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GAIN OF INTEGRABILITY FOR THE BOLTZMANN COLLISIONAL OPERATOR

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In memory of Carlo Cercignani, from whom we have learned so much, with gratitude.

Abstract. In this short note we revisit the gain of integrability property of the gain part of the Boltzmann collision operator. This property implies the $W^{l,r}_k$ regularity propagation for solutions of the associated space homogeneous initial value problem. We present a new method to prove the gain of integrability that simplifies the technicalities of previous approaches by avoiding the argument of gain of regularity estimates for the gain collisional integral. In addition our method calculates explicit constants involved in the estimates.

1. Introduction.

1.1. Motivation.

In recent years there has been a burst of non-linear analysis methods applied to the systematic study of the Boltzmann equation, both in the space homogeneous and inhomogeneous setting. A task of particular relevance is to find estimates in a suitable functional space that will secure compactness to address the problems of existence, propagation of regularity and stability of the solution and its derivatives in L^r_k spaces, for $r \in (1, \infty)$.

Thus, the goal of this short note is to revisit the gain of integrability in terms of L^r-estimates for the binary collisional integral that will yield a $W^{l,r}_k$ propagation property. We use "gain" of integrability (regularity resp.) terminology to refer that the Lebesgue (Sobolev resp.) norm of the collision operator is controlled by a strictly lower order Lebesgue (Sobolev resp.) norm of the input functions.

Up to now, the study the L^r_k-propagation property of the solution to the initial value problem for the Boltzmann equation, has been so far based on the study

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on the gain of regularity of the gain collisional operator. This regularizing effect for the gain operator was fist proved by P.L. Lions [15] in the context of pseudo-differential operators by means of Fourier methods and generalized Radon transform inspired in the work of Sogge and Stein [16, 17]. The technicalities of the argument motivated Wennberg [19] to provide a simplified proof, in the three dimensional case, using the Carleman integral representation [7] and its relation with weighted Radon transform. Later Mouhot and Villani [18] used such gain of regularity to obtain gain of integrability by simply invoking Sobolev embedding theory. In the process they revisited Wennberg technique and extended the estimates to general dimensions. The methods developed both in [18, 19] relied in the use of the Carleman integral representation of the gain collisional operator and classical Fourier transform. We point out that in those works such gain of regularity for the gain operator Q^+ was proven when both the potential and angular collision kernels are smooth and vanishing near the boundaries. Thus, in practical terms when these estimates are applied to more general kernels, a "truncation" argument needs to be implemented on the collision kernels. For details of this truncation process we refer to [18].

In this note we avoid the argument of regularizing effect and prove directly the gain of L^r_k-integrability of Q^+ using elementary techniques for the space dimension $n \geq 3$. Furthermore, our technique only requires boundedness in the angular kernel, that is, the hard potential does not need to be truncated or smoothed in the boundaries. In practical terms, this argument amounts to simplify the truncation argument of [18] as well. We use of the L^r-estimates of the collisional integral for variable hard potentials as recently developed by the first author with Carneiro [1], and extended in [2], were they calculate exact constants by means of radial rearrangement of the angular part of the gain operator.

Furthermore, due to recent studies of numerical methods for the approximation of this equation [11, 6], we are motivated to obtain estimates that not only provide propagation of $W^{l,r}_k$ regularity and compactness but also are accurate in terms of the characterization of their constants.

1.2. **Preliminaries.** Consider the space homogeneous Boltzmann equation [8] modeling the mass density function $f(v,t)$,

$$\frac{\partial}{\partial t} f = Q(f,f)\,, \qquad \text{for} \ \ v \in \mathbb{R}^n, \ t \in \mathbb{R}, \tag{1}$$

with v are velocity and t evolution variables respectively, and $Q(f,f)$ is the quadratic integral operator, expressing the change of f due to instantaneous binary collisions of particles.

The n-dimensional Boltzmann collision operator written in the bilinear form $Q(f,g)(v,t)$, integrated in the scattering direction σ given by the unitary direction in the post-collisional relative velocity, takes the form

$$Q(f,g)\,(v) = \int_{\mathbb{R}^n} \int_{S^{n-1}} \left(f'\, g'_* - f\, g_*\right)\, B(u, \hat{u}\cdot\sigma)\, d\sigma\, dv_*, \tag{2}$$

where, adopting common shorthand notations, $f = f(v)$, $f' = f(v')$, $g_* = g(v_*)$, $g'_* = g(v'_*)$, $\hat{u}$ is the unitary vector in the direction of u and $d\sigma$ is the surface measure on the sphere S^{n-1}. The variables v, v_* denoting post-collision velocities, its corresponding v', v'_* pre-collision velocities associated to an elastic interaction, and u the relative velocity are related by

$$u = v - v_* \quad , \quad v' = v - \frac{1}{2}(u - |u|\sigma) \ \text{ and } \ v + v_* = v' + v'_* \,. \tag{3}$$

For brevity, we shall often omit the t variable from the notation.

Many properties of the solutions of the Boltzmann equation depend crucially on certain features of the kernel B in (2). The kernel B is often assumed as the product of some function of the magnitude of the relative velocity and the effective scattering cross-section (see [14, §18] for terminology and explicit examples); this quantity characterizes the relative frequency of collisions between particles. Our assumptions on B fall in the category of "hard potentials with angular cutoff". More precisely, we assume that

$$B(u, \hat{u}\cdot\sigma) = |u|^{\gamma}\, b(\hat{u}\cdot\sigma), \tag{4}$$

where $0 < \gamma \leq 1$ is a constant and b is a nonnegative bounded function on $(-1, 1)$. For convenience we normalize it by setting

$$\int_{S^{n-1}} b(\hat{u}\cdot\sigma)\, d\sigma = \omega_{n-2}\int_{-1}^{1} b(z)\,(1-z^2)^{\frac{n-3}{2}}\, dz = 1, \tag{5}$$

where ω_{n-2} is the measure of the $(n-2)$-dimensional sphere. For the classical hard-sphere model in $\mathbb{R}^n$ we have $\gamma = 1$ and $b = \frac{1}{4\pi}$. In this particular case, the collisional form $Q(f,g)$ splits into the difference of two positive operators $Q^+(f,g)$ and $Q^-(f,g)$, gain and loss respectively, defined by (nonsymmetric) bilinear forms of the collision terms,

$$Q^+(f,g) = \int_{\mathbb{R}^n}\int_{S^{n-1}} f'\, g'_*\, B(u,\hat{u}\cdot\sigma)\, d\sigma\, dv_*, \quad \text{and} \quad Q^-(f,g) = f\,(g * |v|^{\gamma}). \tag{6}$$

Wherever necessary we explicitly write a subindex γ to clarify the potential used in the operator,

$$Q^+_{\gamma}(f,g) := Q^+(f,g) \text{ with } B(u,\hat{u}\cdot\sigma) = |u|^{\gamma}\, b(\hat{u}\cdot\sigma).$$

Before presenting the main result of this note, let us consent with the following notation. For the distinct Lebesgue norms associated to the spaces L^r_k we use,

$$\|f\|_{r,k} := \left(\int_{\mathbb{R}^n}\left(f(v) < v >^{k}\right)^{r} dv\right)^{1/r}, \quad \text{with} \ \ k \geq 0, \ r \geq 1.$$

Here we introduced the weight $< v > := \sqrt{1+|v|^2}$. The Sobolev spaces $W^{l,r}_k$, for $l \geq 0$ integer, are defined in standard way using these norms. We will use s' to refer the Lebesgue conjugate of $s \geq 1$, that is, $s^{-1} + s'^{-1} = 1$.

The main result of this note is the following theorem proved for space dimension $n \geq 3$.

Theorem 1. *The following quadratic estimate holds for any $\epsilon > 0$, $r \in (1,\infty)$ and $k \geq 0$,*

$$\int_{\mathbb{R}^n} Q^+(f,f)(v)(f(v))^{r-1} < v >^{rk} dv \leq \epsilon^{s'} C_b(r,n)\|f\|_{1,k}\|f\|^{r}_{r,k} + \frac{C_n}{\epsilon^{s}}\|b\|_{\infty}\|f\|^{2-\theta}_{1,\frac{n-2}{1-\theta}+k}\|f\|^{r-1+\theta}_{r,k}. \tag{7}$$

where $s = \frac{n-2}{\gamma}$ *and* $C_b(r,n)$ *and* C_n *are explicit constants. The parameter* $\theta := \theta_{r,n} \in (0,1)$ *satisfies*

$$\theta = \begin{cases} \frac{1}{n} & \text{if } r \in (1,2] \\ \frac{n(r-2)+1}{n(r-1)} & \text{if } r \in [2,\infty). \end{cases}$$

This estimate can readily be used in the general theory of the initial value problem associated to the n-dimensional Boltzmann equation for $n \geq 3$. Truncating the angular kernel $b \in L^1(S^{n-1})$ in the sets $A := \{z : b(z) \leq M\}$ and A^c, we can choose M such that the mass of b in A^c is as small as desired. Thus, a standard argument, see [18] for details, proves

Corollary 1.1. *Let f be a solution to the space homogeneous Boltzmann problem (1-4) with a hard potential model and $b \in L^1(S^{n-1})$ and initial data $f_0 \in L^r_k$. Then, the following propagation of L^r_k regularity estimate holds for any $r \in (1,\infty)$ and $k \geq 0$,*

$$\frac{d}{dt}\|f\|^r_{r,k} \leq C_1 M \, \|f\|^{2-\theta}_{1,\frac{n-2}{1-\theta}+k}\|f\|^{r-1+\theta}_{r,k} - C_2\|f\|^r_{k+r/\gamma}. \tag{8}$$

where C_1 can be explicitly given depending on $\epsilon, (n-2)/\gamma, C_b(r,n), C_n$ and the L^1_k-norm of the initial data. The parameter $\theta \in (0,1)$ is the one described above.

This corollary recovers Theorem 4.1 in [18] with a broader and explicit range of parameters θ and extended to angular functions b without regularity assumption.

In addition, Theorem 3 below presents the bilinear version of (7). Such estimate allows us to obtain the propagation of $W^{l,r}_k$ norms for $r \in (1,\infty)$ at least for bounded angular cross section.

We point out that the propagation of $W^{l,2}_k$ norms and stability estimates have been recently applied to the study of consistence, convergence and error estimates [6] to the spectral-Lagrangian computational method for the Boltzmann equation developed in [11, 12]. This work addresses an open question posed by Carlo Cercignani and collaborators in the conclusions of their book [8]. Currently, the method of proof presented in this note is being used in the study of the regularity properties of the Boltzmann equation for binary collisions in the presence of a cold thermostat [3] (i.e. a scalar kinetic equation with elastic binary collisions added to a fixed linear form whose background temperature is driven by the Dirac delta dynamics.)

Recall the following representation of the gain collision operator, introduced by Carleman [7] in three dimensions for hard spheres, which we take from Appendix C in [10] in their n-dimensional version,

$$Q^+(g,h)(v) = \\ 2^{n-1}\int_{x\in\mathbb{R}^n} \frac{g(v+x)}{|x|}\int_{z\cdot x=0} h(v+z)|z+x|^{2-n}\, B(-(z+x), 1-2\tfrac{|z|^2}{|z+x|^2})\, d\pi_z\, dx. \tag{9}$$

This representation takes a particularly simple form in the case of the hard-sphere model in $\mathbb{R}^3$; in this case $B(u,\hat{u}\cdot\sigma) = \frac{1}{4\pi}|u|$, thus

$$Q^+(g,h)(v) = \frac{1}{\pi}\int_{x\in\mathbb{R}^3} \frac{g(v+x)}{|x|}\int_{z\cdot x=0} h(v+z)\, d\pi_z\, dx. \tag{10}$$

The remainder of the manuscript is organized as follows. In the next section we further explore the Carleman integral representation in the form (9) applied to the linear form $Q^+(\delta_0, h)$, with the δ_0 a Dirac mass localized at the origin, and write a new representation of the bilinear form of gain operator $Q^+(g,h)$ as a double mixing transform of convolutional type. We present its corresponding L^2 estimates, calculated by the methods introduced in [1], depending on the potential exponent γ which yield exact constants and well determined exponents. In the last section we show, by means of the Reisz-Thorin interpolation theorem the extension to the L^r-theory.

2. L^2 Theory.

Let g and h be nonnegative functions. First, taking the Carleman's representation (9) for the $Q^+(g,h)$, calculated in the translated variables $x \to x - v$ yields

$$Q^+(g,h)(v) = 2^{n-1} \int_{\mathbb{R}^n} \frac{g(x)}{|v-x|} \int_{\{(v-x)\cdot z = 0\}} \frac{\tau_{-x} h(z + (v-x))}{|z + (v-x)|^{n-2}} \tilde{B}(v,z,x)\, d\pi_z\, dx \tag{11}$$

$$\tilde{B}(v,z,x) := B\left(|z+(v-x)|, 1 - 2\tfrac{|z|^2}{|z+(v-x)|^2}\right).$$

Thus applying this representation to the linear form $Q^+(\delta_0, h)$ reads the n-dimensional weighted Radon transform

$$Q^+(\delta_0,h)(v) = \frac{2^{n-1}}{|v|} \int_{v\cdot z=0} \frac{h(z+v)}{|z+v|^{n-2}} B\left(|z+v|, 1 - \frac{2|z|^2}{|z+v|^2}\right) d\pi_z. \tag{12}$$

Therefore combining (11) and (12) we obtain the double mixing convolution

$$Q^+(g,h)(v) = \int_{\mathbb{R}^n} g(x) \tau_x Q^+(\delta_0, \tau_{-x} h)(v) dx. \tag{13}$$

This identity (13) shows the similarities and differences between a regular convolution and the gain collision operator. Let us start by computing the L^2 norm using the Minkowski's integral inequality

$$\begin{aligned} \|Q^+(g,h)\|_2 &= \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} g(x) \tau_x Q^+(\delta_0, \tau_{-x}h)(v) dx \right)^2 dv \right)^{1/2} \\ &\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \left(\tau_x Q^+(\delta_0, \tau_{-x}h)(v) \right)^2 dv \right)^{1/2} g(x) dx. \end{aligned} \tag{14}$$

Thus, we need to compute the L^2-norm of the bilinear gain operator with the Dirac point mass.

Fix $n \geq 3$ and take the explicit notation for the collision operator with potential $|v|^\gamma$ to be $Q^+_\gamma(g,h)$. This estimate can be easily done taking advantage of two observations:

First, for $n \geq 3$ and $\gamma \in (0,1]$ one can estimate $|v|^\gamma \leq \frac{\epsilon^{s'}}{s'} + \frac{1}{s\epsilon^s}|v|^{\gamma r}$ valid for any $\epsilon > 0$. As a consequence

$$Q^+_\gamma(\delta_0,h)(v) \leq \frac{\epsilon^{s'}}{s'} Q^+_0(\delta_0,h)(v) + \frac{1}{s\epsilon^r} Q^+_{n-2}(\delta_0,h)(v),$$

by choosing $s = \frac{n-2}{\gamma}$.

The second observation is that it is possible to estimate the L^2-norm of $Q^+_0(\delta_0, h)$ (Maxwell Molecule type kernels) using the recent estimates of [1] or [2]. Furthermore, using that the angular function b is bounded and the cancellation of the singularity in the Carleman integral representation (9) we can estimate $Q^+_{n-2}(\delta_0, h)$, see Lemma 2.1 below.

Indeed, standard estimates for the bilinear gain operator with the Dirac point mass yield

$$\|Q^+_0(\delta_0, h)\|_2 \le C_b(n)\|h\|_2, \tag{15}$$

where the constant is given by,

$$C_b(n) := \int_{S^{n-1}} \left(\tfrac{1+\hat{u}\cdot\sigma}{2}\right)^{-\frac{n}{4}} b(\hat{u}\cdot\sigma)d\sigma < \infty \quad \text{for} \quad n \ge 3.$$

For the corresponding estimate for the bilinear gain operator with the Dirac point mass with an interaction potential $|u|^{n-2}$ yields the striking estimate

Lemma 2.1. *Let $h \in L^1_{\frac{n-3}{n-1}n}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ non-negative function,*

$$\begin{aligned} \|Q^+_{n-2}(\delta_0, h)(v)\|_2 &\le C_n\|b\|_\infty \|h\|_{\frac{2n}{2n-1}, n-3} \\ &\le C_n\|b\|_\infty \|h\|^{1-\theta}_{1,\frac{n-3}{n-1}n}\|h\|^\theta_2, \end{aligned} \tag{16}$$

with C_n some explicit constant and $\theta := \frac{1}{n} \in (0, 1)$.

Before we prove this Lemma we conclude that, by combining with (15) and (16),

$$\|Q^+_\gamma(\delta_0, h)\|_2 \le \epsilon^{s'} C_b(n)\|h\|_2 + \frac{1}{\epsilon^s} C_n\|b\|_\infty \|h\|^{1-\theta}_{1,\frac{n-3}{n-1}n}\|h\|^\theta_2. \tag{17}$$

In addition, since inequality (14) implies

$$\|Q^+_\gamma(g, h)\|_2 \le \epsilon^{s'} C_b(n)\|g\|_1\|h\|_2 + \frac{1}{\epsilon^s} C_n\|b\|_\infty \|g\|_{1,n-3}\|h\|^{1-\theta}_{1,\frac{n-3}{n-1}n}\|h\|^\theta_2. \tag{18}$$

With these estimates we are in conditions to prove the theorem.

Theorem 2. *The collision operator satisfies the estimate for any $\epsilon > 0$ and $k \ge 0$*

$$\|Q^+_\gamma(g, h)\|_{2,k} \le \epsilon^{s'} C_b(n)\|g\|_{1,k}\|h\|_{2,k} + \frac{1}{\epsilon^s} C_n\|b\|_\infty \|g\|_{1,n-3+k}\|h\|^{1-\theta}_{1,\frac{n-3}{n-1}n+k}\|h\|^\theta_{2,k}\ ,$$

where $\theta = \frac{1}{n}$, $s = \frac{n-2}{\gamma}$ and C_n constant depending only on the dimension.

Proof. It remains to include the weight in the norms. To this end, note the pointwise estimate

$$< v > \le < v' > < v'_* > .$$

Thus, for any $k \ge 0$,

$$Q^+_\gamma(g, h)(v) < v >^k \le Q^+_\gamma(\tilde{g}, \tilde{h})(v),$$

where $\tilde{\psi}(v) := \psi(v) < v >^k$. Therefore,

$$\|Q^+_\gamma(g, h)\|_{2,k} = \|Q^+_\gamma(g, h)(v) < v >^k\|_2 \le \|Q^+_\gamma(\tilde{g}, \tilde{h})\|_2.$$

Using this observation in (18) yields the result. □

The following result follows using Cauchy-Schwarz inequality and Theorem 2.

Corollary 2.2. *The following estimate holds for any $\epsilon > 0$ and $k \geq 0$,*

$$\int_{\mathbb{R}^n} Q^+(f,f)(v)f(v) < v >^{2k}\, dv$$
$$\leq \epsilon^{s'} C_b(n) \|f\|_{1,k} \|f\|_{2,k}^2 + \frac{1}{\epsilon^s} \|b\|_\infty \|f\|^{2-\theta}_{1,\frac{n-3}{n-1}n+k} \|f\|^{1+\theta}_{2,k}.$$

The parameters θ and s are defined in Theorem 2.

It remains to present the proof of Lemma 2.1.

Proof. Using Carleman's representation (9) we write the strong formulation for $Q^+(\delta_0, h)$ as

$$Q^+(\delta_0,h)(v) = \frac{2^{n-1}}{|v|} \int_{z\cdot v=0} \frac{h(z+v)}{|z+v|^{n-2}}\, B\left(|z+v|, 1 - \frac{2|z|^2}{|z+v|^2}\right) d\pi_z\,,$$

where $d\pi_z$ is the $\mathbb{R}^{n-1}$ Lebesgue's measure. Clearly taking a $(n-2)$-variable potential produces a cancellation between the potential and the term $|z+v|^{n-2}$. In particular, the Carleman representation of Q^+_{n-2} reduces to

$$Q^+_{n-2}(\delta_0,h)(v) = \frac{2^{n-1}}{|v|} \int_{z\cdot v=0} h(z+v)\, b\left(1 - \frac{2|z|^2}{|z+v|^2}\right) d\pi_z \qquad (19)$$
$$\leq 2^{n-1}\, \|b\|_\infty \frac{1}{|v|} \int_{z\cdot v=0} h(z+v) d\pi_z.$$

Using polar coordinates $v = r\sigma$, it is possible to estimate its L^2-norm in the following way

$$\left(2^{n-1}\, \|b\|_\infty\right)^{-2} \int_{\mathbb{R}^n} \left(Q^+(\delta_0,h)(v)\right)^2 dv$$
$$\leq \int_{S^{n-1}} \int_{\mathbb{R}} \int_{z_1\cdot\sigma=0} \int_{z_2\cdot\sigma=0} h(z_1+r\sigma)h(z_2+r\sigma) d\pi_{z_1} d\pi_{z_2} r^{n-3} dr d\sigma =: I.$$

next, perform the change of variables for fixed σ, $x := z_1 + r\sigma$. Note that $r = x\cdot\sigma$, and thus,

$$I = \int_{S^{n-1}} \int_{\mathbb{R}^n} \int_{z_2\cdot\sigma=0} h(x)h(z_2 + (x\cdot\sigma)\sigma) d\pi_{z_2} (x\cdot\sigma)^{n-3} dx d\sigma.$$

Writing

$$z_2 + (x\cdot\sigma)\sigma = x + (z_2 + (x\cdot\sigma)\sigma - x) := x + z_3,$$

it easy to check that $z_3 \in \{z : z\cdot\sigma = 0\}$. Hence,

$$I = \int_{S^{n-1}} \int_{\mathbb{R}^n} \int_{z_3\cdot\sigma=0} h(x)h(x+z_3) d\pi_{z_3} (x\cdot\sigma)^{n-3} dx d\sigma.$$

Using the identity

$$\int_{\mathbb{R}^n} \delta_0(z\cdot y)\, \varphi(z)\, dz = |y|^{-1} \int_{z\cdot y=0} \varphi(z)\, d\pi_z \qquad (20)$$

valid for any smooth φ, we transform the integration in the hyperplane $\{z_3\cdot\sigma = 0\}$ into an integration in $\mathbb{R}^n$,

$$\int_{z_3\cdot\sigma=0} h(x+z_3) d\pi_{z_3} = \int_{\mathbb{R}^n} \delta(z\cdot\sigma) h(x+z) dz$$
$$= \int_{\mathbb{R}^n} \delta(\hat{z}\cdot\sigma) \frac{h(x+z)}{|z|} dz.$$

Hence,

$$I \leq C_n \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\tilde{h}(x)h(z)}{|z-x|}dzdx,$$

where $\tilde{h}(x) := h(x)|x|^{n-3}$. Furthermore, the constant is explicit,

$$C_n = \int_{S^{n-1}} \delta_0(\hat{z}\cdot\sigma)d\sigma = |S^{n-2}|\int_{-1}^{1}\delta_0(s)(1-s^2)^{\frac{n-3}{2}}ds = |S^{n-2}|.$$

Finally, recalling the Hardy-Littlewood-Sobolev inequality we have for any $r \in (1,\infty)$,

$$\begin{aligned} I &\leq C_n\|\tilde{h}\|_{r'}\left\|h * |x|^{-1}\right\|_r \\ &\leq C_{n,r,p}\|\tilde{h}\|_{r'}\left\|h\right\|_p, \end{aligned}$$

where $1/p + 1/n = 1 + 1/r$. Choosing $p = r'$, that is $p = \frac{2n}{2n-1}$, yields

$$I \leq \tilde{C}_n\|\tilde{h}\|_{\frac{2n}{2n-1}}\left\|h\right\|_{\frac{2n}{2n-1}} \leq \tilde{C}_n\left\|h\right\|^2_{\frac{2n}{2n-1},n-3}. \tag{21}$$

Finally using Lebesgue's interpolation

$$\|h\|_{\frac{2n}{2n-1},n-3} \leq \|h\|^{1-\theta}_{1,\frac{n-3}{1-\theta}}\|h\|^{\theta}_2 = \|h\|^{1-\theta}_{1,\frac{n-3}{n-1}n}\|h\|^{\theta}_2. \tag{22}$$

with $\theta := \frac{1}{n}$. This proves inequality (16) and the proof on Lemma 2.1 is completed. □

3. L^r **Theory.** In order to find estimates for the full gain collision operator for any $r \in (1,\infty)$, the easiest path is to use interpolation theory, more specifically, the Riesz-Thorin theorem first between L^1 and L^2 spaces and second between L^2 and L^∞ ones.

Case 1: $r \in (1,2]$*:* The interpolation is made using the estimates

$$\begin{aligned} \left\|Q^+_{n-2}(\delta_0,h)\right\|_1 &\leq \|b\|_1\|h\|_{1,n-2} \\ \left\|Q^+_{n-2}(\delta_0,h)\right\|_2 &\leq C_n\|b\|_\infty\|h\|_{\frac{2n}{2n-1},n-2}. \end{aligned}$$

We refer to [1, 2] for the first estimate, and Lemma 2.1 for the second one since $L_{\frac{2n}{2n-1},n-3} \hookrightarrow L_{\frac{2n}{2n-1},n-2}$.

The interpolation of L^1 and L^2 by L^1 and $L^{\frac{2n}{2n-1}}$ yields

$$\left\|Q^+_{n-2}(\delta_0,h)\right\|_r \leq C(b,n)\|h\|_{p,n-2}, \tag{23}$$

$$\text{where } \frac{1-1/n}{1} + \frac{1/n}{r} = \frac{1}{p}, \text{ that is, } p = \frac{nr}{r(n-1)+1}.$$

Case 2: $r \in [2,\infty)$*:* The interpolation between L^2 and L^∞ can be done using the additional estimate from [1, 2]

$$\left\|Q^+_{n-2}(\delta_0,h)\right\|_\infty \leq C_b(\infty,n)\|h\|_{\infty,n-2},$$

where the constant is

$$C_b(\infty,n) = \int_{S^{n-1}}\left(\tfrac{1+\hat{u}\cdot\sigma}{2}\right)^{-\frac{n}{2}} b(\hat{u}\cdot\sigma)d\sigma.$$

Note that this constant is finite as long as b is defined in the hemisphere$\{\hat{u}\cdot\sigma \geq 0\}$. This is not a restriction in the elastic theory for the quadratic estimates.

Again, using the Riesz-Thorin theorem, now for the interpolation of L^2 and L^∞ by $L^{\frac{2n}{2n-1}}$ and L^∞ yields,

$$\left\|Q^+_{n-2}(\delta_0,h)\right\|_r \le C(b,n)\|h\|_{p,n-2} \quad \text{with} \quad p=\frac{nr}{2n-1}\,. \tag{24}$$

In order to obtain the gain in integrability, let $\theta\in(0,1)$ be related to the exponents p and r by the convex combination

$$\frac{1-\theta}{1}+\frac{\theta}{r}=\frac{1}{p},$$

then Lebesgue's interpolation reads,

$$\|h\|_{p,n-2}\le \|h\|^{1-\theta}_{1,\frac{n-2}{1-\theta}}\|h\|^{\theta}_r\,.$$

Therefore, we conclude that for any p satisfying the conditions in (23) or in (24), there exist $\theta\in(0,1)$ such that,

$$\left\|Q^+_{n-2}(\delta_0,h)\right\|_r \le C(b,n)\|h\|^{1-\theta}_{1,\frac{n-2}{1-\theta}}\|h\|^{\theta}_r, \tag{25}$$

where the parameter $\theta:=\theta_{r,n}\in(0,1)$ satisfies

$$\theta:=\theta_{r,n}=\begin{cases}\frac{1}{n} & \text{if } \; r\in(1,2] \;\text{ and }\; p=\frac{nr}{r(n-1)+1}\,,\\ \frac{n(r-2)+1}{n(r-1)} & \text{if } \; r\in[2,\infty) \;\text{ and }\; p=\frac{nr}{2n-1}\,.\end{cases} \tag{26}$$

We can now obtain the estimates for the linear gain operator acting on the Dirac point mass, for variable hard potentials rates $\gamma\in(0,1]$, proceeding in a similar form as worked out in (17) for the case $r=2$. In particular, the following inequality is valid for any $\epsilon>0$, $r\in(1,\infty)$ and $s=\frac{n-2}{\gamma}$

$$\|Q^+_\gamma(\delta_0,h)\|_r \le \epsilon^{s'}C_b(r,n)\|h\|_2+\frac{1}{\epsilon^s}C_n\|b\|_\infty\|h\|^{1-\theta}_{1,\frac{n-2}{1-\theta}}\|h\|^{\theta}_r, \tag{27}$$

where $\theta\in(0,1)$ defined in (26). Furthermore, the constant $C_b(r,n)$ is given by

$$C_b(r,n):=\int_{S^{n-1}}\left(\tfrac{1+\hat u\cdot\sigma}{2}\right)^{-\frac{n}{2r'}}b(\hat u\cdot\sigma)d\sigma<\infty \quad \text{for} \quad n\ge 3.$$

A simple use of use of Minkowski's integral inequality

$$\begin{aligned}\|Q^+_\gamma(g,h)\|_r &= \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} g(x)\tau_x Q^+_\gamma(\delta_0,\tau_{-x}h)(v)dx\right)^r dv\right)^{1/r}\\ &\le \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\left(\tau_x Q^+_\gamma(\delta_0,\tau_{-x}h)(v)\right)^r dv\right)^{1/r} g(x)dx\\ &= \int_{\mathbb{R}^n}\|Q^+_\gamma(\delta_0,\tau_{-x}h)\|_r g(x)dx.\end{aligned} \tag{28}$$

Combining (27) and (28) proves the following generalization of Theorem 2.

Theorem 3. *The collision operator satisfies the estimate for any $\epsilon>0$, $r\in(1,\infty)$ and $k\ge 0$*

$$\|Q^+_\gamma(g,h)\|_{r,k}\le \epsilon^{s'}C_b(r,n)\|g\|_{1,k}\|h\|_{r,k}+\frac{1}{\epsilon^s}C_n\|b\|_\infty\|g\|_{1,n-2+k}\|h\|^{1-\theta}_{1,\frac{n-2}{1-\theta}+k}\|h\|^{\theta}_{r,k}\,,$$

where $\theta\in(0,1)$ from (26), $s=\frac{n-2}{\gamma}$ and C_n constant depending only on the dimension.

Finally, using Hölder's inequality and Theorem 3 we obtain the following integral estimate for the quadratic $Q^+(f,f)(v)$,

$$\int_{\mathbb{R}^n} Q^+(f,f)(v)(f(v))^{r-1} < v >^{rk} \, dv$$
$$\leq \epsilon^{s'} C_b(r,n) \|f\|_{1,k} \|f\|^r_{r,k} + \frac{1}{\epsilon^s} \|b\|_\infty \|f\|^{2-\theta}_{1,\frac{n-2}{1-\theta}+k} \|f\|^{r-1+\theta}_{r,k}.$$

which holds for any $\epsilon > 0$, $r \in (1,\infty)$ and $k \geq 0$. The parameter θ is defined in (26) and $s = \frac{n-2}{\gamma}$. This last result concludes the proof of the **Theorem 1**.

Remark. The amount of moments needed in this inequality, namely $\frac{n-2}{1-\theta}$, is not optimal but irrelevant for the general theory of the Boltzmann equation. Furthermore, the interpolating exponent $p = \frac{2n}{2n-1}$ improves, in the range $r \in (1,2]$, the one obtained using the previous technique which invokes the Sobolev injection $H^{\frac{n-1}{2}} \hookrightarrow L^{2n}$. Indeed, the precise exponent using this technique is $\tilde{p} = \frac{4n-2}{3n-2}$ and clearly $p < \tilde{p}$ for any $n \geq 3$.

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