Geometry, topology, and analysis: a mathematical triad

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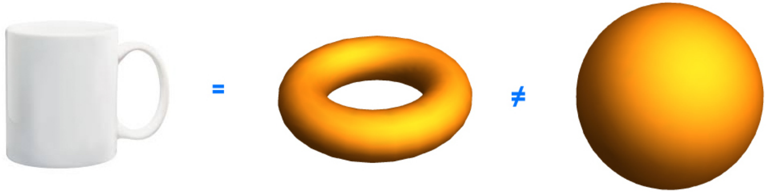
Geometry

Loosely, geometry is the study of shapes made from (possibly high dimensional) paper: it is concerned with properties like distance, which is invariant under translations, rotations, and bending (but not stretching or tearing).

Topology

Loosely, topology is the study of shapes made from a (possibly high dimensional) rubbersheet: it is concerned with properties that are invariant under translations, rotations, bending, and stretching (but not tearing).



Analysis

Analysis is the study of calculus, differential equations, differential operators, etc.

Overview

In this talk I will discuss how these three areas of mathematics are intertwined. We will discuss the following three topics:

1. Gauss-Bonnet: the underlying topology restricts the possible geometry.
2. de Rham's theorem: the number of solutions to a certain differential equation is the number of holes the space has.
3. *Can you hear the shape of a drum?*: How much of the geometry of a space can be recovered from the eigenvalues of the Laplace operator?

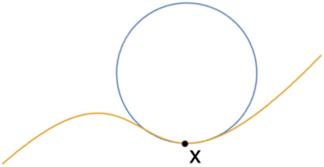
Geometry

Curvature

A key geometric invariant of a shape is its curvature. How should we define this?

Start off with the curvature of a curve:

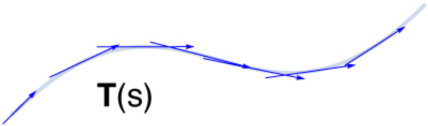
1. Define the curvature of a circle of radius R should be $1/R$ (at any point on the circle): the smaller the radius, the larger the curvature.
2. For an arbitrary curve, we define its curvature at a point x, $\kappa(x)$, to be $1/R$ where R is the radius of the *osculating circle at* x: the circle that is tangent to the curve at x and most closely approximates the curve:



Equivalently:

$$\kappa(x(s)) = \left|\left|\frac{d\mathbf{T}}{ds}\right|\right|$$

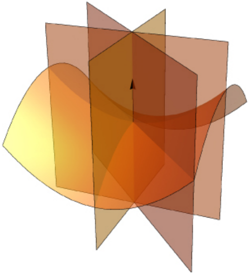
where **T** is the unit tangent vector along the curve, parametrized with respect to arclength s.

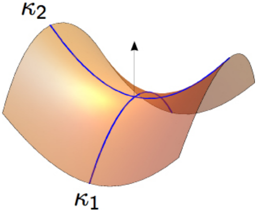


Curvature of a surface

The curvature $\kappa(x)$ of a surface Σ at a point x is defined as follows:

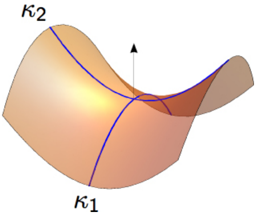
1. Consider all normal planes at x: these are planes that contain the normal vector to Σ at x.
2. Each one will intersect Σ in a curve through x. Consider the curvature of all such curves, taking curvature to be negative if the curve curves away from the normal vector.
3. Define $K(x) = \kappa_1 \kappa_2$ where κ_1 is the smallest such curvature and κ_2 the largest. These curves will always be perpendicular and their curvatures are called *principal curvatures*.



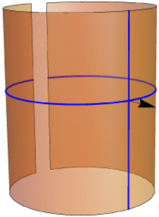


Curvature of a surface

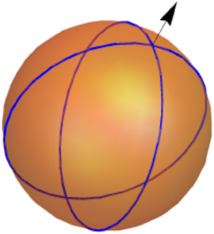
Examples



$K(x) < 0$



$K(x) = 0$



$K(x) > 0$
$(K(x) = \frac{1}{R^2})$

Theorem Egregium and pizza

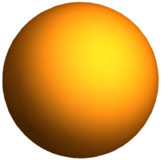
Theorem (Gauss's Theorem Egregium)

The function $K(x)$ is invariant under rigid transformations (rotating, translating, and paper bending). In other words, it is an intrinsic property of the surface's geometry and can be determined by just measuring distance on the surface itself.

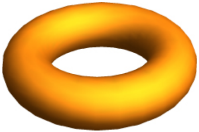
A real world example of this theorem is in how we eat pizza. A slice has $K(x) = 0$ everywhere but as we grip the crust it tries to curve down. Therefore we bend it into a parabola, causing a non-zero principal curvature. Since $K(x)$ must be 0, this means that the orthogonal curve must have principal curvature 0, which causes it to be a straight line.

Topology

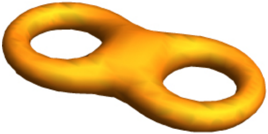
For a compact (closed and bounded) surface Σ, a key invariant is the *genus* g, which is loosely the number of handles it has.



$g = 0$



$g = 1$



$g = 2$

$\chi(\Sigma) = 2 - 2g$ is called the *Euler characteristic of* Σ.

The Euler characteristic and CW Complexes

You may have heard of the Euler characteristic in the context of graphs or polyhedra as

$$\underbrace{V}_{\text{vertices}} - \underbrace{E}_{\text{edges}} + \underbrace{F}_{\text{faces}}$$

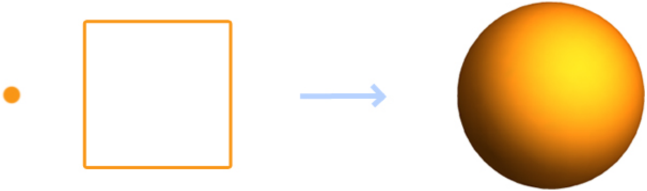
What's the relationship between these two notions?

We can build up a surface Σ by gluing together points (vertices), intervals (edges), and squares (faces). This is called making Σ into a CW complex. Then $\chi(\Sigma)$ is the same as $V - E + F$ where V, E, and F are the number of vertices, edges, and faces used, respectively.

Examples of CW Complexes

The sphere

- For example, the topological sphere can be obtained by starting with one vertex and then attaching to it a rectangle by collapsing the boundary of the rectangle to the vertex:



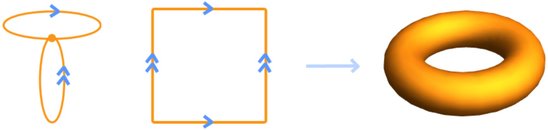
- $V = 1, E = 0, F = 1, g = 0$ so $V - E + F = 2 = 2 - 2g$ ✓

Examples of CW Complexes

The torus

- The torus can be obtained by starting with one vertex and then attaching to it two intervals into circles based at the vertex:

- Then to this we attach a rectangle by attaching its edges according to the diagram:



- $V = 1, E = 2, F = 1, g = 1$ so $V - E + F = 0 = 2 - 2g$ ✓

In general, we can construct a surface Σ of genus g by taking $V = 1, E = 2g, F = 1$ so

$$V - E + F = 2 - 2g = \chi(\Sigma).$$

The Gauss-Bonnet theorem

Topology controlling geometry

Theorem (Gauss-Bonnet)

For any geometric surface Σ that is compact (closed and bounded) and without boundary we have

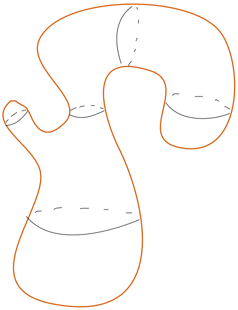
$$\int_\Sigma K d\sigma = 2\pi\chi(\Sigma).$$

The left-hand side is geometric while the right hand side is topological. So the topology gives a constraint on the curvature. As a quick check:

- If Σ is topologically a sphere then $g = 0$ so $\chi(\Sigma) = 2$ and the right hand side is 4π.
- On the other hand, if Σ is geometrically a sphere of radius R, $K(x) = \frac{1}{R^2}$ is constant and $g = 0$. Then

$$\int_\Sigma K d\sigma = \frac{1}{R^2}\int_\Sigma d\sigma = \frac{1}{R^2}(\text{surface area of } \Sigma) = 4\pi \quad \checkmark$$

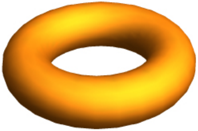
Examples



This shape is topologically the same as a sphere but is very different geometrically. For example there are points where $K(x)$ is negative, zero, or positive. Nonetheless, the integral of $K(x)$ has to be 4π.

Examples

The surface of a donut and mug are the same topologically but they certainly have different curvatures. Nevertheless, the integral of the curvatures must be 0.



One can actually stretch the torus in $\mathbb{R}^4$ so that $K(x) = 0$ everywhere. But by the theorem there is no way, for example, to stretch or bend the torus (in any dimension) that makes it have positive curvature everywhere– any region of positive curvature must be balanced out by an equal amount of negative curvature.

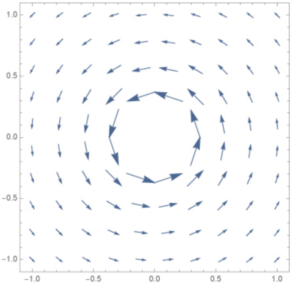
de Rham cohomology

Analysis detecting topology

Recall from multivariable calculus that if **F** is a vector field on $\mathbb{R}^2$ with $\nabla \times \mathbf{F} = 0$, then $\mathbf{F} = \nabla f$ for some function $f : \mathbb{R}^2 \to \mathbb{R}$.

This is *not* true for any subset of $\mathbb{R}^2$. For example, let **F** be the vector field on $\mathbb{R}^2 \backslash \{(0,0)\}$ defined by

$$\mathbf{F}(x, y) = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle .$$



One can check that $\nabla \times \mathbf{F} = 0$ but $\mathbf{F}$ is not a gradient vector field since

$$\oint_{\text{unit circle}} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = 2\pi$$

and the contour integral of any gradient vector field ∇f around a closed loop is zero:

$$\oint_C \nabla f \cdot d\mathbf{r} = f(\text{end point}) - f(\text{beginning point}) = 0.$$

$\mathbf{F}$ is essentially unique: any vector field $\mathbf{G}$ defined on $\mathbb{R}^2 \backslash (0,0)$ with zero curl can be written uniquely in the form

$$\mathbf{G} = c\mathbf{F} + \nabla f$$

for some constant $c \in \mathbb{R}$ and some function f.

More generally, if we remove k-many points $p_1, \ldots, p_k$ from $\mathbb{R}^2$ then:

Theorem

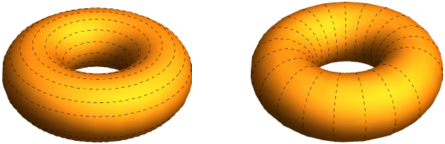
There exists k-many curl-free but non-gradient vector fields $\mathbf{F}_1, \ldots, \mathbf{F}_k$ on $\mathbb{R}^2 \backslash \{p_1, \ldots, p_k\}$ such that any curl-free vector field $\mathbf{G}$ on $\mathbb{R}^2 \backslash \{p_1, \ldots, p_k\}$ can be written uniquely in the form

$$\mathbf{G} = c_1 \mathbf{F}_1 + \cdots + c_k \mathbf{F}_k + \nabla f.$$

So the number of holes is exactly the number of linearly independent vector fields that have zero curl but are not gradients.

This gives an example of analytic information (the number of "non-trivial" solutions to the differential equation $\nabla \times \mathbf{F} = 0$) being equal to topological information (the number of holes the region has).

We can even do this for surfaces. For example, the following two vector fields on a torus have zero curl but are not gradients:



And in general.

Theorem

On a surface of genus g there exists $2g$-many curl-free but non-gradient vector fields $\mathbf{F}_1, \ldots, \mathbf{F}_{2g}$ on Σ such that any curl-free vector field $\mathbf{G}$ on $\mathbb{R}^2 \backslash \{p_1, \ldots, p_k\}$ can be written uniquely in the form

$$\mathbf{G} = c_1\mathbf{F}_1 + \cdots + c_{2g}\mathbf{F}_{2g} + \nabla f.$$

Higher dimensions

All of this generalizes even further to arbitrary manifolds (spaces that locally look like $\mathbb{R}^n$), using the theory of differential forms. This area is called *de Rham cohomology*.

“Can you hear the shape of a drum?”

Analysis determining geometry

Specifically,

If you had perfect pitch, would it be possible for you to tell the shape of a drum just by hearing all of the possible overtones it emits?

Mathematical model

Model a drum as a membrane $\Omega \subset \mathbb{R}^2$ whose amplitude $u(x, y, t)$ at time t and position (x, y) satisfies the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \Delta u,$$

where $\Delta = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ is the Laplace operator.
Since the ends of the drum are fixed, we specify the boundary condition

$$u(x, y, t) = 0 \text{ for all } (x, y) \in \partial\Omega.$$

The overtones are the solutions of the form $u(x, y, t) = U(x, y)e^{\omega t}$ for some $\omega \in \mathbb{R}, \omega \neq 0$. Then U must satisfy

$$\begin{cases} \Delta U + \omega^2 U = 0 \\ U(x, y) = 0 \text{ for } x, y \in \partial\Omega \end{cases}$$

Then the drum can emit the frequency ω if

$$\begin{cases} \Delta U + \omega^2 U = 0 \\ U(x, y) = 0 \text{ for } x, y \in \partial\Omega \end{cases}$$

has a non-zero solution U.

The set of all such ω^2 for such ω is the set of all eigenvalues of the operator $-\Delta$, also called the spectrum of $-\Delta$ on Ω. So our question becomes:

Question (Bochner, Kac)

If I know the spectrum of Δ on Ω (which is just a collection of real numbers), then do I know what Ω must be geometrically?

Warm-up: hearing the length of a string

Let's consider the analogous equation in one-dimensional: "Can one hear the length of a vibrating string?"

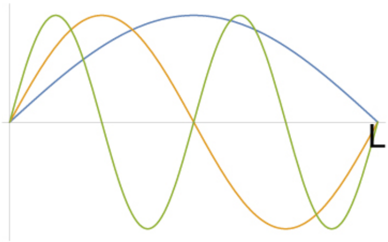
Let $L > 0$ be the length of the string and take our region to be $\Omega = [0, L] \subset \mathbb{R}$ so that $\partial\Omega = \{0, L\}$. Then we need to see for which ω we have non-zero solutions to the initial value problem

$$\begin{cases} U'' + \omega^2 U = 0 \\ U(0) = 0, U(L) = 0. \end{cases}$$

- The general solution to $U'' + \omega^2 U = 0$ is $U(x) = c_1 \sin(\omega x) + c_2 \cos(\omega x)$.
- $U(0) = 0$ gives us $c_2 = 0$ while $U(L) = 0$ implies $c_1 \sin(\omega L) = 0$.
- Thus for c_1 to be non-zero (and to get a non-zero solution U) we need $\omega L = \pi k$ for some integer k.
- Therefore in this case the spectrum of $-\Delta$ is

$$\left\{ \frac{\pi^2 k^2}{L^2} \right\} = \left\{ \frac{\pi^2}{L^2}, \frac{4\pi^2}{L^2}, \frac{9\pi^2}{L^2}, \ldots \right\}.$$

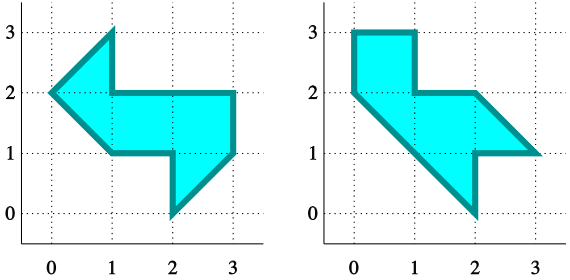
In this case the spectrum determines L since, e.g., we can recover L as $\pi/\sqrt{\lambda}$ where λ is the smallest number in the spectrum. Thus we can hear the length of a string.



Hearing the shape of a drum

Back to the original question, it turns out that one cannot hear the shape of a drum in general.

- The problem was made famous in a 1966 article by Mark Kac.
- Was unsolved until 1992, when Gordon, Webb and Wolpert gave an example of two regions on $\mathbb{R}^2$ that are geometrically different but for which the Laplace operator has the same spectrum:



Hearing the shape of Riemannian manifolds

This question can also be asked on compact Riemannian manifolds without boundary, where the notion of Laplace operator still makes sense. Here the answer is also a no, with Milnor in 1964 coming up with an example of two 16 dimensional tori that have the same spectrum but are different geometrically.

A positive result

Hearing the area of a drum

However, one thing that the spectrum can detect is the area of Ω.
If we order the elements in the spectrum $\lambda_1 < \lambda_2 < \lambda_3 < \ldots$ then

Theorem (Weyl 1911)

As $k \to \infty$, we have

$$\lambda_k \sim \frac{4k\pi}{A}$$

where A is the area of Ω.

Concluding remarks

- We have just scratched the surface of interactions between these areas.
- A ton of mathematics over the past decades lies in the intersection between these three areas such as the Atiyah Singer index theorem, Poincare conjecture, Hodge theory, theoretical physics/string theory, moduli spaces, etc.

Thanks!