GEOMETRIC ANALYSIS SUMMER COURSE

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These are lecture notes from a week-long summer course given at UT Austin in 2018. They're currently missing the material from chapters 1, 2, and 4 of [Li12], and I may update them sometime later to include this content. So all of the information below is from [Lee97].

1 Background

Note: all manifolds are presumed to be smooth (C^∞), Hausdorff, and second countable.

Definition 1.1. Let V be a finite-dimensional vector space. V^* is called the space of **covectors** on V (a.k.a. the dual), and we denote the pairing $V^* \times V \to \mathbb{R}$ by

$$(\omega, X) \mapsto \langle \omega, X \rangle \quad \text{or} \quad (\omega, X) \mapsto \omega(X).$$

A **covariant k-tensor on V** is a multilinear map

$$F : \underbrace{V \times \cdots \times V}_{k \text{ copies}} \to \mathbb{R}.$$

A **contravariant l-tensor on V** is a multilinear map

$$F : \underbrace{V^* \times \cdots \times V^*}_{l \text{ copies}} \to \mathbb{R}.$$

A **tensor of type $\binom{k}{l}$** is a multilinear map

$$F : \underbrace{V^* \times \cdots \times V^*}_{l \text{ copies}} \times \underbrace{V \times \cdots \times V}_{k \text{ copies}} \to \mathbb{R}.$$

The **space of all covariant k-tensors** is denoted by $T^k(V)$, and the **space of all contravariant l-tensors** is denoted by $T_l(V)$. The **space of all mixed $\binom{k}{l}$-tensors** is denoted by $T^k_l(V)$. The **rank** of a tensor is the number of arguments (vectors and/or covectors) it takes.

Definition 1.2. We define the **tensor product** as follows: if $F \in T^k_l(V)$ and $G \in T^p_q(V)$, then the tensor $F \otimes G \in T^{k+p}_{l+q}(V)$ is defined by

$$\begin{aligned} &F \otimes G(\omega^1, \ldots, \omega^{l+q}, X_1, \ldots, X_{k+p}) \\ &\qquad = F(\omega^1, \ldots, \omega^l, X_1, \ldots, X_k)\, G(\omega^{l+1}, \ldots, \omega^{l+q}, X_{k+1}, \ldots, X_{k+p}). \end{aligned}$$

Definition 1.3. If $(E_1, \ldots, E_n)$ is a basis for a vector space V, then $(\varphi^1, \ldots, \varphi^n)$ denotes the **dual basis** for V^*, defined by $\varphi^i(E_j) = \delta^i_j$.

A basis for $T^k_l(V)$ is given by the set of all tensors of the form

$$E_{j_1} \otimes \ldots \otimes E_{j_l} \otimes \varphi^{i_1} \otimes \ldots \otimes \varphi^{i_k},$$

where the indices i_p, j_q range from 1 to n. These tensors act on basis elements by

$$E_{j_1} \otimes \ldots \otimes E_{j_l} \otimes \varphi^{i_1} \otimes \ldots \otimes \varphi^{i_k}(\varphi^{s_1}, \ldots, \varphi^{s_l}, E_{r_1}, \ldots, E_{r_k}) = \delta^{s_1}_{j_1} \ldots \delta^{s_l}_{j_l} \delta^{i_1}_{r_1} \ldots \delta^{i_k}_{r_k}.$$

Any tensor $F \in T^k_l(V)$ can be written as

$$F = F^{j_1 \ldots j_l}_{i_1 \ldots i_k} E_{j_1} \otimes \ldots \otimes E_{j_l} \otimes \varphi^{i_1} \otimes \ldots \otimes \varphi^{i_k},$$

where

$$F^{j_1 \ldots j_l}_{i_1 \ldots i_k} = F(\varphi^{j_1}, \ldots, \varphi^{j_l}, E_{i_1}, \ldots, E_{i_k}).$$

Definition 1.4. A **k-form** is a covariant k-tensor on V that changes sign whenever two arguments are interchanged. The space of all such k-froms is denoted by $\Lambda^k(V)$.

There is a natural bilinear, associative product on forms called the **wedge product**, defined on 1-forms $\omega^1, \ldots, \omega^k$ by setting

$$\omega^1 \wedge \ldots \wedge \omega^k(X_1, \ldots, X_k) \; = \; \det\big(\langle \omega^i, X_j \rangle\big)$$

and extending by linearity.

Definition 1.5. Given a smooth manifold M, **local coordinates for a point** $p \in M$ are functions $x^i : U \subset M \to \mathbb{R}$ for open U containing p.

Definition 1.6. For any point p in a smooth manifold M, we define the **tangent space** T_pM as follows: pick a coordinate chart $\phi : U \to \mathbb{R}^n$, where U is an open subset of M containing p. Suppose that two curves $\gamma_1, \gamma_2 : (-1,1) \to M$ are such that $\gamma_1(0) = p = \gamma_2(0)$ and $\phi \circ \gamma_1$, $\phi \circ \gamma_2 : (-1,1) \to \mathbb{R}^n$ are differentiable. γ_1 and γ_2 are equivalent at 0 iff the derivatives of $\phi \circ \gamma_1$ and $\phi \circ \gamma_2$ at 0 agree. This defines an equivalence relation on the set of all differentiable curves initialized at 0, and the equivalence classes of all such curves are call the **tangent vectors** of M at p. The **tangent space of M at** p, denoted T_pM, is defined as the set of all tangent vectors at p.

Definition 1.7. We can find a **basis of the tangent space at a point** p as follows: given a chart $\varphi = (x^1, \ldots, x^n) : U \to \mathbb{R}^n$ with $p \in U$, we can define an ordered basis

$$\left(\left(\frac{\partial}{\partial x^i} \right)_p \right)_{i=1}^{n}$$

of T_pM by

$$\forall f \in C^\infty(M) : \quad \left(\frac{\partial}{\partial x^i} \right)_p (f) \; := \; \big(\partial_i (f \circ \varphi^{-1}) \big) (\varphi(p)).$$

Then, for every tangent vector $v \in T_pM$, we have

$$v = \sum_{i=1}^{n} v(x^i) \cdot \left(\frac{\partial}{\partial x^i} \right)_p .$$

Definition 1.8. A **vector bundle** consists of:

1. a pair of smooth manifolds E (the *total space*) and M (the *base*)
2. a continuous surjective map $\pi : E \to M$ (the *projection*)
3. a vector space structure on each set $E_p := \pi^{-1}(p)$ (the *fiber* of E over p)

such that for every $p \in M$, there exists an open neighborhood U of p, an integer k, and a diffeomorphism $\varphi : U \times \mathbb{R}^k \to \pi^{-1}(U)$ such that for all $q \in U$

- $(\pi \circ \varphi)(q, v) = q$ for all vectors $v \in \mathbb{R}^k$
- the map $v \mapsto \varphi(q, v)$ is a linear isomorphism between $\mathbb{R}^k$ and $\pi^{-1}(x)$.

Definition 1.9. The **tangent bundle** is given by $TM = \dot{\bigcup}_{p \in M} T_pM$, and the **cotangent bundle** is given by $T^*M = \dot{\bigcup}_{p \in M} (T_pM)^*$.

Definition 1.10. The **bundle of $\binom{k}{l}$-tensors on** M is defined as

$$T^k_l M \ := \ \dot{\bigcup_{p \in M}} T^k_l(T_pM),$$

and the **bundle of k-forms** is

$$\Lambda_k M \ := \ \dot{\bigcup_{p \in M}} \Lambda^k(T_pM).$$

Definition 1.11. If $\pi : E \to M$ is a vector bundle over M, a **section of** E is a map $F : M \to E$ such that $F(p) \in E_p = \pi^{-1}(p)$ for all $p \in M$. It is said to be a **smooth section** if it is smooth as a map between manifolds.

Definition 1.12. A **tensor field** on M is a smooth section of some tensor bundle $T^k_l M$, and a **differentiable k-form** is a smooth section of some $\Lambda^k M$. The **space of all $\binom{k}{l}$-tensor fields** is denoted by $\mathcal{T}^k_l(M)$, and the **space of all covariant k-tensor fields** (smooth sections of $T^k M$) is denoted by $\mathcal{T}^k(M)$.

Definition 1.13. A **vector field** X is a section of the tangent bundle, denoted $X \in \Gamma(TM)$. In coordinates,

$$X = X^i \partial_i, \quad X^i \in C^\infty(M),$$

where $\partial_i = \frac{\partial}{\partial x^i}$ is the coordinate basis as defined above.

Definition 1.14. A **1-form** ω is a section of the cotangent bundle, denoted $\omega \in \Gamma(T^*M)$. In coordinates,

$$\omega = \omega_i dx^i, \quad w_i \in C^\infty(M).$$

2 Some Riemannian geometry

2.1 Riemannian metric

Definition 2.1. A **Riemannian metric** on a smooth manifold M is a family of positive definite inner products

$$g_p : T_pM \times T_pM \to \mathbb{R}, \quad p \in M$$

such that for all vector fields X, Y on M

$$p \mapsto g_p(X(p), Y(p))$$

defines a smooth function $M \to \mathbb{R}$.

Equivalently, a Riemannian metric is a 2-tensor field $g \in \mathcal{T}^2(M)$ that is **symmetric**, e.g. $g(X,Y) = g(Y,X)$, and **positive definite**, e.g. $g(X,X) > 0$ if $X \neq 0$.

In coordinates (x^i), the components of g at a point p are given by

$$g_{ij}(p) := g_p\left(\left(\frac{\partial}{\partial x^i}\right)_p, \left(\frac{\partial}{\partial x^j}\right)_p\right),$$

so that

$$g = g_{ij}\, dx^i \otimes dx^j,$$

where $\{dx^1, \dots, dx^n\}$ is the dual basis of the tangent bundle.

Definition 2.2. A Riemannian metric thus determines an inner product on each tangent space T_pM, which is typically written $\langle X, Y\rangle := g(X,Y)$ for $X, Y \in T_pM$.

We define the **length** of any tangent vector $X \in T_pM$ to be $|X| := \langle X, X\rangle^{1/2}$. We define the **angle** between any two nonzero vectors $X, Y \in T_pM$ to be the unique $\theta \in [0, \pi]$ such that $\cos\theta = \langle X, Y\rangle/(|X|\,|Y|)$. Finally, X and Y are **orthogonal** if $\langle X, Y\rangle = 0$. $E_1, \dots, E_n$ are **orthonormal** if $\langle E_i, E_j\rangle = \delta_{ij}$.

Definition 2.3. Finally, a **Riemannian manifold** is a smooth manifold M together with a Riemannian metric g, written (M, g).

2.2 The musical isomorphisms

Definition 2.4. The metric g gives an isomorphism between TM and T^*M called the **flat operator**,

$$\flat : TM \to T^*M$$

defined by

$$\flat(X)(Y) = g(X,Y).$$

The inverse map is denoted by $\sharp : T^*M \to TM$. The cotangent bundle is then endowed with the metric

$$\langle \omega_1, \omega_2\rangle = g(\sharp\omega_1, \sharp\omega_2).$$

If $X \in \Gamma(TM)$, then

$$\flat(X) = X_i dx^i,$$

where $X_i = g_{ij} X^j$. Hence the flat operator "lowers" an index, or equivalently converts a vector into a covector.

If $\omega \in \Gamma(T^*M)$, then

$$\sharp(\omega) = \omega^i \partial_i,$$

where $\omega^i = g^{ij}\omega_j$ (and g^{ij} are the elements of the inverse matrix $(g_{ij})^{-1}$). Hence the sharp operator "raises" an index, or equivalent converts a covector into a vector.

2.3 Connections on vector bundles

Definition 2.5. Let $\pi : E \to M$ be a vector bundle, and let $\Gamma(E)$ denote the space of all smooth sections of E. A **connection** is a map

$$\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E), \quad (X, Y) \mapsto \nabla_X Y$$

with the properties

1. $\nabla_X Y$ is linear over $C^\infty(M)$ in X:

$$\nabla_{fX_1+gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y \qquad \text{for } f, g \in C^\infty(M)$$

2. $\nabla_X Y$ is linear over $\mathbb{R}$ in Y:

$$\nabla_X(aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2 \qquad \text{for } a, b \in \mathbb{R}$$

3. ∇ satisfies the following product rule:

$$\nabla_X(fY) = f\nabla_X Y + (Xf)Y \qquad \text{for } f \in C^\infty(M).$$

for all $f \in C^\infty(M)$, $X \in \Gamma(TM)$, and $Y \in \Gamma(E)$.

Definition 2.6. A connection on the tangent bundle is called a **linear connection**:

$$\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM).$$

We note that $\nabla_X Y$ is called the **covariant derivative of Y in the direction of X**. In coordinates (x^i), we can expand $\nabla_{\partial_i}\partial_j$ in terms of the coordinates:

$$\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k$$

These functions Γ^k_{ij} are called the **Christoffel symbols** of ∇ with respect these coordinates.

Proposition 2.7. *Let ∇ be a linear connection, and let $X, Y \in \Gamma(TU)$ (a smooth section of the tangent space in an open set U). Write $X = X^i\partial_i$ and $Y = Y^j\partial_j$. Then,*

$$\nabla_X Y = (XY^k + X^i Y^j \Gamma^k_{ij})\partial_k.$$

Hence, the action of a linear connection on U is completely determined by its Christoffel symbols.

2.4 Riemannian connection

Definition 2.8. Let g be a Riemannian metric on a manifold M. A linear connection ∇ is said to be **compatible with** g if it satisfies the following product rule for all vector fields X, Y, Z :

$$\nabla_X \langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle.$$

Definition 2.9. The **torsion tensor** of a linear connection is

$$\tau : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$$

defined by

$$\tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

where $[X, Y]$ is the smooth vector field given by

$$[X, Y](f) = X(Y(f)) - Y(X(f))$$

for all $f \in C^\infty(M)$.

Definition 2.10. A linear connection ∇ is said to be **torsion free** (or symmetric) if its torsion vanishes identically, e.g.

$$\nabla_X Y - \nabla_Y X \equiv [X, Y].$$

Theorem 2.11. **[Fundamental theorem of Riemannian geometry]** *Let (M, g) be a Riemannian manifold. There exists a unique linear connection ∇ on M that is compatible with g and torsion free.*

Proof. We only do a sketch of the proof:

By a computation, the connection is determined by

$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}\Big(X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle Y, [X, Z]\rangle - \langle Z, [Y, X]\rangle + \langle X, [Z, Y]\rangle\Big).$$

Using coordinates (x^i) and letting $X = \partial_i$, $Y = \partial_j$, and $Z = \partial_k$, we can reduce the above to

$$\langle \nabla \partial_i \partial_j, \partial_k\rangle = \frac{1}{2}(\partial_i\langle \partial_j, \partial_k\rangle + \partial_j\langle \partial_k, \partial_i\rangle - \partial_k\langle \partial_i, \partial_j\rangle),$$

where we used the fact that the Lie brackets of coordinate vector fields are zero. Now, recalling the definitions of the metric coefficients and Christoffel symbols:

$$g_{ij} = \langle \partial_i, \partial_j\rangle, \qquad \nabla_{\partial_i}\partial_j = \Gamma_{ij}^m \partial_m,$$

we can insert these above to find

$$\Gamma_{ik}^k = \frac{1}{2} g^{kl}\Big(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\Big).$$

This is the formula for the **Riemannian Christoffel symbols**. $\square$

2.5 Curvature in the tangent bundle

Definition 2.12. The **curvature endomorphsim** is a map

$$R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$$

defined by

$$R(X,Y)\,Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

If $X = \partial/\partial x_i$ and $Y = \partial/\partial x_j$ are coordinate vector fields, then $[X,Y] = 0$, so

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z,$$

i.e. the curvature endomorphism measures the *noncommutativity of the covariant derivative.*

We can write the curvature endomorphism in coordinates (x^i) by

$$R = R^l_{ijk} dx^i \otimes dx^j \otimes dx^k \otimes \partial_l,$$

where the coefficients R^l_{ijk} are determined by

$$R(\partial_i, \partial_j)\partial_k = R^l_{ijk}\partial_l.$$

Definition 2.13. The **curvature tensor** is then given by

$$Rm(X,Y,Z,W) := \langle R(X,Y)Z, W\rangle$$

for vector fields X, Y, Z, W.

We can write it in local coordinates by

$$Rm = R_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l,$$

where $R_{ijkl} = g_{lm} R^m_{ijk}$.

Proposition 2.14. *We have the following symmetries:*

$$\begin{aligned}
R(X,Y)Z &= -R(Y,X)Z \\
0 &= R(X,Y)Z - +R(Y,Z)X - R(Z,X)Y \\
Rm(X,Y,Z,W) &= -Rm(X,Y,W,Z) \\
Rm(X,Y,Z,W) &= Rm(W,Z,X,Y).
\end{aligned}$$

Definition 2.15. The **Ricci tensor** as follows: given an orthonormal basis ∂_i for T_pM with respect to $g(p)$, we define the Ricci curvature by

$$Ric_p(X,Y) = \sum_i \langle R(\partial_i, X)Y, \partial_i\rangle.$$

The components of Ric are denoted by R_{ij}, so that

$$Ric = R_{ij} dx^i \otimes dx^j,$$

where $R_{ij} := R^k_{kij}$ is from the above definition of the curvature endomorphism (we're taking the trace of the curvature endomorphism on its first and last components – hence the k in both the sub- and super-scripts).

Definition 2.16. Finally, the **scalar curvature** is the function S defined as the trace of the Ricci tensor:

$$S := g^{ij} R_{ij}.$$

2.6 Riemannian submanifolds and the second fundamental form

Definition 2.17. Let $(\tilde{M}, \tilde{g})$ be a Riemannian manifold of dimension m, let M be a manifold of dimension n, and let $\iota : M \to \tilde{M}$ be an embedding. If M is given the induced Riemannian metric $g := \iota^* \tilde{g}$, then f is said to be an **isometric embedding**, and M is said to be a **Riemannian submanifold** of $\tilde{M}$. $\tilde{M}$ is called the **ambient manifold**.

We henceforth assume that M is an embedded Riemannian submanifold of $\tilde{M}$. We will use the standard symbols, e.g. ∇, for operations in M, and we will use add tildes, e.g. $\tilde{\nabla}$, to write these operations in $\tilde{M}$. However, we can unambiguously use the inner-product notation $\langle X, Y \rangle$ to refer to either g or $\tilde{g}$, since g is just the restriction of $\tilde{g}$ to TM.

Definition 2.18. We can see that the set

$$T\tilde{M}\big|_M := \dot{\bigcup}_{p \in M} T_p \tilde{M}$$

is a smooth vector bundle over M. We call it the **ambient tangent bundle over** M.

At each $p \in M$, the ambient tangent space $T_p\tilde{M}$ splits as orthogonal direct sum $T_p\tilde{M} = T_pM \bigoplus N_pM$, where $N_pM := (T_pM)^\perp$ is the **normal space** at p with respect to the inner product $\tilde{g}$ on $T_p\tilde{M}$.

The set

$$NM := \dot{\bigcup}_{p \in M} N_pM$$

is called the **normal bundle** of M. This is a smooth vector bundle over M. Given a point $p \in M$, there is an open set $\tilde{U}$ of p in $\tilde{M}$ and a smooth orthonormal frame $(E_1, \ldots, E_m)$ on $\tilde{U}$ called an **adapted orthonormal frame** satisfying the following: the restrictions of $(E_1, \ldots, E_n)$ to M form a local orthonormal frame for TM and the last $m - n$ vectors $(E_{n+1}|_p, \ldots, E_m|_p)$ form a basis for N_pM at each $p \in M$.

Definition 2.19. Projecting orthogonally at each point $p \in M$ onto the subspaces T_pM and N_pM gives maps called the **tangential** and **normal projections**

$$\pi^\top : T\tilde{M}|_M \to TM$$
$$\pi^\perp : T\tilde{M}|_M \to NM.$$

In terms of adapted orthonormals frames, these are the usual projections onto $\mathrm{span}(E_1, \ldots, E_n)$ and $\mathrm{span}(E_{n+1}, \ldots, E_m)$. If X is a section of $T\tilde{M}|_M$, we use the shorthand notation $X^\top := \pi^\top X$ and $X^\perp := \pi^\perp X$ for its tangential and normal projections.

Definition 2.20. If X, Y are vector fields in $\Gamma(TM)$, we can extend them to be vector fields on $\tilde{M}$, apply the ambient covariant derivative $\tilde{\nabla}$, and decompose them at points of M to get

$$\tilde{\nabla}_X Y = \left(\tilde{\nabla}_X Y\right)^\top + \left(\tilde{\nabla}_X Y\right)^\perp.$$

We define the **second fundamental form** of M to be the map $I\!I$ (read "two")

$$I\!I : \Gamma(TM) \times \Gamma(TM) \to \Gamma(NM)$$

given by

$$I\!I(X, Y) := \left(\tilde{\nabla}_X Y\right)^\perp,$$

where X and Y have been extended arbitrarily to $\tilde{M}$. Since $\pi_\perp$ maps smooth sections to smooth sections, $I\!I(X, Y)$ is a smooth section of NM.

Proposition 2.21. *The second fundamental form is*

1. *independent of the extensions X and Y*
2. *bilinear over $C^\infty(M)$*
3. *symmetric in X and Y.*

Theorem 2.22. **[Gauss formula]** *If $X, Y \in \Gamma(TM)$ are extended arbitrarily to vector fields on $\tilde{M}$, the following holds along M:*

$$\tilde{\nabla}_X Y = \nabla_X Y + I\!I(X, Y).$$

Definition 2.23. Give a unit normal vector field N, we can define the **scalar second fundamental form** h to be the symmetric 2-tensor on M defined by

$$h(X, Y) := \langle I\!I(X, Y), N \rangle.$$

Definition 2.24. Raising one index of h, we get a tensor field $s \in \mathcal{T}^1_1(M)$ called the **shape operator** of M. It is characterized by

$$\langle X, sY \rangle = h(X, Y) \quad \forall X, Y \in \Gamma(TM).$$

Because h is symmetric, s is self-adjoint:

$$\langle sX, Y \rangle = \langle X, sY \rangle.$$

At any point $p \in M$, the shape operator s is a self-adjoint linear transformation on the tangent space T_pM. Hence, it has real eigenvalues $\kappa_1, \ldots, \kappa_n$ and an orthonormal basis $(E_1, \ldots, E_n)$ for T_pM such that $sE_i = \kappa_i E_i$ (no summation). In this basis, both h and s are diagonal, and h has the expression

$$h(X, Y) = \kappa_1 X^1 Y^1 + \ldots + \kappa_n X^n Y^n.$$

The eigenvalues of s are called the **principal curvatures** of M at p, and the corresponding eigenspaces are called the **principal directions**.

Definition 2.25. There are two combinations of the principal curvatures that are of particular importance: the **Gaussian curvature** is given by

$$K = \kappa_1 \kappa_2 \cdots \kappa_n,$$

and the **mean curvature** is given by

$$H = \frac{1}{n}(\kappa_1 + \kappa_2 + \ldots + \kappa_n).$$

3 Quick reference of dietary restrictions

Object	**Eats**	**Gives**
Vector fields	Functions	Functions
$\partial/\partial x^i$	Functions	Functions
k-forms	k-many vectors	Scalars
	k-many vector fields	Functions
dx^i	Single vector fields	Functions
Riemannian metric	Pairs of vectors	Scalars
	Pairs of vector fields	Functions
Covectors	Single vectors	Scalars
	Single vector fields	Functions
Covariant tensors	Many vectors	Scalars
	Many vector fields	Functions
Contravariant tensors	Many covectors	Scalars
	Many covector fields	Functions

References

[Lee97] John M. Lee. Riemannian manifolds, 1997.

[Li12] Peter Li. Geometric analysis, 2012.