A NEW APPROACH TO THE CREATION AND PROPAGATION OF EXPONENTIAL MOMENTS IN THE BOLTZMANN EQUATION

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Abstract. We study the creation and propagation of exponential moments of solutions to the spatially homogeneous d-dimensional Boltzmann equation. In particular, when the collision kernel is of the form $|v-v_*|^\beta b(\cos(\theta))$ for $\beta \in (0,2]$ with $\cos(\theta) = |v-v_*|^{-1}(v-v_*)\cdot\sigma$ and $\sigma \in \mathbb{S}^{d-1}$, and assuming the classical cut-off condition $b(\cos(\theta))$ integrable in $\mathbb{S}^{d-1}$, we prove that there exists $a > 0$ such that moments with weight $\exp(a \min\{t,1\}|v|^\beta)$ are finite for $t > 0$, where a only depends on the collision kernel and the initial mass and energy. We propose a novel method of proof based on a single differential inequality for the exponential moment with time-dependent coefficients.

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1. Introduction

We consider the spatially homogeneous Boltzmann equation in dimension $d \geq 2$ with initial condition $f_0 \geq 0$, given by

$$\partial_t f = Q(f,f), \quad f(t,\cdot) = f_0 \tag{1}$$

where $f = f(t,v) \geq 0$ is a non-negative function depending on time $t \geq 0$ and velocity $v \in \mathbb{R}^d$, with $d \geq 2$. We will assume throughout this paper that f_0 has finite mass and energy, i.e.,

$$\|f_0\|_{L^1(1+|v|^2)} := \int_{\mathbb{R}^N} (1+|v|^2) f_0(v)\,\mathrm{d}v < +\infty.$$

For $p \in [1,+\infty]$, we denote by L^p the Lebesgue spaces of p-integrable real functions on $\mathbb{R}^d$, and the notation $L^p(w(v)\,\mathrm{d}v)$ (or simply $L^p(w(v))$) denotes the L^p space with weight $w(v)$. The *collision operator* $Q(f,f)$ is given by

$$Q(f,f)(v) := \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} B(|v-v_*|, \cos\theta)(f'_* f' - f_* f)\,\mathrm{d}v_*\,\mathrm{d}\sigma,$$

representing the total rate of binary interactions due to precollisions taking the direction of v, minus those that were knocked out from the v direction. We follow the usual notation $f \equiv f(v)$, $f_* \equiv f(v_*)$, $f' \equiv f(v')$, $f'_* \equiv f(v'_*)$. The vectors v', v'_*, which denote the velocities after an elastic collision of particles with velocities v, v_*, are given by

$$v' := \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \qquad v'_* := \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma.$$

The variable θ denotes the angle between $v - v_*$ and σ, where σ is the unit vector in the direction of the postcollisional relative velocity. On the collision kernel B we assume that for some $\beta \in (0,2]$

$$B(|v-v_*|, \cos\theta) = |v-v_*|^\beta\, b(\cos\theta), \tag{2}$$

with the following cut-off assumption:

$$b \in L^1\left([-1,1], (1-z^2)^{\frac{d-3}{2}}\,\mathrm{d}z\right). \tag{3}$$

If we define $\tilde{b}(\sigma) := b(e_1 \cdot \sigma)$, with $e_1 \in \mathbb{S}^{d-1}$ any fixed vector, then (3) is equivalent to $\tilde{b} \in L^1(\mathbb{S}^{d-1})$, which can be easily seen by a spherical change of coordinates.

Throughout the paper f always represents a solution to equation (1) on $[0,+\infty)$ (in the sense of, e.g., [10]) and we always write, for $p \geq 0$ (not necessarily an integer),

$$m_p = m_p(t) := \int_{\mathbb{R}^d} f(t,v)\,|v|^p\,\mathrm{d}v. \tag{4}$$

Main results. It is known that moments of order $p > 2$ and exponential moments (L^1-exponentially weighted estimates) with weight up to $\exp(a|v|^2)$ for some $a > 0$ are propagated by equation (1) [5, 12, 2, 3, 6]; that is, they are finite for all times $t > 0$ if they are initially finite, however with a deterioration of the constant a. In [12] it was proved that in fact equation (1) with $\beta > 0$ instantaneously creates all moments of orders $p > 2$, which then remain finite for all times $t > 0$. Here the assumption that $\beta > 0$ is necessary, since the result is not true for Maxwell molecules for instance [7]. Moreover, moments with exponential weight up to $\exp(a|v|^{\beta/2})$ for some constant $a > 0$ were also shown to be instantaneously created in [9, 11]. In all these proofs it was crucial to assume that the angular function b is in $L^q\big([-1,1],(1-z^2)^{\frac{d-3}{2}}\,\mathrm{d}z\big)$ for $q > 1$ as done in [4, 6, 1]. We also refer to the recent preprint [8] for moment production estimates in the so-called non-cutoff case, whose proofs are based on the optimization of the traditional inductive argument [2, 3, 6, 9, 11].

We have several noticeable contributions in this paper. Indeed, we can extend the existing propagation and creation of L^1-exponentially weighted estimates to include the classical cut-off assumption $b \in L^1\big([-1,1],(1-z^2)^{\frac{d-3}{2}}\,\mathrm{d}z\big)$ without using the iterative methods developed in [4, 6, 1]), and also we slightly relax the assumptions on the initial data by requiring only finite mass and energy, and not necessarily finite entropy as in previous works on creation of moments [12]. In addition, we improve the weights for the creation of L^1-exponentially weighted moments, with a weight up to $\exp(a|v|^\beta)$ (hence removing the 1/2 factor) for solutions with finite mass and energy, assuming only an integrability condition on b. More specifically, Theorem 1 gives an explicit rate of appearance of exponential moments by showing that the coefficient multiplying $|v|^\beta$ in the exponential weight can be taken linearly growing in time.

The most important point is that we introduce a new simpler method of proof that not only does not need iterative arguments but also allowed us to achieve all these improvements. This approach is also used in Theorem 2 for the propagation of exponential moments, and extends these results to classical cut-off assumptions on the angular cross section b.

Theorem 1 (Creation of exponential moments)**.** *Let f be an energy-conserving solution to the homogeneous Boltzmann equation* (1) *on* $[0,+\infty)$ *with initial data $f^0 \in L^1(1+|v|^2)$, and assume* (2) *and* (3) *with $\beta \in (0,2]$. Then there are some constants $C, a > 0$ (which depend only on b, β and the initial mass* *and energy) such that*

$$\int_{\mathbb{R}^d} f(t,v)\, \exp\left(a \min\{t,1\}\, |v|^\beta\right) \mathrm{d}v \le C \qquad \textit{for } t \ge 0.$$

We remark that the existence and uniqueness of energy-conserving solutions with initial data $f^0 \in L^1(1+|v|^2)$ was proved in [10].

As we said, our new approach also provides a new simpler proof of the property of *propagation* of exponential moments [6, 4], as stated in the following theorem:

Theorem 2 (Propagation of exponential moments)**.** *Let f be an energy-conserving solution to the homogeneous Boltzmann equation* (1) *on* $[0,+\infty)$ *with initial data $f^0 \in L^1(1+|v|^2)$, and assume* (2) *and* (3) *with $\beta \in (0,2]$. Assume moreover that the initial data satisfies for some $s \in [\beta, 2]$*

$$\int_{\mathbb{R}^d} f_0(v) \exp\left(a_0\, |v|^s\right) \mathrm{d}v \le C_0. \tag{5}$$

Then there are some constants $C, a > 0$ (which depend only on b, β and the initial mass, energy and a_0, C_0 in (5)*) such that*

$$\int_{\mathbb{R}^d} f(t,v)\, \exp\left(a\, |v|^s\right) \mathrm{d}v \le C \qquad \textit{for } t \ge 0.$$

We give in Section 3 a novel argument for proving these results which is based on a differential inequality for the exponential moment itself, and the exploitation of a discrete convolution-type estimate for the exponential moment of the gain part of the collision operator. This avoids the intricate combination of induction and maximum principle arguments in the previous proofs of propagation [6, 4] and appearance [9, 11] of exponential moments. It also clarifies the structure underlying these induction arguments. The starting point of both these previous works and our new approach is the creation and propagation of polynomial moments in [5, 12] and the Povzner inequalities proved in [4, 1].

2. Refresher on the sharp Povzner Lemma

The following lemma reflects the angular averaging property of the spherical integral acting on positive convex test functions evaluated at the post-collisional velocities. These estimates are crucial to be able to control in a sharp form the moments of the gain operator by estimates for lower bounds of the loss operator. They were originally introduced in [4, Corollary 1] and further developed in [6, Lemma 3 and 4] and more recently in [1, Lemma 2.6]. We summarize these results as follows:

Lemma 3 (Sharp Povzner (angular averaging) Lemma)**.** *Assume that* $b : (-1,1) \to [0,\infty)$ *satisfies* (3)*, and impose without loss of generality the following normalization condition*

$$\int_{-1}^{1} b(z)(1-z^2)^{\frac{d-3}{2}}\,\mathrm{d}z = \frac{1}{|\mathbb{S}^{d-2}|}, \tag{6}$$

where $|\mathbb{S}^{d-2}|$ *is the area of the* $(d-2)$*-dimensional unit sphere. Then for* $p \geq 1$ *it holds that*

$$\int_{\mathbb{S}^{d-1}} \left(|v'|^{2p} + |v'_*|^{2p}\right) b(\cos\theta)\,\mathrm{d}\sigma \leq \gamma_p \left(|v|^2 + |v_*|^2\right)^p \tag{7}$$

where $\gamma_p > 0$ *are constants such that* $\gamma_1 = 1$*,* $p \mapsto \gamma_p$ *is strictly decreasing and tends to* 0 *as* $p \to \infty$*.*

Remark 4. In the case when the symmetrization of b is nondecreasing in $[0,1]$, these constants are controlled by

$$\gamma_p \leq \frac{1}{|\mathbb{S}^{d-2}|} \int_{-1}^{1} b(z) \left(\frac{1+z}{2}\right)^p (1-z^2)^{\frac{d-3}{2}}\,\mathrm{d}z\,. \tag{8}$$

Remark 5. In addition, when $b \in L^q([-1,1],(1-z^2)^{(d-3)/2}\,dz)$ with $q > 1$, the decay of γ_p can be estimated and shown to be polynomial: there exists a constant $C > 0$ such that

$$\gamma_p \leq \min\left\{1, \frac{C}{p^{1/q'}}\right\} \qquad (p > 1),$$

with q' the Hölder dual of q (i.e., $1/q + 1/q' = 1$). Furthermore, in the case $q = +\infty$, that is, for b bounded, it holds that

$$\gamma_p \leq \min\left\{1, \frac{16\pi b^*}{p+1}\right\} \qquad (p > 1),$$

with $b^* := \max_{-1 \leq z \leq 1} b(z)$.

Let us now state the key a priori estimate on the polynomial moments, which shall be used in the sequel. For later reference, we define the following quantity for any $s, p > 0$:

$$S_{s,p} = S_{s,p}(t) := \sum_{k=1}^{k_p} \binom{p}{k} \left(m_{sk+\beta}\, m_{s(p-k)} + m_{sk}\, m_{s(p-k)+\beta}\right), \tag{9}$$

with k_p the integer part of $(p+1)/2$.

Lemma 6 (A priori estimate on the polynomial moments)**.** *For $s \in (0,2]$ and $p_0 > 2/s$, the following a priori inequality is true whenever all the terms make sense:*

$$\frac{\mathrm{d}}{\mathrm{d}t} m_{sp} \leq 2\gamma_{sp/2} S_{s,p} - K_1 m_{sp+\beta} + K_2 m_{sp} \quad \text{for } t \geq 0,\ p \geq p_0 > \frac{2}{s}, \tag{10}$$

with S_p given by (9) *and constants*

$$K_1 := 2(1-\gamma_{sp_0/2}) C_\beta m_0 \quad \text{and} \quad K_2 = 2\, m_\beta \tag{11}$$

with $C_\beta := \min\{1, 2^{1-\beta}\}$.

Alternatively in the case $\beta \in (0,1]$, it is possible to get rid of the second constant, and obtain

$$K_1 := 2(1-\gamma_{sp_0/2}) \bar{C}_\beta m_0 \quad \text{and} \quad K_2 = 0 \tag{12}$$

for some constant $\bar{C}_\beta$ depending on β and the initial data.

In both cases, the constant $\gamma_{sp_0/2}$ depends on the initial data (i.e. initial mass and energy), on the integrability of the angular function b and on $p_0 > 2/s$.

Proof. Using Lemma 3 one obtains for any $p \geq 2/s$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} m_{sp} & \\ \leq \gamma_{sp/2} \int_{\mathbb{R}^d\times\mathbb{R}^d} & f f_* \left(\left(|v|^2 + |v_*|^2\right)^{\frac{sp}{2}} - |v|^{sp} - |v_*|^{sp} \right) |v - v_*|^\beta \,\mathrm{d}v\, \mathrm{d}v_* \\ & - 2(1-\gamma_{sp/2}) \int_{\mathbb{R}^d\times\mathbb{R}^d} f f_* |v|^{sp} |v-v_*|^\beta \,\mathrm{d}v\,\mathrm{d}v_*. \end{aligned} \tag{13}$$

In order to estimate the right hand side of (13) we first focus on an upper bound for its positive term. Since $0 < s/2 \leq 1$, then

$$\left(|v|^2 + |v_*|^2\right)^{\frac{sp}{2}} \leq \left(|v|^s + |v_*|^s\right)^p .$$

Hence, using [4, Lemma 2], and the estimate $|v - v_*|^\beta \leq 2|v|^\beta + 2|v_*|^\beta$ we obtain that, for any $p \geq 1$, the first integral in (13) is controlled by

$$\begin{aligned} \gamma_{sp/2} \int_{\mathbb{R}^d\times\mathbb{R}^d} f f_* \left(\left(|v|^2 + |v_*|^2\right)^{\frac{sp}{2}} - |v|^{sp} - |v_*|^{sp} \right) |v - v_*|^\beta \,\mathrm{d}v\, \mathrm{d}v_* & \\ \leq 2\gamma_{sp/2} S_{s,p}. & \end{aligned} \tag{14}$$

The estimate of the negative term in (13) requires a control from below. When $\beta \in (0,1]$ it follows from [6, Lemma 2] and its immediate consequence

[6, (27)] that the lower bound for the negative term in (13) satisfies

$$2(1-\gamma_{sp/2})\int_{\mathbb{R}^d\times\mathbb{R}^d} f f_* |v|^{sp}|v-v_*|^\beta \,\mathrm{d}v\,\mathrm{d}v_* \geq 2\,\bar{C}_\beta\,(1-\gamma_{sp/2})m_0 m_{sp+\beta} \tag{15}$$

for some constant $\bar{C}_\beta$ related to β and the initial data (cf. [6, Lemma 2]). So estimate (10) follows with K_1 and K_2 as in (12).

In the general case $\beta \in (0,2]$, the previous argument does not necessarily follow, yet it is still possible to obtain an easier lower bound that still allows for the control of moments and their summability. We use the fact that $|v-v_*|^\beta \geq 2^{1-\beta}|v|^\beta - |v_*|^\beta$ (which can be obtained from the triangle inequality and the inequality $(x+y)^\beta \leq C_\beta^{-1}(x^\beta + y^\beta)$ for $x, y \geq 0$.) This gives a lower bound for the negative term in (13):

$$2(1-\gamma_{sp/2})\int_{\mathbb{R}^d\times\mathbb{R}^d} f f_* |v|^{sp}|v-v_*|^\beta \,\mathrm{d}v\,\mathrm{d}v_* \geq 2(1-\gamma_{sp/2})C_\beta m_0 m_{sp+\beta} - 2m_\beta m_{sp}. \tag{16}$$

Since γ_{sp} decreases as $p \to \infty$, it follows that $2(1-\gamma_{sp/2})C_\beta m_0 \geq K_1$ for any $p \geq p_0$. Hence, estimate (10) follows with K_1 and K_2 as in (11). $\square$

Remark 7. We note that neither the work in [6, Lemma 2] nor in here the finiteness of the entropy was required, however it was needed in the earlier work [12] in order to obtain lower bounds for the negative term in (13). If the solution has a finite entropy, then these lower bounds may be obtained by the same technique as in [12]. Observe however that the constant $\bar{C}_\beta$ in the case $\beta \in (0,1]$ with $K_2 = 0$ depends on the initial data in a non-trivial way, through the positive constant $C > 0$ such that

$$\int_{\mathbb{R}^d} f_0(v_*)\,|v-v_*|^\beta \,\mathrm{d}v_* \geq C(1+|v|^\beta)$$

which cannot be expressed simply in terms of the mass and energy of f_0. Nevertheless the general argument does provide constants only depending on the initial data through its mass and momentum.

Next, we recall and prove a very similar result to that in [12, Theorem 4.2]. The main difference is that finiteness of the entropy of the initial condition is not required here.

Lemma 8 (Creation and propagation of polynomial moments)**.** *Assume* (2) *and* (3) *with* $0 < \beta \leq 2$. *Set* $s \in (0,2]$, *and let* f *be a solution to the homogeneous Boltzmann equation* (1) *with initial data* $f_0 \in L^1(1+|v|^2)$.

For every $p > 0$ there exists a constant $C_{sp} \geq 0$ depending only on p, s, b and the initial mass and energy, such that

$$m_{sp}(t) \leq C_{sp} \max\{1, t^{-sp/\beta}\} \quad \text{for } t > 0. \tag{17}$$

If $m_{sp}(0)$ is finite, then the control can be improved to simply

$$m_{sp}(t) \leq C_{sp} \quad \text{for } t \geq 0 \tag{18}$$

for some constant C_{sp} depending only on p, s, b, the initial mass and energy, and $m_p(0)$.

Proof. Following a common procedure (see [10, 12]), the estimates can be carried first on a truncated solution (for which all moments are finite and our calculations are rigorously justified), and then proved for the solution to the full problem by relaxing the truncation parameter.

Let us prove (17): observe that by Hölder's inequality

$$S_{s,p} \leq C m_\beta m_{sp} \quad \text{and} \quad m_{sp+\beta} \geq K m_{sp}^{1+\beta/(sp)}$$

for some constants C, $K > 0$ depending only on s, p, the initial mass and energy. Since $\beta \leq 2$, we have $1 \leq 2/\beta$ and therefore m_β is controlled by the mass and energy. We deduce that m_{sp} satisfies the differential inequality

$$\frac{\mathrm{d}m_{sp}}{\mathrm{d}t} \leq C' m_{sp} - K m_{sp}^{1+\beta/(sp)} \tag{19}$$

for some other constant $C' > 0$ depending only on s, p, the initial mass and energy. This readily implies the bound (17) by computing an upper solution to this differential inequality, and the bound (18) by a maximum principle argument. □

Remark 9. Observe that the polynomial bound $O(t^{-sp/\beta})$ on the appearance of m_p is *not* optimal, as can be seen from [10, Theorem 1.1]. Correspondingly we conjecture that the optimal rate of appearance of exponential moment in Theorem 1 has the form

$$\int_{\mathbb{R}^d} f(t,v) \exp\left(a \min\{\alpha(t), 1\} \, |v|^\beta\right) dv \leq C \quad \text{for } t \geq 0$$

with $\alpha(t)/t \to 0$ as $t \to 0$.

3. Proof of the main theorems

In this section we give a proof of Theorems 1 and 2 based on a new argument, and valid for any integrable cross-section b. We first carry out the estimates on a finite sum of polynomial moments, and then pass to the limit.

Our goal is to estimate the quantity

$$E_s(t,z) := \int_{\mathbb{R}^d} f(t,v) \exp\left(z\,|v|^s\right) dv = \sum_{p=0}^{\infty} m_{sp}(t) \frac{z^p}{p!}$$

where $s = \beta$ and $z = at$ for Theorem 1 and $s \in (0,2]$ and $z = a$ for Theorem 2, for some $a > 0$. For use below let us define the truncated sum as

$$E_s^n(t,z) := \sum_{p=0}^{n} m_{sp}(t) \frac{z^p}{p!}$$

for $n \in \mathbb{N}$, $z \geq 0$, and $t \geq 0$. We also define

$$I_s^n(t,z) := \sum_{p=1}^{n} m_{sp}(t) \frac{z^{p-1}}{(p-1)!}.$$

Let us first prove the key lemma, which identifies the discrete convolution structure. This result gives a control for finite sums of the moments associated to the gain operator. It is uniform in $\beta \in (0,2]$:

Lemma 10. *Assume $0 < \beta \leq s \leq 2$. For any $p_0 \geq 2/s$, we have the following functional inequality*

$$\sum_{p=p_0}^{n} \frac{z^p}{p!} S_{s,p}(t) \leq 2 E_s^n(t,z) I_s^n(t,z) \tag{20}$$

where $S_{s,p}$ was defined in (9).

Proof. Let us recall the definition of $S_{s,p}$ from (9):

$$S_{s,p} := \sum_{k=1}^{k_p} \binom{p}{k} \left(m_{sk+\beta}\, m_{s(p-k)} + m_{sk}\, m_{s(p-k)+\beta} \right),$$

where k_p is the integer part of $(p+1)/2$. The first part of the sum can be bounded as:

$$\sum_{p=p_0}^{n} \frac{z^p}{p!} \sum_{k=1}^{k_p} \binom{p}{k} m_{sk+\beta}\, m_{s(p-k)} = \sum_{p=p_0}^{n} \sum_{k=1}^{k_p} m_{sk+\beta} \frac{z^k}{k!}\, m_{s(p-k)} \frac{z^{p-k}}{(p-k)!}$$
$$\leq \sum_{k=1}^{n} m_{s(k+1)} \frac{z^k}{k!} \sum_{p=\max\{p_0, 2k-1\}}^{n} m_{s(p-k)} \frac{z^{p-k}}{(p-k)!} \leq E_s^n(t,z) I_s^n(t,z)$$

where we have used $\beta \le s$ in the last line. We carry out a similar estimate for the other part:

$$\sum_{p=p_0}^{n} \frac{z^p}{p!} \sum_{k=1}^{k_p} \binom{p}{k} m_{sk}\, m_{s(p-k)+\beta} = \sum_{p=p_0}^{n} \tilde{\gamma}_p \sum_{k=1}^{k_p} m_{sk} \frac{z^k}{k!}\, m_{s(p-k)+\beta} \frac{z^{p-k}}{(p-k)!}$$
$$\le \sum_{k=1}^{n} m_{sk} \frac{z^k}{k!} \sum_{p=\max\{p_0,2k-1\}}^{n} m_{s(p-k+1)} \frac{z^{p-k}}{(p-k)!} \le E_s^n(t,z) I_s^n(t,z)$$

which concludes the proof. $\square$

We now can prove both Theorem 1 and Theorem 2. We write the proof first for the case $\beta \in (0,1]$ with the choice of constants (12) in (10) (hence with $K_2 = 0$). Later we show the corresponding estimates for the full range $\beta \in (0,2]$ using the choice of constants (11) in (10).

Proof of Theorem 1. First we notice that it is enough to prove the following (under the same assumptions): there are some constants $T, C, a > 0$ (which depend only on b and the initial mass and energy) such that

$$\int_{\mathbb{R}^d} f(t,v) \exp\left(at\, |v|^\beta\right) \mathrm{d}v \le C \quad \text{for } t \in [0,T]. \tag{21}$$

Indeed, since the assumptions are propagated along the flow, for $t \ge T$ it is then possible to apply (21) starting at time $(t - T/2)$.

Hence, we aim at proving the estimate (21). We set $s = \beta$. Consider $a > 0$ to be fixed later, $n \in \mathbb{N}$ and define $T > 0$ as

$$T := \min\left\{1 \;;\; \sup\left\{t > 0 \;\text{ s.t. }\; E_\beta^n(t,at) < 4m_0\right\}\right\}.$$

The definition is consistent since $E_\beta^n(0,0) = m_0$ and the Lemma 8 ensures that $T > 0$ for each given n. The bound of 1 is not essential, and is included just to ensure that T is finite. We note that a priori such T depends on the index n in the sum E_β^n. However, we will show that T has a lower bound that depends only on b, β and the initial mass and energy, thus proving the theorem. Unless otherwise noted, all equations below which depend on time are valid for $t \in [0,T]$.

Choose an integer $p_0 > 2/\beta$, to be fixed later. Starting from Lemma 6 (inequality (10)), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} m_{\beta p} \le 2\gamma_{\beta p/2} S_{\beta,p} - K_1 m_{\beta(p+1)} \quad \text{for } t \ge 0,\ p \ge p_0, \tag{22}$$

with $S_{\beta,p}$ given by (9) and K_1 defined in (12), independent of p with $p \ge 0$.

In addition, from Lemma 8 (inequality (17)) we know that there exists a constant $C_{p_0} > 0$ (depending on p_0) such that

$$\sum_{p=0}^{p_0} m_{\beta p} t^p \leq C_{p_0} \quad \text{for all } t \in [0, T]. \tag{23}$$

Taking any $a < 1$ and using the product rule,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \sum_{p=p_0}^{n} m_{\beta p} \frac{(at)^p}{p!} \\
&\qquad \leq \sum_{p=p_0}^{n} \frac{(at)^p}{p!} \left(2\gamma_{\beta p/2} S_{\beta,p} - K_1 m_{\beta(p+1)}\right) + a \sum_{p=p_0}^{n} m_{\beta p} \frac{(at)^{p-1}}{(p-1)!} \\
&\quad \leq 2 \sum_{p=p_0}^{n} \frac{(at)^p}{p!} \gamma_{\beta p/2} S_{\beta,p} + (a - K_1) I_\beta^n(t, at) + (K_1 + a) \sum_{p=1}^{p_0} m_{\beta p} \frac{(at)^{p-1}}{(p-1)!} \\
&\qquad\qquad \leq 2 \sum_{p=p_0}^{n} \frac{(at)^p}{p!} \gamma_{\beta p/2} S_{\beta,p} + (a - K_1) I_\beta^n(t, at) + \frac{1}{t}(K_1 + a) C_{p_0},
\end{aligned}$$

where we have used $a < 1$ and (23) in the last step. Hence, from Lemma 10 (inequality (20)) we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \sum_{p=p_0}^{n} m_{\beta p} \frac{(at)^p}{p!} \\
&\qquad\qquad \leq I_\beta^n(t, at) \Big[4\gamma_{\beta p_0/2} E_\beta^n(t, at) + (a - K_1) \Big] + \frac{1}{t}(K_1 + a) C_{p_0}.
\end{aligned}$$

Next, choose p_0 such that $16\gamma_{\beta p_0/2} m_0 \leq (1/4) K_1$ (or equivalently, by using de definition of K_1 in (11), $\gamma_{\beta p_0/2} < (32 + \bar{C}_\beta)^{-1}$) and restrict further the parameter a, so that $a \leq K_1/2$. Then, as $E_\beta^n(t, at) \leq 4m_0$ for $t \in [0, T]$, by the definition of T we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \sum_{p=p_0}^{n} m_{\beta p} \frac{(at)^p}{p!} &\leq -\frac{1}{4} K_1 I_\beta^n(t, at) + \frac{1}{t}(K_1 + a) C_{p_0} \\
&\leq -\frac{1}{t} \left(\frac{K_1}{4a} (E_\beta^n(t, at) - m_0) - (K_1 + a) C_{p_0} \right)
\end{aligned} \tag{24}$$

where for the last inequality we have used that $I_\beta^n(t, at) \geq (E_\beta^n(t, at) - m_0)/(at)$. We make the additional restriction that $a < m_0/(6C_{p_0})$, which together with $a < K_1/2$ implies that

$$\frac{K_1}{4a} m_0 > (K_1 + a) C_{p_0}.$$

Then, whenever $E^n_\beta(t, at) \geq 2m_0$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{p=p_0}^{n} m_{\beta p} \frac{(at)^p}{p!} \leq 0 \tag{25}$$

for any time $t \in [0, T]$ for which $E^n_\beta(t, at) \geq 2m_0$ holds. This is true in particular when $\sum_{p=p_0}^{n} m_{\beta p} \frac{(at)^p}{p!} \geq 2m_0$. We deduce that

$$\sum_{p=p_0}^{n} m_{\beta p} \frac{(at)^p}{p!} \leq 2m_0 \quad \text{for } t \in [0, T]. \tag{26}$$

In order to finish the argument we need to bound the initial part of the full sum (from $p = 0$ to $p_0 - 1$.) Indeed, we note that from (23),

$$\sum_{p=0}^{p_0-1} m_{\beta p} \frac{(at)^p}{p!} \leq m_0 + aC_{p_0} \quad \text{for } t \in [0, T], \tag{27}$$

so, recalling that $6aC_{p_0} < m_0$ and using (26) and (27)

$$E^n_\beta(t, at) = \sum_{p=0}^{p_0-1} m_{\beta p} \frac{(at)^p}{p!} + \sum_{p=p_0}^{n} m_{\beta p} \frac{(at)^p}{p!} \leq 3m_0 + aC_{p_0} \leq \frac{19}{6} m_0$$

for $t \in [0, T]$, uniformly in n. Finally, gathering all conditions imposed along the proof on the parameter a, we choose

$$a := \min \left\{ 1, \frac{K_1}{2}, \frac{m_0}{6C_{p_0}} \right\} \tag{28}$$

independently of n, where K_1 was defined in (12) and C_{p_0} in (23). We conclude, from the definition of T, that $T = 1$ for all n. Sending $n \to \infty$, Theorem 1 follows.

In the general case $\beta \in (0, 2]$, since K_2 in (11) is not zero, equation (30) has an extra term in the right hand side, namely

$$\frac{\mathrm{d}}{\mathrm{d}t} m_{\beta p} \leq 2\gamma_{\beta p/2} S_{\beta,p} - K_1 m_{\beta(p+1)} + K_2 m_{\beta p} \quad \text{for } t \geq 0,\ p \geq p_0.$$

In this case using again that $E^n_\beta(t,at) \le 4m_0$ on $[0,T]$, (24) is now modified as

$$\text{(29)}\quad \begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{p=p_0}^{n} m_{\beta p}\frac{(at)^p}{p!} &\le -\frac{1}{4}K_1 I^n_\beta(t,at) + \frac{1}{t}(K_1+a)C_{p_0} + K_2 E^n_\beta(t,at)\\ &\le -\frac{1}{t}\left(\frac{K_1}{4a}(E^n_\beta(t,at)-m_0) - (K_1+a)C_{p_0}\right) + 4K_2 m_0.\end{aligned}$$

Hence by tuning the constants as before, at any time $t \in [0,T]$ for which $E^n_\beta(t,at) \ge 2m_0$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{p=p_0}^{n} m_{\beta p}\frac{(at)^p}{p!} \le K_3$$

with $K_3 = 4K_2 m_0$. The corresponding to equation (26) is then

$$\sum_{p=p_0}^{n} m_{\beta p}\frac{(at)^p}{p!} \le 2m_0 + K_3 t \qquad t \in [0,T].$$

It follows as before that

$$E^n_\beta(t,at) \le \frac{19}{6}m_0 + K_3 t\,, \qquad t \in [0,T]\,,$$

uniformly in n. Then $T \ge 2m_0/K_3$, where K_3 is a constant which depends only on b, the hard potential exponent β and initial mass and energy. In particular for the same rate a as in (28) the conclusion follows since both a and T are uniform in the index n and the limit in n can be performed as well. $\square$

Proof of Theorem 2. Consider again first the case $\beta \in (0,1]$, and $s \in [\beta,1]$ as in (5), $a > 0$ to be fixed later and $n \in \mathbb{N}$. Define $T > 0$ as

$$T := \sup\left\{t > 0 \ \text{ s.t. } \ E^n_s(t,a) < 4m_0\right\}.$$

This definition is consistent since $E^n_s(0,a) \le E_s(0,a) < 4m_0$ for a small enough thanks to the assumption (5) on the initial data, and the Lemma 8 ensures that $T > 0$ for each given n. We will show that, for a chosen small enough, $T = +\infty$ for any n, thus proving the theorem.

Choose an integer $p_0 > 2/s$, to be fixed later. Starting again from Lemma 6 (inequality (10) with the choice of constants (12)), we have

$$\text{(30)}\qquad \frac{\mathrm{d}}{\mathrm{d}t}m_{sp} \le 2\gamma_{sp/2}S_{s,p} - K_1 m_{sp+\beta} \quad \text{for } t \ge 0,\ p \ge p_0,$$

with $S_{s,p}$ given by (9) and K_1 given by (12), independent of p with $p \geq 0$. Also, from Lemma 8 (inequality (18)) we know that there exists a constant $C_{s,p_0} > 0$ (depending on s, p_0) such that

$$\sum_{p=0}^{p_0} m_{sp} \leq C_{s,p_0} \quad \text{for all } t \in [0,T]. \tag{31}$$

Taking any $a < \min\{1, a_0\}$, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \sum_{p=p_0}^{n} m_{sp} \frac{a^p}{p!} &\leq \sum_{p=p_0}^{n} \frac{a^p}{p!} \left(2\gamma_{sp/2} S_{s,p} - K_1 m_{sp+\beta}\right) \\
&\leq 2 \sum_{p=p_0}^{n} \frac{a^p}{p!} \gamma_{sp/2} S_{s.p} + (a - K_1) I_s^n(t,a) + K_1 \sum_{p=1}^{p_0} m_{sp} \frac{a^{p-1}}{(p-1)!} \\
&\leq 2 \sum_{p=p_0}^{n} \frac{a^p}{p!} \gamma_{sp/2} S_{s,p} + (a - K_1) I_s^n(t,a) + K_1 C_{s,p_0},
\end{aligned}$$

where we have used $a < 1$ and (31) in the last step. Hence, from Lemma 10 (inequality (20)) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{p=p_0}^{n} m_{sp} \frac{a^p}{p!} \leq I_s^n(t,a) \Big[4\gamma_{sp_0} E_s^n(t,a) + (a - K_1) \Big] + K_1 C_{s,p_0}. \tag{32}$$

where, as in the previous proof, we also choose p_0 such that $16\gamma_{sp_0/2} m_0 \leq (1/4) K_1$ and restrict a further, so that $a \leq K_1/2$. Then, as $E_s^n(t,a) \leq 4m_0$ for $t \in [0,T]$ by definition of T we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \sum_{p=p_0}^{n} m_{sp} \frac{a^p}{p!} &\leq -\frac{1}{4} K_1 I_s^n(t,a) + K_1 C_{s,p_0} \\
&\leq -\frac{K_1}{4a} E_s^n(t,a) + \frac{K_1 m_0}{4a} + K_1 C_{s,p_0},
\end{aligned}$$

where for the last inequality we have used that $I_s^n(t,a) \geq (E_s^n(t,a) - m_0)/a$, so that

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{p=p_0}^{n} m_{sp} \frac{a^p}{p!} \leq -\frac{K_1}{4a} E_s^n(t,a) + \frac{K_1 m_0}{4a} + K_1 C_{s,p_0}. \tag{33}$$

Next, recalling estimate (19) in the proof of Lemma 8

$$\frac{\mathrm{d}}{\mathrm{d}t} m_{sp} \leq C' m_{sp}$$

valid for any $p \in \mathbb{N}$ and constant C' depending only on s, p, initial mass and energy. Summing in p, from 0 to $p_0 - 1$, and using estimate (18) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} E_s^n(t,a) \le -\frac{K_1}{4a} E_s^n(t,a) + \frac{K_1 m_0}{4a} + (K_1 + C')C_{s,p_0}\,.$$

This implies, by a maximum principle argument for ODEs, that the bound

$$E_s^n(t,a) \le m_0 + a\left(4 + \frac{C'}{K_1}\right) C_{s,p_0}$$

holds uniformly for $t \in [0,T]$, as the parameters in the right hand side are uniform in time. Choosing a small enough such that

$$m_0 + a\left(4 + \frac{C'}{K_1}\right) C_{s,p_0} < 4m_0\,,$$

or equivalently

$$a < \min\left\{1, a_0, \frac{K_1}{2}, \frac{3m_0}{C_{s,p_0}}\left(4 + \frac{C'}{K_1}\right)^{-1}\right\},$$

where K_1 was defined in (12) and C_{s,p_0} in (31), proves by definition of T that $T = +\infty$ for any n. Passing to the limit $n \to +\infty$ concludes the proof.

In the general case $\beta \in (0,2]$, again as in the previous proof it follows that equation (30) has the extra positive term in the right hand side $K_2 m_{sp}$. The corresponding equation to (32) is now

$$\begin{aligned}(34) \quad \frac{\mathrm{d}}{\mathrm{d}t} \sum_{p=p_0}^{n} m_{sp} \frac{a^p}{p!} \le I_s^n(t,a)\Big[&4\gamma_{sp_0} E_s^n(t,a) \\ &+ (a - K_1)\Big] + K_2 E_s^n(t,a) + K_1 C_{s,p_0} + (K_2 + a)m_1\end{aligned}$$

and consequently,

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{p=p_0}^{n} m_{sp} \frac{a^p}{p!} \le \left(K_2 - \frac{K_1}{4a}\right) E_s^n(t,a) + \frac{K_1 m_0}{4a} + K_1 C_{s,p_0}.$$

In particular, making the additional restriction that $a < K_1/(8K_2)$ we recover exactly equation (33), which implies the bound

$$E_s^n(t,a) \le m_0 + a\left(4 + \frac{C'}{K_1}\right) C_{s,p_0}$$

uniformly for $t \in [0, T]$, where now a is chosen so that

$$a < \min\left\{1, a_0, \frac{K_1}{8K_2}, \frac{3m_0}{C_{s,p_0}}\left(4 + \frac{C'}{K_1}\right)^{-1}\right\},$$

with K_1 and K_2 given in (11), with p_0 such that $\gamma_{sp_0/2} < (32 + 2^{1-\beta})^{-1}1$. The proof is then completed as in the case $\beta \in (0, 1]$ above. □

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