Analisis para la ecuacion de Boltzmann
Soluciones y Approximaciones

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Kinetic modeling: non-linear analysis and deterministic simulations

Some current active areas: Statistical flows with applications to Physics and Social Networks (mesoscopic modeling level)

The non-linear Boltzmann equation and non-equilibrium statistics:

- **Kinetic Statistical modeling and simulation:**
  - Analysis and computations of Boltzmann-Poisson Transport system: applications to Plasmas, semiconductor and solar fuel cells modeling

- **Non-linear Boltzmann equation: classical elastic theory of rarefied gases and granular (inelastic) flows**
  - with R. Alonso, E. Carneiro, S. Harsha Tharskabushanam, Current Postdocs: Jeff Haack, Jingwei Hu, Graduate Students: Maja Taskovic, Chenglong Zhang

- **Kinetic inverse problems: reconstruction of doping profiles in nanostructures**
  - with K. Ren, Y. Cheng

- **Statistical flows in social networks: multiplicativity of interactive stochastic processes, information and opinion dynamics, networks**
  - with F. Bolley, Postdoc: Ravi Srinivasan, Graduate Student: Juan Diego Rodriguez
Consider the Cauchy Boltzmann problem (Maxwell, Boltzmann 1860s-80s; Grad 1950s; Cercignani 60s; Kaniel Shimbrot, Illner Shimbrot 80’s, Toscani and Di Perna-Lions late 80’s):

Find a function $f(t, x, v) \geq 0$ that solves the initial value problem associated to the equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla f = Q(f, f) \text{ in } (0, +\infty) \times \mathbb{R}^{2n}$$

$$Q(f, g) = \int_{\mathbb{R}^n} \int_{S^{n-1}} (f'g' - fg) B(|u|, \hat{u} \cdot \sigma) \, d\sigma \, dv_*,$$

$$v' = \frac{v + v_*}{2} + \frac{|u|\sigma}{2}; \quad v_*' = \frac{v + v_*}{2} - \frac{|u|\sigma}{2}; \quad \sigma \in S^{n-1},$$

where $u = v - v_*$ is the pre-collisional relative velocity and $\hat{u} = u/|u|$. $u' = v' - v_*' = |u|\sigma$ and the angle $\theta \in [0, \pi]$ that measures the deviation from $u$ to $u'$ is given by $\cos \theta = \hat{u} \cdot \sigma$.

Collision kernel $B(|u|, \hat{u} \cdot \sigma) = \Phi(|u|) \, b(\hat{u} \cdot \sigma)$

Non-mollified potential kernel $\Phi(|u|) = |u|^\lambda$ with $-n < \lambda \leq 1$; → soft potentials: $-n < \lambda < 0$

Weak Grad’s cut-off assumption: $b(\cos \theta) \in L^\alpha(S^{n-1})$, for $1 \leq \alpha$

Non cut-off assumption: $b(\cos \theta)(\sin \theta)^2 \in L^1(S^{n-1})$, 
Recall: $Q(\nu)$ operator in weak (Maxwell) form, extended to dissipative (inelastic) collisions

$$\int_{\mathbb{R}^n} Q(f, f) \varphi \, dv = \frac{1}{2} \int_{\mathbb{R}^{2n}} \int_{S^{n-1}} ff^* (\varphi' - \varphi) B(u, \hat{u} \cdot \sigma) \, d\sigma \, dv \, dv$$

**Collision kernel**

$$B(u, \hat{u} \cdot \sigma) = |u|^\lambda b(\hat{u} \cdot \sigma)$$

**Grad Cut-off condition**

$$\int_{S^{n-1}} b(\hat{u} \cdot \sigma) \, d\sigma \leq \infty$$

**Exchange of velocities in center of mass-relative velocity frame**

$$u = v - v_*, \quad v' = v - \frac{\beta}{2}(u + |u| \sigma), \quad v + v_* = v' + v'_*$$

**Energy dissipation parameter $\beta$ or restitution parameter $e(z)$**

$$\beta : [0, \infty) \rightarrow (0, 1] \text{ defined by } \beta(z) := \frac{1 + e(z)}{2} \quad \text{with} \quad z = |u| \sqrt{\frac{1 - \hat{u} \cdot \omega}{2}}$$

(i) $z \rightarrow e(z)$ is absolutely continuous and non-increasing.

(ii) $z \rightarrow z e(z)$ is non-decreasing.

$\lambda = 0$ for Maxwell Type (or Maxwell Molecule) models $\gamma$.

$\lambda = 1$ for hard spheres models.

$0 < \lambda < 1$ for variable hard potential models.

$-d < \lambda < 0$ for variable soft potential models.
In fact we will look into a more general collisional kernel structure classified as follows:

\[ B(|u|, \hat{u} \cdot \sigma) = \Phi(|u|) \ b(\hat{u} \cdot \sigma) \]

**Potential kinetic kernel:**

**H1-A:** (Non-mollified potential) \( \Phi(|u|) = |u|^\lambda \) with \(-n < \lambda\),

\[ C_1 (1 + |u|)^\lambda \leq \Phi(|u|) \leq C_2 (1 + |u|)^\lambda, \]  with \(-n < \lambda\),

**Angular cross section integrability:**

**H2-A** Non integrable angular cross-section \( b(\cos \theta) (\sin \theta)^2 \in L^1(S^{n-1}) \), (non cut-off)

**H2-B** Integrable angular cross-section \( b(\cos \theta) \in L^\alpha(S^{n-1}) \), for \( 1 \leq \alpha \) (cut-off)

**Remark:** Grad’s assumption for \( 1 \leq \alpha \), allows to split the collision operator as a difference of two positive forms:

\[ Q(f, g) = Q^+(f, g) - Q^-(f, g) = \text{Gain} - \text{Loss} \]
\[ \frac{\partial f}{\partial t} + v \cdot \nabla f = Q(f, f) \text{ in } (0, +\infty) \times \mathbb{R}^{2n} \]

The loss operator \( Q^-(f, g) = f \cdot R(g) \), with \( R(g) \), the collision frequency, given by

\[ R(g) = \int_{\mathbb{R}^n} \int_{S^{n-1}} g(v_\ast)|u|^{-\lambda} b(u \cdot \sigma) d\sigma dv_\ast = \|b\|_{L^1(S^{n-1})} \int_{\mathbb{R}^n} g(v_\ast)|u|^{-\lambda} dv_\ast = \|b\|_{L^1(S^{n-1})} g \ast |v|^{-\lambda}. \]

We can show:

1- Existence theory for soft potentials (by Kaniel-Shimbrot methods) only requires \( \alpha = 1 \) and \( -n+1 < \lambda < 0 \) (not too soft potentials) trapped between two local Maxwellians with different decay rates.

2- The loss bilinear form is local in \( f \) and a weighted convolution in \( g \), while the gain is a bilinear form with a weighted symmetric convolution structure.

3- The \( L^p \) regularity estimates need more integrability \( b(s) \in L^a(S^{n-1}) \).

4- Gain of integrability estimates and improved cancellation lemmas.

5- Moments and \( L^p \)-propagation estimates for non-cut-off and cut-off angular kernels respectively for \( -n < \lambda < 0 \) (hard and soft-potentials cases).
Review of Properties of the collisional integral and the equation: conservation of moments

Due to symmetries of the collisional integral one can obtain (after interchanging the variables of integration)

Weak (Maxwell) form for the BTE

\[
\frac{\partial}{\partial t} \int_{\mathbb{R}^N} f \varphi \, dv = \int_{\mathbb{R}^N} Q(f, f) \varphi(v) \, dv = \frac{\kappa(t)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{S}^N} \int_{\mathbb{S}^N} f f_*(\varphi' + \varphi_* - \varphi - \varphi_*) |u|^2 b(\sigma) \, d\sigma \, dv_* \, dv,
\]

Properties: It is easy to see, from the weak formulation:

conservation of mass \( \rho \) and momentum \( J \): set \( \varphi(v) = 1 \) and \( \varphi(v) = v \)

Using local conservation of momentum on the test function: \( v + v_* = v + v_* \)

\[
\frac{\partial}{\partial t} \int_{\mathbb{R}^N} f \{1, v_i\} \, dv = \kappa(t) \int_{\mathbb{R}^N} Q(f, f)(v) \{1, v_i\} \, dv = 0, \quad i = 1, 2, 3.
\]

holds, both for the Elastic and Inelastic cases

Next, set \( \varphi(v) = |v|^2 \Rightarrow \text{It conserves energy for } e = 1 - \text{ELASTIC} \):

Using local conservation of energy on the test function: \( |v|^2 + |v_*|^2 = |v|^2 + |v_*|^2 \)

\[
\frac{\partial}{\partial t} \Theta(t) = \kappa(t) \int_{\mathbb{R}^N} Q(f, f)(v) |v|^2 \, dv = 0
\]
Recall Boltzmann H-Theorem for **ELASTIC** interactions:

\[
\frac{\partial}{\partial t} \int f \log f \, dv = \kappa(t) \int_{\mathbb{R}^N} Q(f, f) \log f \, dv = \\
\frac{\kappa(t)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{S^{N-1}} \left( f f_* - f' f'_* \right) \log \frac{f' f'_*}{f f_*} |u| \gamma b(\sigma) \, d\sigma \, dv \, dv_* \leq 0
\]

*Time irreversibility is expressed in this inequality ➔ stability*

In addition:

**The Boltzmann Theorem:** *there are only N+2 collision invariants*

\[
\int_{\mathbb{R}^N} Q(f, f) \log f \, dv = 0 \iff \log f(\cdot, v) = A + B \cdot v + C|v|^2 \iff \\
f(\cdot, v) = M_{A, B, C}(v) \text{ Maxwellian (Gaussian in v-space) parameterized by } A, B, C
\]

related the first \(N + 2\) moments of the initial probability state of \(f(0, v) = f_0(v)\).
Time Irreversibility and relation to Thermodynamics

- **Stability** \( \lim_{t \to \infty} \| f(t,v) - M_{A,B,C} \|_{L^2} \to 0 \) where \( \{A, B, C\} \leftrightarrow \{\rho, u, w\} \), \( \rho = \int f_0 \, dv \), \( \rho u = \int vf_0 \, dv \) and \( \rho w = \int |v|^2 f_0 \, dv \)

- **Macroscopic balance equations:** For the space inhomogeneous problem:
  Under the ansatz of a Maxwellian state in \( v \)-space
  \[
  f(t,x,v) = M_{a,b,u} = ae^{-(b|v-u|^2)}
  \]
  where the dependence of \( (t,x) \) is only through the parameters \( (a,b,u) \):
  \[
  u = \frac{J}{\rho} \quad \text{mean velocity} \quad \text{and} \quad \Theta = \rho w = \frac{1}{2} \rho u + \rho e \quad \text{kinetic energy}, \quad e = \text{internal energy}
  \]
  choosing
  \[
  a = \frac{3^{3/2} \rho}{(4\pi e)^{3/2}}; \quad b = \frac{3}{4e}
  \]
  plus **equilibrium constitutive relations**:
  \[
  P = \frac{2}{3} \rho e \quad \text{pressure}
  \]
  yields the compressible Euler eqs  \( \rightarrow \) Small perturbations of Maxwellians yield CNS eqs.
Conservation Laws:
Even further, any solution \( F \) of the \textbf{Inhomogeneous Boltzmann equation} formally satisfies the following the local conservation laws:

\[
\partial_t \int_{\mathbb{R}^D} \xi \ F \, dv + \nabla_x \cdot \int_{\mathbb{R}^D} v \xi \ F \, dv = 0
\]

when \( \xi(v, x, t) \) is any quantity that satisfies:

\[
\xi(\cdot, x, t) \in \text{Span}\{1, v_1, v_2, \ldots, v_D, |v|^2\}, \quad \text{and} \quad \partial_t \xi + v \cdot \nabla_x \xi = 0
\]

It has been known (since Boltzmann who worked out the case \( D = 3 \)), that the \textbf{only} such quantities \( \xi \) are linear combinations of the \( 4 + 2D + D(D-1)/2 \) quantities

\[
1, \ v, \ x-vt, \ |v|^2/2, \ v \wedge x = v x^T - x v^T, \ v \cdot (x- vt), \ |x- vt|^2/2
\]

By integrating the corresponding local conservation laws over space and time,

\[
\int \int_{\mathbb{R}^D \times \mathbb{R}^D} \begin{pmatrix} 1 \\ v \\ x- vt \\ \frac{1}{2} |v|^2 \\ v \wedge x \\ v \cdot (x- vt) \\ \frac{1}{2} |x- vt|^2 \end{pmatrix} F(v, x, t) \, dv \, dx = \int \int_{\mathbb{R}^D \times \mathbb{R}^D} \begin{pmatrix} 1 \\ v \\ x \\ \frac{1}{2} |v|^2 \\ v \wedge x \\ v \cdot x \\ \frac{1}{2} |x|^2 \end{pmatrix} F^{\text{in}}(v, x) \, dv \, dx
\]
Local and global Maxwellians

If $\rho$, $u$, and $\theta$ functions of $(x, t)$ the following pdf are called **local Maxwellians**

$$f = \frac{\rho}{(2\pi\theta)^{\frac{D}{2}}} \exp\left(-\frac{|v-u|^2}{2\theta}\right) \quad \text{for some } (\rho, u, \theta) \in \mathbb{R}_+ \times \mathbb{R}^D \times \mathbb{R}_+.$$

The family of all **global Maxwellians** over the spatial domain $\mathbb{R}^D$ with positive mass, zero net momentum, and center of mass at the origin has the form

$$\mathcal{M} = \frac{m}{(2\pi)^D} \sqrt{\det(Q)} \exp\left(-q(v, x, t)\right), \quad Q = (ac - b^2)I + B^2,$$

$$q(v, x, t) = \frac{1}{2} \begin{pmatrix} v \\ x - vt \end{pmatrix}^T \begin{pmatrix} cI & bI + B \\ bI - B & aI \end{pmatrix} \begin{pmatrix} v \\ x - vt \end{pmatrix} = \frac{1}{2\theta(t)} |v - u(x, t)|^2 + \frac{\theta(t)}{2} x^T Q x,$$

with $m > 0$ and $(a, b, c, B) \in \Omega$ where $\Omega$ is the open cone in $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^D \times \mathbb{R}^D$ defined by

$$\Omega = \{ (a, b, c, B) : (ac - b^2) I + B^2 > 0 \}.$$

Any **global Maxwellian** can be written as a **local Maxwellian** form:

$$\mathcal{M} = \frac{\rho(x, t)}{(2\pi\theta(t))^\frac{D}{2}} \exp\left(-\frac{|v-u(x, t)|^2}{2\theta(t)}\right) \quad \text{with} \quad \theta(t) = \frac{1}{at^2 - 2bt + c}$$

$$u(x, t) = \theta(t)(ax - bx - Bx) \quad \text{and} \quad \rho(x, t) = m \left(\frac{\theta(t)}{2\pi}\right)^\frac{D}{2} \sqrt{\det(Q)} \exp\left(-\frac{\theta(t)}{2} x^T Q x\right).$$
Property 1: let \( M_1 \) and \( M_2 \) be global Maxwellians with parameters given by \((m_1, a_1, b_1, c_1, B_1) \in \mathbb{R}_+ \times \Omega \) and \((m_2, a_2, b_2, c_2, B_2) \in \mathbb{R}_+ \times \Omega\) resp. respectively. Then \( M_1 \leq M_2 \) for every \((v, x, t)\) if and only if

\[
\left( \begin{array}{cc}
    c_2 I & b_2 I + B_2 \\
    b_2 I - B_2 & a_2 I \\
\end{array} \right) \preceq \left( \begin{array}{cc}
    c_1 I & b_1 I + B_1 \\
    b_1 I - B_1 & a_1 I \\
\end{array} \right).
\]

Property 2 (for stability): Let the collision kernel \( b \) have the separated form \( b = |u|^{\beta} b \), for some \( \beta \in (-D, 2] \).

Let \( M \) be a global Maxwellian for some \((m, a, b, c, B) \in \mathbb{R}_+ \times \Omega \). Let \( F \) be any measurable function that satisfies the pointwise bounds \( 0 \leq F(v, x, t) x, s + t) \leq M(v, x, s+t), s \in [0, \infty) \). Then for every \([t_1, t_2] \subset [0, \infty)\) one has the \( L^1 \) bound

\[
\int_{t_1}^{t_2} \int \int \int \int |F'_* F' - F_* F| \, d\omega \, dv_* \, dv \, dx \, dt \leq C_1 \int_{t_1}^{t_2} \theta(s + t)^{\frac{\beta + D}{2}} \, dt,
\]

with \( \theta(t) = \frac{1}{at^2 - 2bt + c} \).
Analytical tool: Symmetric weighted convolution structure

The weak formulation of the gain operator is a symmetric weighted convolution

\[
\int_{\mathbb{R}^n} Q^+(f, g)(v) \psi(v) dv = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(v) g(v - u) \mathcal{P}(\tau_v \mathcal{R} \psi, 1)(u) |u|^\lambda dudv
\]

\[
\int_{\mathbb{R}^n} Q^+(f, g)(v) \psi(v) dv = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(u + v) g(v) \mathcal{P}(1, \tau_{-v} \psi)(u) |u|^\lambda du dv
\]

Where the weight is an invariant under rotation operator involving translations and reflections

\[
\tau_v \psi(x) := \psi(x - v) \quad \text{and} \quad \mathcal{R} \psi(x) := \psi(-x)
\]

and the angular mixing operator

\[
\mathcal{P}(\psi, \phi)(u) := \int_{S^{n-1}} \psi(u^-) \phi(u^+) b(\hat{u} \cdot \sigma) d\sigma,
\]

\[
u^- = \frac{\beta}{2} (u - |u|\sigma) \quad \text{and} \quad u^+ = u - u^- = (1 - \beta)u + \frac{\beta}{2} (u + |u|\sigma).
\]

Bobylev’s (’75) The angular operator on Maxwell type interactions \( \lambda=0 \)
is the well know identity for the Fourier transform of the \( Q^+ \)

\[
\widehat{Q^+(f, g)} = \mathcal{P}(\hat{f}, \hat{g})
\]
Young’s inequality for variable hard potentials with exact constants: $0 \leq \lambda \leq 1$


$$\|f\|_{L^p_k(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(v)|^p \left(1 + |v|^{pk}\right) dv\right)^{1/p} \quad \text{and} \quad B(|u|, \hat{u} \cdot \omega) = |u|^\lambda b(\hat{u} \cdot \omega),$$

Theorem 1. Let $1 \leq p, q, r \leq \infty$ with $1/p + 1/q = 1 + 1/r$ and $\lambda \geq 0$. For $\alpha \geq 0$, the bilinear operator $Q^+$ extends to a bounded operator from $L^p_{\alpha+\lambda}(\mathbb{R}^n) \times L^q_{\alpha+\lambda}(\mathbb{R}^n) \to L^r_\alpha(\mathbb{R}^n)$ via the estimate

$$\left\|Q^+(f, g)\right\|_{L^r_\alpha(\mathbb{R}^n)} \leq C \|f\|_{L^p_{\alpha+\lambda}(\mathbb{R}^n)} \|g\|_{L^q_{\alpha+\lambda}(\mathbb{R}^n)}, 0 \leq \lambda \leq 1$$

$$C = |S^{n-2}| \left(\frac{\sqrt{n}}{2^{r'}} \int_{-1}^{1} (\frac{1-s}{2})^{-\frac{n}{2r'}} d\xi^b_n(s)\right)^{r'} \left(\int_{-1}^{1} \left[\left(\frac{1+s}{2}\right) + (1 - \beta_0)^2 \left(\frac{1-s}{2}\right)\right]^{-\frac{n}{2r'}} d\xi^b_n(s)\right)^{r' \left\|\frac{n}{p'} + 1\right\|.}

These estimates resemble a Beckner and Brascamp-Lieb type inequality argument (for a nonlinear weight) with best/exact constants approach to obtain Young’s inequality.

Remarks:

1- Previous $L^p$ estimates by Gustafsson 88, Villani-Mouhot ‘04 for pointwise bounded $b(u \cdot \sigma)$, I.M.G-Panferov-Villani ’03 for $(p,1,p)$ with $\sigma$-integrable $b(u \cdot \sigma)$ in $S^{n-1}$

2-The dependence on the weight $\alpha$ may have room to improvement. We could expect estimates with exponential or polynomial (?) decay rates in $\alpha$, like in $L^1_\alpha$ as shown Bobylev, I.M.G, Panferov and recently with Villani (97, 04,09), Alonso & Lods’11 (also previous work of Wennberg ’94, Desvilletes, 96, without decay rates.)
**Hardy-Littlewood-Sobolev type inequality for soft potentials:** \(-n < \lambda < 0\)

**Theorem 2.** Let \(1 < p, q, r < \infty\) with \(-n < \lambda < 0\) and \(\frac{1}{p} + \frac{1}{q} = 1 + \frac{\lambda}{n} + \frac{1}{r}\). Then the bilinear operator \(Q^+\) extends to a bounded operator from \(L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \to L^r(\mathbb{R}^n)\) via the estimate

\[
\left\| Q^+(f, g) \right\|_{L^r(\mathbb{R}^n)} \leq C \left\| f \right\|_{L^p(\mathbb{R}^n)} \left\| g \right\|_{L^q(\mathbb{R}^n)}.
\]

- In both theorems the constant depends on \(C = C(n, \alpha, p, q, b, \beta, \lambda)\) are explicit and depend on bounds for \(P(\tau \varphi \psi, 1)(u)\), but generally not sharp.

- Only in the cases \(\alpha = \lambda = 0\) (Maxwell type interactions), \((p, q, r) = (2, 1, 2)\) and \((p, q, r) = (1, 2, 2)\) we find sharp constants \(C\) for the Young's inequality.

**The Loss operator** \(Q^-\) **is local in** \(f\) **and a convolution in** \(g\) **(in weak form)**

- The Young’s estimates for \(Q^-\) holds as well \((0 \leq \lambda \leq 1)\)

\[
\frac{1}{p} + \frac{1}{q} = 1 + \frac{\lambda}{n} + \frac{1}{r}.
\]

- The HLS estimates for \(Q^-\) \((-n < \lambda < 0)\) require \(r < p\). \(r = p\) is critical!

We can find a counterexample to the HLS inequality
Inequalities with Maxwellian weights – fundamental estimates for pointwise exponentially weighted estimates

As an application of these ideas one can also show Young type estimates for the non-symmetric Boltzmann collision operator with exponential weights.

**First, for any** $a > 0$ **and** $\gamma \geq 0$ **define** $M_\gamma(v) := \exp(-a|v|^\gamma)$

**Theorem** Let $1 \leq p, q, r \leq \infty$ with $1/p + 1/q = 1 + 1/r$. Assume that

$$B(|u|, \hat{u} \cdot \omega) = |u|^\lambda b(\hat{u} \cdot \omega).$$

with $0 \leq \lambda \leq 2$. Then, for non-increasing restitution coefficient such that $e(z) < 1$ for $z \in (0, \infty)$,

$$\left\| Q^+(f, g) \ M_\lambda^{-1} \right\|_{L^r(\mathbb{R}^n)} \leq C \left\| f \ M_\lambda^{-1} \right\|_{L^p(\mathbb{R}^n)} \left\| g \ M_\lambda^{-1} \right\|_{L^q(\mathbb{R}^n)}$$

The constant $C := C(n, \lambda, p, q, b, \beta)$ is computed in the proof and is similar to the one obtained for Young's inequality proof.

In the important case $(p, q, r) = (\infty, 1, \infty)$ The constant reduces to

$$C = C(n, \lambda) \int_{-1}^1 \left[ \left( \frac{1+s}{2} \right) + (1 - \beta(0))^2 \left( \frac{1-s}{2} \right) \right]^{-n/2} b_\beta(s) \, ds, \quad b_\beta(s) := \left[ 1 - \left( \frac{1+|\theta(s)|}{2} \right)^{\lambda/2} \right]^{-1} b(s),$$

with $|\theta(s)| = \sqrt{(1 - \beta(x))^2 + \beta^2(x) + 2(1 - \beta(x))\beta(x)s}$, and $x = \sqrt{\frac{1-s}{2}}$.

Proof: an elaborated argument of the pre/post collision exchange of coordinates(Alonso, Carneiro, IMG ‘10)
2- Gain of integrability and propagation of $W^r_k$ estimates (R. Alonso and I.M.G., KRM ‘10, and with E. Carneiro in '12)

Improvement for the gain rate of integrability estimates with respect to the ones of Mouhot and Villani (ARMA’04)

without requiring the gain of regularity estimates
(as developed by Lions’94, Wennberg’97 and Mouhot & Villani’04)

We only use an angular function $b(\theta)$ bounded and integrable (no constrains on pointwise truncation at the endpoints) (Here $\gamma := \lambda$ the rate of potentials)

$$\|Q^+_\gamma (g, h)\|_{r,k} \leq \epsilon^s C'_b(r, n) \|g\|_{1,k} \|h\|_{r,k} + \frac{1}{\epsilon^s} C_n \|b\|_\infty \|g\|_{1,n-2+k} \|h\|_{1,\frac{n-2}{1-\theta}+k} \|h\|_{r,k}^\theta,$$

$s = \frac{n-2}{\gamma}$, $C_n$ constant depending only on the dimension and

$$C_b(n) := \int_{S^{n-1}} \left( \frac{1+\hat{\nu} \cdot \sigma}{2} \right)^{-\frac{n}{4}} b(\hat{\nu} \cdot \sigma) d\sigma < \infty \text{ for } n \geq 3,$$

where the parameter $\theta \in (0, 1)$ satisfies

$$\theta = \theta_r = \begin{cases} \frac{1}{n} & \text{if } r \in (1, 2] \\ \frac{n(r-2)+1}{n(r-1)} & \text{if } r \in [2, \infty) \end{cases}$$
Sketch of Gain of integrability estimates

Recall

\[ Q^+(g, h)(v) = \]

Carleman representation of the gain operator in n-dimensions

\[ 2^{n-1} \int_{x \in \mathbb{R}^n} \frac{g(v + x)}{|x|} \int_{z \cdot x = 0} h(v + z)|z + x|^{2-n} B(-(z + x), 1 - 2\frac{|z|^2}{|z+x|^2}) \, d\pi_z \, dx. \]

Is a generalized Radon transform that applied to the linear form below gives a weighted Radon transform

\[ Q^+_{n-2}(\delta_0, h)(v) = \frac{2^{n-1}}{|v|} \int_{v \cdot z = 0} h(z + v)\frac{1}{|z + v|^{n-2}} B\left(|z + v|, 1 - \frac{2|z|^2}{|z + v|^2}\right) d\pi_z \]

\[ Q^+(g, h)(v) = \int_{\mathbb{R}^n} g(x) \tau_x Q^+_{n-2}(\delta_0, \tau_{-x} h)(v) \, dx. \quad \text{a double mixing convolution} \]

By Minkowski's integral inequality

\[ \|Q^+(g, h)\|_2 \leq \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \left( \tau_x Q^+_{n-2}(\delta_0, \tau_{-x} h)(v) \right)^2 \, dv \right)^{1/2} g(x) \, dx. \]

Controlling this term

Plus using exact Young's inequality for variable hard potential γ plus interpolation theorems □

gain of \( L^p_k \) integrability of \( Q^+_{n-2} \)
Sketch Proof: \( n \geq 3 \) and denote the gain operator with potential \( |v|^\gamma \) to be \( Q^+_\gamma (g; h) \).

\[
\|Q_{n-2}^+ (\delta_0, h)\|_2 \leq C_b(n) \|h\|_2,
\]
for

\[
C_b(n) := \int_{c^{n-1}} \left( \frac{1+\hat{u} \cdot \sigma}{2} \right)^{-\frac{n}{4}} b(\hat{u} \cdot \sigma) d\sigma < \infty
\]

By AC-AM’09 or ACG CMP’10

By a cancellation in the Carleman integral/weighted Radon Transform; and using a potential weight \( \gamma = n-2 \) and computing the integral and HLS inequality

\[
\|Q_{n-2}^+ (\delta_0, h) (v)\|_2 \leq C_n \|b\|_\infty \|h\|_{\frac{2n}{2n-1}, n-3}
\]
\[
\leq C_n \|b\|_\infty \|h\|_{1, \frac{n-3}{n-1}, n} \|h\|^\theta_2,
\]

\[
\|Q_{n-2}^+ (g, h)\|_{2,k} \leq \epsilon^{s'} C_b(n) \|g\|_{1, k} \|h\|_{2,k} + \frac{1}{\epsilon^s} C_n \|b\|_\infty \|g\|_{1, n-3+k} \|h\|_{1, \frac{n-3}{n-1}, n+k} \|h\|^\theta_{2,k}
\]

The \( L^r_k \) theory follows interpolating between \( k \)-weighted \( L^1 \& L^2 \) and \( L^2 \& L^\infty \) with

\[
\|Q_{n-2}^+ (\delta_0, h)\|_r \leq C(b, n) \|h\|_{1, \frac{n-2}{1-\theta}} \|h\|^\theta_r,
\]

\[
\frac{1-\theta}{1} + \frac{\theta}{r} = \frac{1}{p},
\]

or, equivalently,

\[
\theta = \theta_r = \left\{ \begin{array}{ll}
\frac{\frac{1}{n}}{n(r-2)+1} & \text{if } r \in (1, 2] \text{ and } p = \frac{nr}{r(n-1)+1}, \\
\frac{\frac{1}{n}}{n(r-1)} & \text{if } r \in [2, \infty) \text{ and } p = \frac{nr}{2n-1}.
\end{array} \right.
\]
Corollaries:

1- \( \int_{\mathbb{R}^n} Q^+(f,f)(v)(f(v))^{r-1} dv \leq \epsilon^s C_b(r,n) \|f\|_{1,k} \|f\|_{r,k}^r + \frac{C_n}{\epsilon^s} \|b\|_{\infty} \|f\|_{1,\frac{n-2}{1-\theta}+k}^{2-\theta} \|f\|_{r,k}^{r-1+\theta} \).

2- Propagation of \( L^r_k \) estimates: \( \gamma \in (0, 1] \), (under \( H1-A \) and \( H2-B \))

Non-mollified and integrable angular cross section

\[
\frac{\partial}{\partial t} \|f\|_{r,k}^r \leq C_1 M \|f\|_{1,\frac{n-2}{1-\theta}+k}^{2-\theta} \|f\|_{r,k}^{r-1+\theta} - C_2 \|f\|_{k+r/\gamma}^r
\]

\[
\theta = \theta_r = \begin{cases} 
\frac{1}{n} & \text{if } r \in (1, 2] \\
\frac{n(r-2)+1}{n(r-1)} & \text{if } r \in [2, \infty)
\end{cases}
\]

3- Using the Leibniz formula on the collisional integral, the propagation of solutions in \( W^{l,r}_k \) follows as in Mouhot-Villani, ARMA 2004.

4- Applications to \( H^s_k \) error estimates to the Spectral-Lagrangian constrained numerical solvers for computations of the BTE (IMG, with R. Alonso, H.Tharskabhusanam '12,
Important properties for non-integrable angular cross-section

A quantitative form of a Cancellation Lemma (R. Alonso, E. Carneiro & I.M.G, ’12)
(originally by Alexandre, Villani, Wennberg and Desvilletes, ’01)

Let $b$ as in hypothesis H2-A (non-integrable angular cross-section) and $s > 2$.

Then

$$
\left| \int_{S^{n-1}} \left[ \langle v' \rangle^s + \langle v'_* \rangle^s - \langle v \rangle^s - \langle v_* \rangle^s \right] b(\hat{u} \cdot \sigma) \, d\sigma \right|
\leq C \left( \int_{S^{n-1}} b(\hat{u} \cdot \sigma) \left[ 1 - \hat{u} \cdot \sigma \right] \, d\sigma \right) |v - v_*|^2 \left[ \langle v \rangle^{s-2} + \langle v_* \rangle^{s-2} \right]
$$

where $C > 0$ is a constant that depends only on $s$.

Local lower estimate: $f$ is a solution of the Boltzmann eq. with bounded mass, energy and entropy. Then, there exists $\delta = \delta(m_0, m_2, H(f_0)) > 0$ and a constant $C = C(m_0, m_2, \lambda) > 0$, such that for any $v \in \mathbb{R}^n$ and $t \geq 0$

$$
\int_{\mathbb{R}^n} f(t, v_*) \Phi(|u|) \chi_{|u| > \delta(|u|)} \, dv_* \geq C \langle v \rangle^\lambda.
$$

Then, for the norms

$$
\|f\|_{L^p_s(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |f(v)|^p \langle v \rangle^{ps} \, dv \right)^{1/p}; \quad \|f\|_{L^\infty_s(\mathbb{R}^n)} = \sup_{v \in \mathbb{R}^n} |f(v)| \langle v \rangle^s
$$

Propagation of Moments and $L^p_s$ estimates:
Theorem 1 (Propagation of Moments) (Alonso, Carneiro & I.M.G'12): Under assumptions 
H1-A (non-mollified potential) and H2-A (non-integrable angular cross-section), let 
$\lambda \in (-2, 0]$ (not too soft potentials) and $s > 2$ (initial moments).
Assume that the initial data $f_0 \in L^1_{s+|\lambda|}(\mathbb{R}^n) \cap L \log L(\mathbb{R}^n)$. Then

$$\sup_{t \geq 0} \|f(t, \cdot)\|_{L^1_{s}(\mathbb{R}^n)} \leq C(m_0, \|f_0\|_{L^1_{s+|\lambda|}}, H(f_0), b, s, \lambda).$$

Comments on the proof: the constrain $\lambda \in (-2, 0]$ is due to the cancellation lemma and 
The local lower estimate and the norms we use:

to obtain

$$\frac{d}{dt} \|f(t, \cdot)\|_{L^1_{s}} \leq C_8 \|f(t, \cdot)\|_{L^1_{s-2}} - C_9 \|f(t, \cdot)\|_{L^1_{s+\lambda}}$$

with $C_8 = C_8(m_0, m_2, H(f_0), b, s, \lambda+2)$ and $C_9 = C_9(m_0, m_2, b, s, \lambda)$, with $\lambda+2 > 0$.

and, by interpolation

$$\|f(t, \cdot)\|_{L^1_{s-2}} \leq m_0^{1-\theta} \|f(t, \cdot)\|_{L^1_{s+\lambda}}^\theta \text{ with } \theta = \frac{s-2}{s+\lambda} < 1$$

$$\|f(T, \cdot)\|_{L^1_{s}} \leq \|f(0, \cdot)\|_{L^1_{s+|\lambda|}} + \int_0^T \left\{ C_8 m_0^{\frac{2+\lambda}{s}} \|f(t, \cdot)\|_{L^1_{s}}^{\frac{s+|\lambda|-2}{s}} - C_9 \|f(t, \cdot)\|_{L^1_{s}} \right\} dt$$
Comments: Our estimate improves the one of Desvilletes & Mouhot Asymp.Anal.'07, where also uniform propagation was also obtained in the same range \( \lambda \in (-2, 0] \) under much stronger assumptions, namely:

1- Mollified potential \( \Phi \),

2- Integrable and bounded-by-below angular cross-section \( b(\theta) \),

3- Initial datum \( f_0 \in L^{1}_{2s} \cap L^{2}_{q_0} \) for some \( q_0 > 0 \).

4- Their method consists of first proving polynomial bounds and then combining these with quantitative results of convergence of the solution to a Maxwellian equilibrium.

Instead we use:

i) Finite entropy hypothesis (which would be implied by \( f_0 \in L^p \), for any \( p > 1 \)),

ii) Diminish the number of additional moments required from \( s \) to \( |\lambda| \).

iii) Drop the smoothness on \( \Phi \) and the integrability on \( b \) assumptions (non-cut-off)

iv) The method for the a priori bound in our proof is direct and based on the use of an appropriate cancellation lemma and local lower estimate.
\textbf{Theorem 2 (Propagation of $L^p$-norms)} (Alonso, Carneiro & I.M.G'12):

Assume the collision kernel satisfies \textit{H1-B (mollified potential kernel), H2-B (integrable angular cross section)}, $\lambda \in (-n, 0]$ (up to very soft potentials) $p \in (1, \infty)$ and $s \geq 0$. Let $f$ be a non-negative solution of the Boltzmann equation such that

$$\sup_{t \geq 0} \| f(t) \|_{L^1_{s+|\lambda|(1+1/p')}} < \infty.$$  

Then,

$$\sup_{t \geq 0} \| f(t) \|_{L^p_{s-|\lambda|/p}} \leq C \left( \| f_0 \|_{L^p_{s}}, \sup_{t \geq 0} \| f(t) \|_{L^1_{s+|\lambda|(1+1/p')}} \right).$$

In particular when $\lambda \in (-2, 0]$, the $L^p_{s-|\lambda|/p}$ norm of $f$ is uniformly propagated for initial data $f_0 \in L^1_{\max\{2,s+|\lambda|(2+1/p')\}} \cap L^p_s$.

\textbf{Theorem 3: (Propagation of $L^p$-norms)} (Alonso, Carneiro & I.M.G'12):

Assume the collision kernel satisfying \textit{H1-A (non-mollified potential)} and $b \in L^a$ with $a > 1$ (this is \textit{H2-B+} condition)

Let $s \geq 0$ and $p \in (n/|\lambda|, \infty)$ and assume $f_0$ belonging to $L= L^1_{2+s} \cap L^p_s$.

Then, there exists $\lambda_0 \in (-2, 0)$ depending on $\| f_0 \|_L$ and such that

$$\sup_{t \geq 0} \| f(t) \|_{L^p_{s-|\lambda|/p}} \leq C \left( \| f_0 \|_{L^1_{2+s}}, \| f_0 \|_{L^p_s} \right); \quad \text{for any } \lambda \in [\lambda_0, 0].$$
Comments on the propagation of $L^p$-norms.

1- These estimates treat mollified potentials and its proof and uses the gain of integrability estimates (Alonso, I.M.Gamba, KRM'11) as done for propagation of $L^p$-integrability in the case of hard potentials ($\lambda \in (0,1]$) (as also done in Mouhot & Villani ARMA'04 for propagation of any high order Sobolev norms)

2- Here we get a unified approach to treat both hard and soft potentials as well as relaxes and simplifies considerably the methods and assumptions Lions, Wennberg and Mouhot & Villani

3- These results also show that propagation of $L^p$ integrability is a consequence of a priori uniform propagation of just a few moments calculated explicitly
The Cauchy Boltzmann problem
(Maxwell, Boltzmann 1860s-80s; Grad 1950s; Cercignani 60s; Kaniel, Illner, Shimbrot 80’s, Toscani ‘82, Di Perna-Lions late 80’s): Find a function \( f(t, x, v) \geq 0 \) that solves the initial value problem associated to the equation
\[
\frac{\partial f}{\partial t} + v \cdot \nabla f = Q(f, f) \quad \text{in} \quad (0, +\infty) \times \mathbb{R}^{2n}
\]

The loss operator \( Q^-(f, g) = f \cdot R(g) \), with \( R(g) \), called the collision frequency, given by
\[
R(g) = \int_{\mathbb{R}^n} \int_{S^{n-1}} g(v_*) |u|^{-\lambda} b(\hat{u} \cdot \sigma) d\sigma dv_*
\]
\[
= ||b||_{L^1(S^{n-1})} \int_{\mathbb{R}^n} g(v_*) |u|^{-\lambda} dv_* = ||b||_{L^1(S^{n-1})} g * |v|^{-\lambda}.
\]

Remarks:
1- Existence theory for soft potentials (by Kaniel-Shimbrot methods) only requires \( \alpha=1 \) and \( n-1 > \lambda > 0 \) (not so soft potentials) (Note for large data near Maxwellians Toscani’88, Mishler & Perthame ’97, Alonso & IMG’09,)

2- The loss bilinear form is local in \( f \) and a weighted convolution in \( g \). while the gain is a bilinear form with a weighted symmetric convolution structure

3- The \( L^p \) regularity estimates needs more integrability \( b(s) \in L^\alpha(S^{n-1}) \)
Kaniel & Shinbrot iteration '78: define the sequences \( \{l_n(t)\} \) and \( \{u_n(t)\} \) as the mild solutions to
(also Illner & Shinbrot '83)

\[
\frac{dl_n(t)}{dt} + Q_-^\#(l_n, u_{n-1})(t) = Q_+^\#(l_{n-1}, l_{n-1})(t)
\]

\[
\frac{du_n(t)}{dt} + Q_-^\#(u_n, l_{n-1})(t) = Q_+^\#(u_{n-1}, u_{n-1})(t)
\]

with \( 0 \leq l_n(0) \leq f_0 \leq u_n(0) \).

which relies in choosing a pair of functions \((l_0, u_0)\) satisfying so called the beginning condition in \([0, T]\):

\[
u_0^\# \in L^\infty(0, T; M_{\alpha,\beta}) \quad \text{and} \quad 0 \leq \hat{l}_0(t) \leq \hat{l}_1^\#(t) \leq \hat{u}_1^\#(t) \leq \hat{u}_0^\#(t) \quad \text{a.e. in } 0 \leq t \leq T.
\]

**Theorem:** Let \( \{l_n(t)\} \) and \( \{u_n(t)\} \) the sequences defined by the mild solutions of the linear system above, such that the beginning condition is satisfied in \([0, T]\), then

(i) The sequences \( \{l_n(t)\} \) and \( \{u_n(t)\} \) are well defined for \( n \geq 1 \). In addition, \( \{l_n(t)\}, \{u_n(t)\} \) are increasing and decreasing sequences respectively, and

\[
l_n^\#(t) \leq u_n^\#(t) \quad \text{a.e. in } 0 \leq t \leq T.
\]

(ii) If \( 0 \leq l_n(0) = f_0 = u_n(0) \) for \( n \geq 1 \), then

\[
\lim_{n \to \infty} l_n(t) = \lim_{n \to \infty} u_n(t) = f(t) \quad \text{a.e. in } [0, T].
\]

The limit \( f(t) \in C(0, T; M^\#_{\alpha,\beta}) \) is the unique distributional solution of the Boltzmann equation in \([0, T]\) and fulfills

\[
0 \leq l_0^\#(t) \leq f^\#(t) \leq u_0^\#(t) \quad \text{a.e. in } [0, T].
\]
**Theorem:** Let $B(u, \hat{u} \cdot \sigma) = |u|^{-\lambda} b(\hat{u} \cdot \sigma)$ with $-n+1 < \lambda \leq 0$ with the Grad’s assumption (i.e. $b \in L^1$)

In addition, assume that $f_0$ is $\varepsilon$–close to the local Maxwellian distribution $M(x, \nu) = C M_{\alpha, \beta}(x - \nu, \nu) (0 < \alpha, 0 < \beta)$.

Then, for sufficiently small $\varepsilon$ the Boltzmann equation has a unique solution satisfying

$$C_1(t) M_{\alpha_1, \beta_1}(x - (t + 1)\nu, \nu) \leq f(t, x - t \nu, \nu) \leq C_2(t) M_{\alpha_2, \beta_2}(x - (t + 1)\nu, \nu)$$

for some positive functions $0 < C_1(t) \leq C \leq C_2(t) < \infty$, and parameters $0 < \alpha_2 \leq \alpha \leq \alpha_1$ and $0 < \beta_2 \leq \beta \leq \beta_1$.

Moreover, the case $\beta = 0$ (infinite mass) is permitted as long as $\beta_1 = \beta_2 = 0$.

(this last part extends the result of Mishler & Perthame ’97 to soft potentials)

As a consequence, one concludes that the distributional solution $f$ is controlled by a traveling Maxwellian, and that

$$\lim_{t \to \infty} f(t, x, \xi) \to 0 \text{ a.e. in } \mathbb{R}^{2n}.$$  

*It behaves like the heat equation, as mass spreads as $t$ grows*
Fundamental estimates that select the range of potential rates \( \lambda \)

Lemma: Assume \(-1 \leq \lambda < n - 1\). Then, for any \(0 \leq s \leq t \leq T\) and functions \(f^#, g^#\) that lie in \(L^\infty(0, T; M^\#, \alpha, \beta)\), then the following inequality holds

\[
\int_s^t |Q^+_+(f, g)(\tau)| \, d\tau \leq k_{\alpha, \beta} \exp\left(-\alpha |x|^2 - \beta |v|^2\right) \left\|f^#\right\|_{L^\infty(0, T; M_{\alpha, \beta})} \left\|g^#\right\|_{L^\infty(0, T; M_{\alpha, \beta})},
\]

with

\[
k_{\alpha, \beta} = \sqrt{\pi} \alpha^{-1/2} \left\|b\right\|_{L^1(S^{n-1})} \left(\frac{|S^{n-1}|}{n - \lambda - 1} + C_n \beta^{-n/2}\right).
\]

This estimate implies global in time control.

This and all estimates that follow hold for \(L^\infty\) - Globa Maxweilians weights of the form

\[
M = \frac{m}{(2\pi)^D} \sqrt{\det(Q)} \exp\left(-q(v, x, t)\right), \quad Q = (ac - b^2)I + B^2,
\]

\[
q(v, x, t) = \frac{1}{2} \begin{pmatrix} v \\ x - vt \end{pmatrix}^T \begin{pmatrix} cI & bI + B \\ bI - B & aI \end{pmatrix} \begin{pmatrix} v \\ x - vt \end{pmatrix} = \frac{1}{2\theta(t)} |v - u(x, t)|^2 + \frac{\theta(t)}{2} x^T Q x.
\]
Distributional solutions near local Maxwellians: \textit{(large data)}: (Ricardo Alonso, IMG’SP’08)

Sketch of proof: \textit{for } 0 \leq \lambda < n-1

Define the \textit{distance} between two Maxwellian distributions \( M_i = C_i M_{\alpha_i \beta_i} \) for \( i = 1, 2 \) as

\[ d(M_1, M_2) := |C_2 - C_1| + |\alpha_2 - \alpha_1| + |\beta_2 - \beta_1|. \]

Second, we say that \( f \) is \( \varepsilon \)-\textit{close} to the Maxwellian distribution \( M = C M_{\alpha, \beta} \) if there exist Maxwellian distributions \( M_i (i = 1, 2) \) such that \( d(M, M) < \varepsilon \) for some small \( \varepsilon > 0 \), and \( M_1 \leq f \leq M_2 \).

Also define

\[
\phi_{\alpha, \beta}(t, x, v) := \|b\|_{L^1(S^{n-1})} \int_{\mathbb{R}^n} \exp \left( -\alpha |x + u|^2 - \beta |v - u/t|^2 \right) |u|^{-\lambda} du.
\]

Then, \textit{it is possible to find a couple of monotone sequences of barrier functions}

\[ l_1^\# < l_n^\# < u_n^\# < u_1^\# \quad \text{such that} \quad l_1^\# = C_1(t) M_{\alpha_1 \beta_1} \quad \text{and} \quad u_1^\# = C_2(t) M_{\alpha_2 \beta_2} \]

where the \textit{‘amplitude’} functions \( C_1(t) \) and \( C_2(t) \) need to be chosen such that

\[ 0 \leq C_1(t) \leq C_2(t) \quad \text{with} \quad C_1(1) = C_1 \leq C_2 = C_2(1); \]

and satisfy the inequations derived from the Kaniel-Shinbrot type iteration
Following the Kaniel-Shimbrrot procedure, $C_1(t)$ and $C_2(t)$ satisfy the following non-linear system of inequations

$$C'_1(t) + \frac{C_1(t) C_2(t)}{t^{n-\lambda}} \phi_2 \leq \frac{C_1^2(t)}{t^{n-\lambda}} \phi_1$$

$$C'_2(t) + \frac{C_1(t) C_2(t)}{t^{n-\lambda}} \phi_1 \geq \frac{C_2^2(t)}{t^{n-\lambda}} \phi_2.$$  

We show it can be solved for a suitable choice of Upper and lower barrier functions for suitable minorized/majorized system of equations for $C_1(t)$ and $C_2(t)$ satisfying for all time $t$

\[ \frac{C_1(t)}{C_1(t_0)} = \frac{C_2(t_0)}{C_2(t)} \]

and the initial data for $t_0=1$ must have that $C_2(1)$ satisfies the following

**Beginning Condition**

\[ |C_2(1) - k| \leq K_1(C, \alpha, \beta) d(M_1, M_2) \leq 2 K_1(C, \alpha, \beta) \varepsilon, \]

\[ \exp \left( k \frac{\|\phi_1 + \phi_2\|_{L^\infty} + \|\phi_1 - \phi_2\|_{L^\infty}}{n-\lambda-1} \right) \leq K_2(C, \alpha, \beta), \]

and \[ C_2(1) + k \geq K_3(C, \alpha, \beta), \]

with $k$ satisfying

\[ k^2 = \frac{\|\phi_1 + \phi_2\|_{L^\infty} - \|\phi_1 - \phi_2\|_{L^\infty}}{\|\phi_1 + \phi_2\|_{L^\infty} + \|\phi_1 - \phi_2\|_{L^\infty}} C_1(1) C_2(1) \]
Gradient estimates

Theorem (R. Alonso, I.M.G, JSP09 and 2010): Assume $b(s) \in L^\alpha(S^{n-1})$ for some $\alpha > 1$, and that the initial state $f_0$ is small or is near a local Maxwellian. Assume $\nabla_x f_0 \in (L^1 \cap L^{p_0})$ for $p_0 > 1$.

Then, there is a unique classical solution $f$ to the Cauchy problem in the interval $[0, T]$ satisfying the $L^\infty$-Maxwellian weighted estimates. Furthermore, there exists $1 < \beta(\alpha) < \min(p_0, \beta_0, n/(n-\lambda))$ with $\beta_0$ satisfying

\[
\beta_0 = \begin{cases} 
\infty & \text{when } \frac{n-\lambda}{n-1} \alpha' \leq 1 \\
\left(\frac{n-\lambda}{n-1} \alpha'\right)' & \text{when } \frac{n-\lambda}{n-1} \alpha' > 1.
\end{cases}
\]

such that for any $p \in [1, \beta(\alpha)]$ the following estimates hold

\[
\||\nabla_x f||_{L^p(\mathbb{R}^{2n})}(t) \leq C' ||\nabla_x f_0||_{L^p(\mathbb{R}^{2n})} \quad \text{for all } t \in [0, T],
\]

with constant $C' = C'(n, p, \lambda, \|b\|_{L^\alpha(S^{n-1})})$.

Remarks:

- $b \in L^\alpha$ secures $C^+$ is finite.
- The range of $\beta_0$ may be enlarge with a modified argument (work in progress)
- There is no constrain on the size of $\nabla_x f_0$
- These estimates resemble those of the kind for $L^p$-regularity of non-linear PDE's.
**Velocity regularity**

**Theorem:** Let $f$ be a classical solution in $[0, T]$ with $f_0$ satisfying the condition of smallness assumption or is near to a local Maxwellian and $\nabla_x f_0 \in L^p(R^{2n})$ for some $1 < p < \beta(\alpha) < \min\{p_0, \beta_0, n/(n-\lambda)\}$. If also $\nabla_v f_0 \in L^p(R^{2n})$, then the solution $f(x,v,t)$ satisfies the estimate

$$\|\nabla_v f(t)\|_{L^p(R^{2n})} \leq C \left( \|\nabla_v f_0\|_{L^p(R^{2n})} + t \|\nabla_x f_0\|_{L^p(R^{2n})} \right),$$

with $C = C(n, p, \lambda, \|b\|_{L^*((S^{n-1})^n)})$ independent of the time.

**$L^p$ and $M_{\alpha, \beta}$ stability**

**Theorem:** Let $f$ and $g$ distributional solutions of problem associated to the initial datum $f_0$ and $g_0$ respectively. Assume that these datum satisfies the condition of theorems for small data or near Maxwellians solutions ($0 < \lambda < n-1$). Then, there exist $C > 0$ independent of time such that

$$\|f - g\|_{L^p} \leq C \|f_0 - g_0\|_{L^p}$$

for $1 \leq p < \beta(\alpha) < \min\{p_0, \beta_0, n/(n-\lambda)\}$

Moreover, for $f_0$ and $g_0$ sufficiently small in $M_{\alpha, \beta}$

$$\|(f - g)^\#\|_{L^\infty(0,T; M_{\alpha, \beta})} \leq C \|f_0 - g_0\|_{L^\infty(0,T; M_{\alpha, \beta})}.$$

Ha&Yun’06 proved $L^1$-stability for $b(\theta)$ bounded Here $b(\hat{u} \cdot \sigma)$ in $L^\gamma(S^{n-1})$
Recall from the classical scattering results: the advection operator \( A = -v \cdot \nabla x \) generates the group \( e^{tA} \) that acts on every function \( F \) that is defined almost everywhere by the formula
\[
e^{tA}F^{\text{in}}(v, x) = F^{\text{in}}(v, x - vt)
\]

When \( F^{\text{in}} \) is locally integrable then \( F = e^{tA}F^{\text{in}} \) is the unique distribution solution of the initial-value problem
\[
\partial_t F + v \cdot \nabla x F = 0, \quad F\big|_{t=0} = F^{\text{in}}.
\]

In the Boltzmann setting we have the following scattering Theorem:
Let \( F(v, x, t) \) be a global mild solution of the Cauchy problem for the Boltzmann eq. with potentials \( \lambda \in (-2, n-1] \) that also satisfying all estimates listed above. Then there exists a unique \( F^{\infty}(v, x) \) integrable such that
\[
\lim_{t \to \infty} \| F(t) - e^{tA}F^{\infty} \|_{L^1(dv \, dx)} = 0
\]

\[
F^{\infty} = F^{\text{in}} + \int_0^\infty e^{-t'A}\mathcal{B}(F(t'), F(t')) \, dt'
\]

and \( F^{\infty} \) satisfies the bound
\[
0 \leq F^{\infty}(v, x) \leq \mathcal{M}(v, x, s) \quad \text{almost everywhere over } \mathbb{R}^n \times \mathbb{R}^n.
\]

with
\[
\mathcal{M}(s) = e^{-tA}\mathcal{M}(s + t)
\]
I- Sketch of proofs for Radial Symmetryzation, Young’s and HLS inequalities for the collisional integrals

1- Radial symmetrization

Radial Symmetrization and the operator $\mathcal{P}$

- $G = SO(n)$ the group of orthonormal rotations in $\mathbb{R}^n$.
- The Haar measure $d\mu$ of this compact topological group re-normalized to $\int_G d\mu(R) = 1$.
- The radial symmetrization $f_p^*$ is defined by

$$f_p^*(x) = \left( \int_G |f(Rx)|^p d\mu(R) \right)^{\frac{1}{p}}, \quad \text{if } f \in L^p(\mathbb{R}^n) \quad 1 \leq p < \infty.$$ 

and

$$f_{\infty}^*(x) = \text{ess sup}_{|y|=|x|} |f(y)|$$

taken over the sphere of radius $|x|$ w.r.t.measure over that sphere

The rearrangement $f_p^*$ can be seen as an $L^p$-average of $f$ over all the rotations $R \in G$:

2- Preservation of the $L^p$-norm

Let $d\nu$ be a rotationally invariant measure on $\mathbb{R}^n$:

$$\int_{\mathbb{R}^n} |f(x)|^p \, d\nu(x) = \int_{\mathbb{R}^n} |f_p^*(x)|^p \, d\nu(x) \quad \text{so} \quad \|f\|_{L^p(\mathbb{R}^n)} = \|f_p^*\|_{L^p(\mathbb{R}^n)}.$$
In particular, for radial function set: \( f(x) = f(|x|) \) \( \alpha \) corresponds to moments weights

\[
\int_{\mathbb{R}^n} f(x)^p \, |x|^\alpha \, dx = \left| S^{n-1} \right| \int_0^\infty \tilde{f}(t)^p \, t^{n-1+\alpha} \, dt.
\]

and for \( d\nu_\alpha(x) = |x|^\alpha dx \), and \( \sigma_n^\alpha \) on \( \mathbb{R}^+ \) by \( d\sigma_n^\alpha(t) = t^{n-1+\alpha} dt \) set

\[
\| f \|_{L^p(\mathbb{R}^n, d\nu_\alpha)} = \left| S^{n-1} \right| \frac{1}{p} \left\| \tilde{f} \right\|_{L^p(\mathbb{R}^+, d\sigma_n^\alpha)}
\]

3- angular mixing operator on radial functions

- For radial functions \( P \) simplifies to a 1-dimensional integral

\[
P(\psi, \varphi)(u) = \int_{S^{n-1}} \tilde{\psi}(|u^-|) \, \tilde{\varphi}(|u^+|) \, b(\tilde{u} \cdot \omega) \, d\omega
\]

\[
= \left| S^{n-2} \right| \int_{-1}^{1} \tilde{\psi}(a_1(|u|, s)) \tilde{\varphi}(a_2(|u|, s)) b(s) \left( 1 - s^2 \right)^{\frac{n-3}{2}} \, ds.
\]

with \( a_1 \) and \( a_2 \) are defined on \( \mathbb{R}^+ \times [-1, 1] \rightarrow \mathbb{R}^+ \) by

\[
a_1(x, s) = \beta \, x \left( \frac{1-s}{2} \right)^{1/2} \quad \text{and} \quad a_2(x, s) = x \left[ \left( \frac{1+s}{2} \right) + (1 - \beta)^2 \left( \frac{1-s}{2} \right) \right]^{1/2}
\]
\[ P(\psi, \varphi)(x) = \left| S^{n-2} \right| \int_{-1}^{1} \tilde{\psi}(a_1(x, s)) \tilde{\varphi}(a_2(x, s)) \, d\xi_n^b(s) \]

where the measure \( \xi_n^b \) on \([-1, 1]\) is defined as \( d\xi_n^b(s) = \left| S^{n-2} \right| b(s)(1 - s^2)^{\frac{n-3}{2}} \)

### 4-

**Angular averaging lemma**

**Lemma 4.** Let \( 1 \leq p, q, r \leq \infty \) with \( \frac{1}{p} + \frac{1}{q} = \frac{1}{r} \). For \( \psi \in L^p(\mathbb{R}^n, d\sigma_n^\alpha) \) and \( \varphi \in L^q(\mathbb{R}^n, d\sigma_n^\alpha) \) we have

\[
\|P(\psi, \varphi)\|_{L^r(\mathbb{R}^n, d\sigma_n^\alpha)} \leq \left\| \overline{P(\psi, \varphi)} \right\|_{L^r(\mathbb{R}^n, d\sigma_n^\alpha)} \leq C \|\psi\|_{L^p(\mathbb{R}^n, d\sigma_n^\alpha)} \|\varphi\|_{L^q(\mathbb{R}^n, d\sigma_n^\alpha)},
\]

where the constant \( C \) is given explicitly as a function of the weight, the inelasticity and the angular integration.

In the case of constant parameter \( \beta = (1 + e)/2 \), one can show that \( C \) is sharp

\[
C(n, \alpha, p, q, b, \beta) = \beta^{-\frac{n+\alpha}{p}} \int_{-1}^{1} \left( \frac{1-s}{2} \right)^{-\frac{n+\alpha}{2p}} \left[ \left( \frac{1+s}{2} \right) + (1 - \beta)^2 \left( \frac{1-s}{2} \right) \right]^{-\frac{n+\alpha}{2q}} \, d\xi_n^b(s)
\]
5- Young's inequality for hard potentials for general \(1 \leq p, q, r \leq \infty\)

The main idea is to establish a connection between the \(Q^*\) and \(P\) operators, and then use the knowledge from the previous estimates. For \(\alpha = 0 = \lambda\) (Maxwell type interactions) no weighted norms

\[
I := \int_{\mathbb{R}^n} Q^+(f, g)(v)\psi(v) \, dv = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(v)g(v-u)\mathcal{P}(\tau_v \mathcal{R}\psi, 1)(u) \, du \, dv.
\]

The exponents \(p, q, r\) in Theorem 1 satisfy \(1/p' + 1/q' + 1/r = 1\),

Regroup and use Holder's inequality and the angular averaging estimates on \(L^{r'/q'}\) to obtain

\[
I = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left( f(v)^{\frac{p}{pq'}} \right)^{\frac{q'}{q}} g(v-u)^{\frac{q}{r}} \left( f(v)^{\frac{p}{pq'}} \mathcal{P}(\tau_v \mathcal{R}\psi, 1)(u)^{\frac{r}{q'}} \right)^{\frac{q'}{q'}} du \, dv \leq C \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)} \|\psi\|_{L^{r'/q'}(\mathbb{R}^n)},
\]

\[
C = |S^{n-2}| \left( \frac{2}{r'} \int_{-1}^1 \left( \frac{1-s}{2} \right)^{-\frac{n}{2r'}} \, ds \right)^{\frac{r'}{q'}} \left( \int_{-1}^1 \left[ \left( \frac{1+s}{2} \right)^{1-\alpha} + (1-\beta_0)^2 \left( \frac{1-s}{2} \right)^{1-\alpha} \right]^{-\frac{n}{2r'}} \, ds \right)^{\frac{r'}{q'}}.
\]

These estimates resemble a Beckner and Brascamp-Lieb type inequality argument (for a nonlinear weight) with best/exact constants approach to obtain Young's inequality.

Remark: 1- Previous \(L^p\) estimates by Gustafsson 88, Villani-Mouhot '04 for pointwise bounded \(b(u \cdot \sigma)\), I.M.G-Panferov-Villani '03 for \((p,1,p)\) with \(\sigma\)-integrable \(b(u \cdot \sigma)\) in \(S^{n-1}\).

2-The dependence on the weight \(\alpha\) may have room to improvement. One may expect estimates with polynomial (?) decay in \(\alpha\), like in \(L^1_{\alpha}\) as shown Bobylev,I.M.G, Panferov and recently with Villani (97, 04,08) (also previous work of Wennberg '94, Desvillettes, 96, without decay rates.)
Maxwell type interactions $\lambda=0$ with $\beta$ constant: the constants are sharp in $(1,2,2)$ and in $(2,1,2)$

**Corollary:** Let $f \in L^1(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R}^n)$. Then

$$\|Q^+(f,g)\|_{L^2(\mathbb{R}^n)} = \|Q^+(\hat{f},\hat{g})\|_{L^2(\mathbb{R}^n)} = \|P(\hat{f},\hat{g})\|_{L^2(\mathbb{R}^n)} \leq C_0 \|\hat{f}\|_{L^\infty(\mathbb{R}^n)} \|\hat{g}\|_{L^2(\mathbb{R}^n)} \leq C_0 \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)}$$

The constant is given by

$$C_0 = |S^{n-2}| \int_{-1}^{1} \left[ \left( \frac{1+s}{2} \right) + (1-\beta)^2 \left( \frac{1-s}{2} \right) \right]^{-\frac{n}{4}} \, d\xi^b_n(s).$$

Similarly, for $f \in L^2(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$ we have

$$\|Q^+(f,g)\|_{L^2(\mathbb{R}^n)} \leq C_1 \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)},$$

where

$$C_1 = |S^{n-2}| \beta^{-\frac{n}{2}} \int_{-1}^{1} (\frac{1-s}{2})^{-\frac{n}{4}} \, d\xi^b_n(s).$$

The constant is achieved by the sequences: $f \geq 0$, $\|\hat{f}\|_{L^\infty(\mathbb{R}^n)} = \|f\|_{L^1(\mathbb{R}^n)}$. So approximate

for $x \geq 0$

$$\hat{f}_\epsilon(x) = e^{-\pi \epsilon^2 x^2}$$

and

for $0 < x < 1$

$$\hat{g}_\epsilon(x) = \begin{cases} \epsilon^{1/2} \ x^{-(n-\epsilon)/2} & \text{for } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(see Alonso and Carneiro, Adv Math 2009)
Hardy-Littlewood-Sobolev inequality for soft potentials $-n < \lambda < 0$:

$$
\int_{\mathbb{R}^n} Q^+(f, g)(v) \psi(v) \, dv = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(v)g(v-u)\mathcal{P}(\tau_v\mathcal{R}\psi, 1)(u) \, |u|^\lambda \, du \, dv
$$

$$
= \int_{\mathbb{R}^n} f(v) \left( \int_{\mathbb{R}^n} \tau_v\mathcal{R}g(u) \, \mathcal{P}(\tau_v\mathcal{R}\psi, 1)(u) \, |u|^\lambda \, du \right) \, dv.
$$

This is a type of Brascamp-Lieb inequality argument (for a non-linear coordinate transformation).

Applying Holder's inequality and then the angular averaging lemma to the inner integral with $(p, q, r) = (a, 1, a)$, $a$ to be determined, one obtains

$$
\int_{\mathbb{R}^n} \tau_v\mathcal{R}g(u) \, \mathcal{P}(\tau_v\mathcal{R}\psi, 1)(u) \, |u|^\lambda \, du \leq C_1 \left\| \tau_v\mathcal{R}\psi \right\|_{L^a(\mathbb{R}^n, d\nu^\lambda)} \left\| \tau_v\mathcal{R}g \right\|_{L^{a'}(\mathbb{R}^n, d\nu^\lambda)}
$$

$$
= C_1 \left[ \left( |\psi|^a \ast |u|^\lambda \right)(v) \right]^{1/a} \left[ \left( |g|^a \ast |u|^\lambda \right)(v) \right]^{1/a'}
$$

$$
C_1 = \left| S^{n-2} \right| 2^{\frac{n+\lambda}{a}} \int_{-1}^{1} \left( \frac{1-s}{2} \right)^{-\frac{n+\lambda}{2a}} \, d\xi_n^b(s).
$$

The choice of integrability exponents allowed to get rid of the integrand singularity at $s = -1$, producing a uniform control with respect to the inelasticity $\beta$.

Is it possible to make such choice of $a$?

Indeed, combining with the complete integral above, using triple Holder's inq. yields

$$
\int_{\mathbb{R}^n} Q^+(f, g)(v) \psi(v) \, dv \leq C_1 \left\| f \right\|_{L^p(\mathbb{R}^n)} \left\| |\psi|^a \ast |u|^\lambda \right\|_{L^{b/a}(\mathbb{R}^n)}^{1/a} \left\| |g|^a \ast |u|^\lambda \right\|_{L^{c/a'}(\mathbb{R}^n)}^{1/a'}
$$
Then: for
\[ \frac{1}{a} + \frac{1}{a'} = 1, \quad 1 \leq a \leq \infty \]
and
\[ \frac{1}{p} + \frac{1}{b} + \frac{1}{c} = 1, \quad 1 < b, c < \infty \]

\[ \int_{\mathbb{R}^n} Q^+(f, g)(v) \psi(v) \, dv \leq C_1 \| f \|_{L^p(\mathbb{R}^n)} \| \psi \|^a * |u|^\lambda \|_{L^{b/a}(\mathbb{R}^n)} \| g |^{a'} * |u|^\lambda \|_{L^{c/a'}(\mathbb{R}^n)}^{1/a'} \]

Using the classical Hardy-Littlewood-Sobolev inequality to obtain (Lieb '83) with explicit constants

\[ \| \psi |^a * |u|^\lambda \|_{L^{b/a}(\mathbb{R}^n)} \leq C_2 \| \psi \|^{a} \| \psi \| \| 
\]

where the exponents satisfy
\[ 1 + \frac{a}{b} = \frac{1}{d} - \frac{\lambda}{n}, \quad b > a, \quad 1 < d < \infty \]

\[ 1 + \frac{a'}{c} = \frac{1}{e} - \frac{\lambda}{n}, \quad c > a', \quad 1 < e < \infty \]

In fact, it is possible to find 1/a in the non-empty interval

\[ \max \left\{ \frac{1}{r'(2 + \frac{\lambda}{n})}, 1 - \frac{1}{q(1 + \frac{\lambda}{n})} \right\} < \frac{1}{a} < \min \left\{ \frac{1}{r'(1 + \frac{\lambda}{n})}, 1 - \frac{1}{q(2 + \frac{\lambda}{n})} \right\} \]

provided \( r < q \) !!

so
\[ \| Q^+(f, g) \|_{L^r(\mathbb{R}^n)} \leq C \| f \|_{L^p(\mathbb{R}^n)} \| g \|_{L^q(\mathbb{R}^n)} \]

with \( C = D_1 = C_1 C_2^{1/a} C_3^{1/a'} \) and \( r < q \)

\[ C_1 = C_1(q) \]

\(-n < \lambda < 0 \) and \( 1/p + 1/q = 1 + \lambda/n + 1/r \).
The case \( r < p \) follows using the symmetric convolution structure \( Q^+(f, g) \) (crucial for relating to the classical HLS inequality for convolutions with singular kernels)

\[
\int_{\mathbb{R}^n} Q^+(f, g)(v) \psi(v) \, dv = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(u+v)g(v)P(1, \tau_{-v}\psi)(u) \, |u|^\lambda \, du \, dv
\]

so \[ \|Q^+(f, g)\|_{L^r(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)} \quad \text{with } C = D_2 = C_4 C_5^{1/a} C_6^{1/a'} \quad \text{and } r < p \]

Finally, the case \( r \leq \max\{p, q\} \) follows using the Riesz-Thorin interpolation theorem

so \[ \|Q^+(f, g)\|_{L^r(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)} \quad \text{with } C = D_1^t D_2^{1-t} \]

\( 1 < p, q, r < \infty \) with \(-n < \lambda < 0\) and \(1/p + 1/q = 1 + \lambda/n + 1/r\).

However, the loss operator \( Q^- (f, g) \) lacks the symmetric convolution structure !!

- The locality in \( f \) does not support a commutative convolution structure in its weak form.

\[
\int_{\mathbb{R}^n} Q^-(f, g)(v)\psi(v)dv = \|b\|_{L^1(S^{n-1})} \int_{\mathbb{R}^n} g(v) \left( \int_{\mathbb{R}^n} f(u)\psi(u)\Phi(v-u) \, du \right) dv
\]

\[ \|Q^- (f, g)\|_{L^r(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)} \quad \text{for } r < q \]

\[ \text{with } -n < \lambda < 0 \text{ and } 1/p + 1/q = 1 + \lambda/n + 1/r. \]
Counterexample for the HLS loss operator \( Q^*(f,g) \) estimates:

One should not expect the HLS type inequality for the loss term \( Q^*(f,g) \) to hold outside the range \( 1 \leq p \leq n/(n-\lambda) \).

For instance, set \( p = q = r = 2 \) and \( \lambda = n/2 \), \( \text{ (so } n/(n-\lambda) = 2 \text{) } \)

Then taking the potential \( \Phi(x) = |x|^{-n/2} \) and the function \( g(x) \)

\[
g(x) = \begin{cases} 
|x|^{-n/2} (\ln |x|)^{-1} & \text{if } |x| \geq e, \\
0 & \text{otherwise.}
\end{cases}
\]

Then, compute

\[
\int_{\mathbb{R}^n} g(v_*) \Phi(u) \, dv_* = \int_{\mathbb{R}^n} g(x) \Phi(v - x) \, dx
\]

\[
= \int_{\{|x| \geq e\}} |x|^{-n/2} (\ln |x|)^{-1} |x - v|^{-n/2} \, dx
\]

\[
\geq \frac{1}{2^{n/2}} \int_{\{|x| \geq \max\{e,|v|\}\}} |x|^{-n}(\log |x|)^{-1} \, dx = \infty
\]

Then \( Q^*(f,g)(v) = f(v) \int_{\mathbb{R}^n} g(v_*) \Phi(u) \, dv_* \) has a pointwise blow up.
II- Sketch of proofs for $L^p$ gradients regularity for solutions to the Boltzmann equation in $L^\infty_{M_{\alpha,\beta}}(\mathbb{R}^n)$ and $L^p$ stability

**Gradient estimates**

**Theorem** (R. Alonso, I.M.G): Assume $b(s) \in L^\alpha(S^{n-1})$ for some $\alpha > 1$, and that the initial state $f_0$ is small or is near a local Maxwellian. Assume also that $\nabla f_0 \in (L^1 \cap L^{p_0})$ for $p_0 > 1$.

Then, there is a unique classical solution $f$ to the Cauchy problem in the interval $[0, T]$ satisfying the $L^\infty$-Maxwellian weighted estimates. Furthermore, there exists $1 < \beta(\alpha) < \min(p_0, \beta_0, n/(n-\lambda))$ with $\beta_0$ satisfying

$$
\beta_0 = \begin{cases} 
\infty & \text{when } \frac{n-\lambda}{n-1} \alpha' \leq 1 \\
\left(\frac{n-\lambda}{n-1} \alpha' \right)' & \text{when } \frac{n-\lambda}{n-1} \alpha' > 1.
\end{cases}
$$

such that for any $p \in [1, \beta(\alpha)]$ the following estimates hold

$$
\|\nabla_x f\|_{L^p(\mathbb{R}^{2n})}(t) \leq C \|\nabla_x f_0\|_{L^p(\mathbb{R}^{2n})} \quad \text{for all } t \in [0, T],
$$

with the constant $C = C(n, p, \lambda, \|b\|_{L^\alpha(S^{n-1})})$.

**Remarks:** $b \in L^\alpha$ secures $C_+$ is finite. There is no constrain on the size $\nabla f_0$
Proof: set
\[
(D_{h,x}f)(x) := \frac{f(x + h\hat{x}) - f(x)}{h}, \quad (\tau_{h,x}f)(x) := f(x + h\hat{x}).
\]

\[
p \left\| (Df)^\# \right\|^{p-1} \text{sgn}((Df)^\#) : \int
\]

to obtain
\[
\frac{d(Df)^\#}{dt}(t) = (DQ(f, f))^\#(t) = Q^\#(Df, f)(t) + Q^\#(\tau f, Df)(t).
\]

\[
\frac{d \left\| Df \right\|_p^p}{dt} \leq p \int_{\mathbb{R}^n} \left\| Df \right\|_p^{p-1} \left( \left\| Q_+(Df, f) \right\|_{L_p^p(\mathbb{R}^n)} + \left\| Q_+(\tau f, Df) \right\|_{L_p^p(\mathbb{R}^n)} + \left\| Q_-(\tau f, Df) \right\|_{L_p^p(\mathbb{R}^n)} \right) dx.
\]
Case $p>1$:
1. Estimate the $Q^+$ part of the operator by HLS

$$\|Q_+(Df, f)\|_{L^p_0(\mathbb{R}^n)} \leq C_+ \|u\|_{L^\infty(\mathbb{R}^n)}^{-\lambda} \|Df\|_{L^p_0(\mathbb{R}^n)} \|f\|_{L^q_0(\mathbb{R}^n)}$$

for

$$C_+ \leq C(n, p, q, r) \left( \int_{-1}^1 \frac{1}{(1-s)^{\frac{n-\lambda}{2a}}(1-s^2)^{\frac{n-3}{2}}} ds \right)^{1/\alpha'} \|b\|_{L^\alpha(S^{n-1})}.$$

now taking $1 < \beta(\alpha) < \min(p_0, \beta_0, n/(n-l))$

such that for any $p \in (1, \beta(\alpha)) \iff (n-\lambda)\alpha'/2p_0 < 1 \iff C_+ \text{ finite}$

2. The corresponding estimate for $Q^+ (\tau f, Df)$ is direct since $b(\cdot)$ was chosen to have support in $[0, 1]$.

3. The estimate for $Q^- (\tau f, Df)$ follows using the HLS theorem for the loss operator by choosing $\beta(\alpha) < \min \{p_0, n/(n-\lambda)\}$.

4. \[ d\|Df\|_{L^p}^p \leq p \int_{\mathbb{R}^n} \|Df\|_{L^p_0(\mathbb{R}^n)}^p (\|f\|_{L^q(\mathbb{R}^n)} + \|\tau f\|_{L^q(\mathbb{R}^n)}) \, dx. \]

with \[
\left\{ \begin{array}{l}
\|f\|_{L^q(\mathbb{R}^n)} \leq \frac{C}{(1+t)^{n/a}} = \frac{C}{(1+t)^{n-\lambda}}, \\
\|\tau f\|_{L^q(\mathbb{R}^n)} \leq \frac{C}{(1+t)^{n-\lambda}}.
\end{array} \right.
\]

By Gronwall inequality,

$$\|Df\|_{L^p_0(\mathbb{R}^{2n})}(t) \leq \|Df_0\|_{L^p_0(\mathbb{R}^{2n})} \exp \left( \int_0^t \frac{C}{(1+s)^{n-\lambda}} \, ds \right),$$

$p \in (1, \beta(\alpha))$.
Case $p = 1$:

1. Separate the potential in two radially symmetric potentials

$$|u|^{-\lambda} = \Phi_1(u) + \Phi_2(u),$$

where $\Phi_1 \in L^s(\mathbb{R}^n)$ for any $1 \leq s < n/\lambda$ and $\Phi_2 \in L^\infty(\mathbb{R}^n)$

2. Then

$$\|Q_-(f, \nabla f)\|_{L^1(\mathbb{R}^n)} \leq \|Q_-,\Phi_1(f, \nabla f)\|_{L^1(\mathbb{R}^n)} + \|Q_-,\Phi_2(f, \nabla f)\|_{L^1(\mathbb{R}^n)}$$

$$\leq C \left( \|\Phi_1\|_{L^1(\mathbb{R}^n)} \|f\|_{L^{s'}(\mathbb{R}^n)} \|\nabla f\|_{L^1(\mathbb{R}^n)} + \|\Phi_2\|_{L^\infty(\mathbb{R}^n)} \|f\|_{L^1(\mathbb{R}^n)} \|\nabla f\|_{L^1(\mathbb{R}^n)} \right).$$

3. Where, using the upper Maxwellian control on $f$

$$\|Q_-(f, \nabla f)\|_{L^1(\mathbb{R}^{2n})} \leq \frac{C}{(1 + t)^{n/s'}} \|\nabla f\|_{L^1(\mathbb{R}^{2n})}$$

with $n/s' > 1$ for $s \in (n/(n-1), n/(n-\lambda))$.

4. Then

$$\frac{d}{dt} \|\nabla f\|_{L^1} \leq \frac{C}{(1 + t)^{n/s'}} \|\nabla f\|_{L^1}.$$

by Gronwall

$$\|\nabla f\|_{L^1}(t) \leq C \|\nabla f_0\|_{L^1} \text{ for all } t \in [0, T].$$
Velocity regularity

**Theorem** Let \( f \) be a classical solution in \([0, T]\) with \( f_0 \) satisfying the condition of smallness assumption or is near to a local Maxwellian and \( \nabla_x f_0 \in L^p(\mathbb{R}^{2n}) \) for some 
\[ 1 < p < \beta(\alpha) < \min\{p_0, n/(n-\lambda)\} \quad \text{and} \quad (n-\lambda)\alpha'/2p_0 < 1 \]
In addition assume that \( \nabla_v f_0 \in L^p(\mathbb{R}^{2n}) \). Then, \( f \) satisfies the estimate

\[
\left\| (\nabla_v f)(t) \right\|_{L^p(\mathbb{R}^{2n})} \leq C \left( \left\| \nabla_v f_0 \right\|_{L^p(\mathbb{R}^{2n})} + t \left\| \nabla_x f_0 \right\|_{L^p(\mathbb{R}^{2n})} \right),
\]

with \( C = C(n, p, \lambda, \|b\|_{L^1(S^{n-1})}) \) independent of the time.

**Proof:** Take 
\[
(D_h \hat{v} f)(v) := \frac{f(v + h\hat{v}) - f(v)}{h}
\]
for a fix \( h > 0 \) and \( \hat{v} \in S^{n-1} \) and the corresp. translation operator and transforming 
\( v_* \rightarrow v_* + h\hat{v} \) in the collision operator.

\[
p \left| (D f)^{p-1} \right| \left| \text{sgn}((D f)) \right| \leq \int
\]

\[
\frac{d(D f)}{dt}(t) + v \cdot \nabla (D f)(t) + \hat{v} \cdot \nabla (\tau f)(t) = Q(D f, f)(t) + Q(\tau f, D f)(t).
\]

\[
\frac{d \left\| D f \right\|_{L^p}^p}{dt}(t) \leq \frac{p C}{(1 + t)^{n-\lambda}} \left\| D f \right\|_{L^p(\mathbb{R}^{2n})}^p + p \left\| D f \right\|_{L^p(\mathbb{R}^{2n})}^{p-1} \left\| \nabla f \right\|_{L^p(\mathbb{R}^{2n})}.
\]
\[
\frac{d \|Df\|_{L^p}^p}{dt}(t) \leq \frac{p \ C}{(1 + t)^{n-\lambda}} \|Df\|_{L^p(\mathbb{R}^{2n})}^p + p \|Df\|_{L^p(\mathbb{R}^{2n})}^{p-1} \|\nabla f\|_{L^p(\mathbb{R}^{2n})}.
\]

Just set

\[X(t) := \|Df\|_{L^p(\mathbb{R}^{2n})}^p(t)\]

then

\[
\frac{dX(t)}{dt} \leq a(t)X(t) + b(t)X^{p-1}(t).
\]

Bernoulli ODE

with \(a(t) = \frac{p \ C}{(1 + t)^{n-\lambda}}\) and \(b(t) = p \|(\nabla f)(t)\|_{L^p(\mathbb{R}^{2n})}^{p-1}\).

Which is solved by

\[
\frac{1}{X^p}(t) \leq \frac{1}{X_0^p} \exp \left(\frac{1}{p} \int_0^t a(s) ds \right) + \frac{1}{p} \int_0^t \exp \left(\frac{1}{p} \int_0^s a(\sigma) d\sigma \right) b(\sigma) d\sigma,
\]

Then, by the regularity estimate

\[
\|Df\|_{L^p(\mathbb{R}^{2n})}(t) \leq \left(\|Df_0\|_{L^p(\mathbb{R}^{2n})} + t \|\nabla f_0\|_{L^p(\mathbb{R}^{2n})}\right) \exp \left(\int_0^t \frac{C}{1 + s^{n-\lambda}} ds \right).
\]

for \(1 \leq p < \beta(\alpha) < \min\{p_0, n/(n-\lambda)\}\) and \((n-\lambda) \alpha'/2p_0 < 1\) with \(0 < \lambda < n-1\).
\[ L^p \text{ and } M_{\alpha,\beta} \text{ stability} \]

Set
\[
\frac{d(f-g)^\#}{dt}(t) = Q^\#(f,f)(t) - Q^\#(g,g)(t) = \frac{1}{2} \left[ Q^\#(f-g, f+g) - Q^\#(f+g, f-g) \right].
\]

Multiplying by \(|(f-g)^\#|^p \text{sgn}((f-g)^\#)|^p\) with \(p > 1\)

\[
\frac{d \|f-g\|_{L^p}^p}{dt}(t) \leq C \int_{\mathbb{R}^n} \|f-g\|_{L^p_{\nu}(\mathbb{R}^n)}^p \|f+g\|_{L^q_{\nu}(\mathbb{R}^n)} d\nu.
\]

Now, since \(f\) and \(g\) are controlled by traveling Maxwellians one has
\[
\|f+g\|_{L^q_{\nu}(\mathbb{R}^n)} \leq \frac{C}{(1+t)^{n-\lambda}}.
\]
with \(0 < \lambda < n-1\)

**Theorem** Let \(f\) and \(g\) distributional solutions of problem associated to the initial datum \(f_0\) and \(g_0\) respectively. Assume that these datum satisfies the condition of theorems for small data or near Maxwellians solutions \((0 < \lambda < n-1)\). Then, there exist \(C > 0\) independent of time such that

\[
\|f-g\|_{L^p} \leq C \|f_0 - g_0\|_{L^p}
\]

Moreover, for \(f_0\) and \(g_0\) sufficiently small in \(M_{\alpha,\beta}\)

\[
\|(f-g)^\#\|_{L^\infty(0,T;M_{\alpha,\beta})} \leq C \|f_0 - g_0\|_{L^\infty(0,T;M_{\alpha,\beta})}.
\]

for \(1 < \beta < p < \beta(\alpha) < \min\{ p_0, n/(n-\lambda) \}\)

and \((n-\lambda)\alpha'/2p_0 < 1\)

Our result is for \(b(\hat{u} \cdot \sigma)\) in \(L^\alpha(\mathbb{S}^{n-1})\)
Approximations by Spectral-Lagrangian Constrained methods

Collaborators:
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The spectral conserved numerical scheme and convergence

(I.M.G. with H. Tharskabhusanam, JCP’09 & JCM,10)

- **The main difficulty is the non-linear nature of the collisional Integral**

- **Crucial point: Find a good transformation in its weak form**

Thus for a suitably regular test function $\psi(v)$, the weak form of the collision integral (operator) takes the form (suppressing the time dependence in $f$):

$$
\int_{v \in \mathbb{R}^d} Q(f, f) \psi(v) dv = \int \int \int_{v, v_*, \sigma \in \mathbb{R}^{2d} \times S^{d-1}} f f^* B(|u|, \mu(\sigma))[\psi(v') - \psi(v)] d\sigma dw dv
$$

Using $e^{-ik.v}$ for $\psi(v)$ and substituting the definition of $v'$, we get the Fourier transformed collision operator:

$$
\hat{Q}(k) = \int \int \int_{v, v_*, \sigma \in \mathbb{R}^{2d} \times S^{d-1}} f f^* B(|u|, \mu(\sigma))[e^{-ik.(v+\frac{\sigma}{2}(|u|\sigma-u))} - e^{-ik.v}] d\sigma dv_* dv
$$
Transformations for an efficient numerical approach

$$B(|u|, \mu(\sigma)) = b_\lambda(\sigma)|u|^\lambda,$$

$$\hat{Q}(k) = \int \int \int_{v, v_\sigma \in \mathbb{R}^{2d+1}} f f^* b_\lambda(\sigma)|u|^\lambda e^{-i k \cdot v} \left[ e^{-i \frac{d}{2} k \cdot (|u|\sigma - u)} - 1 \right] d\sigma dv_* dv$$

With a change of variables $v_* = v - u \Rightarrow dv_* = du$, re-arrangement and re-grouping:

$$\hat{Q}(k) = \int_{\mathbb{R}^d} \hat{f}(y) \hat{f}(k - y) \hat{G}_{\lambda, \beta}(y, k) dy$$

with $\hat{G}(y, k) = \mathcal{F}_{u \rightarrow y} G(u, k)$ and

$$G_{\lambda, \beta}(u, k) = \int_{\mathbb{S}^{d-1}} b_\lambda(\sigma)|u|^\lambda \left[ e^{-i \frac{d}{2} k \cdot (|u|\sigma - u)} - 1 \right] d\sigma$$

is an operator invariant under rotations in $(y, k)$: it has an expansion on a basis of $d$-dimensional spherical harmonics

or, in the best possible case as in hard spheres in $3d$, for uniform in angle scattering

$$G_{\lambda, \beta}(u, k) = b_\lambda c_d |u|^\lambda \left\{ e^{i \frac{\beta}{2} k \cdot u} \text{sinc}\left( \frac{\beta |u| |k|}{2} \right) - 1 \right\}$$

the weight function in the convolutional form is highly oscillatory
To get back the collision integral, one takes the inverse Fourier Transform of $\hat{Q}_{\lambda, \beta}[f](k)$

$$Q_{\lambda, \beta}[f, f](v) = \tilde{Q}_{\lambda, \beta}[f](k) = \int_{k \in \mathbb{R}^d} \left\{ \int_{y \in \mathbb{R}^d} \hat{G}_{\lambda, \beta}(y, k) \hat{f}(y) \hat{f}(y - k) \right\} dy e^{i k \cdot v}$$

with $\beta \in [0, 1], \lambda \in [0, 1]$.

computational cost: FFTW of $f(v) f(v - u)$ for each $u$, with respect to $v$; multiplying this result with $G_{\lambda, \beta}(u, k)$ for each $u$ and $k$. Take Inverse FFTW with respect to $k$:

Total # operations $d O(N^d \log N) + O(N^{2d})$.

Can we do better? No! , if $f(x,v,t)$ is discontinuous in $v$

Remark: An alternative approach that uses the Carlemann integral representation was proposed by Bobylev, Rjasanov 99, Rjasanov& Ibrahimov 02, Pareschi, Russo, Filbet’03,’07

Our approach is easy to implement using the FFTW plus the above integral representation of the total $Q(f,f)$ operator
Collision Integral Algorithm: $O(N^d \log N) + O(M^d N^d)$

1. $(C \ (N^d \ \log \ (N^d \ ))$
   $\hat{f}(\zeta_m) = \text{FFT}_{v_k \rightarrow \zeta_m}[f(v_k)]$

2. $(O(N^3))$
   For $\zeta_m \in C_u$, Do
   
   2.1 $\hat{Q}(\zeta_m) = 0$
   
   2.2 $(O(M^d))$
      For $\xi_1 \in C_u$, Do
      
      2.2.1 $g(\xi_1) = \hat{f}(\xi_1) \times \hat{f}(\zeta_m - \xi_1)$
      2.2.2 $\hat{Q}(\zeta_m) = \hat{Q}(\zeta_m) + \tilde{G}_{1,m} \times \omega[l] \times g(\xi_1)$

      2.2* End Do

3. $(C \ (N^d \ \log(N)))$
   $Q(v_k) = \text{IFFT}_{\zeta_m \rightarrow v_k}[\hat{Q}(\zeta_m)]$

Filbet, Mouhot&Pareschi’07 proposed to solve hard spheres in 3 dimensions (isotropic scattering) by $O(N^d \ \log \ N)$ operations: This approach would imply that step [2.2] is done in $O(M^{d-1} \ \log \ N)$ with $M^{d-1} \ \log \ N \ll N^d$, and so loosing the accuracy of the calculation of $Q^\wedge$ by computing a few approximated weights $G_{lm}$, $l=1,\ldots, M$. 

$\omega[l]$ are the integration weights.
Conservation Procedure  
(I.M.G. with H. Tharskabhushanam, JCP’09)

Discrete version of the conservation scheme

\( M = N^d = \) the total number of Fourier modes. For elastic collisions, \( a \in \mathbb{R}^m \), \( m = \) number of conserved moments (collision invariants)

\[
\begin{align*}
\vec{Q} &= (\vec{Q}_1, \vec{Q}_2, \ldots, \vec{Q}_M)^T \text{, computed } C^O \\
Q &= (Q_1, Q_2, \ldots, Q_M)^T \text{ conserved } C^O
\end{align*}
\]

Let \( \omega_j \) be the integration weights where \( j = 1, 2, \ldots, M \). Define \( C = '\text{vector of moments}, \quad a = '\text{vector of conserved quantities}:

\[
C^e_{(m(d) \times M)} = \begin{pmatrix}
\langle \omega_j \rangle \\
\langle v_j \omega_j \rangle \\
\vdots \\
\langle \times m(d) v_j \omega_j \rangle
\end{pmatrix} \quad \text{and} \quad a^e_{m(d) \times 1} = \begin{pmatrix}
a_1 \\
a_2 \\
\vdots \\
a_m
\end{pmatrix}
\]

Then, the conservation method can be written as a constrained optimization problem: Find \( Q \) such that

\[
(*) \{ \min \left\| \vec{Q} - Q \right\|_2^2 : C^e Q = a^e; C^e \in \mathbb{R}^{d+2 \times M}, \vec{Q} \in \mathbb{R}^M, a^e \in \mathbb{R}^{d+2} \}
\]

To solve \((*)\), one can employ the Lagrange multiplier method.
Let $\gamma \in \mathbb{R}^{d+2}$ be the Lagrange multiplier vector. Then the scalar objective function to be optimized is given by

$$L(\tilde{Q}, \gamma) = \sum_{j=1}^{M} |\tilde{Q}_j - Q_j|^2 + \gamma^T(C^eQ - a^e).$$

Can be solved explicitly for the corrected value and the resulting equation of correction is implemented numerically in the code.

Taking the derivative of $L(\tilde{Q}, \lambda)$ with respect to $f_j, j = 1, ..., M$, and $\gamma_i, i = 1, ..., m(d)$, i.e., gradients of $L$, retrieve the constrains by

$$\frac{\partial L}{\partial \tilde{Q}_j} = 0 \quad j = 1, ..., M, \quad \Rightarrow \quad Q = \tilde{Q} + \frac{1}{2}(C^e)^T \gamma$$

$$\frac{\partial L}{\partial \gamma_1} = 0; \quad i = 1, ..., d+2, \Rightarrow \quad C^eQ = a^e,$$

and solve for $\gamma$,

$$C^e(C^e)^T \gamma = 2(a^e - C^e\tilde{Q}).$$

Now $C^e(C^e)^T$ is symmetric and positive definite so its inverse exists $\Rightarrow$

$$\gamma = 2(C^e(C^e)^T)^{-1}(a^e - C^e\tilde{Q}).$$
\[ Q = \bar{Q} + (C^e)^T (C^e(C^e)^T)^{-1} (a^e - C^e \bar{Q}) \]
\[ = [I_M - (C^e)^T (C^e(C^e)^T)^{-1} C^e] \bar{Q} \]
\[ = \Lambda_M(C^e) \bar{Q}, \quad \text{Discrete Conservation operator} \]

\[ \Rightarrow \text{Define } \Lambda_M(C^e) : I_M - (C^e)^T (C^e(C^e)^T)^{-1} C^e \text{ Then this procedure is} \]
\[ \text{Conserve(} \bar{Q} \text{)} = Q = \Lambda_M(C^e) \bar{Q}. \]

Then, for \( D_t f \) any \textbf{order time discretization} of \( \frac{df}{dt} \) \( \Rightarrow \)

\[ D_t f = \Lambda_M(C^e) \bar{Q}, \quad \text{`conserve' algorithm} \]

\textbf{This identity summarizes the whole conservation process:}

- Required observables are conserved

- The approximate solution to the \textit{elastic homogeneous BE} approaches a stationary state, since

\[ \lim_{n \to \infty} \| \Lambda_M(C) \bar{Q}(f_j^n, f_j^n) \|_\infty = 0 \quad \text{Stabilization property} \]
Consistency and Convergence analysis of the spectral conservation method

Set $\Omega_v = [L, L)^d$ and the set of trigonometric polynomials

$$\mathbb{P}^N = \text{span}\{e^{i\zeta_k \cdot v} \mid -L \leq \zeta_k \leq L, l = 1, 2, 3; -N/2 \leq k < N/2\}$$

and $\Pi : L_2(\Omega_v) \to \mathbb{P}^N$ is the orthogonal projection operator upon $\mathbb{P}^N$

$$f^\Pi(v) = \sum_k \hat{f}_N(\zeta_k) e^{i\zeta_k \cdot v}$$

with $\sum_{k=-N/2}^{N/2+1} = \sum_k$

with $\langle f - \Pi f, \psi \rangle = 0 \quad \forall \ \psi \in \mathbb{P}^N$ and $\langle f, f \rangle = \|f\|_{L_2(\Omega_v)}$.

Main tool: the use of the Extension Operator

$$\mathbf{E} f = f \text{ a.e. in } \Omega_v.$$

$\mathbf{E} : L^2(\Omega_v) \to L^2(\mathbb{R}^d)$ with also $\mathbf{E} : H^\alpha(\Omega_v) \to H^\alpha(\mathbb{R}^d)$ for any $\alpha \leq \alpha_0$. satisfying

- $\|\mathbf{E} f\|_{H^\alpha(\mathbb{R}^d)} \leq C\alpha \|f\|_{H^\alpha(\Omega_v)}$ for $\alpha \leq \alpha_0$ is linear and bounded.
Extension Operator

\[ E f = f \text{ a.e. in } \Omega_v. \]

\[ E : L^2(\Omega_v) \to L^2(\R^d) \text{ with also } E : H^\alpha(\Omega_v) \to H^\alpha(\R^d) \text{ for any } \alpha \leq \alpha_0 \text{ satisfying} \]

- \[ \|Ef\|_{H^\alpha(\R^d)} \leq C_\alpha \|f\|_{H^\alpha(\Omega_v)} \text{ for } \alpha \leq \alpha_0 \text{ is linear and bounded.} \]

- Outside \( \Omega_v \) reflect \( f \) near the boundary \( \partial \Omega_v \) with support in a dilated set \( \delta \Omega_v, \delta \geq 1 \), and

  \[ \|Ef\|_{L^p(\delta \Omega_v \setminus \Omega_v)} \leq C_0 \|f\|_{L^p(\Omega_v \setminus \delta^{-1} \Omega_v)} \text{ for } 1 \leq p \leq 2, \]

  with \( C_0 \) is independent of the support of the extension.

- The two previous properties imply: for any \( \delta \geq 1 \) there is an extension

  \[ \|Ef\|_{L^p_k(\R^d)} \leq 2C_0 \delta^{2k} \|f\|_{L^p_k(\Omega_v)} \text{ for } 1 \leq p \leq 2, \ k \geq 0. \]
The spectral conservation method

Formally, \[ \frac{\partial \Pi^N f}{\partial t}(v, t) = \Pi^N Q(f, f)(v, t) \text{ in } \Omega_v \times [0, T], \]

and it is expected that for large \( N \) it holds \( \Pi^N Q(f, f) \sim \Pi^N Q(\Pi^N f, \Pi^N f) \)

- A good approximation to \( \Pi^N f \) would be the solution of the problem
  \[ \frac{\partial g}{\partial t}(v, t) = \Pi^N Q(g, g)(v, t) \text{ in } \Omega_v \times [0, T] \]
  with the initial condition \( g_0 = \Pi^N f_0 \)

- or more adequate the solution of the Spectral Conservation Form
  \[ \frac{\partial g}{\partial t}(v, t) = Q_c(g)(v, t) \text{ in } \Omega_v \times [0, T], \]
  where \( Q_c(g) \) is defined as the \( L^2(\Omega_v) \)-closest function to \( \Pi^N Q(Eg, Eg) \) conserving the collision invariants.
A Conservation Method - An Extended Isoperimetric (Isomoment) problem

Set \[ Q_u(f)(v) := \Pi^N (Q(E_f, E_f) 1_{\Omega_v})(v). \]

Unconstrained Collision approximation

Elastic Problem: Minimize in \( B^e = \left\{ X \in L^2(\Omega_v) : \int_{\Omega_v} X = \int_{\Omega_v} X v = \int_{\Omega_v} X |v|^2 = 0 \right\} \), the functional

\[ A^e(X) := \int_{\Omega_v} (Q_u(f)(v) - X)^2 dv. \]

(Elastic Lagrange Estimate): The problem has a unique minimizer given by

\[ Q_c(g)(v) := X^* = Q_u(f)(v) - \frac{1}{2} \left( \gamma_1 + \sum_{j=1}^{d} \gamma_{j+1} v_j + \gamma_{d+2} |v|^2 \right), \]

where \( \gamma_j \), for \( 1 \leq j \leq d + 2 \), are Lagrange multipliers satisfying

\[
\begin{align*}
\gamma_1 &= O_d \rho_u + O_d e_u, \\
\gamma_{j+1} &= O_{d+2} \mu^j_u, & j = 1, 2, \ldots, d, \\
\gamma_{d+2} &= O_{d+2} \rho_u + O_{d+4} e_u.
\end{align*}
\]

The parameters \( \rho_u, e_u, \mu^j_u \) are the moments of \( g \), and \( O_r := O(L^{-r}) \) depends on \( |\Omega_v|^{-1} \).

In particular, for dimension \( d = 3 \), the minimized objective function is given by

\[
A^e(X^*) = \|Q_u(f) - X^*\|_{L^2(\Omega_v)}^2 = 2\gamma_1^2 L^3 + \frac{2}{3} (\gamma_2^2 + \gamma_3^2 + \gamma_4^2) L^5 + 4\gamma_1 \gamma_5 L^5 + \frac{38}{15} \gamma_5^2 L^7.
\]
1- Conservation Correction Estimate: Fix $f$ in $L^2$ then the accuracy of the conservation minimization problem is proportional to the spectral accuracy:

$$\| (Q_c(f) - Q_u(f)) |v|^k \|_{L^2(\Omega_v)} \leq \frac{C}{\sqrt{k}} L^k \| Q(Ef, Ef) - Q_u(f) \|_{L^2(\Omega_v)}$$

$$+ \frac{\delta^{2k'}}{\sqrt{k}} O(d/2 + k' - k - \lambda) \| f(1 + |v|^{k'}) \|_{L^1(\Omega_v)} \| f \|_{L^1_{\lambda}(\Omega_v)},$$

where $C$ is a universal constant.

2- Shannon Sampling theorem applied to the collisional form

Recall Fourier Approximation Estimate: Let $u \in H^\alpha_0(\Omega_L) \cap S(\Omega_L)$,

$$u_N = \Pi u = \sum_k \hat{u}_N(\zeta_k) e^{i\zeta_k \cdot v}$$

and $\alpha$ be a multi index

$$\| u - u_N \|_{L^2(\Omega_v)} \leq \frac{C}{N|\alpha|} \| u \|_{H^\alpha_0}.$$
3- $L^1-L^2$ propagation theory and error estimates:

**Main property.** Fix $v$ and let $\{g_N\}$ be a sequence of solutions for the semi-discrete problem. Then there exist $\epsilon (N_0, g_0) > 0$ such that for any $N > N_0$

$$\int_{\{g_N < 0\}} |g_N(t, v)| < v >^2 \, dv \leq \epsilon \int_{\{g_N \geq 0\}} g_N(t, v) < v >^2 \, dv, \quad \text{for } t \in [0, T].$$

with $\epsilon (N_0, g_0) < K << 1$ and as $|\Omega_v| \to \infty$.

This property basically says that if the domain of the approximation sequence $v$ is large enough the sequence of semi-discrete solutions is essentially nonnegative, as expected.
The property on positivity plus propagation of $L^p$ and higher regularity of the continuous solution (Villani & Mouhot '04, Alonso & I.M.G. '10) plus moments of conserved estimates and the Shannon sampling theorem: imply $L^1_k$ and $L^2_k$ stability

$$\sup_{t \in [0,T]} \| g_N \|_{L^1_k(\Omega_v)} \leq C \left( \| g_0 \|_{L^1_k(\Omega_v)} \right)$$

all with $C$ independent of $N$, $L$, and $T$.

$L^2_k$ stability

$$\sup_{t \in [0,T]} \| g_N \|_{L^2_k(\Omega_v)} \leq \max \{ \| g_0 \|_{L^2_k(\Omega_v)}, C \}$$

for $\alpha$ a positive multi-index

Error estimates and asymptotic behavior

Rate of convergence of the approximating solutions $gN$ towards the actual Boltzmann solution assuming smooth initial data: (Alonso, I.M.G., Tharsakabhushanam, 2011)

Theorem ($H^\alpha$-convergence) Fix a sufficiently large $\Omega_v$ and assume that $f_0 \in H^{\alpha_0}(\Omega_v)$ has finite support. Suppose also that the sequence of solutions $\{g_N\}$ of problem that satisfies assumption.

Then, for $\alpha \leq \alpha_0$, $k' \geq 0$ and $k > d/2 + \lambda$ there exists an extension $E$ such that,

$$\sup_{t \in [0,T]} \| f - g_N \|_{H^\alpha_k(\Omega_v)} \leq e^{CT} \left( O \left( \frac{L^{k+|\alpha_0|+|\alpha|\lambda}}{N|\alpha_0|-|\alpha|} \right) + O_{k'} \right).$$
Testing - Maxwell Elastic Collisions

Figure 1: Left four graphs: Momentum Flow - Right graph: pdf evolution
Inelastic collisions evolution

\[ K'(t) = \beta (1 - \beta) \left( \frac{|\mathbf{V}|^2}{2} - K(t) \right) \Rightarrow K(t) = K(0)e^{-\beta(1-\beta)t} + \frac{|\mathbf{V}|^2}{2} (1 - e^{-\beta(1-\beta)t}) \]

where \( K(0) \) is Kinetic Energy at time \( t = 0 \) and \( \mathbf{V} \) - momentum (constant) of the distribution function.

**Figure 2:** Kinetic Energy for \( N = 30 \)

See also I. M. G., S. Rjasanow, and W. Wagner, MCM’05 for DSMC simi
Discontinuous Initial Data – Elastic collisions

Conservative scheme
N=34
Final time: 250 mean free times

We observe no oscillatory behavior
Resolving the discontinuity
**Soft Potential calculations**  (with Jeff Haack ’11)

\[ \hat{Q}(k) = \int_{y \in \mathbb{R}^d} \hat{f}(y) \hat{f}(k-y) \hat{G}_{\lambda,\beta}(y, k) dy \]

with \( \hat{G}(y, k) = \mathcal{F}_{u-y} G(u, k) \) and \( G_{\lambda,\beta}(u, k) = \int_{\sigma \in S^{d-1}} b_{\lambda}(\sigma) |u|^\lambda [e^{-\frac{\beta}{2} k \cdot (|u|\sigma - u)} - 1] d\sigma \)

\( \beta = 1 \) (elastic)

**Very soft potential kernels**

\( \lambda = -2.2; -2.9; -3 \)
Benchmarking self similar asymptotics

A weakly coupled binary mixture problem for Maxwell type interactions: explicit solutions can be constructed and convergence of rescaled solutions is proven (A.V. Bobylev & I.M.G, JSP’06)

\[
\frac{\partial f(v, t)}{\partial t} = \int_{\omega \in \mathbb{R}^3} \int_{\sigma \in S^2} B(|u|, \mu)[f'(v, t)f'(w, t) - f(v, t)f(w, t)]d\sigma dw \\
+ \theta_b \int_{\omega \in \mathbb{R}^3} \int_{\sigma \in S^2} B(|u|, \mu)[f'(v, t)M_T'(w) - f(v, t)M_T(w)]d\sigma dw
\]

with \( M_T(v) = \frac{-|v|^2}{(2\pi T)^{3/2}} \), \( B(|u|, \mu) = C_\lambda = \frac{1}{4\pi}, \beta = 1.0, \theta_b \) - depending on the asymptotics and \( T \) being the background temperature.

様々 A system of two different particles with the same mass is considered. One set of particles is assumed to be at equilibrium i.e., with a Maxwellian distribution with temperature \( T(t) \).

様々 Second set of particles is assumed to collide with themselves (first integral) and the background particles (Linear Boltzmann Collision Integral).

The collisions are assumed to be locally elastic i.e., \( |v|^2 + |v_*|^2 = |v'|^2 + |v_*'|^2 \) but the above form leads to global energy dissipation i.e., \( \int_{\mathbb{R}^3} |v|^2 f(v, t)dv \neq 0 \).
Self-Similar Asymptotics elastic BTE with thermostat

For self similar asymptotics we study \( t \to \infty \) so \( \hat{T} \to T \) in \( f^ss_T(v, t) \) (i.e. the particle distribution temperature approaches the background temperature as expected due to the linear coll. op.)

Interesting NESS behavior can be observed if \( T \to 0 \): Set
\[
\hat{T} = s^2 e^{-2t/3} \quad \text{so} \quad f^ss_0(|v|) \text{ is explicit.}
\]

Then \( f(|v|e^{-t/3}, t) \to_{t \to \infty} e^t f^ss_0(|v|) \) where
\[
f^ss_0(|v|) = \frac{4}{\pi} \int_0^{\infty} e^{-|v|^2/(2s^2)} \frac{e^{-|v|^2/(2s^2)}}{(2\pi s^2)(1+s^2)^2} ds
\]

Finite kinetic energy case
(also solvable for infinity kinetic energy)

\[
f^ss_0(|v|) = O\left(\frac{1}{|v|^3}\right) \quad \text{as} \quad |v| \to \infty,
\]
and
\[
f^ss_0(|v|) = O\left(\frac{1}{|v|^2}\right) \quad \text{as} \quad |v| \to 0
\]

Soft condensed matter phenomena

Remark: The numerical algorithm is based on the evolution of the continuous spectrum of the solution as in Greengard-Lin’00 spectral calculation of the free space heat kernel, i.e. self-similar of the heat eq. in all space.
Self-similar solutions and Power-like Tails

**Theorem:** (Bobylev, Cercignani, I.M.G, 06) The self-similar asymptotic function $F_{\mu(p)}(|v|)$ does **NOT** have finite moments of all orders if the energy dissipates, i.e. $\mu(1) < 0$.

If $0 \leq p \leq 1$ then, $m_q = \int_{\mathbb{R}^3} F_{\mu(p)}(|v|)|v|^q \, dv \leq \infty; \quad 0 \leq q \leq p$

If $p = 1$ (finite initial energy) then, $m_q \leq \infty$ only for $0 \leq q \leq p_*$, where $p_* \geq 1$ is the unique maximal root of the equation $\mu(p_*) = \mu(1)$.

$$\lambda(p) = (\lambda(p) - 1) \, p^{-1} := \text{Spectral function associated to } \Gamma(u) = F(Q^+).$$

for $\Gamma(x^p) = \lambda(p) \, x^p$ with $Q(f)$ a multi-linear operator similar to Maxwell-type interactions.

See also F. Bassetti and L. Ladelli’11 for a recent probabilistic interpretation of martingale theorems, convergence to stable laws for extension of the problem to the real line, and generalizations of Wild sums representations for multi-linear interactions.
In probability space, with the weak convergence of probability measures:

**Theorem** The following statements hold:

[i]: There exists a unique (in the class of probability measures) solution \( f(|v|, t) \) with initial state

\[
 f(|v|, 0) = f_0(|v|) \geq 0, \int_{\mathbb{R}^3} f_0(|v|) dv = 1 \text{ such that, with } x = \frac{|k|^2}{2}
\]

\[ u_0 = F[f_0(|v|)] = 1 + O(x^p), x \to 0, 0 < p \leq 1. \]

[ii]: The solution \( f(|v|, t) \) has self-similar asymptotics in the following sense:

Take \( p = 1 \), \( \mu(1)xu_0' = \Gamma(u_0) + O(x^{1+\varepsilon}) \) with both, \( \mu(1) \) (energy dissipation rate) and \( \mu'(1) < 0 \).

Then there exists a non-negative self-similar solution:

\[
 f(|v|e^{\frac{1}{2}\mu(1)t}, t) = e^{-\frac{d}{2}\mu(1)t} F_1(|v|e^{-\frac{1}{2}\mu(1)t}),\]

\[
 f(|v|e^{\frac{1}{2}\mu(1)t}, t) \to e^{-\frac{d}{2}\mu(1)t} F_1(|v|) \text{ with }
\]

\[
 || \frac{\hat{f}(|x|e^{\frac{1}{2}\mu(1)t}, t) - e^{-\frac{d}{2}\mu(1)t} \hat{F}_1(|v|)}{|x|^{1+\varepsilon}} ||_\infty \leq C || \frac{\hat{f}_0 - \hat{F}_1(|x|)}{|x|^{1+\varepsilon}} ||_\infty e^{-t(1+\varepsilon)(\mu(1) - \mu(1+\varepsilon)).}
\]

1. For the statement with initial \( \frac{p}{2} \) moments (i.e. in Fourier with order \( O(x^{p+\varepsilon}) \); \( p > 1 \)) replace \( \mu(1), \mu(1+\varepsilon) \) by \( \mu(p), \mu(p+\varepsilon) \), respectively.

2. This decay rate was computed first in Bobylev, Cercignani and Toscani, JSP’03, for the elastic collisions models converging to homogeneous cooling states example.

[iii]: However,

\[
 f(|v|e^{\frac{1}{2}\eta t}, t) \to_{t \to \infty} e^{-\frac{d}{2}\eta t} \delta_0(|v|); \quad \eta > \mu(1) \quad \text{ and }
\]

\[
 f(|v|e^{\frac{1}{2}\eta t}, t) \to_{t \to \infty} 0; \quad \mu(p_{\min}) < \mu(1+\delta) < \eta < \mu(1)
\]

estimates hold for any \( p \leq 1 \): initial states with infinite energy as well.
Evolution of the Particle Density $(N = 32, T = 0.25e^{-2t/3})$

Density Evolution of a convex combination of Gaussians with $T = 0.25e^{-2t/3}$

Testing - Mixture Problem

Computed Vs. Analytical Distribution:

$(N = 24, T = 1)$

$(N = 24, T = 0.25)$

$(N = 24, T = 0.125)$

Setting $\hat{T} = e^{-\frac{2}{3}t} \left(\frac{1}{4} + s^2\right)$

Evolution of Moment $(m_q(t))$ of $\mathcal{F}(v(t))$ for $N = 26$
Space inhomogeneous simulations
(I.M.G. with H. Tharskabhushanam, JCM’10)

$\frac{\partial f}{\partial t} + v_1 \frac{\partial f}{\partial x} = Q(f, f)$

- Mean free time := the average time between collisions
- Mean free path := average speed × mft (average distance traveled between collisions)

Set the scaled equation for $1 = Kn := \text{mfp/geometry of length scale}$

Spectral-Lagrangian methods in 3D-velocity space and 1D physical space discretization in the simplest setting:

Finite difference scheme with splitting into a convective and a collision step:

Define $CFL := \Delta t \frac{v_j}{\Delta x}$ and $t^n = n \Delta t$ ⇒ set $f(x^k, v^j, t^n) = f^n_{k,j}$

- Convective Step Space discretization of $O(\Delta x)$: $\frac{\partial f}{\partial t} + v_1 \frac{\partial f}{\partial x} = 0, \quad f(x, v, 0) = f^n_{k,j}$

$$\bar{f}^{j,k} = \begin{cases} (1 - CFL) \cdot f^n_k + CFL \cdot f^n_{k-1,j} & \text{if } v_1 > 0 \\ (1 + CFL) \cdot f^n_k - CFL \cdot f^n_{k+1,j} & \text{if } v_1 < 0 \end{cases}$$

- Collision Step Time discretization of $O(\Delta t)$ - first forward Euler (or second order Runge Kutta)

on the "conserve" algorithm for: $\frac{\partial f}{\partial t} = Q(f, f), \quad f(x, v, 0) = \bar{f}^{j,k}$, uniformly in $x$

\[\bar{Q}_n = \text{Conserve}(Q(f_n, f_n)), \quad \Rightarrow \quad f_{n+1/2}(x, v) = f_n(x, v) + \frac{dt}{2} \bar{Q}_n,\]

\[Q_n = \text{Conserve}(Q(f_{n+1/2}, f_{n+1/2})), \quad \Rightarrow \quad f_{n+1}(x, v) = f_n(x, v) + dtQ_n.\]

Spatial mesh size $\Delta x = 0.01$ mfp Time step $\Delta t = r$ mft, mft = reference time

N= Number of Fourier modes in each j-direction in 3D
Resolution of discontinuity 'near the wall' for diffusive boundary conditions:

**Sudden heating:** Constant moments initial state with a discontinuous pdf at the boundary wall, with wall kinetic temperature increased by twice its magnitude:

Initial state \( f_0(x, v) = \frac{1}{(\pi T(x))^{3/2}} e^{-\frac{|v|^2}{2T(x)}} \) with \( T(0) = T_0 \) and \( T(x) = 2T_0 \) for \( x > 0 \)

Boundary Conditions for sudden heating:

\[
f(0, v, t) = \frac{\sigma_w}{(\pi T_w)^{3/2}} e^{-\frac{|v|^2}{2T_w}} \quad \text{with} \quad \sigma_w = \left(\frac{8\pi}{T_w}\right)^{3/2} \int_{v_1 > 0} v_1 f(0, v, t) dv
\]

with \( T_w(0, 0) = T_0 \) and \( T_w(0, t) = 2T_0 \) for \( t > 0 \)

Calculations in the next two pages:
Mean free path \( l_0 = 1 \).
Number of Fourier modes \( N = 24^3 \).
Spatial mesh size \( \Delta x = 0.01 \ l_0 \).
Time step \( \Delta t = r \ mft \)
Comparisons with K. Aoki, Y. Sone, K. Nijino, H. Sugimoto '91 (finite differences on BGK)

Marginal Distribution at $t = 0.15t_r$ for $N = 16$.

Sudden heating problem

Fig. 5. The reduced velocity distribution function $g$ for $T_1/T_0 = 2$. 

Jump in pdf
Spectral-Lagrangian Boltzmann solver
(with J. Haack)

Formation of a shock wave by an initial sudden change of wall temperature from $T_0$ to $2T_0$.

Sudden heating problem (BGK eq. with finite difference Boltzmann solvers): bulk velocity $v_1$

Heat transfer problem:

Initial state \( f_0(x, v) = \frac{1}{(\pi T(z))^{3/2}} e^{-\frac{|v|^2}{4T(z)}} \) with \( T(0) = T_0 \) and \( T(x) = 2T_0 \) for \( x > 0 \)

Diffusive boundary conditions

\[
\begin{align*}
 f(0, v, t) &= \frac{\sigma_w}{(\pi T_w)^{3/2}} e^{-\frac{|v|^2}{T_w}} \quad \text{with} \quad \sigma_w = \left(\frac{8\pi}{T_w}\right)^{3/2} \int_{v_1 > 0} v_1 f(0, v, t) dv 
\end{align*}
\]

Temperature: \( T_0 \) given at \( x_0 = 0 \) and \( T_1 = 2T_0 \) at \( x_1 = 1 \).

Knudsen \( Kn = 0.1, 0.5, 1, 2, 4 \)

Benchmarked with Aristov, 01

Stationary Temperature Profile for increasing Knudsen number values.

Stationary Density Profile for increasing Knudsen number values.
Thank you very much for your attention!

References: at www.ma.utexas.edu/users/gamba/research and references therein