EXPONENTIALLY-TAILED REGULARITY AND TIME ASYMPTOTIC FOR THE HOMOGENEOUS BOLTZMANN EQUATION

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Abstract. We present in this document the Lebesgue and Sobolev propagation of exponential tails for solutions of the homogeneous Boltzmann equation for hard and Maxwell interactions. In addition, we show the L^p-integrability creation of such tails in the case of hard interactions. The document also presents a result on exponentially-fast convergence to thermodynamical equilibrium and propagation of singularities and regularization of such solutions. All these results are valid under the mere Grad's cut-off condition for the angular scattering kernel. Highlights of this contribution include: (1) full range of L^p-norms with $p \in [1, \infty]$, (2) analysis for the critical case of Maxwell interactions, (3) propagation of fractional Sobolev exponential tails using pointwise conmutators, and (4) time asymptotic and propagation of regularity and singularities under general physical data. In many ways, this work is an improvement and an extension of several classical works in the area [4, 9, 13, 20, 30, 34]; we use known techniques and introduce new and flexible ideas that achieve the proofs in an elementary manner.

Keywords: Boltzmann equation, Lebesgue integrability, fractional regularity, entropic methods, exponential convergence, decomposition theorem.

MSC: 82B40, 45Gxx.

1. Introduction

We study qualitative aspects of the L^p_{exp} (exponentially weighted Lebesgue) integrability propagation and creation, in the sense of tails, and the H^k_{exp} (exponentially weighted Sobolev) propagation for the homogeneous Boltzmann equation with hard potentials and Maxwell molecules. In addition, we study the exponentially fast asymptotic convergence of solutions towards thermodynamical equilibrium and propagation of regularity and singularities under general physical initial data. The proof is valid for any dimension $d \geq 2$, integrability $p \in [1, \infty]$, regularity $k \geq 0$, and Grad's cut-off assumption. More precisely, the scattering kernel is assumed to have the form

$$B(x, y) = x^{\gamma}\, b(y)\,, \quad \text{with} \;\; x \geq 0,\; y \in [0, 1],\; \gamma \in [0, 2]\,,$$

where we can set, without loss of generality, $\|b\|_{L^1(\mathbb{S}^{d-1})} = 1$. When the dependence of the angular scattering kernel b is important, say in the constants involved in the estimates, it will be explicitly stated. We note that the support of b is assumed in $[0, 1]$ thanks to a symmetrization argument (thus, b is assumed symmetrized [20, pg. 3]). Throughout the paper, we use the notation $Q_{\gamma,b}(f, f)$ to refer to the collision operator corresponding to such scattering kernel unless it is clear from the context. In the latter we simply write $Q(f, f)$. We recall the definition of such operator

$$\begin{aligned} &Q_{\gamma,b}(f, g)(v) = Q^{+}_{\gamma,b}(f, g) - Q^{-}_{\gamma,b}(f, g) \\ &:= \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} f(v')g(v'_*)B(|u|, \hat{u} \cdot w)\mathrm{d}w\mathrm{d}v_* - f(v) \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} g(v_*)B(|u|, \hat{u} \cdot w)\mathrm{d}w\mathrm{d}v_*\,, \end{aligned}$$

where the collisional variables are defined as

$$v' := v - u^{-}, \qquad v'_* := v_* + u^{-}, \qquad u := v - v_*, \qquad u^{\pm} := \frac{u \pm |u|w}{2}\,.$$

The scattering angle θ is simply defined as $\cos(\theta) := \hat{u} \cdot w$. Unitary vectors will always be denoted as $\hat{u} := u/|u|$. In this way, the homogeneous Boltzmann equation simply writes

$$\partial_t f(v) = Q_{\gamma,b}(f, f)(v)\,, \quad (t, v) \in \mathbb{R}^+ \times \mathbb{R}^d\,. \tag{1}$$

The initial data considered in this document is *nonnegative* and has finite mass and energy,

$$\int_{\mathbb{R}^d} f_0 \langle v \rangle^2 \mathrm{d}v < \infty, \qquad \text{where} \;\; \langle v \rangle := \sqrt{1 + |v|^2}, \tag{2}$$

moreover, without loss of generality we can set the initial mass equal to unity. In this case, equation (1) has a unique solution $f(t, v) \geq 0$ that conserves initial mass, momentum and energy, see [27].

We will commonly use, for technical reasons, the decomposition of the angular scattering kernel to estimate distinct parts of the collision operator

$$\begin{gathered} b\big(\cos(\theta)\big) = b\big(\cos(\theta)\big)\big(1_{|\sin(\theta)| \geq \varepsilon} + 1_{|\sin(\theta)| < \varepsilon}\big) =: b^{\varepsilon}_1\big(\cos(\theta)\big) + b^{\varepsilon}_2\big(\cos(\theta)\big)\,, \\ \text{or, in terms of } y \\ b(y) = b(y)\big(1_{y \leq \sqrt{1-\varepsilon^2}} + 1_{y > \sqrt{1-\varepsilon^2}}\big) =: b^{\varepsilon}_1(y) + b^{\varepsilon}_2(y)\,. \end{gathered} \tag{3}$$

Before starting with the technical details, let us mention that propagation of L^p-integrability for the Boltzmann equation, with different degrees of cut-off and weights, has been studied for quite some time now. Classical papers for the hard potential case are [17, 9] for $p = \infty$, [25, 30] for $p \in (1, \infty)$, and [35, Prop. 1.4] for $p \in [1, \infty]$ with the Maxwell molecules model. The propagation of exponentially weighted norms is more rare in the literature, however, reference [20] is a beautiful example of it, where pointwise gaussian estimates are shown to be propagated in the case of hard potentials using a comparison principle. These techniques have been applied later for the Maxwell case in [13] after nontrivial modifications. See also [30, 4] for similar results for propagation of Sobolev norms in the hard potential case.

One of the main contributions of the present document is to unify all previous works with a relatively simple line of reasoning that includes all ranges of integrability $p \in [1,\infty]$ and weights, polynomial and exponential, including gaussian. It also includes both, Maxwell and hard interactions. The program to prove Lebesgue exponential tail propagation or creation consists in 3 main steps, (1) prove propagation or creation, respectively, of exponential moments [11, 12, 14, 2, 31], (2) prove a so-called "gain of integrability" inequality for the gain collision operator in the spirit of [30, 5] and, (3) use Young's inequality [25, 3] for the gain collision operator to deal with the case $p = \infty$. In addition to these main steps, we will need an explicit lower bound for the negative part of the collision operator which seems to be classical in the literature, at least when finite initial entropy is assumed, see Lemma 2.1 below. Contrary to the hard potential case, the critical case of Maxwell interactions will need propagation of entropy. That is, the additional assumption

$$\int_{\mathbb{R}^d} f_0 \ln(f_0)\mathrm{d}v < \infty$$

will be required on the initial condition. This is harmless in our context since we will impose more restrictive conditions on f_0, namely, $f_0 \in \big(L^1_2 \cap L^p\big)(\mathbb{R}^d)$, for some $p > 1$.

In the Section 4, we prove exponentially-tailed Sobolev regularity propagation for solutions of the homogeneous Boltzmann equation. Propagation of regularity has been discussed before in [30, 4]. One of the central ingredients for the proof of propagation of Sobolev norms is the estimate [15, Theorem 2.1] which is valid under the assumption $b \in L^2(\mathbb{S}^{d-1})$. This result was used in [30] to prove propagation of regularity in the case of hard potentials with polynomial weights and later, indirectly, in [4] for exponential weights. Three extensions are given in this section with respect to [30, 4], (1) we relax the assumption on b to mere Grad's cut-off, (2) we are able to prove propagation of fractional regularity, and (3) Maxwell molecules are considered, all in the context of exponential tails. Although our main goal is to prove all previous extensions in the context of exponential weights in the spirit of [4], similar results follow with the same line of reasoning for polynomial weights of any order.

Finally, in Section 5 we show the exponentially fast convergence towards thermodynamical equilibrium and prove a decomposition theorem for propagation of smoothness and singularities in the case of hard potentials. The effort is concentrated in two fronts: (1) provide a proof in the context of mere Grad's cut-off hypothesis assuming (2) general physical data, that is, initial data having only finite mass, energy, and entropy. These results generalize classical references in the topic such as [10, 28, 30, 34] at the level of the model and the initial data. An entropic method [33] together with the technique of spectral space enlargement [28, 24] will lead to the desired results. The commutators developed in Section 4 will also play a role in the proof of the decomposition theorem. In contrast to the usual argument made in the proof of exponential convergence that uses a decomposition theorem first and then an entropy method, our argument eliminates the need of the decomposition theorem. This considerably reduces the technicalities here and in other contexts where entropic methods are used in Boltzmann-like equations.

1.1. Notation.

We work with classical Lebesgue spaces $L^p(\mathbb{R}^d)$ for $p \in [1,p]$. The addition of polynomial or exponential weights are central throughout the manuscript. No particular notation will be used, however, in some places we adopt the following standard notation for convenience

$$L^p_\mu(\mathbb{R}^d) : = \Big\{ f \text{ measurable} \,\big|\, \|f\|_{L^p_\mu(\mathbb{R}^d)} := \|f \langle \cdot \rangle^\mu\|_{L^p(\mathbb{R}^d)} < +\infty \Big\}, \text{ and}$$
$$L^p_{exp}(\mathbb{R}^d) : = \Big\{ f \text{ measurable} \,\big|\, \|f\|_{L^p_{exp}(\mathbb{R}^d)} := \|f\, e^{r\langle \cdot \rangle^\alpha}\|_{L^p(\mathbb{R}^d)} < +\infty \Big\},$$

for some $\mu \geq 0$, $r > 0$, and $\alpha > 0$. We restrict ourself to the Sobolev spaces $H^k(\mathbb{R}^d)$, with $k \geq 0$, defined as

$$H^k(\mathbb{R}^d) := \Big\{ f \in L^2(\mathbb{R}^d) \,\big|\, \big\| \big(1 + (-\Delta)\big)^{\frac{k}{2}} f \big\|_{L^2(\mathbb{R}^d)} < +\infty \Big\}.$$

Here, the operator $\big(1 + (-\Delta)\big)^{\frac{k}{2}}$ is defined using the Fourier transform $\mathcal{F}$,

$$\mathcal{F}\Big\{ \big(1 + (-\Delta)\big)^{\frac{k}{2}} f \Big\}(\xi) = \langle \xi \rangle^k \mathcal{F}\{f\}(\xi)\,,$$

where, we recall, the brackets stand for $\langle \xi \rangle := \sqrt{1 + |\xi|^2}$. Polynomial and exponential weights will also be used in these spaces,

$$H^k_\mu(\mathbb{R}^d) := \Big\{ f \in H^k(\mathbb{R}^d) \,\big|\, \big\| \langle \cdot \rangle^\mu \big(1 + (-\Delta)\big)^{\frac{k}{2}} f \big\|_{L^2(\mathbb{R}^d)} < +\infty \Big\},\ \mu \geq 0,$$

and,

$$H^k_{exp}(\mathbb{R}^d) := \Big\{ f \in H^k(\mathbb{R}^d) \,\big|\, \big\| e^{r\langle \cdot \rangle^\alpha} \big(1 + (-\Delta)\big)^{\frac{k}{2}} f \big\|_{L^2(\mathbb{R}^d)} < +\infty \Big\},$$

for some nonnegative r and α. These are the spaces of functions enjoying Sobolev regularity with polynomial and exponential tails respectively. Observe that weights were chosen to be outside the differentiation operator. Special care has to be made with this choice when fractional differentiation is performed in the particular case of exponential weights. Finally, we will commonly use the norm

$$\|\cdot\|_{(L^p \cap L^q)(\mathbb{R}^d)} := \max\big\{ \|\cdot\|_{L^p(\mathbb{R}^d)}, \|\cdot\|_{L^q(\mathbb{R}^d)} \big\}$$

for the intersection space $L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$, with p, $q \in [1, \infty]$.

2. Important lemmas and application to L^∞-propagation

We begin this section with a classical inequality used to obtain a lower estimate for the loss operator Q^-. This estimate is well know in the community when entropy is assumed to be finite, see for example [9], and it has been used in a central way in the analysis of moments [37] and the propagation of L^p-norms [30]. The case where no finite entropy is assumed is presented in [20], however, the proof is strongly based in the fact that $f(t,v)$ solves the Boltzmann equation. Here, we present an elementary proof based only in functional arguments.

Lemma 2.1 (Lower bound)**.** *Fix $\gamma\in[0,2]$, and assume $0\leq\{f(t)\}_{t\geq0}\subset L^1_2(\mathbb{R}^d)$ satisfies*

$$C\geq\int_{\mathbb{R}^d}f(t,v)dv\geq c\,,\quad C\geq\int_{\mathbb{R}^d}f(t,v)|v|^2dv\geq c\,,\quad \int_{\mathbb{R}^d}f(t,v)\,v\,dv=0\,,$$

for some positive constants C and c. Assume also the boundedness of some 2^+ moment

$$\int_{\mathbb{R}^d}f(t,v)|v|^{2^+}dv\leq B.$$

Then, there exists $c_o:=c_o(B,C,c)>0$ such that

$$\big(f(t,\cdot)*|\cdot|^{\gamma}\big)(v)\geq c_o\langle v\rangle^{\gamma}\,.$$

Remark 2.1. *The symbol $a^{\pm}$, with $a>0$, denotes a fixed real number bigger (+) or smaller (-) than a.*

Proof. The case $\gamma=0$ is trivial, thus, assume $\gamma\in(0,2]$. Take $v\in B(0,r)\subset\mathbb{R}^d$, the open ball centered at the origin and radius $r>0$, and note that for any $R>0$,

$$\begin{aligned}\int_{\{|v-w|\leq R\}}f(t,w)|v-w|^2\mathrm{d}w&=\int_{\mathbb{R}^d}f(t,w)|v-w|^2\mathrm{d}w-\int_{\{|v-w|\geq R\}}f(t,w)|v-w|^2\mathrm{d}w\\&\geq c\,\langle v\rangle^2-\int_{\{|v-w|\geq R\}}f(t,w)|v-w|^2\mathrm{d}w\\&\geq c\,\langle v\rangle^2-\frac{1}{R^{2^+-2}}\int_{\{|v-w|\geq R\}}f(t,w)|v-w|^{2^+}\mathrm{d}w\,.\end{aligned}\tag{4}$$

Since,

$$\int_{\{|v-w|\geq R\}}f(t,w)|v-w|^{2^+}\mathrm{d}w\leq 2^{2^+-1}\max\{C,B\}\langle v\rangle^{2^+}\leq 2^{2^+-1}\max\{C,B\}\langle r\rangle^{2^+},$$

we conclude from (4)

$$\int_{\{|v-w|\leq R\}}f(t,w)|v-w|^2\mathrm{d}w\geq c\,\langle v\rangle^2-\frac{2^{2^+-1}}{R^{2^+-2}}\max\{C,B\}\langle r\rangle^{2^+}\geq\frac{c}{2}\,,\quad\forall\,v\in B(0,r)\,,\tag{5}$$

provided $R:=R(r)$ is sufficiently large. More precisely, for any $R>0$ such that

$$\frac{2^{2^+-1}}{R^{2^+-2}}\max\{C,B\}\langle r\rangle^{2^+}\leq\frac{c}{2}\,.$$

Thus, one infers from (5) that for any $\gamma\in(0,2]$

$$\begin{aligned}\int_{\mathbb{R}^d}f(t,w)|v-w|^{\gamma}\mathrm{d}w&\geq\int_{\{|v-w|\leq R\}}f(t,w)|v-w|^{\gamma}\mathrm{d}w\\&\geq\frac{1}{R^{2-\gamma}}\int_{\{|v-w|\leq R\}}f(t,w)|v-w|^2\mathrm{d}w\geq\frac{c}{2R^{2-\gamma}}\,,\quad\forall\,v\in B(0,r)\,,\end{aligned}\tag{6}$$

is valid for the aforementioned choice of $R(r)$. Additionally,

$$\begin{aligned}\int_{\mathbb{R}^d} f(t,w)|v-w|^\gamma \mathrm{d}w &\geq c_\gamma \int_{\mathbb{R}^d} f(t,w)\big(|v|^\gamma - |w|^\gamma\big)\mathrm{d}w \\ &\geq c_\gamma\big(c\,|v|^\gamma - C\big), \qquad \forall\, v\in\mathbb{R}^d,\ \gamma\in(0,2]\,. \end{aligned} \tag{7}$$

As a consequence of (6) and (7),

$$\int_{\mathbb{R}^d} f(t,w)|v-w|^\gamma \mathrm{d}w \geq \Big(\frac{c}{2R(r)^{2-\gamma}}\,\mathbf{1}_{B(0,r)}(v) + c_\gamma\big(c\,|v|^\gamma - C\big)\,\mathbf{1}_{B(0,r)^c}(v)\Big).$$

Choosing $r := r_* = \big(2C/c\big)^{\frac{1}{\gamma}}$ one ensures that $c\,|v|^\gamma - C \geq \frac{c}{2}|v|^\gamma$ for any $|v|\geq r$. Then, there exists an explicit $c_o := c_o(B,C,c)$ such that

$$\int_{\mathbb{R}^d} f(t,w)|v-w|^\gamma \mathrm{d}w \geq \Big(\frac{c}{2R(r_*)^{2-\gamma}}\,\mathbf{1}_{B(0,r_*)}(v) + \frac{c_\gamma\, c}{2}\,|v|^\gamma\,\mathbf{1}_{B(0,r_*)^c}(v)\Big) \geq c_o\langle v\rangle^\gamma\,.$$

□

Lemma 2.2 (Upper bound)**.** *Write $b = b_1^\varepsilon + b_2^\varepsilon$ as in (3). Then, for every $\gamma\geq 0$,*

$$\begin{aligned} &\|Q^+_{o,b_1^\varepsilon}\big(f, f\langle\cdot\rangle^\gamma\big)\|_{L^\infty(\mathbb{R}^d)} \leq \varepsilon^{-\frac{d}{2}}\,C(b)\|f\|_{L^2(\mathbb{R}^d)}\|f\langle\cdot\rangle^\gamma\|_{L^2(\mathbb{R}^d)}\,, \\ &\|Q^+_{o,b_2^\varepsilon}\big(f, f\langle\cdot\rangle^\gamma\big)\|_{L^\infty(\mathbb{R}^d)} \leq \mathfrak{m}(b_2^\epsilon)\|f\|_{L^\infty(\mathbb{R}^d)}\|f\langle\cdot\rangle^\gamma\|_{L^1(\mathbb{R}^d)}\,. \end{aligned} \tag{8}$$

The constants are such that $C(b)\sim\|b\|_{L^1(\mathbb{S}^{d-1})}$ and $\mathfrak{m}(b_2^\epsilon)\sim\|b_2^\varepsilon\|_{L^1(\mathbb{S}^{d-1})}$. In particular, $\lim_{\varepsilon\to 0}\mathfrak{m}(b_2^\epsilon)=0$.

Proof. Both estimates are a direct consequence of Young's inequality for the gain part of the collision operator [3, Theorem 1] (see Theorem 6.1 in the appendix for a clear statement of such theorem). Indeed, recalling that b has support in $[0,1]$, we use for the first estimate in (8) the case $(p,q,r)=(2,2,\infty)$ with constant (94)

$$\begin{aligned} C &= K\left(\int_0^1\Big(\frac{1-s}{2}\Big)^{-\frac{d}{2}}\big(1-s^2\big)^{\frac{d-3}{2}}b_1^\varepsilon(s)\mathrm{d}s\right)^{\frac{1}{2}}\left(\int_0^1\Big(\frac{1+s}{2}\Big)^{-\frac{d}{2}}\big(1-s^2\big)^{\frac{d-3}{2}}b_1^\varepsilon(s)\mathrm{d}s\right)^{\frac{1}{2}} \\ &\leq \varepsilon^{-\frac{d}{2}}2^{\frac{d}{2}}K\int_0^1\big(1-s^2\big)^{\frac{d-3}{2}}b(s)\mathrm{d}s = \varepsilon^{-\frac{d}{2}}2^{\frac{d}{2}}K\|b\|_{L^1(\mathbb{S}^{d-1})} =: \varepsilon^{-\frac{d}{2}}C(b)\,. \end{aligned}$$

For the inequality we used the fact that $(1-s)^{-1}\leq 2\varepsilon^{-2}$ in the support of $b_1^\varepsilon\leq b$. For the second estimate we use the case the case $(p,q,r)=(\infty,1,\infty)$ with constant (95)

$$C = K\int_0^1\Big(\frac{1+s}{2}\Big)^{-\frac{d}{2}}\big(1-s^2\big)^{\frac{d-3}{2}}b_2^\varepsilon(s)\mathrm{d}s \leq 2^{\frac{d}{2}}K\int_0^1\big(1-s^2\big)^{\frac{d-3}{2}}b_2^\varepsilon(s)\mathrm{d}s =: \mathfrak{m}(b_2^\epsilon)\,.$$

The fact that $\lim_{\varepsilon\to 0}\mathfrak{m}(b_2^\epsilon)=0$ is a direct consequence of the monotone convergence theorem. □

Remark 2.2. *In the sequel the symbol $\mathfrak{m}(b)$ will be interpreted, more generally, as a quantity proportional to $\|b\|_{L^1(\mathbb{S}^{d-1})}$ having a constant of proportionality that depends only on the dimension $d\geq 2$ and $\gamma\geq 0$. Such symbol will be reserved to a quantity that will be taken sufficiently small at some point in the argument in question.*

A direct application of these two lemmas gives an elementary proof of the L^∞-norm propagation for the homogeneous Boltzmann equation.

Theorem 2.1 (Propagation of L^∞)**.** *Take $\gamma \in (0,1]$, $b \in L^1(\mathbb{S}^{d-1})$ be the angular kernel (with mass normalized to unity) and*

$$\|f_0\|_{(L^1_{2+}\cap L^\infty)(\mathbb{R}^d)} = C_o\,,$$

for some positive constant C_o. Then, there exist constant $C(f_0) > 0$ depending on C_o, γ and b such that

$$\|f(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \le C(f_0)\,, \quad t \ge 0\,,$$

for the solution $f(t,v)$ of the Boltzmann equation.

Proof. Due to the propagation of moments for the homogeneous Boltzmann equation, see [2, Theorem 2]

$$\sup_{t\ge 0} \|f(t,\cdot)\|_{L^1_{2+}(\mathbb{R}^d)} < \infty\,. \tag{9}$$

Since $\|f_0\|_{L^2_1(\mathbb{R}^d)} \le \|f_0\|^{\frac12}_{L^1_2(\mathbb{R}^d)} \|f_0\|^{\frac12}_{L^\infty(\mathbb{R}^d)} < \infty$, classical theory of L^p-propagation implies that, see [30, Theorem 4.1] or [5, Corollary 1.1],

$$\sup_{t\ge 0} \|f(t,\cdot)\|_{L^2_1(\mathbb{R}^d)} < \infty\,. \tag{10}$$

Recall that $b(\hat u\cdot w)$ is supported in $\hat u \cdot w \in [0,1]$. Thus, using the definition of the collisional law $v - v'_* = \frac{|u|}{2}\big(\hat u + w\big)$, it follows that

$$2\sqrt{2}\langle v\rangle\langle v'_*\rangle \ge 2|v - v'_*| \ge \big(v - v'_*\big)\cdot(\hat u + w) = |u|\big(1 + \hat u\cdot w\big) \ge |u|\,.$$

Then,

$$Q^+(f,f)(v) \le 2^{\frac{3\gamma}{2}} Q^+_{o,b}\big(f, f\langle\cdot\rangle^\gamma\big)(v)\,\langle v\rangle^\gamma\,.$$

Using Lemma 2.1 and previous observation in the Boltzmann equation (1), we conclude that

$$\partial_t f(v) = Q^+(f,f)(v) - f(v)\big(f * |\cdot|^\gamma\big)(v) \le \Big(2^{\frac{3\gamma}{2}} Q^+_{o,b}\big(f, f\langle\cdot\rangle^\gamma\big)(v) - c_o\, f(v)\Big)\langle v\rangle^\gamma\,. \tag{11}$$

Applying the splitting (3) as $Q^+_{o,b} = Q^+_{o,b^\varepsilon_1} + Q^+_{o,b^\varepsilon_2}$, invoking Lemma 2.2, and keeping in mind estimates (9) and (10), we conclude from estimate (11) that for some constants $C_i(f_0)$, $i = 1,2$, depending on f_0 through the norm $\|f_0\|_{L^1_{2+}\cap L^\infty(\mathbb{R}^d)}$ it holds

$$\begin{aligned} \partial_t f(t,v) &\le \Big(\varepsilon^{-\frac{d}{2}} C_1(f_0) + \mathfrak{m}(b^\epsilon_2)\, C_2(f_0)\|f(t)\|_{L^\infty(\mathbb{R}^d)}\Big)\langle v\rangle^\gamma - c_o\, f(t,v)\langle v\rangle^\gamma \\ &= \Big(\tilde C_1(f_0) + \frac{c_o}{4}\|f(t)\|_{L^\infty(\mathbb{R}^d)}\Big)\langle v\rangle^\gamma - c_o\, f(t,v)\langle v\rangle^\gamma\,, \end{aligned} \tag{12}$$

where, for the latter, we simply took $\varepsilon > 0$ sufficiently small such that $\mathfrak{m}(b_2^\epsilon)C_2(f_0) \le \frac{c_o}{4}$. Now, we can integrate estimate (12) in $[0,t]$ to obtain

$$\begin{aligned} f(t,v) &\le f_0(v)e^{-c_o\langle v\rangle^\gamma t} + \int_0^t e^{-c_o\langle v\rangle^\gamma (t-s)}\Big(\tilde{C}_1(f_0) + \frac{c_o}{4}\|f(s)\|_{L^\infty(\mathbb{R}^d)}\Big)\langle v\rangle^\gamma \mathrm{d}s \\ &\le \|f_0\|_{L^\infty(\mathbb{R}^d)} + \Big(\tilde{C}_1(f_0) + \frac{c_o}{4}\sup_{0\le s\le t}\|f(s)\|_{L^\infty(\mathbb{R}^d)}\Big)\langle v\rangle^\gamma \int_0^t e^{-c_o\langle v\rangle^\gamma (t-s)}\mathrm{d}s \\ &\le \frac{3}{4}C(f_0) + \frac{1}{4}\sup_{0\le s\le t}\|f(s)\|_{L^\infty(\mathbb{R}^d)}\,, \qquad \text{a.e. in } v\in\mathbb{R}^d\,. \end{aligned}$$

where $C(f_0) := \frac{4}{3}\|f_0\|_{L^\infty(\mathbb{R}^d)} + \frac{4}{3}\tilde{C}_1(f_0)$. As a consequence, for any $0 \le t \le T$ it holds

$$f(t,v) \le \frac{3}{4}C(f_0) + \frac{1}{4}\sup_{0\le s\le T}\|f(s)\|_{L^\infty(\mathbb{R}^d)}\,, \qquad \text{a.e. in } v\in\mathbb{R}^d\,. \tag{13}$$

The proof is complete after computing the essential supremum in $v \in \mathbb{R}^d$ and, then, the supremum in $t \in [0,T]$ for $f(t,v)$ in estimate (13). □

3. Propagation and creation of L^p-exponential tail integrability for cut-off Boltzmann

In this section we study the propagation and creation of L^p-exponential tails. The argument for the case $p \in [1,\infty)$ follows closely the standard theory used in the literature for polynomial weights. The reasoning for the case $p = \infty$ is novel and follows the one given in previous section. We divide the proof in the hard potentials $\gamma \in (0,2]$ and Maxwell molecules $\gamma = 0$ cases as they are slightly different. For instance, as opposed to the hard potentials model, the Maxwell molecules model does not create tail only propagates it.

3.1. Hard potential case.

Theorem 3.1 (Propagation of L^p-exponential tails)**.** *Let $\gamma \in (0,2]$ be the potential exponent, $b \in L^1(\mathbb{S}^{d-1})$ be the angular kernel (with mass normalized to unity), and*

$$\|f_0(\cdot)\, e^{a_o\langle\cdot\rangle^\alpha}\|_{(L^1\cap L^p)(\mathbb{R}^d)} = C_o < \infty\,,$$

for some $\alpha \in (0,2]$, $p \in [1,\infty]$ and positive constants a_o and C_o. Then, there exist positive constants a and C depending on the initial mass, energy, a_o, C_o, γ and b such that

$$\|f(t,\cdot)\, e^{a\langle\cdot\rangle^\alpha}\|_{L^p(\mathbb{R}^d)} \le C\,, \quad t \ge 0\,,$$

for the solution $f(t,v)$ of the Boltzmann equation.

Proof. One first notices that thanks to the propagation of moments of [2, Theorem 2]

$$\|f(t,\cdot)\, e^{a\langle\cdot\rangle^\alpha}\|_{L^1(\mathbb{R}^d)} \le C\,, \quad t \ge 0\,, \tag{14}$$

for some positive a and C with dependence as stated. Next, note that for any $\alpha \in [0,2]$

$$\begin{aligned} \langle v\rangle^\alpha = \big(1+|v|^2\big)^{\frac{\alpha}{2}} \le \big(1+|v|^2+|v_*|^2\big)^{\frac{\alpha}{2}} = \big(1+|v'|^2+|v'_*|^2\big)^{\frac{\alpha}{2}} \\ \le \big(1+|v'|^2\big)^{\frac{\alpha}{2}} + |v'_*|^\alpha \le \langle v'\rangle^\alpha + \langle v'_*\rangle^\alpha\,. \end{aligned}$$

and therefore, $e^{r\langle v\rangle^{\alpha}} \leq e^{r\langle v'\rangle^{\alpha}} e^{r\langle v'_*\rangle^{\alpha}}$ for any $r>0$. As a consequence

$$\begin{aligned} Q(f,f)(v)\, e^{r\langle v\rangle^{\alpha}} &= Q^{+}(f,f)(v)\, e^{r\langle v\rangle^{\alpha}} - Q^{-}(f,f)(v)\, e^{r\langle v\rangle^{\alpha}} \\ &\leq Q^{+}(f e^{r\langle \cdot\rangle^{\alpha}}, f e^{r\langle \cdot\rangle^{\alpha}})(v) - f(v) e^{r\langle v\rangle^{\alpha}} \big(f * |\cdot|^{\gamma}\big)(v)\,. \end{aligned}$$

Thus, defining $g := f\, e^{r\langle \cdot\rangle^{\alpha}}$, the Boltzmann equation implies that

$$\partial_t g(v) \leq Q^{+}(g,g)(v) - g(v)\big(f * |\cdot|^{\gamma}\big)(v)\,. \tag{15}$$

As a consequence of Lemma 2.1, estimate (15) implies that

$$\partial_t g(v) \leq Q^{+}(g,g)(v) - c_o\, g(v)\langle v\rangle^{\gamma}\,. \tag{16}$$

Estimate (16) suffices to conclude using the techniques of [30, Theorem 4.1] or [5, Corollary 1.1] that

$$\sup_{t\geq 0} \|g(t,\cdot)\|_{L^p(\mathbb{R}^d)} \leq C_p(g_0)\,, \qquad \text{with } \ p \in (1,\infty)\,, \tag{17}$$

where $C_p(g_0)$ depends on an upper bound of $\|g_0\|_{L^p} + \sup_{t\geq 0}\|g(t,\cdot)\|_{L^1_2}$. Of course, such upper bound is finite for any $r<\min\{a,a_o\}$ thanks to (14) and the weighted L^p integrability of f_0. This proves the result for any $p\in(1,\infty)$.

Let us prove the case $p=\infty$. Recall that

$$Q^{+}(f,f)(v) \leq 2^{\frac{3\gamma}{2}} Q^{+}_{o,b}\big(f, f\langle\cdot\rangle^{\gamma}\big)(v)\, \langle v\rangle^{\gamma}\,.$$

Using estimate (16), we conclude that

$$\partial_t g(v) \leq \Big(2^{\frac{3\gamma}{2}} Q^{+}_{o,b}\big(g, g\langle\cdot\rangle^{\gamma}\big)(v) - c_o\, g(v)\Big)\langle v\rangle^{\gamma}\,. \tag{18}$$

Furthermore, Lemma 2.2 leads to

$$\begin{aligned} &\|Q^{+}_{o,b_1^{\varepsilon}}\big(g, g\langle\cdot\rangle^{\gamma}\big)\|_{L^\infty(\mathbb{R}^d)} \leq \varepsilon^{-\frac{d}{2}}\, C(b) \|g\|_{L^2(\mathbb{R}^d)} \|g\langle\cdot\rangle^{\gamma}\|_{L^2(\mathbb{R}^d)}\,, \\ &\|Q^{+}_{o,b_2^{\varepsilon}}\big(g, g\langle\cdot\rangle^{\gamma}\big)\|_{L^\infty(\mathbb{R}^d)} \leq \mathfrak{m}(b_2^{\epsilon}) \|g\|_{L^\infty(\mathbb{R}^d)} \|g\langle\cdot\rangle^{\gamma}\|_{L^1(\mathbb{R}^d)}\,. \end{aligned} \tag{19}$$

In addition, estimates (14) and (17) assure that

$$\sup_{t\geq 0}\Big(\|g(t)\langle\cdot\rangle^{\gamma}\|_{L^1(\mathbb{R}^d)} + \|g(t)\langle\cdot\rangle^{\gamma}\|_{L^2(\mathbb{R}^d)}\Big) \leq C(g_0)\,, \qquad \text{for any } \ r<\min\{a,a_o\}\,. \tag{20}$$

Using again the splitting (3) as $Q^{+}_{o,b} = Q^{+}_{o,b_1^{\varepsilon}} + Q^{+}_{o,b_2^{\varepsilon}}$, the conclusion from the estimates (18), (19) and (20) is that for some constants $C_i(g_0,b)$, $i=1,2$, depending on g_0 through the norm $\|g_0\|_{L^1\cap L^2(\mathbb{R}^d)}$ it holds

$$\begin{aligned} \partial_t g(t,v) &\leq \Big(\varepsilon^{-\frac{d}{2}} C_1(g_0) + \mathfrak{m}(b_2^{\epsilon})\, C_2(g_0)\|g(t)\|_{L^\infty(\mathbb{R}^d)}\Big)\langle v\rangle^{\gamma} - c_o\, g(t,v)\langle v\rangle^{\gamma} \\ &= \Big(\tilde{C}_1(g_0) + \frac{c_o}{4}\|g(t)\|_{L^\infty(\mathbb{R}^d)}\Big)\langle v\rangle^{\gamma} - c_o\, g(t,v)\langle v\rangle^{\gamma}\,, \end{aligned} \tag{21}$$

where, for the latter, we simply took $\varepsilon > 0$ sufficiently small such that $\mathfrak{m}(b_2^{\epsilon})C_2(g_0) \le \frac{c_o}{4}$. Finally, let us integrate estimate (21) to conclude that

$$\sup_{t\ge 0}\|g(t)\|_{L^\infty(\mathbb{R}^d)} \le \frac{4}{3}\Big(\|g_0\|_{L^\infty(\mathbb{R}^d)} + \frac{\tilde{C}_1(g_0)}{c_o}\Big) =: G(g_0)\,. \tag{22}$$

Indeed, bear in mind that $\partial_t g$ exist a.e. in $\mathbb{R}^+\times\mathbb{R}^d$ since f is solution of the Boltzmann equation. Thus, we can integrate estimate (21) in $[0,t]$ to obtain

$$\begin{aligned} g(t,v) &\le g_0(v)e^{-c_o\langle v\rangle^\gamma t} + \int_0^t e^{-c_o\langle v\rangle^\gamma (t-s)}\Big(\tilde{C}_1(g_0) + \frac{c_o}{4}\|g\|_{L^\infty(\mathbb{R}^d)}\Big)\langle v\rangle^\gamma \mathrm{d}s \\ &\le \|g_0\|_{L^\infty(\mathbb{R}^d)} + \Big(\tilde{C}_1(g_0) + \frac{c_o}{4}\sup_{0\le s\le t}\|g(s)\|_{L^\infty(\mathbb{R}^d)}\Big)\langle v\rangle^\gamma\int_0^t e^{-c_o\langle v\rangle^\gamma(t-s)}\mathrm{d}s \\ &\le \frac{3}{4}G(g_0) + \frac{1}{4}\sup_{0\le s\le t}\|g(s)\|_{L^\infty(\mathbb{R}^d)}\,,\qquad \text{a.e. in } v\in\mathbb{R}^d\,. \end{aligned}$$

As a consequence, for any $0\le t\le T$ it holds

$$g(t,v) \le \frac{3}{4}G(g_0) + \frac{1}{4}\sup_{0\le s\le T}\|g(s)\|_{L^\infty(\mathbb{R}^d)}\,,\qquad \text{a.e. in } v\in\mathbb{R}^d\,. \tag{23}$$

Estimate (22) readily follows after computing the essential supremum in $v\in\mathbb{R}^d$ and, then, the supremum in $t\in[0,T]$ in estimate (23). □

It is well know that instantaneous creation of tails at the level of moments occur for the Boltzmann equation for hard potentials. This is at odds with the Maxwell molecules model where only propagation of tails is possible. This was first noticed in [37]. More recently, L^1-exponential tail creation has been studied in [2]. We show here that from L^1-exponential tail creation, it is possible to deduce L^p-exponential tail creation following the argument presented previously for propagation of exponential integrability. We should stress that integrability is not created only the tails are.

Theorem 3.2 (Creation of L^p-tails)**.** *Let $\gamma\in(0,2)$ be the potential exponent, $b\in L^1(\mathbb{S}^{d-1})$ be the angular kernel (with mass normalized to unity), and assume that for some $p\in(1,\infty)$*

$$\|f_0(\cdot)\|_{(L^1_2\cap L^p)(\mathbb{R}^d)} = C_o < \infty\,,$$

for some constant $C_o>0$. Then, there exist positive constants a and C depending on the initial mass, energy, C_o, γ and b such that

$$\|f(t,\cdot)\,e^{a\min\{1,t\}\langle\cdot\rangle^\gamma}\|_{L^p(\mathbb{R}^d)} \le C\,,\quad t\ge 0\,, \tag{24}$$

for the solution $f(t,v)$ of the Boltzmann equation. This estimate is also true in the case $\gamma=2$ if a moment 2^+ is assumed to be finite for f_0.

For the particular case $p=+\infty$, with $\gamma\in(0,2)$, assume

$$\|f_0(\cdot)\|_{(L^1_2\cap L^2_2\cap L^\infty)(\mathbb{R}^d)} = C_o < \infty\,.$$

Then, estimate (24) *holds true. Again, the validity of this estimate holds for* $\gamma = 2$ *provided the* $L^1_{2^+} \cap L^2_{2^+}$ *norm is assumed finite for the initial datum.*

Proof. Let us handle first the case $p \in (1, \infty)$. Note that for any $r > 0$, $\gamma \in (0, 2]$ it follows that

$$e^{r\,\min\{1,t\}\langle v\rangle^{\gamma}}\partial_t f(t,v) = \partial_t g(t,v) - r\,\chi_{[0,1)}(t)\langle v\rangle^{\gamma} g(t,v)\,,\quad t \geq 0\,,$$

where $g(t,v) := f(t,v)e^{r\,\min\{1,t\}\langle v\rangle^{\gamma}}$. Thus, arguing as in the proof of Theorem 3.1, it follows that

$$\partial_t g(v) \leq Q^{+}(g,g)(v) - (c_o - r)\,g(v)\langle v\rangle^{\gamma}\,.$$

Moreover, in [2, Theorem 1] it is proved that if $f_0 \in L^1_2(\mathbb{R}^d)$ it follows that

$$\|f(t,\cdot)\,e^{a_o\,\min\{1,t\}\langle\cdot\rangle^{\gamma}}\|_{L^1(\mathbb{R}^d)} \leq C_o,\quad t \geq 0\,,$$

for some positive constants a_o and C_o depending on the initial mass, energy, γ and b. Choosing $r < \min\{a_o, c_o\}$ sufficiently small, this estimate suffices to conclude using the techniques of [30, Theorem 4.1] or [5, Corollary 1.1] that

$$\sup_{t\geq 0}\|g(t,\cdot)\|_{L^p(\mathbb{R}^d)} \leq C_p(g_0)\,,\qquad \text{with } \gamma \in (0,2]\,, \tag{25}$$

where $C_p(g_0)$ depends on an upper bound of $\|g_0\|_{L^p} + \sup_{t\geq 0}\|g(t,\cdot)\|_{L^1_{\gamma^+}} < +\infty$.

The case $p = \infty$ is obtained following the respective argument of Theorem 3.1 for this case. Indeed, one easily arrives to the estimate

$$\partial_t g(v) \leq \Big(2^{\frac{3\gamma}{2}} Q^{+}_{o}\big(g, g\langle\cdot\rangle^{\gamma}\big)(v) - (c_o - r)\,g(v)\Big)\langle v\rangle^{\gamma}\,.$$

We will choose $r < c_o$ sufficiently small, thus, we only need to guarantee the finiteness of the term

$$\sup_{t\geq 0}\Big(\big\|g(t)\langle\cdot\rangle^{\gamma}\|_{L^1} + \big\|g(t)\langle\cdot\rangle^{\gamma}\|_{L^2}\Big) < \infty.$$

This holds for any $\gamma \in (0, 2)$, recall that $f_0 \in L^1_2 \cap L^2_2$, due to the propagation of polynomial integrability [30, Theorem 4.1] and creation of exponential integrability Theorem 3.2 for the case $p = 2$. The case $\gamma = 2$ holds true by assuming $f_0 \in L^1_{2^+} \cap L^2_{2^+}$ and invoking the same theorems. □

3.2. **Maxwell molecules case.** Maxwell molecules, $\gamma = 0$, is a critical case for uniform propagation of L^p integrability for general initial data. Indeed, as soon as $\gamma < 0$, i.e. soft potential case, the uniform propagation of moments is lost for general initial data, refer to [18] for an interesting discussion. In order to compensate for this issue, we will use propagation of entropy. Thus, we implicitly have the additional requirement on the initial data

$$\int_{\mathbb{R}^d} f_0(v)\ln\big(f_0(v)\big)\mathrm{d}v < \infty\,,$$

that is, we work with initial data having finite initial entropy. Clearly, this is harmless since more restrictive conditions on f_0 are imposed in our context, namely, $f_0 \in \big(L^1_2 \cap L^p\big)(\mathbb{R}^d)$, for $p > 1$. As a consequence, initial finite entropy is satisfied since for any $1 < p < \frac{d}{d-1} \leq 2$

$$\begin{aligned}\Big|\int_{\mathbb{R}^d} f_0(v)\ln\big(f_0(v)\big)\mathrm{d}v\Big| &\leq \int_{\{f_0\leq 1\}} \big|f_0(v)\ln\big(f_0(v)\big)\big|\mathrm{d}v + \int_{\{f_0\geq 1\}} |f_0(v)\ln\big(f_0(v)\big)|\mathrm{d}v \\ &\leq C_p \int_{\{f_0\leq 1\}} \big|f_0(v)\big|^{\frac{1}{p}}\mathrm{d}v + \int_{\{f_0\geq 1\}} |f_0(v)|^p \mathrm{d}v \\ \leq C_{p,d}\bigg(\int_{\{f_0\leq 1\}} f_0(v)\langle v\rangle^p \mathrm{d}v\bigg)^{\frac{1}{p}} &+ \int_{\{f_0\geq 1\}} |f_0(v)|^p\mathrm{d}v =: C\Big(d, \|f_0\|_{(L^1_2\cap L^p)(\mathbb{R}^d)}\Big) < +\infty.\end{aligned}$$

In the second estimate we used that $|x\ln(x)| \leq C_p\, x^{\frac{1}{p}}$, for any $p > 1$ and $x \in [0,1]$. Furthermore, it is well known that using energy conservation and entropy dissipation it follows for $f(t,v)$, the solution of the homogeneous Boltzmann equation (see [19, page 329] or more recently [7, Lemma A.1]), that

$$\sup_{t\geq 0}\int_{\mathbb{R}^d} f(t,v)\big|\ln\big(f(t,v)\big)\big|\mathrm{d}v \leq C\bigg(\int f_0\ln(f_0), \int f_0|\cdot|^2\bigg). \tag{26}$$

Lemma 3.1. *For any $K > 1$, $\varepsilon > 0$ and $p \in [1,\infty]$ one has*

$$\begin{aligned}\|Q^+_{o,b}(f,f)\|_{L^p(\mathbb{R}^d)} &\leq \frac{C(b)}{\ln(K)}\|f\ln(f)\|_{L^1(\mathbb{R}^d)}\|f\|_{L^p(\mathbb{R}^d)} \\ &\quad + \varepsilon^{-\frac{d}{2p'}} K^{\frac{1}{2p'}} C(b)\|f\|^{1+\frac{1}{2p}}_{L^1(\mathbb{R}^d)}\|f\|^{\frac{1}{2}}_{L^p(\mathbb{R}^d)} + \mathfrak{m}(b^{\epsilon}_2)\|f\|_{L^1(\mathbb{R}^d)}\|f\|_{L^p(\mathbb{R}^d)}\,.\end{aligned}$$

Proof. Use the usual decomposition (3) to write $Q^+_{o,b}(f,f) = Q^+_{o,b^\varepsilon_1}(f,f) + Q^+_{o,b^\varepsilon_2}(f,f)$. On the one hand, we know that

$$\|Q^+_{o,b^\varepsilon_2}(f,f)\|_{L^p(\mathbb{R}^d)} \leq \mathfrak{m}(b^\epsilon_2)\|f\|_{L^p(\mathbb{R}^d)}\|f\|_{L^1(\mathbb{R}^d)}\,.$$

On the other hand, note that bilinearity implies

$$\begin{aligned}Q^+_{o,b^\varepsilon_1}(f,f) &= Q^+_{o,b^\varepsilon_1}(f, f\,1_{\{f\leq K\}}) + Q^+_{o,b^\varepsilon_1}(f, f\,1_{\{f>K\}}) \\ &\leq Q^+_{o,b^\varepsilon_1}(f, f\,1_{\{f\leq K\}}) + \ln(K)^{-1}\, Q^+_{o,b^\varepsilon_1}(f, f\,\ln(f))\,.\end{aligned}$$

Moreover, Young's inequality for the gain collision operator, see [3, Theorem 1] or Theorem 6.1, implies

$$\begin{aligned}\|Q^+_{o,b^\varepsilon_1}(f, f\,\ln(f))\|_{L^p(\mathbb{R}^d)} &\leq C(b)\|f\|_{L^p(\mathbb{R}^d)}\|f\,\ln(f)\|_{L^1(\mathbb{R}^d)}\,, \\ \|Q^+_{o,b^\varepsilon_1}(f, f\,1_{\{f\leq K\}})\|_{L^p(\mathbb{R}^d)} &\leq \varepsilon^{-\frac{d}{2p'}} C(b)\|f\|_{L^{\frac{2p}{p+1}}(\mathbb{R}^d)}\|f\,1_{\{f\leq K\}}\|_{L^{\frac{2p}{p+1}}(\mathbb{R}^d)} \\ &\leq \varepsilon^{\frac{d}{2p'}}\, K^{\frac{1}{2p'}}\, C(b)\|f\|_{L^{\frac{2p}{p+1}}(\mathbb{R}^d)}\|f\|^{\frac{p+1}{2p}}_{L^1(\mathbb{R}^d)}\,.\end{aligned}$$

The estimate follows using, in the last inequality, the interpolation

$$\|f\|_{L^{\frac{2p}{p+1}}(\mathbb{R}^d)} \leq \|f\|^{\frac{1}{2}}_{L^1(\mathbb{R}^d)}\|f\|^{\frac{1}{2}}_{L^p(\mathbb{R}^d)}\,. \tag{27}$$

□

Proposition 3.1. *Consider Maxwell molecules $\gamma = 0$ with angular kernel $b \in L^1(\mathbb{S}^{d-1})$ (with mass normalized to unity). Assume that the initial data satisfies*

$$\|f_0(\cdot)\|_{(L^1_2\cap L^p)(\mathbb{R}^d)} < \infty\,,$$

for some $p \in [1,\infty]$. Then, there exist positive constant C depending on the initial mass, energy, entropy, $\|f_0\|_{L^p(\mathbb{R}^d)}$, γ and b such that

$$\|f(t,\cdot)\|_{L^p(\mathbb{R}^d)} \leq C\,, \quad t \geq 0\,,$$

for the solution $f(t,v)$ of the Boltzmann equation.

Proof. Without loss of generality assume $\|f(t,\cdot)\|_{L^1(\mathbb{R}^d)} = 1$. The case $p \in [1,\infty)$ is a direct consequence of Lemma 3.1. Indeed, multiply the Boltzmann equation by f^{p-1} and integrate in velocity to obtain

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^p_{L^p(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} Q^+(f,f)(v)f^{p-1}(v)\mathrm{d}v - \|f\|^p_{L^p(\mathbb{R}^d)}\\ &\leq \|Q^+(f,f)\|_{L^p(\mathbb{R}^d)}\|f\|^{p-1}_{L^p(\mathbb{R}^d)} - \|f\|^p_{L^p(\mathbb{R}^d)}\,.\end{aligned}$$

Taking $K > 1$ sufficiently large and $\varepsilon > 0$ sufficiently small in Lemma 3.1, we simply obtain that

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^p_{L^p(\mathbb{R}^d)} \leq C(f_0)\|f\|^{p-\frac{1}{2}}_{L^p(\mathbb{R}^d)} - \frac{1}{2}\|f\|^p_{L^p(\mathbb{R}^d)}\,,$$

where the cumulative constant $C(f_0)$ depends on mass, energy, entropy and scattering kernel b. This is enough to conclude that

$$\sup_{t\geq 0}\|f(t)\|_{L^p(\mathbb{R}^d)} \leq \max\left\{\|f_0\|_{L^p(\mathbb{R}^d)}, 4C(f_0)^2\right\}. \tag{28}$$

The case $p = \infty$ is straightforward from here. Indeed, using (19) which is valid for $\gamma = 0$, it follows that

$$\begin{aligned}\partial_t f(v) &\leq \varepsilon^{-\frac{d}{2}}C(b)\|f\|^2_{L^2(\mathbb{R}^d)} + \mathfrak{m}(b_2^{\epsilon})\|f\|_{L^1(\mathbb{R}^d)}\|f\|_{L^\infty(\mathbb{R}^d)} - f(v)\\ &\leq C(f_0) + \frac{1}{4}\|f\|_{L^\infty(\mathbb{R}^d)} - f(v)\,.\end{aligned}$$

We argue as in Theorem 3.1 to conclude that

$$\sup_{t\geq 0}\|f(t)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{4}{3}\Big(\|f_0\|_{L^\infty(\mathbb{R}^d)} + C(f_0)\Big)\,.$$

□

Remark 3.1. *Since Lemma 3.1 is valid for $p = \infty$, estimate (28) does not degenerate as $p \to \infty$. Thus, the L^∞-estimate can be obtained by simply sending $p \to \infty$ in (28). This is at odds with the usual estimates for hard potentials which use interpolation and degenerate as p increases to infinity. Of course, the reason for this difference is that Lemma 3.1 explicitly uses the propagation of entropy which was avoided in the context of hard potentials. This approach using the entropy leads to an alternative argument for propagation of L^p-norms for hard potentials as long as propagation of entropic moments is at hand. Such propagation of entropic moments in fact happens as we prove later in the last section.*

Lemma 3.2. *Define $g := f\, e^{r\langle\cdot\rangle^\alpha}$, and set $a > 0$, $r \in (0,a)$, $\alpha \in (0,2]$. Then, for any $R > 0$, $\varepsilon > 0$ and $p \in [1,\infty]$ it holds*

$$\|Q^+(g,g)\|_{L^p(\mathbb{R}^d)} \le e^{-(a-r)R^\alpha} C(b) \|f\, e^{a\langle\cdot\rangle^\alpha}\|_{L^1(\mathbb{R}^d)} \|g\|_{L^p(\mathbb{R}^d)} + e^{rR^\alpha} \varepsilon^{-\frac{d}{2p'}} C(b) \sqrt{\|f\|_1 \|f\|_p \|g\|_1 \|g\|_p} + \mathfrak{m}(b_2^\epsilon) \|g\|_{L^1(\mathbb{R}^d)} \|g\|_{L^p(\mathbb{R}^d)}\,.$$

Proof. Note that for any $R > 0$ we can write

$$Q^+(g,g) = Q^+(g, g\, 1_{|\cdot|\le R}) + Q^+(g, g\, 1_{|\cdot|>R})\,.$$

On the one hand, for the latter one simply estimates

$$\begin{aligned}\|Q^+(g, g\, 1_{|\cdot|>R})\|_{L^p(\mathbb{R}^d)} &\le C(b)\|g\|_{L^p(\mathbb{R}^d)} \|g\, 1_{|\cdot|>R}\|_{L^1(\mathbb{R}^d)} \\ &\le e^{-(a-r)R^\alpha} C(b) \|g\|_{L^p(\mathbb{R}^d)} \|f\, e^{a\langle\cdot\rangle^\alpha}\|_{L^1(\mathbb{R}^d)}\,.\end{aligned}$$

On the other hand, for the former one uses the decomposition (3)

$$Q^+(g, g\, 1_{|\cdot|\le R}) = Q^+_{b^1_\varepsilon}(g, g\, 1_{|\cdot|\le R}) + Q^+_{b^2_\varepsilon}(g, g\, 1_{|\cdot|\le R})\,.$$

Each of the terms on the right side is easily controlled by the Young's inequality for the gain collision operator. Indeed, for the operator with b^1_ε

$$\begin{aligned}\|Q^+_{b^1_\varepsilon}(g, g\, 1_{|\cdot|\le R})\|_{L^p(\mathbb{R}^d)} &\le \varepsilon^{-\frac{d}{2p'}} C(b) \|g\|_{L^{\frac{2p}{p+1}}(\mathbb{R}^d)} \|g\, 1_{|\cdot|\le R}\|_{L^{\frac{2p}{p+1}}(\mathbb{R}^d)} \\ &\le e^{rR^\alpha} \varepsilon^{-\frac{d}{2p'}} C(b) \|g\|_{L^{\frac{2p}{p+1}}(\mathbb{R}^d)} \|f\|_{L^{\frac{2p}{p+1}}(\mathbb{R}^d)} \\ &\le e^{rR^\alpha} \varepsilon^{-\frac{d}{2p'}} C(b) \sqrt{\|g\|_{L^1(\mathbb{R}^d)} \|g\|_{L^p(\mathbb{R}^d)} \|f\|_{L^1(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d)}}\,,\end{aligned}$$

where the last inequality follows from (27). Similarly, for the operator with b^2_ε

$$\|Q^+_{b^2_\varepsilon}(g, g\, 1_{|\cdot|\le R})\|_{L^p(\mathbb{R}^d)} \le \mathfrak{m}(b_2^\epsilon) \|g\|_{L^p(\mathbb{R}^d)} \|g\, 1_{|\cdot|\le R}\|_{L^1(\mathbb{R}^d)} \le \mathfrak{m}(b_2^\epsilon) \|g\|_{L^p(\mathbb{R}^d)} \|g\|_{L^1(\mathbb{R}^d)}\,.$$

The result follows after gathering these estimates. □

Theorem 3.3. *Consider Maxwell molecules $\gamma = 0$, let $b \in L^1(\mathbb{S}^{d-1})$ be the angular kernel (with mass normalized to unity) and suppose*

$$\|f_0(\cdot)\, e^{a_o\langle\cdot\rangle^\alpha}\|_{(L^1\cap L^p)(\mathbb{R}^d)} = C_o < \infty\,,$$

for some $\alpha \in (0,2]$, $p \in [1,\infty]$ *and positive constants* a_o *and* C_o. *Then, there exist positive constants* a *and* C *depending on the initial mass, energy, entropy,* a_o, C_o *and* b *such that*

$$\| f(t,\cdot)\, e^{a\langle\cdot\rangle^{\alpha}} \|_{L^p(\mathbb{R}^d)} \leq C\,, \quad t \geq 0\,,$$

for the solution $f(t,v)$ *of the Boltzmann equation.*

Proof. Again, our first step consists in noticing the propagation of exponential moments for Maxwell molecules [31, Theorem 4.1] for Maxwell molecules

$$\| f(t,\cdot)\, e^{a\langle\cdot\rangle^{\alpha}} \|_{L^1(\mathbb{R}^d)} \leq C\,, \quad t \geq 0\,, \tag{29}$$

for some positive a and C with dependence as stated. Now, following the notation and argument of Theorem 3.1, we arrive to the equivalent of equation (16)

$$\partial_t g(v) \leq Q^+(g,g)(v) - g(v)\,, \quad \text{recall that } g := f e^{a\langle\cdot\rangle^{\alpha}}. \tag{30}$$

After multiplying (30) by g^{p-1} and integrating in velocity one concludes, as usual, that

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|g\|_{L^p(\mathbb{R}^d)} \leq \|Q^+(g,g)\|_{L^p(\mathbb{R}^d)} \|g\|^{p-1}_{L^p(\mathbb{R}^d)} - \|g\|^{p}_{L^p(\mathbb{R}^d)}\,. \tag{31}$$

Furthemore, the norms $\|f\, e^{r\langle\cdot\rangle^{\alpha}}\|_{L^1(\mathbb{R}^d)}$, with $r \in (0,a]$, are uniformly bounded by (29), and the norm $\|f\|_{L^p(\mathbb{R}^d)}$ is uniformly bounded by Proposition 3.1. As a consequence, using Lemma 3.2 for $R > 0$ sufficiently large and $\varepsilon > 0$ sufficiently small, we can find a cumulative constant depending only on g_0 such that

$$\|Q^+(g,g)\|_{L^p(\mathbb{R}^d)} \leq C(g_0)\|g\|^{\frac{1}{2}}_{L^p(\mathbb{R}^d)} + \frac{1}{2}\|g\|_{L^p(\mathbb{R}^d)}\,. \tag{32}$$

From (31) and (32) it readily follows that

$$\sup_{t\geq 0}\|g(t)\|_{L^p(\mathbb{R}^d)} \leq \max\left\{\|g_0\|_{L^p(\mathbb{R}^d)}\,,\, 4C(g_0)^2\right\}. \tag{33}$$

Finally, the case $p = \infty$ goes in the same manner as in Theorem 3.1, or simply by taking $p \to \infty$ in (33). □

4. Propagation of exponentially-tailed Sobolev regularity

In the sequel, the Fourier transform $\mathcal{F}$ is denoted by the shorthand $\widehat{f} := \mathcal{F}(f)$ for any tempered distribution f. We continue using the shorthand notation $\hat{\xi} := \xi/|\xi|$, with $\xi \in \mathbb{R}^d$, to denote unitary vectors since it should not present any confusion with that of the Fourier transform. An essential identity in the analysis is [15, equation (2.15)] which is a generalization of that of Bobylev for Maxwell molecules [11]

$$\mathcal{F}\Big\{\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} F_{\hat{u}\cdot\sigma}(v',v'_*)\mathrm{d}\sigma\mathrm{d}v_*\Big\}(\xi) = \int_{\mathbb{S}^{d-1}} \widehat{F_{\hat{\xi}\cdot\sigma}}(\xi^+,\xi^-)\mathrm{d}\sigma\,. \tag{34}$$

Here $\widehat{F_x}$ is the Fourier transform in the variables (v',v'_*) keeping $x \in [-1,1]$ fixed, and where we continue using the shorthand notation $\xi^{\pm} := \frac{\xi\pm|\xi|\sigma}{2}$.

Let us just mention that the proofs in this section are explicitly written for propagation of Sobolev regularity with exponential weights. Similar results, shown with analogous arguments, hold for polynomial weights. Also, we will restrict ourselves only to the physical range $\gamma \in [0,1]$; this is not central in the argument, but it simplifies some statements. More important is the restriction $\alpha \in (0,1]$ in the exponential tail, central for the validity of some estimates with fractional commutators.

4.1. **Commutators.** We prove in this section that fractional differentiation commutes with the collision operator up to a lower order remainder. This result is typical of convolution like operators and happens for both, the gain and loss Boltzmann operators. These commutator estimates are new in the context of Grad's cut-off Boltzmann equation, however, the idea has been used before in the context of Boltzmann equation without cut-off to develop an $L^2(\mathbb{R}^d)$ regularity theory, see for example [1, Section 3]. The technique used to prove these estimates is usually pseudo-differential calculus; here we take a more elementary approach exploiting formula (34). This method is flexible and it could be adapted to study propagation in general $L^p(\mathbb{R}^d)$ spaces because our commutators hold in the pointwise sense. We do not explore this path and content ourselves with the presentation of the $L^2(\mathbb{R}^2)$ theory.

Lemma 4.1 (Commutator for the loss operator)**.** *Take any* $\gamma \in [0,1]$*,* $s \in (0,1]$*,* $r \in [0,\frac{1}{2})$ *and* $\alpha \in (0,1]$*. Then,*

$$\big(1+(-\Delta)\big)^{\frac{s}{2}} Q^-_{\gamma,b}\big(f,g\big) = Q^-_{\gamma,b}\big(\big(1+(-\Delta)\big)^{\frac{s}{2}} f, g\big) + \mathcal{I}^-(f,g)\,,$$

where

$$\begin{cases} \mathcal{I}^-(f,g) = 0 & \text{for } \ \gamma = 0\,, \\ \|\mathcal{I}^-(f,g)\, e^{r\langle\cdot\rangle^{\alpha}}\|_{L^2(\mathbb{R}^d)} \leq C\|b\|_{L^1(\mathbb{S}^{d-1})}\|f\, e^{r\langle\cdot\rangle^{\alpha}}\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2_{\frac{d-(1-\gamma)}{2}}(\mathbb{R}^d)} & \text{for } \ \gamma \in (0,1]\,, \end{cases}$$

with constant $C := C(d,s,\gamma,r,\alpha)$*.*

Proof. The case $\gamma = 0$ is trivial, thus, consider $\gamma \in (0,1]$. Using Lemma 6.3 one has

$$\big(1+(-\Delta)\big)^{\frac{s}{2}} Q^-_{\gamma,b}\big(f,g\big)(v) = Q^-_{\gamma,b}\big(\big(1+(-\Delta)\big)^{\frac{s}{2}} f, g\big)(v) + \int_{\mathbb{R}^d} g(v_*)\mathcal{R}_{v_*}(f)(v)\mathrm{d}v_*\,.$$

Then, we define

$$\begin{aligned} \mathcal{I}^-(f,g)(v) &:= \int_{\mathbb{R}^d} g(v_*)\mathcal{R}_{v_*}(f)(v)\mathrm{d}v_* \\ &= s\int_{\mathbb{R}^d} f(v-x)\nabla\varphi(x)\cdot\int_{\mathbb{R}^d} g(v_*)\bigg(\int_0^1 \nabla|\cdot|^{\gamma}(v-v_*+\theta\, x)\mathrm{d}\theta\bigg)\mathrm{d}v_*\mathrm{d}x\,. \end{aligned} \tag{35}$$

Recall that $\varphi := \mathcal{F}^{-1}\big\{\langle\cdot\rangle^{s-2}\big\}$ is the inverse Fourier transform of the Bessel potential of order $s-2$. Now, let us first consider the case $s \in (0,1)$. Using Lemma 6.2 we control the integral in θ in (35)

$$\begin{aligned} \big|\mathcal{I}^-(f,g)(v)\big| &\leq 3s\int_{\mathbb{R}^d} \big|f(v-x)\big|\big|\nabla\varphi(x)\big|\bigg(\int_{\mathbb{R}^d} \big|g(v_*)\big||v-v_*|^{-(1-\gamma)}\mathrm{d}v_*\bigg)\mathrm{d}x \\ &\leq s\, C_{d,\gamma}\|g\|_{L^2_{\frac{d-(1-\gamma)}{2}}(\mathbb{R}^d)}\int_{\mathbb{R}^d} \big|f(v-x)\big|\big|\nabla\varphi(x)\big|\mathrm{d}x\,. \end{aligned}$$

The last inequality follows by breaking the v_*-integral in the sets $\{|v-v_*|\leq 1\}$ and $\{|v-v_*|>1\}$ and using Cauchy-Schwarz inequality in the former. Additionally, $|v|^\alpha \leq |v-x|^\alpha + |x|^\alpha$ for any $\alpha\in(0,1]$. As a consequence,

$$\big|\mathcal{I}^-(f,g)(v)\big|e^{r\langle v\rangle^\alpha} \leq s\,C_{d,\gamma}\|g\|_{L^2_{\frac{d-(1-\gamma)}{2}}(\mathbb{R}^d)}\int_{\mathbb{R}^d}\big|f(v-x)e^{r\langle v-x\rangle^\alpha}\big|\big|\nabla\varphi(x)e^{r\langle x\rangle^\alpha}\big|\mathrm{d}x\,, \tag{36}$$

and, invoking Young's inequality for convolutions yields

$$\|\mathcal{I}^-(f,g)\,e^{r\langle\cdot\rangle^\alpha}\|_{L^2(\mathbb{R}^d)} \leq s\,C_{d,\gamma}\|g\|_{L^2_{\frac{d-(1-\gamma)}{2}}(\mathbb{R}^d)}\|f\,e^{r\langle\cdot\rangle^\alpha}\|_{L^2(\mathbb{R}^d)}\|\nabla\varphi\,e^{r\langle\cdot\rangle^\alpha}\|_{L^1(\mathbb{R}^d)}\,.$$

It is well known that φ is smooth, except at the origin, and decaying as $e^{-\frac{|x|}{2}}$. Moreover,

$$\varphi(x) = \frac{1}{\Gamma(2-s)}|x|^{(2-s)-d} + 1 + O(|x|^{(2-s)-d+2})\,,\quad x\approx 0\,. \tag{37}$$

Thus, $e^{r\langle\cdot\rangle^\alpha}\nabla\varphi\in L^1(\mathbb{R}^d)$ for any $s\in(0,1)$ and $r\in[0,\frac{1}{2})$. This proves the lemma in this case. The case $s=1$ needs a special treatment, but the essential idea remains intact. Indeed, using Remark 6.1 in (35) it readily follows that

$$\begin{aligned}\big|\mathcal{I}^-(f,g)(v)\big| \leq &\left|\int_{\mathbb{R}^d} g(v_*)\nabla|\cdot|^\gamma(v-v_*)\mathrm{d}v_*\right|\big|\big(\nabla\varphi * f\big)(v)\big| \\ &+ C\int_{\mathbb{R}^d}\big|f(v-x)\big|\big||x|^\varepsilon\nabla\varphi(x)\big|\left(\int_{\mathbb{R}^d}\big|g(v_*)\big||v-v_*|^{-(1-\gamma+\varepsilon)}\mathrm{d}v_*\right)\mathrm{d}x\,.\end{aligned}$$

The lemma follows from here using previous arguments and the fact that

$$\|e^{r\langle\cdot\rangle^\alpha}\,\nabla\varphi * f\|_{L^2(\mathbb{R}^d)} \leq C(s,r,\alpha)\|f\,e^{r\langle\cdot\rangle^\alpha}\|_{L^2(\mathbb{R}^d)}\,.$$

□

Lemma 4.2 (Commutator for the gain operator). *Let the potential be $\gamma\in[0,1]$, $s\in(0,1]$, $r\in[0,\frac{1}{4})$ and $\alpha\in(0,1]$. Then,*

$$\big(1+(-\Delta)\big)^{\frac{s}{2}}Q^+_{\gamma,b}\big(f,g\big) = Q^+_{\gamma,b_s}\big(\big(1+(-\Delta)\big)^{\frac{s}{2}}f,g\big) + \mathcal{I}^+(f,g)\,,$$

where $b_s(\cdot) = \left(\frac{2}{1+\cdot}\right)^{\frac{s}{2}}b(\cdot)$, and

$$\|\mathcal{I}^+(f,g)\,e^{r\langle\cdot\rangle^\alpha}\|_{L^2(\mathbb{R}^d)} \leq C\,\|b\|_{L^1(\mathbb{S}^{d-1})}\|f\,e^{r\langle\cdot\rangle^\alpha}\|_{L^2_\gamma(\mathbb{R}^d)}\|g\,e^{r\langle\cdot\rangle^\alpha}\|_{L^2_{\gamma+\frac{d^+}{2}}(\mathbb{R}^d)}\,,$$

with constant $C:=C(d,s,\gamma,r,\alpha)$.

Proof. Using formula (34) with $F(v',v'_*) = f(v')|v'-v'_*|^\gamma g(v'_*)$ one has

$$\begin{aligned}\mathcal{F}\big\{\big(1+(-\Delta)\big)^{\frac{s}{2}}Q^+_{\gamma,b}(f,g)\big\}(\xi) &= \langle\xi\rangle^{\frac{s}{2}}\int_{\mathbb{S}^{d-1}}\widehat{F}(\xi^+,\xi^-)b(\hat{\xi}\cdot\sigma)\mathrm{d}\sigma \\ &= \int_{\mathbb{S}^{d-1}}\langle\xi^+\rangle^{\frac{s}{2}}\widehat{F}(\xi^+,\xi^-)b_s(\hat{\xi}\cdot\sigma)\mathrm{d}\sigma + \widehat{\mathcal{I}^+_1(f,g)}(\xi)\,,\end{aligned} \tag{38}$$

where

$$\widehat{\mathcal{I}_1^{+}(f,g)}(\xi) := \int_{\mathbb{S}^{d-1}} \Big(\big(\tfrac{1+\hat{\xi}\cdot\sigma}{2} + |\xi^{+}|^2\big)^{\frac{s}{2}} - \big(1+|\xi^{+}|^2\big)^{\frac{s}{2}} \Big) \widehat{F}(\xi^{+},\xi^{-}) b_s(\hat{\xi}\cdot\sigma)\mathrm{d}\sigma\,.$$

Now,

$$\langle \xi^{+}\rangle^{\frac{s}{2}} \widehat{F}(\xi^{+},\xi^{-}) = \mathcal{F}\big\{ \big(1+(-\Delta)\big)^{\frac{s}{2}} (f\,\tau_{v'_*}|\cdot|^{\gamma})(v') g(v'_*)\big\}(\xi^{+},\xi^{-})\,.$$

We continue by using Lemma 6.3

$$\begin{aligned} &\langle \xi^{+}\rangle^{\frac{s}{2}} \widehat{F}(\xi^{+},\xi^{-}) = \\ &\mathcal{F}\big\{ \big(1+(-\Delta)\big)^{\frac{s}{2}} f(v') \times |v'-v'_*|^{\gamma} g(v'_*)\big\}(\xi^{+},\xi^{-}) + \mathcal{F}\big\{ \mathcal{R}_{v'_*}(f)(v')g(v'_*)\big\}(\xi^{+},\xi^{-})\,. \end{aligned} \tag{39}$$

The conclusion of (38) and (39) is

$$\begin{aligned} \mathcal{F}\big\{ \big(1+(-\Delta)\big)^{\frac{s}{2}} Q^{+}_{\gamma,b}(f,g)\big\}(\xi) &= \widehat{\mathcal{I}_1^{+}(f,g)}(\xi) + \widehat{\mathcal{I}_2^{+}(f,g)}(\xi) \\ &+ \int_{\mathbb{S}^{d-1}} \mathcal{F}\big\{ \big(1+(-\Delta)\big)^{\frac{s}{2}} f(v') \times |v'-v'_*|^{\gamma} g(v'_*)\big\}(\xi^{+},\xi^{-}) b_s(\hat{\xi}\cdot\sigma)\mathrm{d}\sigma\,, \end{aligned}$$

where

$$\widehat{\mathcal{I}_2^{+}(f,g)}(\xi) := \int_{\mathbb{S}^{d-1}} \mathcal{F}\big\{ \mathcal{R}_{v'_*}(f)(v')g(v'_*)\big\}(\xi^{+},\xi^{-}) b_s(\hat{\xi}\cdot\sigma)\mathrm{d}\sigma\,.$$

As a consequence, taking the inverse Fourier transform

$$\big(1+(-\Delta)\big)^{\frac{s}{2}} Q^{+}_{\gamma,b}(f,g) = Q^{+}_{\gamma,b_s}\big(\big(1+(-\Delta)\big)^{\frac{s}{2}} f, g\big)(v) + \mathcal{I}_1^{+}(f,g)(v) + \mathcal{I}_2^{+}(f,g)(v)\,. \tag{40}$$

Controlling the term $\mathcal{I}_2^{+}$: Assume $\gamma > 0$, otherwise, this term is zero. Also, assume first $s \in (0,1)$ since the case $s = 1$ needs special treatment. Thanks to formula (34), the remainder term $\mathcal{I}_2^{+}$ is

$$\mathcal{I}_2^{+}(f,g)(v) = \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} \mathcal{R}_{v'_*}(f)(v')g(v'_*)b_s(\hat{u}\cdot\sigma)\mathrm{d}\sigma\mathrm{d}v_*\,.$$

Thus, using the pre-post collisional change of variables and Lemmas 6.3 and 6.2 one obtains that for any test function ϕ

$$\begin{aligned} \Big| \int_{\mathbb{R}^d} \mathcal{I}_2^{+}(f,g)(v)\phi(v)\mathrm{d}v \Big| &= \Big| \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} \mathcal{R}_{v_*}(f)(v)g(v_*)\phi(v')b_s(\hat{u}\cdot\sigma)\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v \Big| \\ &\le 3s\int_{\mathbb{R}^d} |\nabla\varphi(x)| \bigg(\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|\tau_x f(v)||\phi(v')|}{|v-v_*|^{1-\gamma}} |g(v_*)| |b(\hat{u}\cdot\sigma)\mathrm{d}v_*\mathrm{d}v\mathrm{d}\sigma \bigg)\mathrm{d}x\,. \end{aligned}$$

Now, let us introduce the exponential weight by choosing $\phi(v) = e^{r\langle v\rangle^{\alpha}}\tilde{\phi}(v)$. Since $|v'|^2 \le |v|^2 + |v_*|^2$, if follows that

$$e^{r\langle v'\rangle^{\alpha}} \le e^{r\langle x\rangle^{\alpha}} e^{r\langle v-x\rangle^{\alpha}} e^{r\langle v_*\rangle^{\alpha}}, \qquad \alpha\in(0,1], \quad r\ge 0\,.$$

As a consequence,

$$\begin{aligned}&\Big|\int_{\mathbb{R}^d}\mathcal{I}_2^+(f,g)(v)\phi(v)\mathrm{d}v\Big|\\&\leq 3s\int_{\mathbb{R}^d}|\tilde{\varphi}(x)|\bigg(\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|\tau_x\tilde{f}(v)||\tilde{\phi}(v')|}{|v-v_*|^{1-\gamma}}|\tilde{g}(v_*)|b(\hat{u}\cdot\sigma)\mathrm{d}v_*\mathrm{d}v\mathrm{d}\sigma\bigg)\mathrm{d}x\,. \end{aligned}\tag{41}$$

where $\tilde{\varphi}=e^{r\langle\cdot\rangle^\alpha}\nabla\varphi$, $\tilde{f}=e^{r\langle\cdot\rangle^\alpha}f$ and $\tilde{g}=e^{r\langle\cdot\rangle^\alpha}g$. For the integral in parenthesis in (41) we use Cauchy-Schwarz inequality

$$\begin{aligned}&\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|\tau_x\tilde{f}(v)||\tilde{\phi}(v')|}{|v-v_*|^{1-\gamma}}|\tilde{g}(v_*)|b(\hat{u}\cdot\sigma)\mathrm{d}v_*\mathrm{d}v\mathrm{d}\sigma\\&\qquad\leq\bigg(\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|\tau_x\tilde{f}(v)|^2}{|v-v_*|^{1-\gamma}}|\tilde{g}(v_*)|b(\hat{u}\cdot\sigma)\mathrm{d}v_*\mathrm{d}v\mathrm{d}\sigma\bigg)^{\frac12}\\&\qquad\quad\times\bigg(\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|\tilde{\phi}(v')|^2}{|v-v_*|^{1-\gamma}}|\tilde{g}(v_*)|b(\hat{u}\cdot\sigma)\mathrm{d}v_*\mathrm{d}v\mathrm{d}\sigma\bigg)^{\frac12}\,.\end{aligned}\tag{42}$$

As for the first integral on the right side, one notes that

$$\begin{aligned}\int_{\mathbb{R}^d}\frac{|\tilde{g}(v_*)|}{|v-v_*|^{1-\gamma}}\mathrm{d}v_*&\leq\bigg(\int_{\mathbb{R}^d}|\tilde{g}(v_*)|^2\langle v_*\rangle^{n-(1-\gamma)}\mathrm{d}v_*\bigg)^{\frac12}\\&\quad\times\bigg(\int_{\mathbb{R}^d}\frac{\langle v_*\rangle^{-n+(1-\gamma)}}{|v-v_*|^{2(1-\gamma)}}\mathrm{d}v_*\bigg)^{\frac12}\leq C_{d,\gamma}\|g\,e^{r|\cdot|^\alpha}\|_{L^2_{\frac{n-(1-\gamma)}{2}}(\mathbb{R}^d)}\,.\end{aligned}$$

Therefore,

$$\begin{aligned}&\bigg(\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|\tau_x\tilde{f}(v)|^2}{|v-v_*|^{1-\gamma}}|\tilde{g}(v_*)|b(\hat{u}\cdot\sigma)\mathrm{d}v_*\mathrm{d}v\mathrm{d}\sigma\bigg)^{\frac12}\\&\qquad\leq C_{d,\gamma}\|b\|^{\frac12}_{L^1(\mathbb{S}^{d-1})}\|g\,e^{r|\cdot|^\alpha}\|^{\frac12}_{L^2_{\frac{n-(1-\gamma)}{2}}(\mathbb{R}^d)}\|f\,e^{r|\cdot|^\alpha}\|_{L^2(\mathbb{R}^d)}\,.\end{aligned}\tag{43}$$

For the second integral on the right side, just recall the definition $v'=v_*+u^+$ which gives us the identity

$$|v'-v_*|=|u^+|=|u|\sqrt{\frac{1+\hat{u}\cdot\sigma}{2}}\,.$$

Then, using the classical change of variables $y=u^+$ (with fixed σ) having Jacobian $J=\frac{1}{2^d}(1+\hat{u}\cdot\sigma)$, it follows that

$$\int_{\mathbb{R}^d}\frac{|\tilde{\phi}(v')|^2}{|u|^{1-\gamma}}b(\hat{u}\cdot\sigma)\mathrm{d}u=\int_{\mathbb{R}^d}\frac{|\tilde{\phi}(v_*+y)|^2}{|y|^{1-\gamma}}\tilde{b}(\hat{u}(y,\sigma)\cdot\sigma)\mathrm{d}y\,,\qquad\tilde{b}(\cdot):=\frac{2^{d-\frac{1-\gamma}{2}}b(\cdot)}{(1+\cdot)^{\frac{1+\gamma}{2}}}\,.\tag{44}$$

Furthermore, using that $\hat{u}(y,\sigma)\cdot\sigma = 2(y\cdot\sigma)^2-1$, it follows by a direct computation that

$$\int_{\mathbb{S}^{d-1}} \tilde{b}(\hat{u}(y,\sigma)\cdot\sigma)\mathrm{d}\sigma = 2^{-\frac{d}{2}}\int_{\mathbb{S}^{d-1}} \frac{\tilde{b}(e_1\cdot\sigma)}{(1+e_1\cdot\sigma)^{\frac{n-2}{2}}}\mathrm{d}\sigma \le C_{d,\gamma}\|b\|_{L^1(\mathbb{S}^{d-1})}\,. \tag{45}$$

Here e_1 is any fixed unitary vector. As a consequence of (44) and (45) we obtain

$$\begin{aligned}\Big(\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|\tilde{\phi}(v')|^2}{|v-v_*|^{1-\gamma}}|\tilde{g}(v_*)|b(\hat{u}\cdot\sigma)\mathrm{d}v_*\mathrm{d}v\mathrm{d}\sigma\Big)^{\frac12}&\\ \le C_{d,\gamma}\|b\|^{\frac12}_{L^1(\mathbb{S}^{d-1})}\|g\,e^{r\langle\cdot\rangle^\alpha}\|^{\frac12}_{L^2_{\frac{n-(1-\gamma)}{2}}(\mathbb{R}^d)}\|\tilde{\phi}\|_{L^2(\mathbb{R}^d)}\,.&\end{aligned} \tag{46}$$

Therefore, after gathering (42), (43) and (46) and plugging into (41) we have

$$\begin{aligned}&\Big|\int_{\mathbb{R}^d}\mathcal{I}_2^+(f,g)(v)\,e^{r\langle\cdot\rangle^\alpha}\tilde{\phi}(v)\mathrm{d}v\Big|\\ &\le C_{d,\gamma}\,s\,\|b\|_{L^1(\mathbb{S}^{d-1})}\|\tilde{\varphi}\|_{L^1(\mathbb{R}^d)}\|f\,e^{r\langle\cdot\rangle^\alpha}\|_{L^2(\mathbb{R}^d)}\|g\,e^{r\langle\cdot\rangle^\alpha}\|_{L^2_{\frac{n-(1-\gamma)}{2}}(\mathbb{R}^d)}\|\tilde{\phi}\|_{L^2(\mathbb{R}^d)}\,.\end{aligned} \tag{47}$$

Since $\|\tilde{\varphi}\|_{L^1(\mathbb{R}^d)} = \|\nabla\varphi\,e^{r\langle\cdot\rangle^\alpha}\|_{L^1(\mathbb{R}^d)} < \infty$, for any $\alpha\in(0,1]$ and $r\in[0,\frac12)$, the result follows by duality.

The case $s=1$ needs special mention due to the fact that $\nabla\varphi$ is not integrable at the origin. However, this issue presents no obstacle for the estimate. Indeed, using Remark 6.1 in the Appendix, one has

$$\begin{aligned}\Big|\int_{\mathbb{R}^d}\mathcal{I}_2^+(f,g)(v)\phi(v)\mathrm{d}v\Big| &= \Big|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}\mathcal{R}_{v_*}(f)(v)g(v_*)\phi(v')b_s(\hat{u}\cdot\sigma)\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v\Big|\\ &\le \gamma\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|(\nabla\varphi * f)(v)||\phi(v')|}{|v-v_*|^{1-\gamma}}|g(v_*)|b(\hat{u}\cdot\sigma)\mathrm{d}v_*\mathrm{d}v\mathrm{d}\sigma\\ &+C\int_{\mathbb{R}^d}\big||x|^\varepsilon\nabla\varphi(x)\big|\Big(\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|\tau_x f(v)||\phi(v')|}{|v-v_*|^{1+\varepsilon-\gamma}}|g(v_*)|b(\hat{u}\cdot\sigma)\mathrm{d}v_*\mathrm{d}v\mathrm{d}\sigma\Big)\mathrm{d}x\,.\end{aligned}$$

The first term on the right side can be estimated by adapting previous argument with $\nabla\varphi * f$ instead of f. While for the second term, one simply repeats the argument with $|\cdot|^\varepsilon\nabla\varphi$ instead of $\nabla\varphi$.

Controlling the term $\mathcal{I}_1^+$: Using the fact that

$$-\Big(\big(1+|\xi^+|^2\big)^{\frac{s}{2}} - \big(a(\hat{\xi}\cdot\sigma)+|\xi^+|^2\big)^{\frac{s}{2}}\Big) = -\frac{s}{2}\int_0^1\frac{1-a(\hat{\xi}\cdot\sigma)}{\big((1-\theta)a(\hat{\xi}\cdot\sigma)+\theta+|\xi^+|^2\big)^{\frac{2-s}{2}}}\mathrm{d}\theta\,,$$

one can conclude, for $a(\hat{\xi}\cdot\sigma) := \frac{1+\hat{\xi}\cdot\sigma}{2}$ and $\ell(\theta,\hat{\xi}\cdot\sigma) := (1-\theta)a(\hat{\xi}\cdot\sigma)+\theta$, that

$$\widehat{\mathcal{I}_1^+(f,g)}(\xi) = -\frac{s}{2}\int_0^1\int_{\mathbb{S}^{d-1}}\Big\langle\frac{\xi^+}{\sqrt{\ell(\theta,\hat{\xi}\cdot\sigma)}}\Big\rangle^{-(2-s)}\widehat{F}(\xi^+,\xi^-)\frac{\big(1-a(\hat{\xi}\cdot\sigma)\big)b_s(\hat{\xi}\cdot\sigma)}{\ell(\theta,\hat{\xi}\cdot\sigma)^{\frac{2-s}{2}}}\mathrm{d}\sigma\mathrm{d}\theta\,.$$

In addition, note that

$$\Big\langle \tfrac{\xi^+}{\sqrt{\ell(\theta,\hat{\xi}\cdot\sigma)}} \Big\rangle^{-(2-s)} \widehat{F}(\xi^+,\xi^-) = \\ \ell(\theta,\hat{\xi}\cdot\sigma)^{\frac{n}{2}} \mathcal{F}\Big\{ \varphi\big(\ell(\theta,\hat{\xi}\cdot\sigma)^{\frac12}\ \cdot\big) * \big(f(\cdot)\tau_{v'_*}|\cdot|^{\gamma}\big)(v')\, g(v'_*) \Big\}(\xi^+,\xi^-)\,.$$

As a consequence, we deduce from formula (34) that

$$\mathcal{I}_1^+(f,g)(v) = -\frac{s}{2}\int_0^1 \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} \varphi\big(\ell(\theta,\hat{u}\cdot\sigma)^{\frac12}\ \cdot\big) * \big(f(\cdot)\tau_{v'_*}|\cdot|^{\gamma}\big)(v')\, g(v'_*) \\ \times \ell(\theta,\hat{u}\cdot\sigma)^{\frac{n+s-2}{2}}\big(1-a(\hat{\xi}\cdot\sigma)\big) b_s(\hat{\xi}\cdot\sigma)\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}\theta\,.$$

Now, $\ell\in[\frac12,1]$ and φ is monotone decreasing, thus $\varphi(\ell\,\cdot)\le\varphi(\frac{\cdot}{2})$. As a consequence,

$$e^{r\langle v\rangle^{\alpha}}\Big|\mathcal{I}_1^+(f,g)(v)\Big| \le \frac{s\,e^{r\langle v\rangle^{\alpha}}}{2}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} \varphi(\tfrac{\cdot}{2}) * (f(\cdot)\tau_{v'_*}|\cdot|^{\gamma})(v')\, g(v'_*)\tilde{b}(\hat{u}\cdot\sigma)\mathrm{d}\sigma\mathrm{d}v_* \\ \le \frac{s}{2}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} (\tilde{\varphi}*\tilde{f})(v')\,\tilde{g}(v'_*)\tilde{b}(\hat{u}\cdot\sigma)\mathrm{d}\sigma\mathrm{d}v_* = \frac{s}{2}\,Q^+_{0,\tilde{b}}(\tilde{\varphi}*\tilde{f},\tilde{g})(v)\,,$$

where

$$\tilde{\varphi} := \varphi(\frac{\cdot}{2})e^{r\langle\cdot\rangle^{\alpha}},\quad \tilde{f}(\cdot) := f(\cdot)\,e^{r\langle\cdot\rangle^{\alpha}}\langle\cdot\rangle^{\gamma},\quad \tilde{g}(\cdot) := g(\cdot)\,e^{r\langle\cdot\rangle^{\alpha}}\langle\cdot\rangle^{\gamma},\quad \tilde{b}(\cdot) := (1-a(\cdot))b_s(\cdot).$$

Therefore, using Young's inequality for the $Q^+_{0,\tilde{b}}$ it readily follows that

$$\big\|\mathcal{I}_1^+(f,g)\,e^{r\langle\cdot\rangle^{\alpha}}\big\|_{L^2(\mathbb{R}^d)} \le c_d\, s\,\|\tilde{b}\|_{L^1(\mathbb{S}^{d-1})}\|\tilde{\varphi}*\tilde{f}\|_{L^2(\mathbb{R}^d)}\|\tilde{g}\|_{L^1(\mathbb{R}^d)} \\ \le c_d\, s\,\|b\|_{L^1(\mathbb{S}^{d-1})}\|\tilde{\varphi}\|_{L^1(\mathbb{R}^d)}\|f\,e^{r\langle\cdot\rangle^{\alpha}}\|_{L^2_{\gamma}(\mathbb{R}^d)}\|g\,e^{r\langle\cdot\rangle^{\alpha}}\|_{L^1_{\gamma}(\mathbb{R}^d)}\,. \tag{48}$$

The result follows using (47), (48), the estimate $\|g\|_{L^1(\mathbb{R}^d)}\le C_d\|g\|_{L^2_{d^+/2}(\mathbb{R}^d)}$ valid for general function g, and $\|\tilde{\varphi}\|_{L^1(\mathbb{R}^d)}<\infty$ for $\alpha\in(0,1]$, $r\in[0,\frac14)$. □

Remark 4.1. *Although the method of proof of Lemmas 4.1 and 4.2 uses Fourier transform, the argument is essentially made in the velocity space. Thus, the method is flexible and it can be modified to obtain more general $L^p(\mathbb{R}^d)$ estimates. Observe also that polynomial weights can be readily handled using the same argument and leading to analogous estimates.*

4.2. **Regularization of the Boltzmann gain operator.** The regularizing effect of the gain Boltzmann operator was first established in [26] and later, with a more elementary proof, polynomial weights were added in [36, 30]. The following theorem is a version of the regularizing effect using exponential weights, and the proof is a simple adaptation of the technique given in [8, Lemma 2.3] for polynomial weights.

Theorem 4.1. *Assume that collision kernel is of the form*

$$B(x,y) = \Phi(x)b(y), \quad \text{with} \quad b\in\mathcal{C}_0^{\infty}\big([0,1)\big)\,,$$

and $\Phi \in \mathcal{C}_0^\infty\big((0,\infty]\big)$*, with* $\Phi(x) \approx x^\gamma$ *for large* x *(*$\gamma \geq 0$*). Then, for* $d \geq 2$*,* $s \geq 0$*,* $r \geq 0$*, and* $\alpha \in (0,1]$

$$\|e^{r\langle\cdot\rangle^\alpha} Q^+(f,g)\|_{H^{s+\frac{d-1}{2}}(\mathbb{R}^d)} \leq C\, \|e^{r\langle\cdot\rangle^\alpha}\langle\cdot\rangle^\mu f\|_{H^s(\mathbb{R}^d)} \|e^{2r\langle\cdot\rangle^\alpha}\langle\cdot\rangle^\mu g\|_{L^1(\mathbb{R}^d)}\,.$$

where $\mu := \mu(s,\gamma) = s^+ + \gamma + \frac{3}{2}$*, and the constant* $C > 0$ *depends, in particular, on the distance from the support of* Φ *and* b *to zero and one respectively.*

Proof. One uses the argument of [8, Lemma 2.3] which is a relaxation of [30, Theorem 3.1]; the central step is to make sure that the expression

$$\max_{|\nu|\leq s+\frac{d-1}{2}} \sup_{w\in\mathbb{S}^{d-1}} \left\| \mathcal{B}\big(|z|,|z\cdot w|\big)\frac{e^{r\langle z\cdot w\rangle^\alpha} z^\nu}{e^{r\langle z\rangle^\alpha}\langle z\rangle^\mu} \right\|_{H^S_z(\mathbb{R}^d)}$$

remains finite. Here

$$\mathcal{B}(x,y) := \frac{\Phi(x)\, b\Big(2\frac{y^2}{x^2}-1\Big)}{x^{d-2}\, y}, \qquad x,\, y > 0\,.$$

Also, ν is a multi-index and $S := s + \lfloor d/2 \rfloor + 1$. This holds true if $\mu := s^+ + \gamma + \frac{3}{2}$. □

Consider now the following decomposition of the gain collision operator, in the spirit of [30], where n stand for “nice” and r for “remainder” terms:

$$x^\gamma = \Phi_n(x) + \Phi_r(x), \qquad b(y) = b_n(y) + b_r(y), \tag{49}$$

with nonnegative functions satisfying the following properties for $\varepsilon, \delta > 0$ sufficiently small

(1) $\Phi_n \in \mathcal{C}^\infty\big((0,\infty)\big)$ vanishing in $(0,\delta)$.
(2) $\Phi_r \in L^2 \cap L^\infty\big((0,\infty)\big)$ vanishing in $(2\delta,\infty)$. Note that

$$\|\Phi_r\|_{L^\infty} \leq (2\delta)^\gamma, \qquad \|\Phi_r\|_{L^2} \leq (2\delta)^{\gamma+1/2}.$$

(4) $b_n \in \mathcal{C}^\infty\big((0,1)\big)$ vanishing in $(1-\varepsilon,1)$.
(3) $b_r \in L^1\big((1-y^2)^{\frac{d-3}{2}}\mathrm{d}y\big)$, with $\|b_r\|_{L^1\left((1-y^2)^{\frac{d-3}{2}}\mathrm{dy}\right)} \sim \mathfrak{m}(b_r)$ sufficiently small.

Decomposition of the collision kernel (49) leads to the decomposition of the gain operator

$$Q^+(f,g) = Q^+_{nn}(f,g) + Q^+_{nr}(f,g) + Q^+_{rn}(f,g) + Q^+_{rr}(f,g)\,,$$

where nr stands for nice kinetic potential with the remainder in the angular scattering. In similar fashion we denote the other terms.

Lemma 4.3 (Remainder terms)**.** *Fix* $\varepsilon, \delta > 0$ *sufficiently small. The following estimates hold for any* $\gamma \in [0,1]$

$$\big\|\langle\cdot\rangle^{-\gamma/2} Q^+_{nr}(f,g)\big\|_{L^2(\mathbb{R}^d)} \leq \mathfrak{m}(b_r)\|f\langle\cdot\rangle^{\gamma/2}\|_{L^2(\mathbb{R}^d)}\|g\langle\cdot\rangle^\gamma\|_{L^1(\mathbb{R}^d)}\,,$$
$$\big\|\langle\cdot\rangle^{-\gamma/2} Q^+_{rr}(f,g)\big\|_{L^2(\mathbb{R}^d)} \leq \mathfrak{m}(b_r)\|f\langle\cdot\rangle^{\gamma/2}\|_{L^2(\mathbb{R}^d)}\|g\langle\cdot\rangle^\gamma\|_{L^1(\mathbb{R}^d)}\,.$$

For the rn term we have

$$\big\|\langle\cdot\rangle^{-\gamma/2}Q^+_{rn}(f,g)\big\|_{L^2(\mathbb{R}^d)} \leq C(b)\,\delta^\gamma\,\|f\langle\cdot\rangle^{\gamma/2}\|_{L^2(\mathbb{R}^d)}\|g\langle\cdot\rangle^\gamma\|_{L^1(\mathbb{R}^d)}, \quad \gamma>0\,,$$
$$\big\|Q^+_{rn}(f,g)\big\|_{L^2(\mathbb{R}^d)} \leq C(b,\varepsilon)\delta^{1/2}\|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}, \quad \gamma=0\,.$$

Proof. The estimates for the terms nr, rr (for $\gamma\geq 0$), and rn (for $\gamma>0$) are classical and can be found, for example, in [30]. They can be obtained as a simple consequence of a standard application of Young's inequality for the gain collision operator Theorem 6.1. The constant in front of the inequality for the terms nr and rr is small due to the small mass of the remainder angular kernel b_r. For term rn in the case of Maxwell molecules, i.e. $\gamma=0$, one can use a generalized version of the Young's inequality that includes the kinetic kernel potential, see [3, Corollary 8]

$$\begin{aligned}\big\|Q^+_{rn}(f,g)\big\|_{L^2(\mathbb{R}^d)} &\leq C(b,\varepsilon)\|\Phi_r\|_{L^2(\mathbb{R}^d)}\|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}\\ &\leq C(b,\varepsilon)\delta^{1/2}\|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}\,.\end{aligned}$$

The finiteness of the constant $C(b,\varepsilon)>0$ is ensured by the truncation of b_n near 1. □

4.3. **Propagation of regularity with exponential tails.**

Proposition 4.1. *Consider Grad's cut-off assumption $b\in L^1(\mathbb{S}^{d-1})$. Assume that for some $\gamma\in[0,1]$, $s\in(0,\min\{1,\frac{d-1}{2}\}]$, and $\alpha\in(0,1]$, one has*

$$e^{a_o\langle v\rangle^\alpha}\big(1+(-\Delta)\big)^{\frac{s}{2}}f_0\in L^2(\mathbb{R}^d).$$

Then, for the solution $f(t,v)$ of the Boltzmann equation, there exists $a\in(0,a_o]$ such that for any $r\in(0,a)$ it holds

$$\big\|e^{r\langle\cdot\rangle^\alpha}\big(1+(-\Delta)\big)^{\frac{s}{2}}f(t,\cdot)\big\|_{L^2(\mathbb{R}^d)} \leq C(f_0)+\big\|e^{r\langle\cdot\rangle^\alpha}\big(1+(-\Delta)\big)^{\frac{s}{2}}f_0\big\|_{L^2(\mathbb{R}^d)},$$

where $C(f_0):=C(f_0,\gamma,s,a_o,\alpha)$ depends on lowers norms of f_0.

Proof. Fix $\varepsilon,\delta>0$ and write the equation as

$$\partial_t f = Q^+_{nn}(f,f)+Q^+_{nr}(f,f)+Q^+_{rn}(f,f)+Q^+_{rr}(f,f)-Q^-(f,f). \tag{50}$$

We will estimate each of these terms on the right side starting from the remainder terms. Note that for the term nr it follows, invoking the commutator Lemma 4.2, that

$$e^{r\langle v\rangle^\alpha}\big(1+(-\Delta)\big)^{s/2}Q^+_{nr}(f,f)=e^{r\langle v\rangle^\alpha}Q^+_{nr}\big(\big(1+(-\Delta)\big)^{s/2}f,f\big)+e^{r\langle v\rangle^\alpha}\mathcal{I}^+_{nr}(f,f)\,.$$

Using Cauchy-Schwarz inequality and Lemma 4.3

$$\begin{aligned}&\int_{\mathbb{R}^d} e^{r\langle v\rangle^\alpha}Q^+_{nr}\big(\big(1+(-\Delta)\big)^{s/2}f,f\big)(v)\,e^{r\langle v\rangle^\alpha}\big(1+(-\Delta)\big)^{s/2}f(v)\mathrm{d}v\\ &\leq \big\|\langle\cdot\rangle^{-\gamma/2}e^{r\langle v\rangle^\alpha}Q^+_{nr}\big(\big(1+(-\Delta)\big)^{s/2}f,f\big)\big\|_{L^2(\mathbb{R}^d)}\big\|\langle\cdot\rangle^{\gamma/2}e^{r\langle v\rangle^\alpha}\big(\big(1+(-\Delta)\big)^{s/2}f\big\|_{L^2(\mathbb{R}^d)}\\ &\quad\leq \mathfrak{m}(b_r)\big\|\langle\cdot\rangle^{\gamma/2}e^{r\langle\cdot\rangle^\alpha}\big(1+(-\Delta)\big)^{s/2}f\big\|^2_{L^2(\mathbb{R}^d)}\big\|\langle\cdot\rangle^\gamma e^{r\langle\cdot\rangle^\alpha}f\big\|_{L^1(\mathbb{R}^d)}.\end{aligned}$$

Also, using Cauchy-Schwarz and Lemma 4.2 again

$$\int_{\mathbb{R}^d} e^{r\langle v\rangle^\alpha}\mathcal{I}^+_{nr}\big(f,f\big)(v)\,e^{r\langle v\rangle^\alpha}\big(1+(-\Delta)\big)^{s/2}f(v)\mathrm{d}v \\ \leq C(b)\big\|\langle\cdot\rangle^{\gamma+\frac{d^+}{2}}e^{r\langle\cdot\rangle^\alpha}f\big\|^2_{L^2(\mathbb{R}^2)}\big\|e^{r\langle\cdot\rangle^\alpha}\big(1+(-\Delta)\big)^{s/2}f\big\|_{L^2(\mathbb{R}^d)}.$$

Thus, thanks to propagation of exponential moments and Theorems 3.1 and 3.3, there exists $0<a\leq a_o$ such that for any $r\in(0,a)$

$$\int_{\mathbb{R}^d} e^{r\langle v\rangle^\alpha}\big(1+(-\Delta)\big)^{s/2}Q^+_{nr}\big(f,f\big)(v)\,e^{r\langle v\rangle^\alpha}\big(1+(-\Delta)\big)^{s/2}f(v)\mathrm{d}v \\ \leq \mathfrak{m}(b_r)\big\|e^{r\langle\cdot\rangle^\alpha}\big(1+(-\Delta)\big)^{s/2}f\big\|^2_{L^2_{\gamma/2}(\mathbb{R}^d)}+C\big\|e^{r\langle\cdot\rangle^\alpha}\big(1+(-\Delta)\big)^{s/2}f\big\|_{L^2_{\gamma/2}(\mathbb{R}^d)}. \tag{51}$$

The same estimate holds for the term rr as well. Furthermore, similar argument leads to the control for the rn term

$$\int_{\mathbb{R}^d} e^{r\langle v\rangle^\alpha}\big(1+(-\Delta)\big)^{s/2}Q^+_{rn}\big(f,f\big)(v)\,e^{r\langle v\rangle^\alpha}\big(1+(-\Delta)\big)^{s/2}f(v)\mathrm{d}v \\ \leq C(b,\varepsilon)\delta^{1/2}\big\|e^{r\langle\cdot\rangle^\alpha}\big(1+(-\Delta)\big)^{s/2}f\big\|^2_{L^2_{\gamma/2}(\mathbb{R}^d)}+C\big\|e^{r\langle\cdot\rangle^\alpha}\big(1+(-\Delta)\big)^{s/2}f\big\|_{L^2_{\gamma/2}(\mathbb{R}^d)}. \tag{52}$$

These last two estimates will handle the remainder term. For the term nn, one uses the commutator formula of Lemma 6.4

$$e^{r\langle v\rangle^\alpha}\big(1+(-\Delta)\big)^{s/2}Q^+_{nn}\big(f,f\big)(v)=\big(1+(-\Delta)\big)^{s/2}e^{r\langle\cdot\rangle^\alpha}Q^+_{nn}\big(f,f\big)(v)-\mathcal{R}\big(Q^+_{nn}(f,f)\big)\,.$$

Thus, invoking Theorem 4.1 and Lemma 6.4 to estimate Q^+_{nn} and $\mathcal{R}$ respectively

$$\int_{\mathbb{R}^d} e^{r\langle v\rangle^\alpha}\big(1+(-\Delta)\big)^{s/2}Q^+_{nn}\big(f,f\big)(v)\,e^{r\langle v\rangle^\alpha}\big(1+(-\Delta)\big)^{s/2}f(v)\mathrm{d}v = \\ \int_{\mathbb{R}^d}\Big[\big(1+(-\Delta)\big)^{s/2}e^{r\langle\cdot\rangle^\alpha}Q^+_{nn}\big(f,f\big)(v)-\mathcal{R}\big(Q^+_{nn}(f,f)\big)\Big]e^{r\langle v\rangle^\alpha}\big(1+(-\Delta)\big)^{s/2}f(v)\mathrm{d}v \\ \leq C(\delta,\varepsilon)\big\|e^{r\langle\cdot\rangle^\alpha}f\big\|_{L^2_\mu(\mathbb{R}^d)}\big\|e^{2r\langle\cdot\rangle^\alpha}f\big\|_{L^1_\mu(\mathbb{R}^d)}\big\|e^{r\langle\cdot\rangle^\alpha}\big(1+(-\Delta)\big)^{s/2}f\big\|_{L^2(\mathbb{R}^d)} \\ +C\big\|e^{r\langle\cdot\rangle^\alpha}Q^+_{nn}(f,f)\big\|_{L^2(\mathbb{R}^d)}\big\|e^{r\langle\cdot\rangle^\alpha}\big(1+(-\Delta)\big)^{s/2}f\big\|_{L^2(\mathbb{R}^d)}\,.$$

As a consequence, one concludes that

$$\int_{\mathbb{R}^d} e^{r\langle v\rangle^\alpha}\big(1+(-\Delta)\big)^{s/2}Q^+_{nn}\big(f,f\big)(v)\,e^{r\langle v\rangle^\alpha}\big(1+(-\Delta)\big)^{s/2}f(v)\mathrm{d}v \\ \leq C(f_0,\delta,\varepsilon)\big\|e^{r\langle\cdot\rangle^\alpha}\big(1+(-\Delta)\big)^{s/2}f\big\|_{L^2(\mathbb{R}^d)}\,. \tag{53}$$

Finally, using Lemma 4.1 and Lemma 2.1

$$\int_{\mathbb{R}^d} e^{r\langle v\rangle^\alpha}\big(1+(-\Delta)\big)^{s/2}Q^-\big(f,f\big)(v)\, e^{r\langle v\rangle^\alpha}\big(1+(-\Delta)\big)^{s/2}f(v)\mathrm{d}v =$$

$$\int_{\mathbb{R}^d} e^{r\langle v\rangle^\alpha}\Big(Q^-\big(\big(1+(-\Delta)\big)^{s/2}f,f\big)(v)+\mathcal{I}^-\big(f,f\big)\Big)e^{r\langle v\rangle^\alpha}\big(1+(-\Delta)\big)^{s/2}f(v)\mathrm{d}v \tag{54}$$

$$\geq c_0\big\|e^{r\langle\cdot\rangle^\alpha}\big(1+(-\Delta)\big)^{s/2}f\big\|^2_{L^2_{\gamma/2}(\mathbb{R}^d)} - C(f_0)\big\|e^{r\langle\cdot\rangle^\alpha}\big(1+(-\Delta)\big)^{s/2}f\big\|_{L^2(\mathbb{R}^d)}\,.$$

In summary, gathering estimates (51), (52), (53) and (51), it follows from (50) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big\|e^{r\langle\cdot\rangle^\alpha}\big(1+(-\Delta)\big)^{s/2}f\big\|^2_{L^2(\mathbb{R}^d)} &\leq C(f_0,\varepsilon,\delta)\\ &+\Big(\mathfrak{m}(b_r)+C(b,\varepsilon)\delta^{\frac12}-c_0\Big)\big\|e^{r\langle\cdot\rangle^\alpha}\big(1+(-\Delta)\big)^{s/2}f\big\|^2_{L^2_{\gamma/2}(\mathbb{R}^d)}\\ &\leq C(f_0,\varepsilon,\delta)-\frac{c_0}{2}\big\|e^{r\langle\cdot\rangle^\alpha}\big(1+(-\Delta)\big)^{s/2}f\big\|^2_{L^2_{\gamma/2}(\mathbb{R}^d)}\,.\end{aligned}$$

For the last inequality we choose ε and, then, $\delta := \delta(\varepsilon)$ sufficiently small. This estimate proves the result. □

Theorem 4.2 (Propagation of exponentially-tailed regularity)**.** *Assume $\gamma\in[0,1]$, $k\geq 0$, and $\alpha\in(0,1]$. Also,*

$$e^{a_o\langle v\rangle^\alpha}\big(1+(-\Delta)\big)^{\frac{k}{2}}f_0\in L^2(\mathbb{R}^d).$$

Then, for the solution $f(t,v)$ of the Boltzmann equation, there exists $a\in(0,a_o]$ such that for any $r\in(0,a)$ it holds

$$\big\|e^{r\langle\cdot\rangle^\alpha}\big(1+(-\Delta)\big)^{\frac{k}{2}}f(t,\cdot)\big\|_{L^2(\mathbb{R}^d)}\leq C(f_0)+\big\|e^{r\langle\cdot\rangle^\alpha}\,\big(1+(-\Delta)\big)^{\frac{k}{2}}f_0\big\|_{L^2(\mathbb{R}^d)}.$$

The constant depends as $C(f_0):=C(f_0,\gamma,k,r,\alpha)$ and lower order $H^{k^-}_{exp}(\mathbb{R}^d)$ Sobolev norms of f_0.

Proof. Here we only show the proof for $d\geq 3$. The same argument with slight modifications will do the job for the case $d=2$. The proof uses induction by first considering the case $k\in\mathbb{N}$. When $k=\{0,1\}$ the result follows using Theorem 3.1 or 3.3 for $k=0$, and Proposition 4.1 for $k=1$. For $k\geq 2$, assume the validity of the result for k and conclude it for $k+1$. Write $k=2n+i$, with $n\in\mathbb{N}$ and $i\in\{0,1\}$, and consider the operator $\mathrm{D}^k:=(-\Delta)^n\nabla^i$. Using classic Leibniz formula for integer differentiation, it is not difficult to check that

$$\begin{aligned}&\mathrm{D}^kQ(f,f)=Q(\mathrm{D}^kf,f)+Q(f,\mathrm{D}^kf)\\ &+\sum_{|j_1|\leq(k-i-1)}\ \sum_{|j_2|\leq(k-i-1)} C_{j_1,j_2}\Big(Q\big(\partial^{j_1}\nabla^i f,\partial^{j_2}f\big)+Q\big(\partial^{j_1}f,\partial^{j_2}\nabla^i f\big)\Big),\end{aligned}$$

for some coefficients C_{j_1,j_2} and with j_1 and j_2 multi-indexes with order ranging as described in the sums. Thus, applying $\big(1+(-\Delta)\big)^{\frac12}\mathrm{D}^k$ to the Boltzmann equation one gets

$$\begin{aligned}\partial_t\big(1+(-\Delta)\big)^{\frac12}\mathrm{D}^k f &= \big(1+(-\Delta)\big)^{\frac12}Q(\mathrm{D}^k f,f)+\big(1+(-\Delta)\big)^{\frac12}Q(f,\mathrm{D}^k f)\\ +\sum_{|j_1|\le(k-i-1)}\ \sum_{|j_2|\le(k-i-1)} &C_{j_1,j_2}\big(1+(-\Delta)\big)^{\frac12}\Big(Q\big(\partial^{j_1}\nabla^i f,\partial^{j_2}f\big)+Q\big(\partial^{j_1}f,\partial^{j_2}\nabla^i f\big)\Big).\end{aligned}\tag{55}$$

Let us control the terms on the right side of (55) starting with the sum. Using the commutator Lemmas 4.1 and 4.2

$$\begin{aligned}\big(1+(-\Delta)\big)^{\frac12}Q\big(\partial^{j_1}\nabla^i f,\partial^{j_2}f\big) &= Q\big(\big(1+(-\Delta)\big)^{\frac12}\partial^{j_1}\nabla^i f,\partial^{j_2}f\big)\\ &\quad+\mathcal{I}^{+}\big(\partial^{j_1}\nabla^i f,\partial^{j_2}f\big)-\mathcal{I}^{-}\big(\partial^{j_1}\nabla^i f,\partial^{j_2}f\big).\end{aligned}$$

Therefore, for $r\in(0,a)$ and $\alpha\in(0,1]$ it holds

$$\begin{aligned}\Big\|e^{r\langle\cdot\rangle^\alpha}\big(1+(-\Delta)\big)^{\frac12}Q\big(\partial^{j_1}\nabla^i f,\partial^{j_2}f\big)\Big\|_{L^2(\mathbb{R}^d)} &\le \Big\|e^{r\langle\cdot\rangle^\alpha}Q\big(\big(1+(-\Delta)\big)^{\frac12}\partial^{j_1}\nabla^i f,\partial^{j_2}f\big)\Big\|_{L^2(\mathbb{R}^d)}\\ +\Big\|e^{r\langle\cdot\rangle^\alpha}\mathcal{I}^{+}\big(\partial^{j_1}\nabla^i f,\partial^{j_2}f\big)\Big\|_{L^2(\mathbb{R}^d)} &+\Big\|e^{r\langle\cdot\rangle^\alpha}\mathcal{I}^{-}\big(\partial^{j_1}\nabla^i f,\partial^{j_2}f\big)\Big\|_{L^2(\mathbb{R}^d)}\\ &\le C\|b\|_{L^1(\mathbb{S}^{d-1})}\big\|e^{r^+\langle\cdot\rangle^\alpha}\big(1+(-\Delta)\big)^{\frac12}\partial^{j_1}\nabla^i f\big\|_{L^2(\mathbb{R}^d)}\big\|e^{r^+\langle\cdot\rangle^\alpha}\partial^{j_2}f\big\|_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Furthermore, using the commutator Lemma 6.4 it follows for $\alpha\in(0,1]$,

$$\begin{aligned}\big\|e^{r^+\langle\cdot\rangle^\alpha}\big(1+(-\Delta)\big)^{\frac12}\partial^{j_1}\nabla^i f\big\|_{L^2(\mathbb{R}^d)} &\lesssim \big\|e^{r^+\langle\cdot\rangle^\alpha}\big(1+(-\Delta)\big)^{\frac{|j_1|+i+1}{2}}f\big\|_{L^2(\mathbb{R}^d)}\\ &\lesssim \big\|e^{r^+\langle\cdot\rangle^\alpha}\big(1+(-\Delta)\big)^{\frac{k}{2}}f\big\|_{L^2(\mathbb{R}^d)}.\end{aligned}$$

For the last inequality we used that $|j_1|\le k-i-1$. Choosing $r^+<a$ one has, by induction hypothesis, that

$$\Big\|e^{r\langle\cdot\rangle^\alpha}\big(1+(-\Delta)\big)^{\frac12}Q\big(\partial^{j_1}\nabla^i f,\partial^{j_2}f\big)\Big\|_{L^2(\mathbb{R}^d)}\le C(f_0),\tag{56}$$

with constant depending on k^{th}-Sobolev regularity of f_0. This controls the sum. In the same fashion, using the commutator lemmas, it follows for the second term

$$\big(1+(-\Delta)\big)^{\frac12}Q(f,\mathrm{D}^k f)=Q\big(\big(1+(-\Delta)\big)^{\frac12}f,\mathrm{D}^k f\big)+\mathcal{I}^{+}\big(f,\mathrm{D}^k f\big)-\mathcal{I}^{-}\big(f,\mathrm{D}^k f\big).$$

As a consequence,

$$\begin{aligned}&\big\|e^{r\langle\cdot\rangle^\alpha}\big(1+(-\Delta)\big)^{\frac12}Q(f,\mathrm{D}^k f)\big\|_{L^2(\mathbb{R}^d)}\\ &\qquad\lesssim C\|b\|_{L^1(\mathbb{S}^{d-1})}\big\|e^{r^+\langle\cdot\rangle^\alpha}\big(1+(-\Delta)\big)^{\frac12}f\big\|_{L^2(\mathbb{R}^d)}\big\|e^{r^+\langle\cdot\rangle^\alpha}\mathrm{D}^k f\big\|_{L^2(\mathbb{R}^d)}\le C(f_0).\end{aligned}\tag{57}$$

In the last inequality we used the induction hypothesis again (valid for $r^+<a$). Finally, following the same argument as in the proof of Proposition 4.1 (with $s=1$)[1], the first term

[1]Indeed, the careful reader observes that such argument is bilinear.

on the right side can be estimated as

$$\begin{aligned}\int_{\mathbb{R}^d} e^{r\langle v\rangle^\alpha}\big(1+(-\Delta)\big)^{\frac12} Q\big(\mathrm{D}^k f, f\big)(v)\, e^{r\langle v\rangle^\alpha}\big(1+(-\Delta)\big)^{\frac12}\mathrm{D}^k f(v)\mathrm{d}v \\ \le C(f_0) - \frac{c_o}{2}\big\| e^{r\langle\cdot\rangle^\alpha}\big(1+(-\Delta)\big)^{\frac12}\mathrm{D}^k f\big\|^2_{L^2(\mathbb{R}^d)}.\end{aligned} \tag{58}$$

As a consequence, multiplying equation (55) by $e^{2r\langle\cdot\rangle}\big(1+(-\Delta)\big)^{\frac12}\mathrm{D}^k f$ and using Cauchy-Schwarz inequality together with estimates (56), (57) and (58), one finds a constant $C(f_0)$ depending on lower order norms of f_0 such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\| e^{r\langle\cdot\rangle^\alpha}\big(1+(-\Delta)\big)^{\frac12}\mathrm{D}^k f\big\|^2_{L^2(\mathbb{R}^d)} \le C(f_0) - \frac{c_o}{2}\big\| e^{r\langle\cdot\rangle^\alpha}\big(1+(-\Delta)\big)^{\frac12}\mathrm{D}^k f\big\|^2_{L^2(\mathbb{R}^d)}.$$

Then,

$$\big\| e^{r\langle\cdot\rangle^\alpha}\big(1+(-\Delta)\big)^{\frac12}\mathrm{D}^k f(t)\big\|_{L^2(\mathbb{R}^d)} \le \max\Big\{\sqrt{\frac{2}{c_o}C(f_0)}, \big\| e^{r\langle\cdot\rangle^\alpha}\big(1+(-\Delta)\big)^{\frac12}\mathrm{D}^k f_0\big\|_{L^2(\mathbb{R}^d)}\Big\}.$$

This proves the case $k\in\mathbb{N}$ after observing that

$$\big\| e^{r\langle\cdot\rangle^\alpha}\big(1+(-\Delta)\big)^{\frac{k+1}{2}} f(t)\big\|_{L^2(\mathbb{R}^d)} \sim \big\| e^{r\langle\cdot\rangle^\alpha}\big(1+(-\Delta)\big)^{\frac12}\mathrm{D}^k f(t)\big\|_{L^2(\mathbb{R}^d)} + \big\| e^{r\langle\cdot\rangle^\alpha} f(t)\big\|_{L^2(\mathbb{R}^d)}.$$

When $k\in\mathbb{R}^+\setminus\mathbb{N}$, write $k=\lfloor k\rfloor + s$ with $s\in(0,1)$. Then, by previous argument

$$\big\| e^{r\langle\cdot\rangle^\alpha}\mathrm{D}^{\lfloor k\rfloor} f(t)\big\|_{L^2(\mathbb{R}^d)} \le C(f_0) + \big\| e^{r\langle\cdot\rangle^\alpha}\mathrm{D}^{\lfloor k\rfloor} f_0\big\|_{L^2(\mathbb{R}^d)}.$$

Perform, again, previous argument for the operator $\big(1+(-\Delta)\big)^{\frac{s}{2}}\mathrm{D}^{\lfloor k\rfloor}$ to conclude the proof. □

Corollary 4.1. *Assume $\gamma\in[0,1]$, $k\ge 0$, and $\alpha\in(0,1]$. Also,*

$$e^{a_o\langle v\rangle^\alpha}\big(1+(-\Delta)\big)^{\frac{k}{2}} f_0 \in L^\infty(\mathbb{R}^d).$$

Then, for the solution $f(t,v)$ of the Boltzmann equation, there exists $a\in(0,a_o]$ such that for any $r\in(0,a)$ it holds

$$\big\| e^{r\langle\cdot\rangle^\alpha}\big(1+(-\Delta)\big)^{\frac{k}{2}} f(t,\cdot)\big\|_{L^\infty(\mathbb{R}^d)} \le C(f_0) + \big\| e^{r\langle\cdot\rangle^\alpha}\,\big(1+(-\Delta)\big)^{\frac{k}{2}} f_0\big\|_{L^\infty(\mathbb{R}^d)}.$$

The constant depends as $C(f_0):=C(f_0,\gamma,k,r,\alpha)$ and lower order $H^{k^-}_{exp}(\mathbb{R}^d)$ Sobolev norms of f_0.

Proof. This is a direct consequence of Theorem 4.2 and the techniques presented in Theorems 3.1 and 3.3, for hard potentials and Maxwell molecules respectively, to show L^∞-norm propagation based on the L^2-norm propagation. □

5. Convergence towards equilibrium and decomposition theorem

In this last section we are interested in studying the convergence of the solution of the homogeneous Boltzmann equation with hard potentials towards the Maxwellian distribution and its propagation of smoothness and singularities. The literature is ample in this respect, notable examples are [10, 28, 30, 34] and the references therein. It is well established that such convergence will occur with exponential rate in L^p-norms, even Sobolev norms, provided some assumptions are satisfied by the angular scattering kernel and the initial data. In all aforementioned references such angular scattering kernel must be at least bounded. In references [10, 34] a clever approach is introduced using a dyadic splitting of the collision operator. In [28, 30] the analysis relies on the decomposition theorem, that is, propagation of smoothness and singularities, and the entropic methods, see for instance [33]. In both approaches spectral theory is also key, in particular, in reference [28] was introduced a powerful quantitative technique known as enlargement of the spectral functional space.

The contributions of this section are given in terms of the requirement of the scattering angle kernel and the generality of the initial data. In particular, we will only assume

$$\text{Angular kernel:} \qquad b \in L^1(\mathbb{S}^{d-1})\,, \qquad b \geq b_o > 0\,, \quad \text{and} \tag{59}$$

$$\text{Initial data:} \qquad \int_{\mathbb{R}^d} f_0 \langle v\rangle^2 \mathrm{d}v < \infty, \qquad \int_{\mathbb{R}^d} f_0 \ln(f_0)\mathrm{d}v < \infty\,. \tag{60}$$

The strategy follows the entropic methods. One of main contributions are the relaxation of the result given in [33] with respect to the dissipation of entropy for hard potentials. This will allow us to prescind of the decomposition theorem. This is important since the most general proof of the decomposition theorem available requires $b \in L^2(\mathbb{S}^{d-1})$, see [30]. Once this is achieved, the result will follow after a fine decomposition of the Boltzmann linearized operator and the spectral enlargement result given in [24]. As an application of this convergence result, we prove the decomposition theorem under mere (59) and (60).

Let us start introducing the relative entropy

$$\mathcal{H}(f|\mathcal{M}_f) = \int_{\mathbb{R}^d} f \log\Big(\frac{f}{\mathcal{M}_f}\Big)\mathrm{d}v\,,$$

and the entropy production

$$\mathcal{D}(f) = \tfrac{1}{4}\int_{\mathbb{R}^{2d}}\int_{\mathbb{S}^{d-1}} \big(f' f'_* - f f_*\big)\log\Big(\frac{f' f'_*}{f f_*}\Big) B(u, \widehat{u}\cdot\sigma)\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v\,.$$

The function $\mathcal{M}_f$ is the thermodynamical equilibrium

$$\mathcal{M}_f(v) := \frac{\rho_f}{(2\pi\, T_f)^{\frac{d}{2}}} e^{-\frac{|v-\mu_f|^2}{2T_f}}\,,$$

where $\rho_f := \int f$ is the density, $\mu_f := \frac{1}{\rho}\int f v$ is the momentum, and $T_f := \frac{1}{d\rho}\int f|v-\mu|^2$ the temperature associated to f. Clearly, for the Boltzmann flow $\mathcal{M}_f = \mathcal{M}_{f_0}$ and

$$0 \leq \mathcal{H}(f|\mathcal{M}_f) \leq \mathcal{H}(f_0|\mathcal{M}_{f_0})\,.$$

In the sequel, and without loss of generality, we assume $\rho = 1$ and $\mu = 0$. We refer the reader to [33] for additional details and references.

5.1. **Dissipation of Entropy.**

Theorem 5.1. *Let the scattering kernel satisfy*

$$B(u, \widehat{u}\cdot\sigma) \geq K_B \min\left\{|u|^{\gamma}, |u|^{-\beta}\right\}, \quad \gamma \geq 0,\, \beta \geq 0,$$

and let $f \geq 0$ be a function with sufficiently high number of moments and entropic moments, and such that

$$f(v) \geq K_o\, e^{-A_o|v|^{q_o}}, \quad K_o > 0,\, A_o > 0,\, q_o \geq 2\,.$$

Then, for any $\varepsilon \in (0,1)$ we have:

(i) If $f \in L^p(\mathbb{R}^d)$, with $p \in (1,\infty]$,

$$\mathcal{D}(f) \geq A_{\varepsilon,p}(f)\mathcal{H}(f|\mathcal{M}_f)^{(1+\varepsilon)(1+\frac{\gamma p'}{d})},$$

where the constant $A_{\varepsilon,p}(f)$ is given in (65)*.*

(ii) If $f \in L\log L(\mathbb{R}^d)$, then

$$\mathcal{D}(f) \geq A_{\varepsilon,L\log L}(f)\,\mathcal{H}(f|\mathcal{M}_f)^{(1+\varepsilon)(1+\frac{\gamma}{d})}e^{-\frac{2\gamma\widetilde{K}_{L\log L}(f)}{dK_\varepsilon(f)\mathcal{H}(f|\mathcal{M}_f)^{1+\varepsilon}}}\,.$$

where the constant $A_{\varepsilon,L\log L}(f)$ is given in (69)*.*

Proof. Note that

$$\begin{aligned} B(u,\widehat{u}\cdot\sigma) &= \Big(K_B\,R^{\gamma}\,1_{|u|\leq R} + B(u,\widehat{u}\cdot\sigma)\Big) - K_B\,R^{\gamma}\,1_{|u|\leq R} \\ &\geq K_B\Big(R^{\gamma}\,1_{|u|\leq R} + \min\left\{|u|^{\gamma},|u|^{-\beta}\right\}\Big) - K_B\,R^{\gamma}\,1_{|u|\leq R} \\ &\geq K_B\,R^{\gamma}\langle u\rangle^{-\beta} - K_B\,R^{\gamma}\,1_{|u|\leq R}\,. \end{aligned} \tag{61}$$

As a consequence, $\mathcal{D}(f) \geq \mathcal{D}_1(f) - \mathcal{D}_2(f)$ where $\mathcal{D}_i(f)$, with $i \in \{1,2\}$, corresponds to each term in the right side of (61) respectively. Using [33, Theorem 3.1] one has

$$\mathcal{D}_1(f) \geq R^{\gamma}K_\varepsilon(f)\mathcal{H}(f|\mathcal{M}_f)^{1+\varepsilon}\,. \tag{62}$$

An explicit form for $K_\varepsilon(f)$ can be found in [33]. We just mention here that it depends on mass and temperature (energy) of f, the parameters A_o, K_o, q_o, and

$$\int_{\mathbb{R}^d} f(v)\langle v\rangle^{2+\frac{2+\beta}{\varepsilon}}\big|\log(f)\big|\mathrm{d}v\,, \qquad \int_{\mathbb{R}^d} f(v)\langle v\rangle^{2+q_o+\frac{2+\beta}{\varepsilon}}\mathrm{d}v\,.$$

For $\mathcal{D}_2(f)$ one can proceed in similar fashion to the proof of [33, Theorem 3.1] to obtain

$$\begin{aligned} \mathcal{D}_2(f) \leq 2K_B|\mathbb{S}^{d-1}|R^{\gamma}\Bigg(&4\int_{\{|u|\leq R\}} f\log(f)f_*\mathrm{d}v_*\mathrm{d}v \\ &+ 2^{\frac{q_o}{2}+1}\Big(\log\Big(\frac{1}{K_o}\Big)+A_o\Big)\int_{\{|u|\leq R\}}\langle v\rangle^{q_0}ff_*\mathrm{d}v_*\mathrm{d}v\Bigg)\,. \end{aligned} \tag{63}$$

Case $f\in L^p(\mathbb{R}^d)$*:* Since, by Hölders inequality,

$$\int_{\{|u|\le R\}} f(v)\mathrm{d}v \le \frac{1}{d}\big|\mathbb{S}^{d-1}\big|^{\frac{1}{p'}} R^{\frac{d}{p'}}\|f\|_{L^p(\mathbb{R}^d)}$$

one concludes from (63) that

$$\mathcal{D}_2(f)\le C K_B R^{\gamma+\frac{d}{p'}}\|f\|_{L^p(\mathbb{R}^d)}\Big(\int_{\mathbb{R}^d} f\big|\log(f)\big|\mathrm{d}v+\int_{\mathbb{R}^d} f\langle v\rangle^{q_o}\mathrm{d}v\Big) =: R^{\gamma+\frac{d}{p'}}\widetilde{K}_p(f)\,, \tag{64}$$

where $C>0$ is a universal constant. Gathering (62) and (64) it follows that

$$\mathcal{D}(f)\ge R^{\gamma}\Big(K_\varepsilon(f)\mathcal{H}(f|\mathcal{M}_f)^{1+\varepsilon}-R^{\frac{d}{p'}}\widetilde{K}_p(f)\Big).$$

Choosing $R^{\frac{d}{p'}}:=\frac{K_\varepsilon(f)}{2\widetilde{K}_p(f)}\mathcal{H}(f|\mathcal{M}_f)^{1+\varepsilon}$ it follows

$$\mathcal{D}(f)\ge \frac{K_\varepsilon(f)^{1+\frac{\gamma p'}{d}}}{2^{1+\frac{\gamma p'}{d}}\widetilde{K}_p(f)^{\frac{\gamma p'}{d}}}\mathcal{H}(f|\mathcal{M}_f)^{(1+\varepsilon)(1+\frac{\gamma p'}{d})} =: A_{\varepsilon,p}(f)\,\mathcal{H}(f|\mathcal{M}_f)^{(1+\varepsilon)(1+\frac{\gamma p'}{d})}\,. \tag{65}$$

Case $f\in L\log L(\mathbb{R}^d)$: Using the generalized Young's inequality, see for example the proof of [7, Proposition A.1], one concludes that

$$\int_{\{|u|\le R\}} f(v)\mathrm{d}v \le \frac{\int_{\mathbb{R}^d} f|\log(f)|\mathrm{d}v}{\mathcal{W}\Big(\frac{\int_{\mathbb{R}^d} f|\log(f)|\mathrm{d}v}{|\{|v|\le R\}|}\Big)}\,, \tag{66}$$

where $\mathcal{W}(x)$ is the Lambert function, that is, $\mathcal{W}^{-1}(x)=x\,e^{x}$. Therefore, recalling (63)

$$\begin{aligned}\mathcal{D}_2(f)&\le C K_B R^{\gamma}\Big(\int_{\mathbb{R}^d} f\big|\log(f)\big|\mathrm{d}v+\int_{\mathbb{R}^d} f\langle v\rangle^{q_o}\mathrm{d}v\Big)\frac{\int_{\mathbb{R}^d} f\big|\log(f)\big|\mathrm{d}v}{\mathcal{W}\Big(\frac{\int_{\mathbb{R}^d} f|\log(f)|\mathrm{d}v}{|\{|v|\le R\}|}\Big)}\\ &=: \frac{R^{\gamma}\,\widetilde{K}_{L\log L}(f)}{\mathcal{W}\Big(\frac{\int_{\mathbb{R}^d} f|\log(f)|\mathrm{d}v}{|\{|v|\le R\}|}\Big)}\,.\end{aligned} \tag{67}$$

Using (62) and (67)

$$\mathcal{D}(f)\ge R^{\gamma}\left(K_\varepsilon(f)\mathcal{H}(f|\mathcal{M}_f)^{1+\varepsilon}-\frac{\widetilde{K}_{L\log L}(f)}{\mathcal{W}\Big(\frac{\int_{\mathbb{R}^d} f|\log(f)|\mathrm{d}v}{|\{|v|\le R\}|}\Big)}\right). \tag{68}$$

Now choose, $R>0$ such that

$$\frac{\widetilde{K}_{L\log L}(f)}{\mathcal{W}\Big(\frac{\int_{\mathbb{R}^d} f|\log(f)|\mathrm{d}v}{|\{|v|\le R\}|}\Big)} = \frac{K_\varepsilon(f)}{2}\mathcal{H}(f|\mathcal{M}_f)^{1+\varepsilon}\,,$$

or more precisely,

$$R^d := \frac{\int_{\mathbb{R}^d} f|\log(f)|\mathrm{d}v}{2\widetilde{K}_{L\log L}(f)}\,K_\varepsilon(f)\mathcal{H}(f|\mathcal{M}_f)^{1+\varepsilon}e^{-\frac{2\tilde{K}_{L\log L}(f)}{K_\varepsilon(f)\mathcal{H}(f|\mathcal{M}_f)^{1+\varepsilon}}}$$

we obtain

$$
\begin{aligned}
\mathcal{D}(f) &\geq \frac{R^{\gamma}}{2} K_{\varepsilon}(f)\mathcal{H}(f|\mathcal{M}_f)^{1+\varepsilon} \\
&= \frac{K_{\varepsilon}(f)^{1+\frac{\gamma}{d}}}{2^{1+\frac{\gamma}{d}}} \left(\frac{d\int_{\mathbb{R}^d} f|\log(f)|\mathrm{d}v}{|\mathbb{S}^{d-1}|\widetilde{K}_{L\log L}(f)} \right)^{\frac{\gamma}{d}} \mathcal{H}(f|\mathcal{M}_f)^{(1+\varepsilon)(1+\frac{\gamma}{d})} e^{-\frac{2\gamma\widetilde{K}_{L\log L}(f)}{dK_{\varepsilon}(f)\mathcal{H}(f|\mathcal{M}_f)^{1+\varepsilon}}} \\
&=: A_{\varepsilon,L\log L}(f)\,\mathcal{H}(f|\mathcal{M}_f)^{(1+\varepsilon)(1+\frac{\gamma}{d})} e^{-\frac{2\gamma\widetilde{K}_{L\log L}(f)}{dK_{\varepsilon}(f)\mathcal{H}(f|\mathcal{M}_f)^{1+\varepsilon}}}\,.
\end{aligned} \tag{69}
$$

□

5.2. **Entropic moments and transitory relaxation.**

Proposition 5.1. *Assume f is solution of the homogeneous Boltzmann problem for hard potentials with initial data satisfying* (60). *Then, for any $t_o > 0$ and $s \in [0,\infty)$*

$$
\sup_{t\geq t_o} \int_{\mathbb{R}^d} f(v)\langle v\rangle^s |\log(f)| dv < \infty
$$

Proof. The case $s = 0$ is clear. Thus, multiply the equation by $\langle v\rangle^s|\log(f)|$, with $s > 0$, to obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} f(v)\langle v\rangle^s|\log(f)|\mathrm{d}v &= \int_{\mathbb{R}^d} Q(f,f)(v)\langle v\rangle^s|\log(f)|\mathrm{d}v \\
&\quad + \int_{\mathbb{R}^d} Q(f,f)(v)\langle v\rangle^s \mathrm{sgn}\big(\log(f)\big)\mathrm{d}v\,.
\end{aligned}
$$

For the second term in the right side readily follows that

$$
\Big|\int_{\mathbb{R}^d} Q(f,f)(v)\langle v\rangle^s \mathrm{sgn}\big(\log(f)\big)\mathrm{d}v\Big| \leq 2\|b\|_{L^1(\mathbb{S}^{d-1})}\|f\langle v\rangle^{\gamma+s}\|_{L^1(\mathbb{R}^d)}\|f\langle v\rangle^{\gamma+s}\|_{L^1(\mathbb{R}^d)}\,. \tag{70}
$$

The first term has a positive and negative parts. For the negative, one concludes using the entropy uniform boundedness that

$$
\int_{\mathbb{R}^d} Q^-(f,f)(v)\langle v\rangle^s|\log(f)|\mathrm{d}v \geq c_o \int_{\mathbb{R}^d} f(v)\langle v\rangle^{s+\gamma}|\log(f)|\mathrm{d}v\,, \tag{71}
$$

where $c_o > 0$ is a constant depending of the initial entropy, mass and energy. For the positive part, one has

(72)

$$
\begin{aligned}
\int_{\mathbb{R}^d} Q^+(f,f)(v)\langle v\rangle^s|\log(f)|\mathrm{d}v &= \int_{\mathbb{R}^{2d}} f(v)f(v_*)|u|^{\gamma}\int_{\mathbb{S}^{d-1}}\big|\log(f(v'))\big|\langle v'\rangle^s b(\widehat{u}\cdot\sigma)\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v \\
&\leq C_s \int_{\mathbb{R}^{2d}}\sum_{(i,j)} f(v)\langle v\rangle^i f(v_*)\langle v_*\rangle^j \int_{\mathbb{S}^{d-1}}\big|\log(f(v'))\big|b(\widehat{u}\cdot\sigma)\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v\,.
\end{aligned}
$$

The sum is performed on $(i,j) \in \{(s+\gamma,0),(s,\gamma),(\gamma,s),(0,s+\gamma)\}$. We estimate the right side in (72) controlling the integral in the integration sets $\{f' \leq 1\}$ and $\{f' > 1\}$ separately.

For the former recall the classical result proved in [32]: for any $t_o > 0$ there exists positive K_o, A_o depending only on the initial mass, energy, entropy and t_o such that

$$f(t,v) \geq K_o e^{-A_o|v|^2}\,, \qquad t \geq t_o > 0\,.$$

As a consequence,

$$\int_{\mathbb{S}^{d-1}} \big|\log(f(v'))\big| 1_{f'\leq 1} b(\widehat{u}\cdot\sigma)\mathrm{d}\sigma \leq \|b\|_{L^1(\mathbb{S}^{d-1})}\Big(\log\Big(\frac{1}{K_o}\Big) + A_o\Big)\langle v\rangle^2\langle v_*\rangle^2\,. \tag{73}$$

The latter set $\{f' > 1\}$ is trickier. We concentrate first in the combination $(i,j) = (0, s+\gamma)$ since the other follow a simpler argument. First, we fix $\varepsilon > 0$ and use the usual angular split (3). In each component we use the generalized Young's inequality

$$xy \leq x\log x - x + e^y\,, \qquad x \geq 0,\ y \in \mathbb{R}\,,$$

in slightly, but crucially, different way. For the good part, the one with $b_1^{\varepsilon}\big(\cos\theta\big)$, we choose $x = f(v)$ and $y = |\log(f')|1_{f'>1}$ to get

$$f\big|\log(f')\big|1_{f'>1} \leq f\log(f) - f + f'\,.$$

Thus,

$$\begin{aligned} &\int_{\mathbb{R}^{2d}} f(v_*)\langle v_*\rangle^{s+\gamma} f(v)\int_{\mathbb{S}^{d-1}}\big|\log(f(v'))\big|1_{f'>1} b_1^{\varepsilon}(\widehat{u}\cdot\sigma)\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v \\ &\qquad\leq \|b_1^{\varepsilon}\|_{L^1(\mathbb{S}^{d-1})}\|f\langle v\rangle^{s+\gamma}\|_{L^1(\mathbb{R}^d)}\|f\log(f)\|_{L^1(\mathbb{R}^3)} \\ &\qquad\quad + 2^d\Big\|\frac{b_1^{\varepsilon}(\cos(\theta))}{1-\cos(\theta)}\Big\|_{L^1(\mathbb{S}^{d-1})}\|f\langle v\rangle^{s+\gamma}\|_{L^1(\mathbb{R}^d)}\|f\|_{L^1(\mathbb{R}^d)}, \end{aligned} \tag{74}$$

where we have use the singular change of variable $v \to v'$ in the integral containing f'. For the bad part, the one with $b_2^{\varepsilon}\big(\cos\theta\big)$, we choose $x = \langle v_*\rangle^{s+\gamma} f(v_*)$ and $y = |\log(f')|1_{f'>1}$ to get

$$\langle v_*\rangle^{s+\gamma} f_*\big|\log(f')\big|1_{f'>1} \leq \langle v_*\rangle^{s+\gamma} f_*\log\big(\langle v_*\rangle^{s+\gamma} f_*\big) - \langle v_*\rangle^{s+\gamma} f_* + f'\,.$$

Therefore,

$$\begin{aligned} &\int_{\mathbb{R}^{2d}} f(v_*)\langle v_*\rangle^{s+\gamma} f(v)\int_{\mathbb{S}^{d-1}}\big|\log(f(v'))\big|1_{f'>1} b_2^{\varepsilon}(\widehat{u}\cdot\sigma)\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v \\ &\qquad\leq \|b_2^{\varepsilon}\|_{L^1(\mathbb{S}^{d-1})}\Big(\|f\langle v\rangle^{s+\gamma}\log(f)\|_{L^1(\mathbb{R}^d)}\|f\|_{L^1(\mathbb{R}^3)} + \\ &(s+\gamma)\|f\langle v\rangle^{s+\gamma}\log\langle v\rangle\|_{L^1(\mathbb{R}^d)}\|f\|_{L^1(\mathbb{R}^3)}\Big) + 2^d\Big\|\frac{b_2^{\varepsilon}(\cos(\theta))}{1+\cos(\theta)}\Big\|_{L^1(\mathbb{S}^{d-1})}\|f\|^2_{L^1(\mathbb{R}^d)}, \end{aligned} \tag{75}$$

where we used the singular change of variable $v_* \to v'$ in the integral containing f'. Of course, in this case it is harmless since $b(\cdot)$ is supported in $[0,1]$. Furthermore, we recall that $\|b_2^{\varepsilon}\|_{L^1(\mathbb{S}^{d-1})} \sim \mathfrak{m}(b_2^{\varepsilon})$ can be made as small as desired.

For the rest of the cases one does not need to split the kernel in two. It suffices to choose $x = f(v_*)\langle v_*\rangle^s$ when $(i,j) = (\gamma, s)$, $x = f(v_*)\langle v_*\rangle^{\gamma}$ when $(i,j) = (s,\gamma)$, and $x = f(v_*)$

when $(i,j)=(s+\gamma,0)$. In all cases $y=|\log(f')|1_{f'>1}$. Furthermore, the resulting lower order entropic moments can be controlled as

$$\|f\log(f)\langle v\rangle^{\gamma}\|_{L^1(\mathbb{R}^d)} \le \|f\log(f)\langle v\rangle^{s+\gamma}\|^{\frac{\gamma}{s+\gamma}}_{L^1(\mathbb{R}^d)}\|f\log(f)\|^{\frac{s}{s+\gamma}}_{L^1(\mathbb{R}^d)}\,. \tag{76}$$

In summary, after gathering (72), (73), (74), (75) and (76) one concludes

$$\begin{aligned}\int_{\mathbb{R}^d} Q^+(f,f)(v)\langle v\rangle^s|\log(f)|\mathrm{d}v \le \mathfrak{m}(b_2^{\varepsilon})\|f\log(f)\langle v\rangle^{s+\gamma}\|_{L^1(\mathbb{R}^d)}+\\ C_1(f)\|f\log(f)\langle v\rangle^{s+\gamma}\|^{\frac{\gamma}{s+\gamma}}_{L^1(\mathbb{R}^d)}+C_2(f)\|f\log(f)\langle v\rangle^{s+\gamma}\|^{\frac{s}{s+\gamma}}_{L^1(\mathbb{R}^d)}+C_3^{\varepsilon}(f)\,,\end{aligned} \tag{77}$$

where the constants $C_i(f)$, with $i=1,2,3$, depend on the mass, temperature, initial entropy and $\|f\langle v\rangle^{s+\gamma+2}\|_{L^1(\mathbb{R}^d)}$. Now, choose ε such that $\mathfrak{m}(b_2^{\varepsilon})\le\frac{c_o}{2}$ and use the estimates (70), (71) and (72) to get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|f\log(f)\langle v\rangle^{s}\|_{L^1(\mathbb{R}^d)} &\le C_1(f)\|f\log(f)\langle v\rangle^{s+\gamma}\|^{\frac{\gamma}{s+\gamma}}_{L^1(\mathbb{R}^d)}\\ &+C_2(f)\|f\log(f)\langle v\rangle^{s+\gamma}\|^{\frac{s}{s+\gamma}}_{L^1(\mathbb{R}^d)}+C_3^{\varepsilon}(f)-\frac{c_o}{2}\|f\log(f)\langle v\rangle^{s+\gamma}\|_{L^1(\mathbb{R}^d)}\,.\end{aligned}$$

The result follows from here after invoking the instantaneous appearance of moments proven in [37]. □

Proposition 5.2. *Assume f is solution of the homogeneous Boltzmann problem for hard potentials with initial data satisfying* (60)*. Then, for any $\varepsilon>0$*

$$\mathcal{H}(f|\mathcal{M}_f)\le C_{\varepsilon}(f_0)\big(\ln(e+t)\big)^{-\frac{1}{1+\varepsilon}}\,,$$

with constant $C_{\varepsilon}(f_0)$ depending on mass, temperature, and entropy of f_0. Furthermore, if additionally $f_0\in L^p(\mathbb{R}^d)$, with $p\in(1,\infty]$

$$\mathcal{H}(f|\mathcal{M}_f)\le C_{\varepsilon}(f_0)(1+t)^{-\frac{1}{\frac{\gamma p'}{d}+\varepsilon\left(1+\frac{\gamma p'}{d}\right)}}\,,$$

with constant $C_{\varepsilon}(f_0)$ depending additionally of the L^p-norm of f_0.

Proof. Let us prove the first statement. Using the appearance of moments [37] and entropic moments Proposition 5.1, there exist constants $\mathcal{A}_o(f_0)$ and $\mathcal{B}_o(f_0)$ only depending on initial mass, temperature, entropy, and $t_o>0$ (and $\varepsilon>0$) such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(f|\mathcal{M}_f)+\mathcal{A}_o(f_0)\mathcal{H}(f|\mathcal{M}_f)^{(1+\varepsilon)(1+\frac{\gamma}{d})}e^{-\frac{\mathcal{B}_o(f_0)}{\mathcal{H}(f|\mathcal{M}_f)^{1+\varepsilon}}}\le 0\,,\qquad t\ge t_o\,.$$

It is not difficult to check that $X(t)=C\big(\ln(e+t)\big)^{-\frac{1}{1+\varepsilon}}$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}X+\mathcal{A}_o(f_0)X^{(1+\varepsilon)(1+\frac{\gamma}{d})}e^{-\frac{\mathcal{B}_o(f_0)}{X^{1+\varepsilon}}}\ge 0\,,$$

provided $C>0$ is taken large enough depending on $\mathcal{A}_o(f_0)$ and $\mathcal{B}_o(f_0)$. Denote any such value as C_* and choose $C=C(f_0):=\max\{\mathcal{H}(f_0|\mathcal{M}_{f_0}),C_*\}$. Since $\mathcal{H}(f(t_o)|\mathcal{M}_{f(t_o)})\le\mathcal{H}(f_0|\mathcal{M}_{f_0})$ the result follows by a comparison principle. The second statement follows similar argument and it is left to the reader. □

5.3. **Exponential convergence.** After fixing the initial mass, momentum, and temperature, one can rewrite the Boltzmann equation (1) by taking $f = \mathcal{M}_{f_0} + h$, where h is understood as a perturbation of the thermodynamical equilibrium. The equation for h reads

$$\partial_t h(v) = \mathcal{L}(h)(v) + Q(h,h)(v)\,, \qquad (t,v) \in \mathbb{R}^+ \times \mathbb{R}^d\,. \tag{78}$$

Here, the linear component of the dynamics is generated by the operator

$$\mathcal{L}(h)(v) = Q(\mathcal{M}_{f_0}, h) + Q(h, \mathcal{M}_{f_0})\,. \tag{79}$$

This operator was shown to be self-adjoint non-positive with a spectral gap in $L^2(\mathcal{M}_{f_0}^{-1/2})$ in the references [17], [21] and [22] in the grad cut-off case. Later, allowing more general kernels, an explicit estimate of the spectral gap in this same space was made in [29]. This is the starting point at which the spectral enlargement technique works [28, 24] to obtain spectral gap in more general spaces.

Based on [24, Theorem 2.3] or [16, Theorem 3.1], if an spectral enlargement from $L^2(\mathcal{M}_{f_0}^{-1/2})$ to L^1_k is desired, we need to decompose the linearized operator $\mathcal{L}$ in two operators with suitable properties. More precisely, $\mathcal{L} = \mathcal{A} + \mathcal{B}$ where $\mathcal{B} : L^1_k(\mathbb{R}^d) \to L^1_k(\mathbb{R}^d)$ is dissipative [2] and $\mathcal{A} : L^1_k(\mathbb{R}^d) \to L^2(\mathcal{M}_{f_0}^{-1/2};\mathbb{R}^d)$ is bounded. Here we stress that the space $L^2(\mathcal{M}_{f_0}^{-1/2};\mathbb{R}^d)$ is taken as baseline space since a detailed quantification of the spectrum is available in the aforementioned references.

The decomposition is based on truncation of small and large velocities, and glancing angles (similar but simpler to the decomposition given in [24, subsection 4.3.3] or [28]). For the scattering kernel, recall the decomposition we have used along the document

$$b\big(\cos(\theta)\big) = b\big(\cos(\theta)\big)\big(1_{|\sin(\theta)|\geq\varepsilon} + 1_{|\sin(\theta)|<\varepsilon}\big) =: b_1^{\varepsilon}\big(\cos(\theta)\big) + b_2^{\varepsilon}\big(\cos(\theta)\big)\,. \tag{80}$$

For the kinetic potential write $|\cdot|^\gamma =: \Phi_1 + \Phi_2$, where

$$\Phi_1(|u|) := |u|^\gamma 1_{\delta\leq|u|\leq\delta^{-1}}\,, \qquad \Phi_2(|u|) := |u|^\gamma\big(1 - 1_{\delta\leq|u|\leq\delta^{-1}}\big)\,. \tag{81}$$

With the notation $\mathcal{L}_{\Phi,b}$ to express the dependence of the collision kernel, one can write

$$\begin{aligned} \mathcal{L}_{x^\gamma,b} h &= \mathcal{L}_{x^\gamma,b_1} + \mathcal{L}_{x^\gamma,b_2} = \mathcal{L}^o_{x^\gamma,b_1} + \mathcal{L}_{x^\gamma,b_2} - h\int_{\mathbb{R}^d}\mathcal{M}_{f_0}(v_*)|u|^\gamma \mathrm{d}v_* \\ &= \mathcal{L}^o_{\Phi_1,b_1} + \Big(\mathcal{L}^o_{\Phi_2,b_1} + \mathcal{L}_{x^\gamma,b_2} - h\int_{\mathbb{R}^d}\mathcal{M}_{f_0}(v_*)|u|^\gamma \mathrm{d}v_*\Big) =: \mathcal{A}_{\delta,\varepsilon} + \mathcal{B}_{\delta,\varepsilon}\,. \end{aligned} \tag{82}$$

Of course,

$$\mathcal{L}^o_{\Phi_1,b_1} := Q^+_{\Phi_1,b_1}(\mathcal{M}_{f_0}, h) + Q^+_{\Phi_1,b_1}(h, \mathcal{M}_{f_0}) - Q^-_{\Phi_1,b_1}(h, \mathcal{M}_{f_0})\,. \tag{83}$$

Let us prove that $\mathcal{B}_{\delta,\varepsilon}$ is dissipative for sufficiently small parameters $\delta > 0, \varepsilon > 0$ and that $\mathcal{A}_{\delta,\varepsilon}$ has the stated "regularizing" property for any $\delta > 0, \varepsilon > 0$. For the first statement, one essentially needs the following lemma controlling moments of the linearized operator.

[2] That is, the operator $\mathcal{B}$ is closed with domain $L^1_{k+\gamma}(\mathbb{R}^d)$ and satisfying $\langle \mathcal{B}f, f\rangle \leq 0$.

Lemma 5.1. *Consider angular kernel* $b \in L^1(\mathbb{S}^d)$ *and potential* $0 \leq \Phi(|u|) \leq |u|^\gamma$. *For any* $h \in L^1_{k+\gamma}(\mathbb{R}^d)$, *it follows for any* $k \geq 2$

$$\int_{\mathbb{R}^d} sgn(h)\big(1+|v|^k\big)\mathcal{L}_{\Phi,b}(h)(v)dv \leq C_k\big\|b\|_{L^1(\mathbb{S}^{d-1})}\|\langle v\rangle^{k-1}\beta_\Phi(v)h\|_{L^1(\mathbb{R}^d)} - (1-\gamma_k)\int_{\mathbb{R}^{2d}}|h(v)|\mathcal{M}_{f_0}(v_*)\Phi(|u|)\big(|v|^k+|v_*|^k\big)dv_*dv\,.$$

Here above, $0<\gamma_k<\big\|b\|_{L^1(\mathbb{S}^{d-1})}=1$ *is such that* $\gamma_k \searrow 0$ *as* $k\to\infty$. *And,*

$$\beta_\phi(v) := \int_{\mathbb{R}^d}\mathcal{M}_{f_0}(v_*)\langle v_*\rangle^k\Phi(|u|)dv_*\,.$$

Proof. Using the weak representation

$$\int_{\mathbb{R}^d}\varphi\,\mathcal{L}_{\Phi,b}h(v)\mathrm{d}v = \int_{\mathbb{R}^{2d}}\int_{\mathbb{S}^{d-1}}\mathcal{M}_{f_0}(v_*)h\big(\varphi'+\varphi'_*-\varphi_*-\varphi\big)\Phi(|u|)b(\widehat{u}\cdot\sigma)\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v\,.$$

Thus, for $\varphi(v)=\mathrm{sgn}(h)\big(1+|v|^k\big)$ one readily checks that

$$\int_{\mathbb{R}^d}\mathrm{sgn}(h)\big(1+|v|^k\big)\mathcal{L}_{\Phi,b}h(v)\mathrm{d}v \leq \int_{\mathbb{R}^d}|v|^k\mathcal{L}_{\Phi,b}|h|(v)\mathrm{d}v + 2\big\|b\|_{L^1(\mathbb{S}^{d-1})}\|\beta_\Phi(v)h\|_{L^1}\,.$$

The result follows using a Povzner angular averaging lemma in the first term of the right side, see for example [6, Lemma 2.6]. □

Lemma 5.2. *Consider angular kernel* $b \in L^1(\mathbb{S}^d)$ *and potentials with* $\gamma\in(0,1]$. *For* $\varepsilon\in(0,1)$, $\delta\in(0,1)$ *and* $k\geq 2$, *it follows that*

$$\int_{\mathbb{R}^d} sgn(h)\big(1+|v|^k\big)\mathcal{B}_{\delta,\varepsilon}(h)\,dv = C_k\mathfrak{m}(b_2^\varepsilon)\,\|h\langle v\rangle^{k+\gamma}\|_{L^1(\mathbb{R}^d)} + C_k\,\delta^\gamma\,\|h\langle v\rangle^{k+\gamma}\|_{L^1(\mathbb{R}^d)} - c_o\|(1+|v|^k)\langle v\rangle^\gamma h\|_{L^1(\mathbb{R}^d)}\,.$$

The constant $C_k>0$ *in addition to* k, *depend on mass and temperature. The constant* c_o *depends only on mass and temperature.*

Proof. First, let us estimate $\mathcal{L}_{x^\gamma,b_2}$ and $\mathcal{L}^o_{\Phi_2,b_1}$ separately. For the former, using Lemma 5.1 on readily concludes

$$\int_{\mathbb{R}^d} sgn(h)\big(1+|v|^k\big)\mathcal{L}_{x^\gamma,b_2^\varepsilon}(h)(v)dv \leq C_k\big\|b_2^\varepsilon\|_{L^1(\mathbb{S}^{d-1})}\|\langle v\rangle^{k-1}\beta_{x^\gamma}(v)h\|_{L^1(\mathbb{R}^d)} \leq C_k\mathfrak{m}(b_2^\varepsilon)\|\langle v\rangle^{k+\gamma-1}h\|_{L^1(\mathbb{R}^d)}\,. \tag{84}$$

We used, for the second inequality, that $\beta_{x^\gamma}(v)\leq\widetilde{C}_k\langle v\rangle^\gamma$ and $\|b_2^\varepsilon\|_{L^1(\mathbb{S}^{d-1})}\sim\mathfrak{m}(b_2^\varepsilon)$. Similarly for the latter, note that $\mathcal{L}^o_{\Phi_2,b_1^\varepsilon}=\mathcal{L}_{\Phi_2,b_1^\varepsilon}+Q^-_{\Phi_2,b_1^\varepsilon}(h,\mathcal{M}_{f_0})$, thus, using Lemma 5.1 it follows that

$$\int_{\mathbb{R}^d} sgn(h)\big(1+|v|^k\big)\mathcal{L}_{\Phi_2,b_1^\varepsilon}(h)(v)dv \leq C_k\big\|b_2^\varepsilon\|_{L^1(\mathbb{S}^{d-1})}\|\langle v\rangle^{k-1}\beta_{\Phi_2}(v)h\|_{L^1(\mathbb{R}^d)} \leq C_k\,\delta^\gamma\,\|b\|_{L^1(\mathbb{S}^{d-1})}\|\langle v\rangle^{k+\gamma}h\|_{L^1(\mathbb{R}^d)}\,. \tag{85}$$

We used, in the second inequality, the fact that $b_2^\varepsilon \le b$ and that $\beta_{\Phi_2}(v) \le \widetilde{C}_k \delta^\gamma \langle v\rangle^{1+\gamma}$. Second, for the dissipation term, one uses that

$$\int_{\mathbb{R}^d} \mathcal{M}_{f_0}(v_*)|u|^\gamma \mathrm{d}v_* \ge c_o\langle v\rangle^\gamma \,, \tag{86}$$

for constant c_o depending only on mass and temperature. Then,

$$\int_{\mathbb{R}^d} \big(1+|v|^k\big)|h| \int_{\mathbb{R}^d} \mathcal{M}_{f_0}(v_*)|u|^\gamma \mathrm{d}v_* \ge c_o\|(1+|v|^k)\langle v\rangle^\gamma h\|_{L^1(\mathbb{R}^d)} \,. \tag{87}$$

The result follows after gathering (84), (85), (87). □

Corollary 5.1. *There exists positive $(\delta_k, \varepsilon_k)$ depending on mass and temperature such that $\mathcal{B}_{\delta,\varepsilon}$ is dissipative in $L^1_k(\mathbb{R}^d)$ for any $\delta \in (0,\delta_k)$ and $\varepsilon \in (0,\varepsilon_k)$. Furthermore, for any $c_o^- < c_o$, given in* (86)*, it is possible to find (δ,ε) in such ranges and such that*

$$\|e^{t\mathcal{B}_{\delta,\varepsilon}}\|_{L^1_k(\mathbb{R}^d)} \le 2^{\frac{k}{2}-1} e^{-c_o^- t}. \tag{88}$$

Proof. Consider the problem $h' = \mathcal{B}_{\delta,\epsilon}(h)$ in $L^1_k(\mathbb{R}^d)$. Using Lemma 5.2 and the elementary inequality $\langle v\rangle^k \le 2^{\frac{k}{2}-1}(1+|v|^k)$, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(1+|v|^k)h\|_{L^1(\mathbb{R}^d)} + c_o(1-\widetilde{\delta}-\widetilde{\varepsilon})\|(1+|v|^k)h\|_{L^1(\mathbb{R}^d)} \le 0\,, \qquad 0<\widetilde{\delta}+\widetilde{\varepsilon}<1\,,$$

provided we choose $\delta >$ and $\varepsilon > 0$ such that $C_k 2^{\frac{k}{2}-1}\delta^\gamma \le c_o\widetilde{\delta}$ and $C_k 2^{\frac{k}{2}-1}\mathfrak{m}(b_2^\varepsilon) \le c_o\widetilde{\varepsilon}$. Here $\widetilde{\delta}+\widetilde{\varepsilon}$ is allowed to be as small as desired. As a consequence, choosing $(\widetilde{\delta},\widetilde{\varepsilon})$ such that $c_o^- = c_o(1-\widetilde{\delta}-\widetilde{\varepsilon})$ it follows

$$\begin{aligned} 2^{1-\frac{k}{2}}\|\langle v\rangle^k h\|_{L^1(\mathbb{R}^d)} &\le \|(1+|v|^k)h\|_{L^1(\mathbb{R}^d)} \\ &\le \|(1+|v|^k)h_0\|_{L^1(\mathbb{R}^d)} e^{-c_o^- t} \le \|\langle v\rangle^k h_0\|_{L^1(\mathbb{R}^d)} e^{-c_o^- t}\,, \qquad t>0\,. \end{aligned}$$

This is exactly (88). □

Lemma 5.3. *For any $\delta \in (0,1)$ and $\varepsilon \in (0,1)$ and $k \ge 2$, the operator $\mathcal{A}_{\delta,\varepsilon} : L^1_k(\mathbb{R}^d) \to L^2(\mathcal{M}_{f_0}^{-1/2})$ is bounded.*

Proof. Let us estimate each term separately in $\mathcal{A}_{\delta,\varepsilon} = \mathcal{L}^o_{\Phi_1,b_1}$, recall (83). Is direct that the last term is controlled by

$$\|\mathcal{M}_{f_0}^{-1/2} Q^-_{\Phi_1,b_1}(h,\mathcal{M}_{f_0})\|_{L^2(\mathbb{R}^d)} \le \|b\|_{L^1(\mathbb{S}^{d-1})}\|\langle v\rangle^\gamma \mathcal{M}_{f_0}^{1/2}\|_{L^2(\mathbb{R}^d)}\|\langle v\rangle^\gamma h\|_{L^1(\mathbb{R}^d)}\,. \tag{89}$$

For the first term, we use the elementary inequality

$$\mathcal{M}_{f_0}^{-1/2}(v) \le \mathcal{M}_{f_0}^{-3/4}(v'_*)\mathcal{M}_{f_0}^{-3/2}(u)\,,$$

to discover that

$$\mathcal{M}_{f_0}^{-1/2}(v) Q^+_{\Phi_1,b_1}\big(\mathcal{M}_{f_0},|h|\big)(v) \le \|\mathcal{M}_{f_0}^{-3/2}(u)\Phi_1\|_{L^\infty(\mathbb{R}^d)} Q^+_{1,b_1}\big(\mathcal{M}_{f_0}^{1/4},|h|\big)(v)\,.$$

Thus, using Young's inequality for the Q^+,

$$\begin{aligned} &\|\mathcal{M}_{f_0}^{-1/2} Q^+_{\Phi_1,b_1}\big(\mathcal{M}_{f_0},|h|\big)\|_{L^2(\mathbb{R}^d)} \\ &\qquad \le C(b_1)\|\mathcal{M}_{f_0}^{-3/2}(u)\Phi_1\|_{L^\infty(\mathbb{R}^d)}\|\mathcal{M}_{f_0}^{1/4}\|_{L^2(\mathbb{R}^d)}\|h\|_{L^1(\mathbb{R}^d)}\,. \end{aligned} \tag{90}$$

Since, b_1 is cut-off near zero angle, we also have

$$\begin{aligned} &\|\mathcal{M}_{f_0}^{-1/2} Q^+_{\Phi_1,b_1}\big(|h|,\mathcal{M}_{f_0}\big)\|_{L^2(\mathbb{R}^d)} \\ &\qquad \le \widetilde{C}(b_1)\|\mathcal{M}_{f_0}^{-3/2}(u)\Phi_1\|_{L^\infty(\mathbb{R}^d)}\|\mathcal{M}_{f_0}^{1/4}\|_{L^2(\mathbb{R}^d)}\|h\|_{L^1(\mathbb{R}^d)}\,. \end{aligned} \tag{91}$$

The result follows, after gathering (89), (90) and (91). □

Proposition 5.3. *Let $b \in L^1(\mathbb{S}^{d-1})$ and $\gamma \in (0,1]$ and $k \ge 2$. The linear operator $\mathcal{L}$ defined in $L^1_k(\mathbb{R}^d)$ satisfies*

$$Spec(\mathcal{L}) \subset \{z \in \mathbb{C}\big|\Re e(z) \le -\lambda\} \cup \{0\}\,,$$
$$Ker(\mathcal{L}) = Span\{\mathcal{M}_{f_0}, v_1\mathcal{M}_{f_0}, \cdots, v_d\mathcal{M}_{f_0}, |v^2|\mathcal{M}_{f_0}\}\,,$$

for any $\lambda < \lambda_o$ where λ_o is the spectral gap of the restriction of $\mathcal{L}$ to $L^2(\mathcal{M}_{f_0}^{-1/2};\mathbb{R}^d)$. Furthermore, it generates a a strongly continuous semigroup $e^{t\,\mathcal{L}}$ which satisfies

$$\|e^{t\mathcal{L}}h_0 - \pi h_0\|_{L^1_k(\mathcal{R}^d)} \le C_k e^{-\lambda\,t}\|h_0 - \pi h_0\|_{L^1_k(\mathcal{R}^d)}\,.$$

Here π stands for the projection onto $Ker(\mathcal{L})$ defined as

$$\pi g := \sum_{\varphi\in\{1,v_1,\cdots,v_d,|v|^2\}}\left(\int_{\mathbb{R}^d} g\,\varphi\,dv\right)\varphi\,\mathcal{M}_{f_0}\,.$$

Proof. This is a direct consequence of [24, Theorem 2.1] after taking the spaces

$$E := L^2(\mathcal{M}_{f_0}^{-1/2};\mathbb{R}^d) \subset L^1_k(\mathbb{R}^d) =: \mathcal{E}\,.$$

□

Finally, we arrive to the exponential convergence of the full homogeneous Boltzmann equation (78). Let $f := f(t,v)$ be solution of the Boltzmann equation, $\mathcal{M}_{f_0}$ its thermodynamical equilibrium, and set also the perturbation $f := \mathcal{M}_{f_0} + h$. The first step is to invoke Csiszár-Kullback-Pinsker inequality to deduce that

$$\|f - \mathcal{M}_{f_0}\|_{L^1(\mathbb{R}^d)} \le \sqrt{2\,\mathcal{H}(f|\mathcal{M}_{f_0})}\,.$$

As a consequence, using Proposition 5.2 and the property of creation of moments it follows for any $k \ge 0$

$$\begin{aligned} \|f - \mathcal{M}_{f_0}\|_{L^1_k(\mathbb{R}^d)} &\le \|f + \mathcal{M}_{f_0}\|^{\frac12}_{L^1_{2k}(\mathbb{R}^d)}\|f - \mathcal{M}_{f_0}\|^{\frac12}_{L^1(\mathbb{R}^d)} \\ &\le C_k(f_0)\|f - \mathcal{M}_{f_0}\|^{\frac12}_{L^1(\mathbb{R}^d)} \le C_k(f_0)\big(\ln(e+t)\big)^{-\frac13}\,, \qquad t \ge 1\,. \end{aligned}$$

Here, the constant $C_k(f_0) > 0$ only depends on the initial datum through its mass, temperature and entropy. In this way, for every $\varepsilon \in (0,1)$ there exists time $t_k(f_0)$ such that

$$\|f - \mathcal{M}_{f_0}\|_{L^1_{k+2\gamma}(\mathbb{R}^d)} \leq \varepsilon\,, \qquad t \geq t_k(f_0)\,. \tag{92}$$

Define $\tilde{h}(t) := h(t + t_k(f_0))$. Then, from equation (78), it follows that such perturbation satisfies

$$\tilde{h}(t) = e^{t\mathcal{L}}\tilde{h}_0 + \int_0^t e^{(t-s)\mathcal{L}} Q(\tilde{h},\tilde{h})(s)\mathrm{d}s\,, \qquad t \geq 0\,.$$

Since $\pi\tilde{h}_0 = 0$ and $\pi Q(\tilde{h},\tilde{h})(s) = 0$ for every $s \geq 0$, it follows from Proposition 5.3 that

$$\begin{aligned}\|\tilde{h}(t)\|_{L^1_k(\mathbb{R}^d)} &\leq \|e^{t\mathcal{L}}\tilde{h}_0\|_{L^1_k(\mathbb{R}^d)} + \int_0^t \|e^{(t-s)\mathcal{L}}Q(\tilde{h},\tilde{h})(s)\|_{L^1_k(\mathbb{R}^d)}\mathrm{d}s \\ &\leq e^{-\lambda t}\|\tilde{h}_0\|_{L^1_k(\mathbb{R}^d)} + \int_0^t e^{-\lambda(t-s)}\|Q(\tilde{h},\tilde{h})(s)\|_{L^1_k(\mathbb{R}^d)}\mathrm{d}s\,.\end{aligned} \tag{93}$$

Furthermore, using (92) we have

$$\|Q(\tilde{h},\tilde{h})(s)\|_{L^1_k(\mathbb{R}^d)} \leq \|\tilde{h}(s)\|^2_{L^1_{k+\gamma}(\mathbb{R}^d)} \leq \|\tilde{h}(s)\|_{L^1_k(\mathbb{R}^d)}\|\tilde{h}(s)\|_{L^1_{k+2\gamma}(\mathbb{R}^d)} \leq \varepsilon\|\tilde{h}(s)\|_{L^1_k(\mathbb{R}^d)}\,.$$

As a consequence, setting $X(t) := e^{\lambda t}\|\tilde{h}(t)\|_{L^1_k(\mathbb{R}^d)}$, we obtain from (93)

$$X(t) \leq X_0 + \varepsilon\int_0^t X(s)\mathrm{d}s\,, \qquad t \geq 0\,.$$

Using Gronwall's lemma one concludes that $X(t) \leq X_0\, e^{\varepsilon t}$, or equivalently,

$$\|\tilde{h}(t)\|_{L^1_k(\mathbb{R}^d)} \leq \|\tilde{h}_0\|_{L^1_k(\mathbb{R}^d)} e^{-(\lambda-\varepsilon)t}\,, \qquad t \geq 0\,.$$

We have proven the main result of this section

Theorem 5.2. *Let the angular kernel satisfy* (59) *and potential* $\gamma \in (0,1]$. *Assume the initial datum* f_0 *satisfies* (60). *Then, for every* $\lambda < \lambda_o$, *there exists time* $t_k(f_0)$ *and* $C_k(f_0)$ *depending on the initial datum through its mass, temperature and entropy, such that*

$$\|f - \mathcal{M}_{f_0}\|_{L^1_k(\mathbb{R}^d)} \leq C_k(f_0)e^{-\lambda t}\,, \qquad t \geq t_k(f_0)\,.$$

Here $\lambda_o > 0$ *is the spectral gap of* $\mathcal{L}$ *in* $L^2(\mathcal{M}_{f_0}^{-1/2};\mathbb{R}^d)$.

5.4. **Decomposition theorem.** We have now the tools to prove the decomposition theorem. We present here a generalization of [30, Theorem 5.5] for true Grad-cutoff kernel.

Theorem 5.3. *Let the angular kernel satisfy* (59) *and potential* $\gamma \in (0,1]$. *Assume the initial datum* f_0 *satisfies* (60). *Additionally, take* $f_0 \in L^2(\mathbb{R}^d)$. *For any* $s \geq 0$ *and* $t_o > 0$ *there exist functions* $f^S \geq 0$ *and* f^R *such that*

$$f = f^S + f^R\,, \qquad t \geq t_o > 0\,,$$

satisfying the estimates

$$\sup_{t\geq t_o}\|f^S(t)\|_{H^s_k} < C_{s,k}(t_o)\|f_0\|_{L^2}\,, \qquad \|f^R(t)\|_{L^1_k} \leq C_k(t_o)e^{-\lambda t}\,, \quad \lambda < \lambda_o\,.$$

The constants depend on mass and energy of f_0 as well.

Proof. Assume $s \in \mathbb{N}$ (otherwise, we may consider $\lceil s \rceil$). We use induction on the regularity parameter $l \geq 0$ with $l \in \{0, 1, \cdots, s\}$. For $l = 0$, the theorem follows with $f^{S_0} := f$ and $f^{R_0} := 0$ due to the creation of the L^2_k-norm (analog of Theorem 3.2 for polynomial weight). Assume now that the theorem holds for l with $1 \leq l \leq s-1$. Thus, we can write $f = f^{S_l} + f^{R_l}$ having the stated properties. In turn, we can split the operator as

$$
\begin{aligned}
Q(f,f) &= Q^+(f,f) - Q^-(f,f) \\
&= Q^+(\mathcal{M}_{f_0}, f) + Q^+(f - \mathcal{M}_{f_0}, f) - Q^-(f, f - \mathcal{M}_{f_0}) - Q^-(f, \mathcal{M}_{f_0}) \\
&= Q^+(\mathcal{M}_{f_0}, f^{S_l}) + Q^+(\mathcal{M}_{f_0}, f^{R_l}) + Q^+(f - \mathcal{M}_{f_0}, f) - Q^-(f, f - \mathcal{M}_{f_0}) - Q^-(f, \mathcal{M}_{f_0})\,.
\end{aligned}
$$

Using Duhamel formula one gets for any $t \geq t_o$

$$
\begin{aligned}
f(t) = \bigg(S(t, t_o, v) f(t_o) &+ \int_{t_o}^{t} S(t,s,v)\Big(Q^+(\mathcal{M}_{f_0}, f^{R_l}(s)) \\
&+ Q^+(f(s) - \mathcal{M}_{f_0}, f(s)) - Q^-(f(s), f(s) - \mathcal{M}_{f_0})\Big)\mathrm{d}s \bigg) \\
&+ \int_{t_o}^{t} S(t,s,v) Q^+(\mathcal{M}_{f_0}, f^{S_l}(s))\mathrm{d}s =: f^{R_{l+1}} + f^{S_{l+1}}\,,
\end{aligned}
$$

where $0 \leq S(t,s,v) := e^{-(\mathcal{M}_{f_0} * |\cdot|^{\gamma})(v)(t-s)} \leq e^{-\lambda_o (t-s)\langle v \rangle^{\gamma}}$. Since $f^{S_l} \geq 0$ by induction hypothesis, it follows that $f^{S_{l+1}} \geq 0$. Furthermore, we have noticed in the regularity section that for any $k \geq 0$

$$
\begin{aligned}
\|Q^+\big(\mathcal{M}_{f_0}, f^{S_l}(s)\big)\|_{H^{l+1}_k} &= \|(1 + (-\Delta))^{\frac{l+1}{2}} Q^+\big(\mathcal{M}_{f_0}, f^{S_l}(s)\big)\|_{L^2_k} \\
&\sim \|(1 + (-\Delta))^{\frac{1}{2}} Q^+\big((1 + (-\Delta))^{\frac{l}{2}}\mathcal{M}_{f_0}, (1 + (-\Delta))^{\frac{l}{2}} f^{S_l}(s)\big)\|_{L^2_k} \\
&\leq \|Q^+\big((1 + (-\Delta))^{\frac{l+1}{2}}\mathcal{M}_{f_0}, (1 + (-\Delta))^{\frac{l}{2}} f^{S_l}(s)\big)\|_{L^2_k} \\
&+ \|\mathcal{I}^+\big((1 + (-\Delta))^{\frac{l}{2}}\mathcal{M}_{f_0}, (1 + (-\Delta))^{\frac{l}{2}} f^{S_l}(s)\big)\|_{L^2_k} \leq C\|f^{S_l}(s)\|_{H^l_{k+\frac{d+}{2}}}\,,
\end{aligned}
$$

where we used the commutator Lemma 4.2 in the last two inequalities. Also, using the chain rule

$$
\begin{aligned}
\Big|\partial_v^{l+1} S(t,s,v)\Big| &\leq S(t,s,v)(t-s)^{l+1}\Pi_{i=1}^{l+1}\Big|(\partial_v^i \mathcal{M}_{f_0}) * |\cdot|^{\gamma}(v)\Big| \\
&\leq S(t,s,v)(t-s)^{l+1}\langle v \rangle^{(l+1)\gamma} \leq C_l e^{-\frac{\lambda_o}{2}(t-s)}\,.
\end{aligned}
$$

In this way,

$$
\begin{aligned}
\|f^{S_{l+1}}\|_{H^{l+1}_k} &\leq \Big\| \int_{t_o}^{t} S(t,s,v) Q^+(\mathcal{M}_{f_0}, f^{S_l}(s))\mathrm{d}s \Big\|_{H^{l+1}_k} \leq \int_{t_o}^{t} \big\| S(t,s,v) Q^+(\mathcal{M}_{f_0}, f^{S_l}(s))\big\|_{H^{l+1}_k}\mathrm{d}s \\
&\leq C_l \int_{t_o}^{t} e^{-\frac{\lambda_o}{2}(t-s)} \|f^{S_l}(s)\|_{H^l_{k+\frac{d+}{2}}}\mathrm{d}s \leq C_l \sup_{t \geq t_o} \|f^{S_l}(t)\|_{H^l_{k+\frac{d+}{2}}} \leq C_{s,k}(t_o)\|f_0\|_{L^2}\,,
\end{aligned}
$$

where we used the induction hypothesis in the last inequality. Now, the exponential decay in $f^{R_{l+1}}$ is a direct consequence of the induction hypothesis and Theorem 5.2. Indeed, the second term defining $f^{R_{l+1}}$ is controlled, thanks to the induction hypothesis, as

$$\Big\| \int_{t_o}^{t} S(t,s,v) Q^+(\mathcal{M}_{f_0}, f^{R_l}(s)) \mathrm{d}s \Big\|_{L^1_k} \leq \int_{t_o}^{t} \big\| S(t,s,v) Q^+(\mathcal{M}_{f_0}, f^{R_l}(s)) \big\|_{L^1_k} \mathrm{d}s$$

$$\leq \|\mathcal{M}_{f_0}\|_{L^1_{k+\gamma}} \int_{t_o}^{t} e^{-\lambda_o(t-s)} \|f^{R_l}(s)\|_{L^1_{k+\gamma}} \mathrm{d}s \leq C_k(t_o) \int_{t_o}^{t} e^{-\lambda_o(t-s)} e^{-\lambda s} \mathrm{d}s \leq C_{s,k}(t_o) e^{-\beta t}$$

for any $\beta < \lambda$. For the third and fourth terms defining $f^{R_{l+1}}$ one uses the creation of moments property and Theorem 5.2

$$\Big\| \int_{t_o}^{t} S(t,s,v) Q^+(f(s) - \mathcal{M}_{f_0}, f(s)) \mathrm{d}s \Big\|_{L^1_k} \leq \int_{t_o}^{t} \big\| S(t,s,v) Q^+(f(s) - \mathcal{M}_{f_0}, f(s)) \big\|_{L^1_k} \mathrm{d}s$$

$$\leq \sup_{t \geq t_o} \|f(t)\|_{L^1_{k+\gamma}} \int_{t_o}^{t} e^{-\lambda_o(t-s)} \|f(s) - \mathcal{M}_{f_0}\|_{L^1_{k+\gamma}} \mathrm{d}s$$

$$\leq C_k(t_o) \int_{t_o}^{t} e^{-\lambda_o(t-s)} e^{-\lambda s} \mathrm{d}s \leq C_{s,k}(t_o) e^{-\beta t}, \quad \beta < \lambda_o .$$

As a consequence, $\|f^{R_{l+1}}(t)\|_{L^1_k} \leq C_k(t_o) e^{-\lambda t}$ for any $\lambda < \lambda_o$. This proves the induction and the theorem. □

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6. Appendix

6.1. **Young's inequality for the gain collision operator.** The proof of the following theorem can be found in [3, Theorem 1].

Theorem 6.1. *Let $1 \leq p, q, r \leq \infty$ with $1/p + 1/q = 1 + 1/r$. Assume that*

$$B(x,y) = x^{\gamma}\, b(y), \qquad \gamma \geq 0 .$$

Then, for any $k \geq 0$

$$\|Q^+(f,g)\|_{L^r_k} \leq C \|f\|_{L^p_{k+\gamma}} \|g\|_{L^q_{k+\gamma}},$$

where, whenever finite, the constant $C := C(b)$ can be taken as

(94)

$$C = K \left(\int_0^1 \left(\frac{1-s}{2}\right)^{-\frac{d}{2r'}} \left(1-s^2\right)^{\frac{d-3}{2}} b(s)\, ds \right)^{\frac{r'}{q'}} \left(\int_0^1 \left(\frac{1+s}{2}\right)^{-\frac{d}{2r'}} \left(1-s^2\right)^{\frac{d-3}{2}} b(s)\, ds \right)^{\frac{r'}{p'}},$$

with $K := 2^{k+\gamma+3}\big|\mathbb{S}^{d-2}\big|$. *In the particular cases that* $p = 1$ *or* $q = 1$, *the constant is interpreted as*

$$
\begin{aligned}
C &= K\int_0^1 \Big(\frac{1-s}{2}\Big)^{-\frac{d}{2q'}}\big(1-s^2\big)^{\frac{d-3}{2}} b(s)\,ds\,, \quad \textit{when } p=1 \\
C &= K\int_0^1 \Big(\frac{1+s}{2}\Big)^{-\frac{d}{2p'}}\big(1-s^2\big)^{\frac{d-3}{2}} b(s)\,ds\,, \quad \textit{when } q=1\,.
\end{aligned} \tag{95}
$$

6.2. **Fractional differentiation lemmas.** In this appendix some identities and estimates are presented when operating with fractional differentiation of products of functions. They will be handy when proving commutator formulas for the collision operator and dealing with exponential or polynomial weights in the Sobolev estimates.

Lemma 6.1. *For any* $\varepsilon \in (0,1]$ *and* $a \geq 0$ *it holds*

$$
\left|\frac{x}{|x|^{1+a}} - \frac{y}{|y|^{1+a}}\right| \leq 2(1+a)|x-y|^{\varepsilon}\bigg(\frac{1}{|x|^{a+\varepsilon}} + \frac{1}{|y|^{a+\varepsilon}}\bigg)\,, \qquad x,\, y \in \mathbb{R}^d\,.
$$

Proof. Since the role of the vectors x and y is interchangeable, we assume $|y| \geq |x|$ without loss of generality. Thus, estimating directly the difference yields

$$
\begin{aligned}
\left|\frac{x}{|x|^{1+a}} - \frac{y}{|y|^{1+a}}\right| &= \left|\frac{(|y|^{1+a} - |x|^{1+a})x}{|x|^{1+a}|y|^{1+a}} + \frac{|x|^{1+a}(x-y)}{|x|^{1+a}|y|^{1+a}}\right| \\
&\leq \frac{\big||y|^{1+a} - |x|^{1+a}\big|}{|x|^{a}|y|^{1+a}} + \frac{|x-y|}{|y|^{1+a}}\,.
\end{aligned} \tag{96}
$$

Note that

$$
|x-y| = |x-y|^{\varepsilon}|x-y|^{1-\varepsilon} \leq |x-y|^{\varepsilon}\big(|x|^{1-\varepsilon} + |y|^{1-\varepsilon}\big)\,, \qquad \varepsilon \in (0,1]\,.
$$

As a consequence, the second term on the right side in (96) can be readily estimated as

$$
\frac{|x-y|}{|y|^{1+a}} \leq |x-y|^{\varepsilon}\frac{|x|^{1-\varepsilon} + |y|^{1-\varepsilon}}{|y|^{1+a}} \leq 2\frac{|x-y|^{\varepsilon}}{|y|^{a+\varepsilon}}\,. \tag{97}
$$

For the first term on the right side in (96), note that

$$
\big||y|^{1+a} - |x|^{1+a}\big| \leq (1+a)\max\{|x|^{a}, |y|^{a}\}|x-y| \leq (1+a)|y|^{a}|x-y|\,.
$$

Thus, bearing in mind that $|y| \geq |x|$, we conclude that

$$
\begin{aligned}
\frac{\big||y|^{1+a} - |x|^{1+a}\big|}{|x|^{a}|y|^{1+a}} &\leq (1+a)\frac{|x-y|}{|x|^{a}|y|} \leq (1+a)|x-y|^{\varepsilon}\frac{|x|^{1-\varepsilon} + |y|^{1-\varepsilon}}{|x|^{a}|y|} \\
&\leq 2(1+a)\frac{|x-y|^{\varepsilon}}{|x|^{a}|y|^{\varepsilon}} \leq 2(1+a)\frac{|x-y|^{\varepsilon}}{|x|^{a+\varepsilon}}\,.
\end{aligned} \tag{98}
$$

The lemma follows by gathering (96), (97) and (98). ∎

Lemma 6.2. *Let* $0 \le A < B$ *and* $a \in (0,1)$. *Then,*

$$\int_0^1 \big| -A+\theta B\big|^{-a} d\theta = \frac{A^{1-a}+(B-A)^{1-a}}{(1-a)B}\,. \tag{99}$$

In particular, for any $\gamma \in (0,1]$

$$\gamma\int_0^1 \big|v-v_*+\theta\, x\big|^{-(1-\gamma)}d\theta \le \frac{|v-v_*|^{\gamma}+|v-v_*+x|^{\gamma}}{|v-v_*|+|v-v_*+x|} \le 3|v-v_*|^{-(1-\gamma)}\,. \tag{100}$$

Proof. Estimate (99) is clear. Indeed, break the integral where the integrand is nonpositive and nonnegative, namely, in the intervals $(0,\frac{A}{B})$ and $(\frac{A}{B},1)$. Then, perform the integration.

Regarding estimate (100), note that the integration is performed on the line segment ℓ between the vectors $v-v_*$ and $v-v_*+x$. Since the integrand $|\cdot|^{-(1-\gamma)}$ is radially decreasing, this integral is controlled by the integration on the line segment ℓ_1 between the vectors $-|v-v_*|e$ and $|v-v_*+x|e$ with e an arbitrary unitary vector. Thus,

$$\begin{aligned}\int_0^1 \big|v-v_*+\theta\, x\big|^{-(1-\gamma)}\mathrm{d}\theta &= \int_0^1 |\ell(\theta)|^{-(1-\gamma)}\mathrm{d}\theta \le \int_0^1 |\ell_1(\theta)|^{-(1-\gamma)}\mathrm{d}\theta\\ &= \int_0^1 \big|-|v-v_*|+\theta\big(|v-v_*+x|+|v-v_*|\big)\big|^{-(1-\gamma)}\mathrm{d}\theta\,.\end{aligned}$$

Estimate (100) follows using (99) with $A=|v-v_*|$ and $B=|v-v_*+x|+|v-v_*|$. □

Lemma 6.3. *Let* $d\ge 2$, $\gamma \ge 0$, $s\in(0,1]$ *and* $v_*\in\mathbb{R}^d$. *Then, for any suitable function* f *it follows that*

$$\big(1+(-\Delta)\big)^{\frac{s}{2}}(f\,\tau_{v_*}|\cdot|^{\gamma}) = \big(1+(-\Delta)\big)^{\frac{s}{2}} f\times \tau_{v_*}|\cdot|^{\gamma} + \mathcal{R}_{v_*}(f)\,,$$

where τ *is the translation operator. The remainder term is given by*

$$\mathcal{R}_{v_*}(f)(v) := s\int_{\mathbb{R}^d}\left(\int_0^1 \nabla|\cdot|^{\gamma}(v-v_*+\theta\, x)d\theta\right)\cdot\nabla\varphi(x)f(v-x)\,dx\,,$$

where $\varphi := \mathcal{F}^{-1}\big\{\langle\cdot\rangle^{s-2}\big\}$ *is the inverse Fourier transform of the Bessel potential of order* $s-2$.

Remark 6.1. *In Lemma 6.3 the case* $\gamma\in(0,1]$ *and* $s=1$ *is special. In this case write*

$$\begin{aligned}\mathcal{R}_{v_*}(f)(v) &= \nabla|\cdot|^{\gamma}(v-v_*)\cdot\big(\nabla\varphi * f\big)(v)\\ &+\int_{\mathbb{R}^d}\left(\int_0^1\Big(\nabla|\cdot|^{\gamma}(v-v_*+\theta\, x)-\nabla|\cdot|^{\gamma}(v-v_*)\Big)d\theta\right)\cdot\nabla\varphi(x)f(v-x)\,dx\,.\end{aligned}\tag{101}$$

The first term on the right side is a singular integral. Indeed, the conditions

$$\widehat{\nabla\varphi}(\xi) = \frac{i\xi}{\langle\xi\rangle}\in L^{\infty}(\mathbb{R}^d)\,,\quad \nabla\varphi\in\mathcal{C}^1\big(\mathbb{R}^d\setminus\{0\}\big)\,,\quad |\Delta\varphi(x)|\le\frac{C}{|x|^{d+1}}\,,$$

satisfied by the convolution kernel are sufficient to properly define the convolution as a bounded operator in $L^p(\mathbb{R}^d)$ *for any* $p\in(1,\infty)$, *see* [23, Chapter 4] *for details. We just* *notice that for the case in question here, the boundedness in $L^2(\mathbb{R}^d)$ is trivial since $\widehat{\nabla\varphi} \in L^\infty(\mathbb{R}^d)$, thus*

$$\|f * \nabla\varphi\|_{L^2(\mathbb{R}^d)} = \big\| \widehat{\nabla\varphi}\, \widehat{f} \big\|_{L^2(\mathbb{R}^d)} \leq \|f\|_{L^2(\mathbb{R}^d)} .$$

The second term is properly defined as well. Using Lemma 6.1 with $x = v - v_* + \theta x$ and $y = v - v_*$ one has

$$\Big|\nabla|\cdot|^\gamma(v - v_* + \theta\, x) - \nabla|\cdot|^\gamma(v - v_*)\Big|$$
$$\leq 2\gamma(2-\gamma)|\theta x|^\varepsilon \left(\frac{1}{|v - v_* + \theta x|^{1-\gamma+\varepsilon}} + \frac{1}{|v - v_*|^{1-\gamma+\varepsilon}} \right), \qquad \varepsilon \in (0,1].$$

Choosing $\varepsilon \in (0,\gamma)$ one also has $1 - \gamma + \varepsilon < 1$, thus, applying (100) it follows that

$$\Big| \int_{\mathbb{R}^d} \left(\int_0^1 \Big(\nabla|\cdot|^\gamma(v - v_* + \theta\, x) - \nabla|\cdot|^\gamma(v - v_*)\Big) d\theta \right) \cdot \nabla\varphi(x) f(v - x)\, dx \Big|$$
$$\leq \frac{C}{|v - v_*|^{1-\gamma+\varepsilon}} \Big(\big||\cdot|^\varepsilon \nabla\varphi\big| * |f| \Big)(v) . \tag{102}$$

Then, this term is well-defined since $\big||\cdot|^\varepsilon\nabla\varphi\big| \in L^1(\mathbb{R}^d)$.

Proof. Let $g(\cdot) = \tau_{v_*}|\cdot|^\gamma$ and note that

$$\mathcal{F}\big\{(1 + (-\Delta))^{\frac{s}{2}}(f\, g)\big\}(\xi) = \langle\xi\rangle^s \big(\widehat{f} * \widehat{g}\big)(\xi)$$
$$= \mathcal{F}\big\{(1 + (-\Delta))^{\frac{s}{2}} f \times g\big\}(\xi) + \int_{\mathbb{R}^d} \big(\langle\xi\rangle^s - \langle\xi - x\rangle^s\big)\widehat{g}(x)\widehat{f}(\xi - x)\mathrm{d}x .$$

Now, the identity

$$\langle\xi\rangle^s - \langle\xi - x\rangle^s = -\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta}\langle\xi - \theta x\rangle^s \mathrm{d}\theta = -s\int_0^1 \frac{(\xi - \theta x)\cdot x}{\langle\xi - \theta x\rangle^{2-s}}\mathrm{d}\theta ,$$

leads to the definition of the remainder

$$\int_{\mathbb{R}^d} \big(\langle\xi\rangle^s - \langle\xi - x\rangle^s\big)\widehat{g}(x)\widehat{f}(\xi - x)\mathrm{d}x$$
$$= s\int_0^1 \int_{\mathbb{R}^d} \widehat{\nabla g}(x) \cdot \frac{-i(\xi - \theta x)}{\langle\xi - \theta x\rangle^{2-s}} \widehat{f}(\xi - x)\mathrm{d}x\mathrm{d}\theta =: \widehat{\mathcal{R}_{v_*}(f)}(\xi) .$$

In addition, using properties of the Fourier transform yields

$$\mathcal{F}^{-1}\Big\{\frac{-i(\cdot - \theta x)\widehat{f}(\cdot - x)}{\langle\cdot - \theta x\rangle^{2-s}}\Big\}(v) = e^{ix\cdot v}\big(f * \phi\big)(v) , \qquad \phi := e^{i(1-\theta)x\cdot}\, \nabla\mathcal{F}^{-1}\big\{\langle\cdot\rangle^{s-2}\big\} .$$

Thus, plugging in the definition of $\mathcal{R}_{v_*}(f)$ one gets

$$\mathcal{R}_{v_*}(f)(v) = s\int_0^1 \int_{\mathbb{R}^d} \widehat{\nabla g}(x) \cdot (f * \phi)(v) e^{ix\cdot v}\mathrm{d}x\mathrm{d}\theta$$
$$= s\int_{\mathbb{R}^d} \left(\int_0^1 \nabla g(v + \theta\, x)\mathrm{d}\theta \right) \cdot \nabla\varphi(x) f(v - x)\mathrm{d}x , \tag{103}$$

where $\varphi := \mathcal{F}^{-1}\big\{\langle\cdot\rangle^{s-2}\big\}$. □

Lemma 6.4. *Let $d \geq 2$, $s \in (0,1]$, $r \in (0,\frac{1}{4})$, $\alpha \in (0,1]$. Then, for any suitable function f, it follows that*

$$\big(1+(-\Delta)\big)^{\frac{s}{2}}(f\,e^{r\langle\cdot\rangle^{\alpha}}) = \big(1+(-\Delta)\big)^{\frac{s}{2}} f \times e^{r\langle\cdot\rangle^{\alpha}} + \mathcal{R}(f)\,.$$

The remainder term is controled by

$$\big\|\mathcal{R}(f)\big\|_{L^2(\mathbb{R}^d)} \leq C(r,\varphi)\|e^{r\langle\cdot\rangle^{\alpha}} f\|_{L^2(\mathbb{R}^d)}\,.$$

Proof. Use formula (103) with $g := e^{r\langle\cdot\rangle^{\alpha}}$. The validity of such formula for this choice of g is shown by standard approximation procedure. Note that, for $\alpha \in (0,1]$, one has

$$\int_0^1 \nabla g(v+\theta\,x)\mathrm{d}\theta \leq Ce^{2r\langle x\rangle^{\alpha}} e^{r\langle v-x\rangle^{\alpha}}\,.$$

Thus, when $s \in (0,1)$

$$\big|\mathcal{R}(f)(v)\big| \leq s\,C\int_{\mathbb{R}^d}\big|e^{2r\langle x\rangle^{\alpha}}\nabla\varphi(x)\big|\big|e^{r\langle v-x\rangle^{\alpha}} f(v-x)\big|\mathrm{d}x\,.$$

As a consequence,

$$\big\|\mathcal{R}(f)\big\|_{L^2(\mathbb{R}^d)} \leq C\|e^{2r\langle\cdot\rangle^{\alpha}}\nabla\varphi\|_{L^1(\mathbb{R}^d)}\|e^{r\langle\cdot\rangle^{\alpha}} f\|_{L^2(\mathbb{R}^d)}\,.$$

Note that $e^{2r\langle\cdot\rangle^{\alpha}}\nabla\varphi \in L^1(\mathbb{R}^d)$ for $r \in [0,\frac{1}{4})$. For the case $s=1$, follow the argument of Remark 6.1 □

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