicity Formula.

We conclude by showing the second statement of the theorem. Define $f(\rho) := \rho^{-k}\mu_V(B_\rho)$. We use the Monotonicity Formula to bound

$$\frac{f(\rho)-f(\sigma)}{\rho-\sigma} \geq -\frac{||H||_{L^\infty}}{k}\int_{B_\rho} |x| \frac{\max\{|x|,\sigma\}^{-k}-\rho^{-k}}{\rho-\sigma} d\mu_V(x) \geq -\frac{||H||_{L^\infty}}{k}\rho\frac{\sigma^{-k}-\rho^{-k}}{\rho-\sigma}\mu_V(B_\rho).$$

Since the map $\rho \mapsto \rho^{-k}$ is convex, setting $\rho = \sigma+\epsilon$ we conclude

$$\frac{f(\sigma+\epsilon)-f(\sigma)}{\epsilon} \geq -\mu_V(B_\rho)||H||_{L^\infty}(\sigma+\epsilon)\sigma^{-k-1} = -||H||_{L^\infty} f(\sigma+\epsilon)\frac{(\sigma+\epsilon)^{k+1}}{\sigma^{k+1}}. \tag{3.10}$$

If ψ_δ is a standard smooth nonnegative mollifier, we take the convolution of both sides of (3.10) as functions of σ, and then let $\epsilon \downarrow 0$ to conclude $(f \star \psi_\delta)' + ||H||_{L^\infty}(f\star\psi_\delta) \geq 0$. Hence the function $\rho \mapsto e^{\rho||H||_{L^\infty}}(f\star\psi_\delta)(\rho)$ is monotone increasing. Letting $\delta\downarrow 0$, we conclude that $\rho \mapsto e^{\rho||H||_{L^\infty}}\rho^{-k}\mu_V(B_\rho)$ is also monotone increasing. □

Remark 3.2. Looking at the proof above, we never fully needed the strong assumption $\vec{H} \in L^\infty$ until we proved the specific map $\rho \mapsto e^{\rho||H||_{L^\infty}}\rho^{-k}\mu_V(B_\rho)$ is monotone increasing. Up until that point, we only needed $\vec{H} \in L^p_{loc}$ for some p and the generalized boundary measure $\sigma \equiv 0$ for the divergence theorem

$$\int_U \operatorname{div}_{T_x\Gamma} X \, d\mu_V \;=\; -\int_U X\cdot\vec{H}\, d\mu_V$$

to hold.

It turns out that the right condition to assume is $\vec{H} \in L^p_{loc}$ for some $p > n$, as in this case we have the monotonicity formula

$$(\sigma^{-n}\mu_V(B_\sigma(\xi)))^{1/p} \;\leq\; (\rho^{-n}\mu_V(B_\rho(\xi)))^{1/p} + \frac{||\vec{H}||_{L^p(B_R(\xi))}}{p-n}(\rho^{1-n/p}-\sigma^{1-n/p}),$$

for $B_R(\xi) \subset\subset U$ and $0 < \sigma < \rho \leq R$, and Corollary 3.3 below holds in this case. These two results are Theorem 17.7 and Corollary 17.8 in [Sim84].

Corollary 3.3. [Lel12]

Let $V = (\Gamma, f)$ be an integral varifold of dimension k in U with bounded generalized mean curvature. Then,

(i) the limit

$$\theta_V(x) := \lim_{\rho\downarrow 0}\frac{\mu_V(B_\rho(x))}{\omega_k\rho^k}$$

exists for all $x \in U$ and coincides with $f(x)$ μ_V-a.e.,

(ii) $\theta_V(x)$ is upper semicontinuous.

Proof. The existence of the limit is guaranteed by the montonicity of $\rho \mapsto e^{\rho||H||_{L^\infty}}\rho^{-k}\mu_V(B_\rho)$. Moreover, $\theta_V = f$ μ_V-a.e. by the standard density theorems. For upper semicontinuity, fix $x \in U$ and $\epsilon > 0$. Let $0 < 2\rho < \operatorname{dist}(x, \partial U)$ be such that

$$e^{r||H||_{L^\infty}}\frac{\mu_V(B_r(x))}{\omega_k r^k} \;\leq\; \theta_V(x) + \frac{\epsilon}{2} \qquad \forall\, r < 2\rho. \tag{3.11}$$

Then, if $\delta < \rho$ and $|x-y| < \delta$, we then conclude

$$\begin{aligned}
\theta_V(y) &\leq e^{\rho||H||_{L^\infty}} \frac{\mu_V(B_\rho(y))}{\omega_k \rho^k} \\
&\leq e^{(\rho+\delta)||H||_{L^\infty}} \frac{\mu_V(B_{\rho+\delta}(x))}{\omega_k \rho^k} \\
&= e^{(\rho+\delta)||H||_{L^\infty}} \frac{\mu_V(B_{\rho+\delta}(x))}{\omega_k (\rho+\delta)^k} \frac{(\rho+\delta)^k}{\rho^k} \\
&= e^{(\rho+\delta)||H||_{L^\infty}} \frac{\mu_V(B_{\rho+\delta}(x))}{\omega_k (\rho+\delta)^k} \left(1+\frac{\delta}{\rho}\right)^k \\
&\stackrel{(3.11)}{\leq} \left(\theta_V(x) + \frac{\epsilon}{2}\right)\left(1+\frac{\delta}{\rho}\right)^k .
\end{aligned}$$

If δ is sufficiently small, then we conclude $\theta_V(y) \leq \theta_V(x) + \epsilon$, which proves (ii). □

Corollary 3.4. [Sim84]

Let V be a k-dimensional integral varifold in $U \subset \mathbb{R}^n$, and let $\xi \in U$. Let $\eta_{\xi,\lambda} : x \mapsto \frac{x-\xi}{\lambda}$. Suppose for some sequence of $\lambda_j \to 0$ that $V_j := (\eta_{\xi,\lambda_j})_\# V$ converges to W in the sense of Radon measures, where W is a k-dimensional integral varifold which is stationary in all of $\mathbb{R}^n$. Then, W is a cone, in the sense that $W = (C, \psi)$, where

- (i) C is $\mathcal{H}^k$-rectifiable and is invariant under all homotheties $x \mapsto \lambda^{-1}x$ for $\lambda > 0$,
- (ii) ψ is an $\mathcal{H}^k$ locally integrable, integer-valued function such that $\psi(x) = \psi(\lambda^{-1}x)$ for all $x \in C$ and $\lambda > 0$.

Proof. (Sketch) The theorem is proved by showing that taking

$$\begin{aligned}
C &= \{x : \theta_W(x) > 0\} \\
\psi &= \theta_W(x)
\end{aligned}$$

satisfy the conclusions.

First, we note that

$$\begin{aligned}
\frac{\mu_W(B_\sigma(0))}{\sigma^k} &= \lim_{j\to\infty} \frac{\mu_{V_j}(B_\sigma(0))}{\sigma^k} \\
&= \lim_{j\to\infty} \frac{\mu_V(B_{\lambda_j \sigma}(\xi))}{(\lambda_j\sigma)^k} \\
&= \omega_k \theta_V(\xi)
\end{aligned}$$

independent of σ. Therefore, since W is stationary, the monotonicity formula reduces to

$$\sigma^{-n}\mu_W(B_\sigma(0)) = \rho^{-n}\mu_W(B_\rho(0)) - \int_{B_\rho(0)\setminus B_\sigma(0)} \frac{|\nabla^\perp r|^2}{r^n}\, d\mu_W,$$

which implies from above that $|\nabla^\perp r|^2 = 0$ μ_W-a.e. Since $\xi = 0$, we have $r(x) = |x|$, which has gradient $\nabla r(x) = \frac{x}{|x|}$. Since the normal part of ∇r is 0 everywhere, we conclude $x \in T_x W$ μ_W-a.e. Then, by a careful argument, we conclude that if h is homogeneous degree zero function, e.g. $h(x) = h\left(\frac{x}{|x|}\right)$, then

$$\frac{1}{\rho^n}\int_{B_\rho(0)} h\, d\mu_W = \text{const.} \quad \text{independent of } \rho.$$

This is enough to deduce that $\lambda^{-n}\mu_W(\lambda A) = \mu_W(A)$ for any Borel set A. This gives the invariance of the function θ_W under homotheties, which in turn implies the invariance of C under homotheties. □

Corollary 3.5. Let V be a stationary integral varifold. Then, for every sequence $\lambda_j \to 0$ there exists a subsequence $\lambda_{j'}$ such that $(\eta_{\xi,\lambda_{j'}})_{\#}V$ converges in the sense of varifolds to a cone.

Proof. Follows by the Compactness Theorem 2.1 and Corollary 3.4. □

References

[Lel12] Camillo De Lellis. Allard's interior regularity theorem: an invitation to stationary varifolds, 2012.

[Sim84] L. M. Simon. Lectures on geometric measure theory, 1984.