VISCOSITY TECHNIQUES IN GEOMETRIC VARIATIONAL PROBLEMS

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1 Introduction

We want to study the energy

$$\mathcal{P}(E) + \int_E g \, dx, \tag{1.1}$$

where $\mathcal{P}$ is the perimeter functional and $E \subset \mathbb{R}^d$ is a set of finite perimeter. Physically, $\mathcal{P}$ is the isotropic surface tension, and g accounts for external forces. In particular, we consider the unconstrained variational problem:

$$\inf_E \ \mathcal{P}(E) + \int_E g \, dx. \tag{P}$$

Question of Almgren: *If E is a minimizer of (P) for g convex, is E convex?*

The goal of this talk is to highlight the techniques used to prove this. First, we use the following function:

$$S_{\partial E}(x) = \sup_{y \in \partial E} \langle \nu_{\partial E}(x), y - x \rangle, \tag{1.2}$$

which measures the "non-convexity" of ∂E. We can see how it measures convexity by expanding

$$\langle \nu_{\partial E}(x), y - x \rangle = |y - x| \cos \theta,$$

where θ is the angle between $\nu_{\partial(E)}$ and $y - x$. If E is, say, a sphere, then $\theta \in [\pi/2, 3\pi/2]$, so $\langle \nu_{\partial(E)}(x), y - x \rangle \leq 0$, in which case $S_{\partial E} \equiv 0$. If E is not convex, then we can find x and y such that $\cos \theta > 0$ and, hence, $S_{\partial E}(x) > 0$.

We will prove that this function is a viscosity subsolution of one equation, which we will use to then show it's a viscosity subsolution of a different equation. This second equation combined with a stability inequality is what allows us to conclude that $S_{\partial E} \equiv 0$.

2 Preliminaries

2.1 Geometric measure theory

Definition 2.1. Given a Lebesgue measurable set $E \subset \mathbb{R}^n$, we define the **perimeter** of E to be

$$\mathcal{P}(E) = \sup \left\{ \int_E \operatorname{div} T(x)\, dx \ :\ T \in C_c^1(\mathbb{R}^n;\mathbb{R}^n),\ \sup_{\mathbb{R}^n} |T(x)| \le 1 \right\}.$$

In the case that E has C^1 boundary, $\mathcal{P}(E) = \mathcal{H}^{n-1}(\partial E)$.

2.2 Differential geometry

Let $M \subset \mathbb{R}^{n+1}$ be a C^2, n-dimensional manifold oriented by its normal ν_M. Let D denote the flat connection in $\mathbb{R}^{n+1}$, e.g. the normal gradient operator. For vector fields X, Y, the **covariant derivative** on M is

$$\nabla_X Y \ = \ \mathbf{p}_{T_x M}\, D_X Y,$$

where $\mathbf{p}_{T_x M}$ is the orthogonal projection onto the tangent space $T_x M$. The **second fundamental form** A_M is defined on tangent vectors X, Y by

$$A_M(x)[X(x), Y(x)] \ = \ -\langle D_X Y(x), \nu_M(x)\rangle.$$

For a C^1 function f defined on a neighborhood of M, the **tangential gradient** $\nabla f(x) \in T_x M$ is defined by

$$\nabla f \ = \ \mathbf{p}_{T_x M} D f.$$

If X is a tangent vector field, then

$$\nabla_X f = \langle Df, X\rangle.$$

The **tangential Hessian** $\nabla^2 f(x) : T_x M \times T_x M \to \mathbb{R}$ is given by its action on tangent vectors X, Y defined in a neighborhood of x by

$$\nabla^2 f(x)[X(x), Y(x)] \ = \ D^2 f(x)[X(x), Y(x)] - A_M[X, Y](x)\, \langle Df(x), \nu_M(x)\rangle.$$

Note that $\nabla^2 f$ only depends on the values of f on M and of X and Y at x, so it is tensorial. The **Laplace-Beltrami opertator** of f is given by

$$\Delta_M f \ = \ \operatorname{tr} \nabla^2 f.$$

The **mean curvature** H_M is given by

$$H_M = \operatorname{div}_M(\nu_M) = \Big(\operatorname{tr} A_M = \sum_i A_M[\tau_i, \tau_i] \text{ in geodesic coordinates } \tau_i\Big).$$

3 Background results

Theorem 3.1. *Assume that $g \in C^{1,\alpha}(\mathbb{R}^d)$ satisfies*

$$\lim_{|x|\to+\infty} g(x) = +\infty$$

Then, there exists a minimizer of (P) satisfying:

(i) E is equivalent to an open bounded set.

(ii) Let Σ be the singular set of ∂E. Then, Σ is closed, $\mathcal{H}^{d-8+\epsilon}(\Sigma) = 0$ for all $\epsilon > 0$, and for all $x \in \partial E \setminus \Sigma$ there exists a neighborhood U_x such that $E \cap U_x$ can be locally written as the epi-graph of a C^3 function. In particular, $\partial E \setminus \Sigma$ is a (relatively open) C^3 manifold.

(iii) We have

$$H_{\partial E} + g = 0 \qquad \text{for all } x \in \partial E \setminus \Sigma, \tag{3.1}$$

and

$$\int_{\partial E\setminus\Sigma} |\nabla\varphi|^2 - |A_{\partial E}|^2\varphi^2 + \varphi^2\, D_\nu g \geq 0, \tag{3.2}$$

for all $\varphi \in C^1_c(\partial E \setminus \Sigma)$.

Remark 1. (i) and (ii) follow from showing E is a (Λ, r_0)-minimizer and using density arguments. The first part of (iii) follows from the first variation of perimeter and Proposition 17.8 in [Mag12], and the second part of (iii) follows from the second variation of perimeter.

4 Proof of theorem

Let $E \subset \mathbb{R}^{n+1}$ be a bounded open set such that its boundary can be split as

$$\partial E = R_{\partial E} \cup \Sigma_{\partial E},$$

where $R_{\partial E}$ is a C^3 manifold oriented by $\nu_{\partial E}$ and $\Sigma_{\partial E}$ is a closed singular set with empty relative interior. Given $x \in R_{\partial E}$ and $y \in \partial E$, define

$$S_{\partial E}(x,y) = \langle \nu_{\partial E}(x), y - x\rangle$$

and

$$S_{\partial E}(x) = \max_{y\in\partial E} S_{\partial E}(x,y) \geq 0, \tag{4.1}$$

where the maximum is attained since ∂E is compact.

For $\varphi \in C^2(R_{\partial E})$, define the Jacobi operator

$$\mathcal{L}\varphi = \Delta_{\partial E}\varphi + |A_{\partial E}|^2\varphi, \tag{4.2}$$

Notice that by integration by parts, if $\varphi \in C^2_c(\partial E)$, then

$$\int_{R_{\partial E}} \varphi\,(-\mathcal{L}\varphi) + \varphi^2\, D_\nu g \;=\; \int_{R_{\partial E}} |\nabla\varphi|^2 - |A_{\partial E}|^2\varphi^2 + D_\nu g\varphi^2, \tag{4.3}$$

which is the stability inequality from (3.2).

Roadmap.

1. We show $S_{\partial E}$ is a viscosity solution of

$$\mathcal{L}S_{\partial E}(\bar{x}) \;\geq\; H_{\partial E}(\bar{x}) - H_{\partial E}(\bar{y}) + \langle \nabla H_{\partial E}(\bar{x}), \bar{y} - \bar{x}\rangle$$

   using the doubling of variables trick.

2. We use this to show that $S_{\partial E}$ is a viscosity solution of

$$\mathcal{L}S_{\partial E} - D_\nu g\, S_{\partial E} \;\geq\; \omega(S_{\partial E}),$$

   where ω is a modulus of convexity (to be defined below). Note that the left-hand-side is exactly the operator in the stability inequality.

3. We show that $S_{\partial E} \in H^1(\partial E)$. This tells us that it can be approximated by test functions, so, by a limit argument, we can plug it into the stability inequality.

4. We use the stability inequality combined with the second PDE to derive a contradiction unless $S_{\partial E} \equiv 0$, e.g. E is convex.

Remark 2. Why do we focus on viscosity solutions? A priori, we have no reason to expect S to be C^2 or even C^1. Viscosity solutions only need to be continuous, so it's a sufficiently weak framework in which to put this function, because we only care to show that it must be identically 0. This allows us to use only basic differential geometry to prove that its a solution – no analysis is needed. Then, we can chain together these two equations to get exactly the operator in the stability inequality without ever touching regularity, and the only regularity work we have to do to directly plug S into the stability inequality as our test function is show it's in H^1. So, by using viscosity solutions, we don't have to worry about regularity except in the final step when we need to show H^1, which turns out to be simple.

Remark 3. The use of viscosity solutions in geometric problems was introduced by Ben Andrews to show the preservation of the interior ball condition along the mean curvature flow. There's a nice survey paper relating to these techniques:

"Moduli of continuity, isoperimetric profiles, and multi-point estimates in geometric heat equations" Andrews 2014.

Lemma 4.1. *Let E be as above and $\bar{x} \in R_{\partial E}$. Assume that*

$$S_{\partial E}(\bar{x}) = S_{\partial E}(\bar{x}, \bar{y}),$$

with $\bar{y} \in R_{\partial E}$. Then,

$$\mathcal{L}S_{\partial E}(\bar{x}) \geq H_{\partial E}(\bar{x}) - H_{\partial E}(\bar{y}) + \langle \nabla H_{\partial E}(\bar{x}), \bar{y} - \bar{x}\rangle$$

in the viscosity sense, meaning that for all $\varphi \in C^2(R_{\partial E})$ such that

$$\varphi(x) - S_{\partial E}(x) \geq \varphi(\bar{x}) - S_{\partial E}(\bar{x}) = 0$$

then

$$\mathcal{L}\varphi(\bar{x}) \geq H_{\partial E}(\bar{x}) - H_{\partial E}(\bar{y}) + \langle \nabla H_{\partial E}(\bar{x}), \bar{y} - \bar{x} \rangle.$$

Proof. Since the set E is fixed, we drop the dependence on E in the various quantities for notational simplicity. We exploit the **doubling of variables trick** from the theory of viscosity solutions. To do so, let φ be as in the statement of the theorem, and define

$$G(x,y) := \varphi(x) - S(x,y),$$

which achieves its local minimum at $(\bar{x}, \bar{y})$. Moreover, by assumption, it is C^2 in a neighborhood of $(\bar{x}, \bar{y})$. In order to prove the theorem, we will exploit the first and second order critical point conditions of G at the point $(\bar{x}, \bar{y})$.

We omit the tedious computations of the identites below, but they follow from nothing more than simplifications coming from choosing a geodesic frame $\{\tau_i\}$.

Since G achieves a minimum at $(\bar{x}, \bar{y})$, we have

$$0 = \nabla_i^x G(\bar{x}, \bar{y}) = \nabla_i^x \varphi(\bar{x}) - \nabla_i^x S(\bar{x}, \bar{y}), \tag{4.4}$$
$$0 = \nabla_i^y G(\bar{x}, \bar{y}) = -\nabla_i^y S(\bar{x}, \bar{y}). \tag{4.5}$$

We now compute the Hessian of G:

$$\begin{aligned}
\nabla_i^x \nabla_j^x G(\bar{x}, \bar{y}) &= \nabla_i^x \nabla_j^x \varphi(\bar{x}) - \nabla_i^x A_{jk}(\bar{x}) \langle \tau_k^x(\bar{x}), \bar{y} - \bar{x} \rangle + A_{jk}(\bar{x}) A_{ki}(\bar{x}) \varphi(\bar{x}) + A_{ij}(\bar{x}), \\
\nabla_i^x \nabla_j^y G(\bar{x}, \bar{y}) &= -A_{ij}(\bar{x}), \\
\nabla_i^y \nabla_j^y G(\bar{x}, \bar{y}) &= A_{ij}(\bar{y}).
\end{aligned} \tag{4.6}$$

We note that if a symmetric block matrix is positive semi-definite,

$$\begin{bmatrix} A & B \\ B & -C \end{bmatrix} \geq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

then $\mathrm{tr}(A) \geq \mathrm{tr}(C) - 2\mathrm{tr}(B)$. We see this by taking $\{e_i\}$ any standard basis for $\mathbb{R}^n$ and computing:

$$0 \;\leq\; \begin{bmatrix} A & B \\ B & -C \end{bmatrix} \begin{bmatrix} e_1 \\ \vdots \\ e_n \\ e_1 \\ \vdots \\ e_n \end{bmatrix} \cdot \begin{bmatrix} e_1 \\ \vdots \\ e_n \\ e_1 \\ \vdots \\ e_n \end{bmatrix} \;=\; \sum e_i \cdot A e_i + 2 \sum e_i \cdot B e_i - \sum e_i \cdot C e_i \;=\; \mathrm{tr} A + 2\mathrm{tr} B - \mathrm{tr} C,$$

Since G has a minimum at $(\bar{x}, \bar{y})$,

$$0 \leq \begin{bmatrix} \nabla^x \nabla^x G(\bar{x}, \bar{y}) & \nabla^x \nabla^y G(\bar{x}, \bar{y}) \\ \nabla^x \nabla^y G(\bar{x}, \bar{y}) & \nabla^y \nabla^y G(\bar{x}, \bar{y}) \end{bmatrix} = \begin{bmatrix} \nabla^x \nabla^x G(\bar{x}, \bar{y}) & -A(\bar{x}) \\ -A(\bar{x}) & A(\bar{y}). \end{bmatrix}$$

Hence,

$$\operatorname{tr}(\nabla^x\nabla^x G(\bar{x},\bar{y})) \geq -\operatorname{tr} A(\bar{y}) + 2\operatorname{tr} A(\bar{x}).$$

Therefore, using the fact that $A_{ii} = W_i^i$ in our choice of frame and the Codazzi equations

$$\sum_i \nabla_i^x A_{ik} \;=\; \sum_i \nabla_k^x A_{ii} \;=\; \nabla_k^x H,$$

we find

$$\begin{aligned} &\Delta_{\partial E}\varphi(\bar{x}) - \nabla_k^x H(\bar{x})\langle \tau_k^x(\bar{x}),\ \bar{y}-\bar{x}\rangle + |A|^2\varphi(\bar{x}) + H(\bar{x}) \;\geq\; -H(\bar{y}) + 2H(\bar{x}) \\ \implies\;& \Delta_{\partial E}\varphi(\bar{x}) + |A|^2\varphi(\bar{x}) \;\geq\; H(\bar{x}) - H(\bar{y}) + \langle \nabla H(\bar{x}),\ \bar{y}-\bar{x}\rangle \end{aligned}$$

□

Remark 4. If we hadn't used the doubling of variables trick, then we would have had

$$0 \;\leq\; \operatorname{tr}\nabla_i^x\nabla_j^x G(\bar{x}) \;=\; \Delta_{\partial E}\varphi(\bar{x}) + |A|^2\varphi(\bar{x}) + H(\bar{x}) - \langle \nabla H(\bar{x}), \bar{y}-\bar{x}\rangle.$$

By doubling the variables, we could change the sign of $H(\bar{x})$ and introduce a $+H(\bar{y})$. In the next Lemma, this will allow us to upgrade S to be a viscosity solution of the operator in the stability inequality with a modulus of convexity right hand side.

Also notice that we were able to prove this lemma using only basic differential geometry – no PDE theory was required.

Definition 4.2. For $g \in C^1(\mathbb{R}^d)$, we say that an increasing function ω is a *modulus of convexity* for g if

$$g(y) - g(x) - \langle Dg(x), y-x\rangle \geq \omega(|y-x|) \qquad \forall x,y \in \mathbb{R}^d.$$

We note that every convex function has zero as a modulus of convexity and that every strictly convex function has a strictly positive modulus of convexity.

Lemma 4.3. *Let $g \in C^{1,\alpha}$ be convex and coercive with modulus of convexity ω and let E be a minimizer of either (P). Let $S_{\partial E}$ be the corresponding function defined in* (3.1)*. Then,*

(i) For every $x \in \partial E \setminus \Sigma$ the $\bar{y}$ achieving the maximum in the definition of $S_{\partial E}$ is in $\partial E \setminus \Sigma$.

(ii) $S_{\partial E} \in H^1(\partial E \setminus \Sigma)$.

(iii) $S_{\partial E}$ solves

$$\mathcal{L}S_{\partial E} - D_\nu g\, S_{\partial E} \geq \omega(S_{\partial E}) \qquad \text{on } \partial E \setminus \Sigma \tag{4.7}$$

both in the viscosity and distributional sense.

Proof. Since the set E is fixed, we drop the dependence on E in the various quantities.

Proof of (i): Easy hyperplane argument that uses some lemmas I don't want to bother with.

Proof of (ii): Bound $|\nabla S(x)| \le \mathrm{diam(E)}\,|A|(x)$, so the proof boils down to showing $A \in L^2(\partial E \setminus \Sigma)$.

Proof of (iii): That viscosity solutions are distributional solutions was proved by Ishii in 1995. Hence, we just show S is a viscosity solution of (4.7). Thanks to (i), we know that for every $\bar{x} \in \partial E \setminus \Sigma$ the point $\bar{y}$ achieving the maximum in S is in $\partial E \setminus \Sigma$. Therefore, Lemma 3.2 implies that

$$\mathcal{L}\, S(\bar{x}) \ge H(\bar{x}) - H(\bar{y}) + \langle \nabla H(\bar{x}), \bar{y} - \bar{x} \rangle$$

in the viscosity sense. Differentiating the first variation equation,

$$H + g = 0,$$

we see $\nabla H = -\nabla g$. We subtract from both sides of the above inequality the identity

$$D_\nu g(\bar{x}) S(\bar{x}) = D_\nu g(\bar{x}) \langle \nu(\bar{x}), \bar{y} - \bar{x} \rangle$$

to obtain

$$\begin{aligned}
&\mathcal{L}\, S(\bar{x}) - D_\nu(\bar{x}) S(\bar{x}) \\
&\qquad \ge H(\bar{x}) - H(\bar{y}) - \langle \nabla g(\bar{x}), \bar{y} - \bar{x} \rangle - D_\nu g(\bar{x}) \langle \nu(\bar{x}), \bar{y} - \bar{x} \rangle \\
&\qquad = g(\bar{y}) - g(\bar{x}) - \langle Dg(\bar{x}), (\bar{y} - \bar{x}) \rangle \\
&\qquad \ge \omega(|\bar{x} - \bar{y}|)
\end{aligned}$$

where the second to last inequality follows from (3.1) and the last inequality follows from the definition of the modulus of convexity for g. Since by definition $S(\bar{x}) \le |\bar{x} - \bar{y}|$ and since ω is increasing, this concludes the proof. ☐

Theorem 1.4. *Let E be a minimizer of (P) and assume that $g \in C^{1,\alpha}$ is convex and coercive. Then E is convex.*

Proof. Our goal is to prove $S \equiv 0$, which will imply the convexity of E. As usual, we use S as a test function, and we use the equivalence of (3.2) and (4.3) to see

$$\int_{\partial E \setminus \Sigma} (-\mathcal{L} S) S + D_\nu g\, S^2 \ge 0. \tag{4.8}$$

Multiplying (4.7) by $-S$ we obtain the inequality

$$(-\mathcal{L} S) S + D_\nu g\, S^2 \le -\omega(S) S,$$

which after integration gives

$$-\int_{\partial E \setminus \Sigma} \omega(S) S \ge \int_{\partial E \setminus \Sigma} (-\mathcal{L} S) S + D_\nu g\, S^2. \tag{4.9}$$

If g is strictly convex, then we can take $\omega > 0$, which gives a contradiction with (4.8) unless $S \equiv 0$. If not, then we compute the Euler-Lagrange equation of a certain functional and use a different contradiction to conclude. ☐

5 Conclusion

By using viscosity solutions, we were able to do a lot of work with only basic differential geometry, and the doubling of variables trick changed the sign of a term and introduced a new term that allowed us to show our function was a solution of exactly the operator in the stability inequality. The only regularity work we needed to do was show it was in H^1, which was so simple we omitted the proof.