The Cauchy problem and BEC stability for the quantum Boltzmann-Condensation system for bosons at very low temperature

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Abstract

We study a quantum Boltzmann-Condensation system that describes the evolution of the interaction between a well formed Bose-Einstein condensate and the quasi-particles cloud. The kinetic model is valid for a dilute regime at which the temperature of the gas is very low compared to the Bose-Einstein condensation critical temperature. In particular, our system couples the density of the condensate from a Gross-Pitaevskii type equation to the kinetic equation through the dispersion relation in the kinetic model and the corresponding transition probability rate from pre to post collision momentum states. We rigorously show the following three properties (1) the well-posedness of the Cauchy problem for the system in the case of a radially symmetric initial configuration, (2) find qualitative properties of the solution such as instantaneous creation of exponential tails and, (3) prove the uniform condensate stability related to the initial mass ratio between condensed particles and quasi-particles. The stability result from (3) leads to global in time existence of the initial value problem for the quantum Boltzmann-Condensation system.

Keywords Quantum kinetic theory, low-temperature Bose particles, stability of BECs, spin-Peierls model, moments method, abstract ODE theory.

MSC: 82C10, 82C22, 82C40.

Contents

1 Introduction

After the first Bose-Einstein Condensate (BEC) was produced by Cornell, Wieman, and Ketterle [3, 4], there has been an immense amount of research on BECs and cold bosonic gases. Above the condensation temperature, the dynamic of a bose gas is determined by the Uehling-Uhlenbeck kinetic equation introduced in [31]; see for instance [15] for interesting results and list of references. The first proof of BECs was done in [22]. Below the condensation temperature, the bosonic gas dynamics is governed by a system modeling the coupling of quantum Boltzmann and a model of condensation, such us the Gross-Pitaevskii, equations. In such a system, the wave function of the BEC follows the Gross-Pitaevskii equation and the quantum Boltzmann equation describes the evolution of the density function of the excitations (quasi-particles). The system was first derived by Kirkpatrick and Dorfmann in [20, 21], using a Green function approach and was revisited by Zaremba-Nikuni-Griffin and Gardiner-Zoller et. al. in [17, 18, 32]. It has, then, been developed and studied extensively in the last two decades by several authors from the application perspective (see [6, 25, 29], and references therein). In [28], Spohn gave a heuristic derivation for the one-dimensional version of the system, using a perturbation argument for the Uehling-Uhlenbeck equation. A more formal derivation, for the full three dimensional case, is done in [26] where some ideas from the works [8, 12] were taken together with techniques from quantum field theory.

In this work, we focus on the rigorous mathematical study of the dynamics of dilute Bose gases modeled by the quantum Boltzmann equation at very low temperature coupled to the condensation model at the quantum level. The quantum Boltzmann model that we referred to was introduced in [11, 13, 20, 21], that is, the BEC is well formed and the interaction between excited atoms is secondary relative to the interaction between excited atoms with the BEC. The condensation at the quantum level may be described by classical models such as Gross-Pitaevskii [8, 12, 22]. At this quantum level the BEC mass is given by $n_c = n_c(t) := |\Psi|^2(t)$, where Ψ is the wave function of the quantum condensation satisfying a Gross-Pitaevskii type equation with an absorption term proportional to the averaged of the interacting particle (collision) operator from the quantum kinetic model, and the corresponding quantum probability density of the excited states evolves according to the quantum Boltzmann equation with interacting particle (collision) operator proportional to the condensate $n_c(t)$ (cf. [5, 28, 24]).

Under these assumptions, the evolution of the space homogeneous probability density distribution function $f := f(t,p)$, with $(t,p) \in [0,\infty) \times \mathbb{R}^3$, for p the momenta state variable, of the excited bosons and the condensate mass $n_c := n_c(t)$ can be described by the following Boltzmann-Gross-Pitaevskii system

$$\begin{cases} \frac{\mathrm{d}f}{\mathrm{d}t} &= n_c\, Q[n_c, f]\,, \qquad\qquad f(0,\cdot) = f_0\,, \\ \frac{\mathrm{d}n_c}{\mathrm{d}t} &= -n_c \int_{\mathbb{R}^3} \mathrm{d}p\, Q[n_c, f]\,, \quad n_c(0) = n_0\,, \end{cases} \tag{1.1}$$

where the interaction operator is defined as

$$
\begin{aligned}
&Q[n_c, f] := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathrm{d}p_1 \mathrm{d}p_2 \big[R(p,p_1,p_2) - R(p_1,p,p_2) - R(p_2,p_1,p)\big]\,, \\
&R(p,p_1,p_2) := \\
&\qquad |\mathcal{M}(p,p_1,p_2)|^2 \big[\delta\left(\frac{\omega(p)}{k_BT} - \frac{\omega(p_1)}{k_BT} - \frac{\omega(p_2)}{k_BT}\right)\delta(p-p_1-p_2)\big] \\
&\qquad\qquad \times \big[f(p_1)f(p_2)(1+f(p)) - (1+f(p_1)(1+f(p_2))f(p)\big]\,,
\end{aligned}
\tag{1.2}
$$

where $\beta := \frac{1}{k_BT} > 0$ is a physical constant depending on the Boltzmann constant k_B, and the temperature of the quasiparticles T at equilibrium. The particle energy $\omega(p)$ is given by the Bogoliubov dispersion law

$$
\omega(p) = \left[\frac{gn_c}{m}|p|^2 + \left(\frac{|p|^2}{2m}\right)^2\right]^{1/2}, \tag{1.3}
$$

where $p \in \mathbb{R}^3$ is the momenta, m is the mass of the particles, g is an interaction "excited-condensate" coupling constant and n_c is the condensate mass, as introduced earlier.

The term $\mathcal{M}(p,p_1,p_2)$ is referred as the transition probability or matrix element (as much as collision kernel). Its constitutive relation depends on the dispersion relation $\omega(p)$ and, consequently, strongly couples the quantum Boltzmann equation to the quantum condensate.

In the regime treated in this document, the transition probability can be approximated up to first order to a workable expression. Indeed, we restrict the range of the temperature T, the condensate density n_c, and the interaction coupling constant g to values for which k_BT is much smaller than $(gn_c/m)^{1/2}$, i.e. a *cold gas regime*. Under this condition, the dispersion law $\omega(p)$ in (1.3) is approximated by

$$
\frac{1}{k_BT}\left[\frac{gn_c}{m}|p|^2 + \left(\frac{|p|^2}{2m}\right)^2\right]^{1/2} \approx \frac{c}{k_BT}|p|, \quad \text{where } c := \sqrt{\frac{gn_c}{m}}\,,
$$

as long as $|p| \ll 2\sqrt{gn_cm}$. In particular, the energy will now be defined by the phonon dispersion law (still using the same notation), see [11, 19]

$$
\omega(p) = c|p|, \quad \text{for } c := c(t) = \sqrt{\frac{gn_c(t)}{m}}\,. \tag{1.4}
$$

Under the *cold gas regime*, the transition probability $\mathcal{M}$ is approximated by (see, for instance [13, eq. (7)], [19, eq. (83)], [11, eq. (42)])

$$
|\mathcal{M}|^2 = \kappa|p||p_1||p_2| \tag{1.5}
$$

where

$$\kappa = \frac{9c}{64\pi^2 m n_c^2} = \frac{9}{64\pi^2 (m g n_c)^{3/2}}. \tag{1.6}$$

Note that the transition probability could also be approximated as (cf. [5])

$$|\mathcal{M}|^2 = \frac{\omega(p)\omega(p_1)\omega(p_2)}{32 g^3 n_c^3}.$$

We perform the analysis in the whole momentum space, not in a piece of it or the torus [27], requiring a detailed control of the solution's tails and low temperature behavior.

Using that $\delta(\cdot)$ is homogeneous of degree -1, the reduced phonon dispersion law (1.4) is implemented as $\delta(c|p|) = c^{-1}\delta(|p|)$, and so the quantum collisional integral (1.2) becomes

$$\begin{aligned} Q[n_c, f] :&= \kappa c^{-1} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathrm{d}p_1 \mathrm{d}p_2 \big[R(p,p_1,p_2) - R(p_1,p,p_2) - R(p_2,p_1,p)\big] \\ R(p,p_1,p_2) &:= \mathcal{K}(|p|,|p_1|,|p_2|)\big[\delta\left(|p|-|p_1|-|p_2|\right)\delta(p-p_1-p_2)\big] \\ &\quad \times \big[f(p_1)f(p_2)(1+f(p)) - (1+f(p_1)(1+f(p_2))f(p)\big]. \end{aligned} \tag{1.7}$$

Here we introduced $\mathcal{K}(|p|,|p_1|,|p_2|) := |p||p_1||p_2|$. Clearly, from the interaction law $p = p_1 + p_2$ and $|p| = |p_1| + |p_2|$ modeled in the collision operator by the singular Dirac delta masses, this trilinear collisional form (1.7) is reduced into a bilinear one, that can be split in the difference of two positive quadratic operators, as will be shown in the existence result.

In addition, the low temperature quantum collisional form (1.7) can be split into *gain* and *loss* operator forms

$$\begin{aligned} Q[n_c, f](t,p) &= Q^+[n_c,f](t,p) - Q^-[n_c,f](t,p) \\ &= \kappa c^{-1}\Big(\mathcal{Q}^+[f](t,p) - f(t,p)\,\nu[f](t,p)\Big) =: \kappa c^{-1}\mathcal{Q}[f](t,p), \end{aligned} \tag{1.8}$$

as is done with the classical Boltzmann operator. Here, the gain operator is also defined by the positive contributions in the total rate of change in time of the collisional form $Q[n_c,f](t,p)$ in (1.7), that is, $Q^+[n_c,f] = \kappa c^{-1}\mathcal{Q}^+[f]$ where

$$\begin{aligned} \mathcal{Q}^+[f](t,p) :&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathrm{d}p_1 \mathrm{d}p_2\, \mathcal{K}(|p|,|p_1|,|p_2|)\delta(p-p_1-p_2) \\ &\times \delta(|p|-|p_1|-|p_2|) f(t,p_1) f(t,p_2) + 2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathrm{d}p_1 \mathrm{d}p_2\, \mathcal{K}(|p|,|p_1|,|p_2|) \\ &\times \delta(p_1-p-p_2)\delta(|p_1|-|p|-|p_2|)\big[2f(t,p)f(t,p_1) + f(t,p_1)\big]. \end{aligned} \tag{1.9}$$

Similarly, the loss operator models the negative contributions in the total rate of change in time of same collisional form $Q[n_c, f](t,p)$. It is local in $f(t,p)$ and so written $Q^-[n_c, f] := \kappa c^{-1}\, f\, \nu[f]$, where $\nu[f](t,p)$, referred as the *collision frequency or attenuation coefficient*, is defined by

$$
\begin{aligned}
\nu[f](t,p) := &\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathrm{d}p_1 \mathrm{d}p_2\, \mathcal{K}(|p|,|p_1|,|p_2|)\delta(p-p_1-p_2) \\
&\times \delta(|p|-|p_1|-|p_2|)\big[2f(t,p_1)+1\big] + 2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathrm{d}p_1 \mathrm{d}p_2\, \mathcal{K}(|p|,|p_1|,|p_2|) \\
&\quad \times \delta(p_1-p-p_2)\delta(|p_1|-|p|-|p_2|)f(t,p_2)\,,
\end{aligned} \tag{1.10}
$$

and it is nonlocal in $f(t,p)$. Note that the collisional operator $\mathcal{Q}[f] := \mathcal{Q}^+[f] - f\,\nu[f]$ is independent of n_c.

In summary, our goal is to study the Cauchy problem of radial solutions for the Boltzmann-Gross-Pitaevskii system (1.1) at low temperature, which, with the definitions of (1.8), (1.9) and (1.10), reads

$$
\begin{cases}
\frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{\kappa_0}{n_c}\mathcal{Q}[f]\,, \qquad\qquad\quad f(0,\cdot) = f_0\,, \\
\frac{\mathrm{d}n_c}{\mathrm{d}t} &= -\frac{\kappa_0}{n_c}\int_{\mathbb{R}^3}\mathrm{d}p\,\mathcal{Q}[f]\,, \quad n_c(0) = n_0\,,
\end{cases} \tag{1.11}
$$

where the resulting constant $\kappa_0 = \frac{9}{64\pi^2 m}$.

The organization of the paper is as follows.

· In section 2 and 3 we present the weak and strong formulations of the collision operator and use them to recall the main conservation laws as well as the entropy estimate corresponding to an H-Theorem for (1.1) in the low temperature regime collisional form (1.7).

· Section 4 considers *a priori* estimates on the observables or moments of solutions. These are related to high energy tail behavior and will be developed in context of radially symmetric solutions. Moment propagation techniques have been developed for the classical Boltzmann equation in [2, 16, 30].

· In section 5 we address the central issue of the BEC stability. It is clear that the condition $n_c > 0$ is essential for the validity of the approximations that have been made in the derivation of the model. In this section we take advantage of the nonlinear nature of the equation to derive L^∞-estimates that allow us to show the BEC uniform stability.

Natural conditions in terms of the ratio between the initial mass of the condensate and quasi-particles are necessary for the sustainability of the condensate in the long run. This result formalizes the validity of the decomposition of the total density of the gas between a singular part (condensate) and a regular part (quasi-particles) and leads to global in time well-posedness of the problem.

· The existence and uniqueness arguments given in section 6 are based on the *a priori* estimates on the solution's moments and the L^∞-estimate provided for BEC stability. When such estimates are combined with classical abstract ODE theory, the result is a robust and elegant technique to prove well-posedness for collisional integral equations.

· Finally, in section 7, we show that solution to the Cauchy problem have exponential decaying tails in the sense of $L^1(\mathbb{R}^3)$, which are referred to as Mittag-Leffler tails that were introduced for the Boltzmann equation in [30]. This result formalizes, at least qualitatively, the approximations that are made in the low temperature regime were narrow distribution profiles are assumed.

2 Weak and strong formulation of collisional forms

The following properties hold for the low temperature quantum collisional form (1.7) remarking that, for notational convenience, we will usually omit the time variable t unless some stress is necessary in the context.

Proposition 2.1 (Weak Formulation) *For any suitable test function φ, the following weak formulation holds for the collision operator $\mathcal{Q}$*

$$\begin{aligned}
&\int_{\mathbb{R}^3} dp\, \mathcal{Q}[f](p)\varphi(p) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} dp\, dp_1\, dp_2\, \mathcal{K}(|p|,|p_1|,|p_2|)\delta(p-p_1-p_2) \\
&\qquad \times \delta(|p|-|p_1|-|p_2|)\Big[f(p_1)f(p_2)-f(p_1)f(p)-f(p_2)f(p)-f(p)\Big] \\
&\qquad\qquad \times \Big[\varphi(p)-\varphi(p_1)-\varphi(p_2)\Big] \\
&= 2\pi\int_{\mathbb{R}^3} dp_1 \int_{\mathbb{R}^+} |p_2|^2 d|p_2|\, \frac{|p_1|+|p_2|}{|p_1||p_2|}\mathcal{K}\big(|p_1|+|p_2|,|p_1|,|p_2|\big)\Big[f(p_1)f(|p_2|\widehat{p_1}) \\
&\qquad - f(p_1)f(p_1+|p_2|\widehat{p_1}) - f(|p_2|\widehat{p_1})f(p_1+|p_2|\widehat{p_1}) - f(p_1+|p_2|\widehat{p_1})\Big] \\
&\qquad \times \Big[\varphi(p_1+|p_2|\widehat{p_1})-\varphi(p_1)-\varphi(|p_2|\widehat{p_1})\Big],
\end{aligned} \tag{2.1}$$

As a consequence, for radially symmetric functions $f(p) := f(|p|)$ *and* $\varphi(p) := \varphi(|p|)$*, the following holds true*

$$\int_{\mathbb{R}^3} dp\, \mathcal{Q}[f](p)\varphi(p) = 8\pi^2 \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} d|p_1|\, d|p_2|\, \mathcal{K}_0\big(|p_1|+|p_2|, |p_1|, |p_2|\big)\times \Big[f(|p_1|)f(|p_2|) - f(|p_1|)f(|p_1|+|p_2|) - f(|p_2|)f(|p_1|+|p_2|) - f(|p_1|+|p_2|)\Big] \times \Big[\varphi(|p_1|+|p_2|) - \varphi(|p_1|) - \varphi(|p_2|)\Big], \tag{2.2}$$

where $\mathcal{K}_0(|p|,|p_1|,|p_2|) := |p||p_1||p_2|\mathcal{K}(|p|,|p_1|,|p_2|) = |p|^2|p_1|^2|p_2|^2$.

Proof. In this proof we use the short-hand $\int := \int_{\mathbb{R}^9} \mathrm{d}p\, \mathrm{d}p_1\, \mathrm{d}p_2$. First, observe that

$$\begin{aligned}\int_{\mathbb{R}^3} \mathrm{d}p\, \mathcal{Q}[f](p)\varphi(p) = &\int \mathcal{K}(|p|,|p_1|,|p_2|)\delta(p-p_1-p_2)\delta(|p|-|p_1|-|p_2|)R(p,p_1,p_2)\varphi(p)\\ &- \int \mathcal{K}(|p|,|p_1|,|p_2|)\delta(p-p_1-p_2)\delta(|p|-|p_1|-|p_2|)R(p_1,p,p_2)\varphi(p)\\ &- \int \mathcal{K}(|p|,|p_1|,|p_2|)\delta(p-p_1-p_2)\delta(|p|-|p_1|-|p_2|)R(p_2,p_1,p)\varphi(p)\,.\end{aligned} \tag{2.3}$$

Second, interchanging variables $p \leftrightarrow p_1$ and $p \leftrightarrow p_2$,

$$\int \mathcal{K}(|p|,|p_1|,|p_2|)R(p_1,p,p_2)\varphi(p) = \int \mathcal{K}(|p|,|p_1|,|p_2|)R(p,p_1,p_2)\varphi(p_1)\,, \tag{2.4}$$

and

$$\int \mathcal{K}(|p|,|p_1|,|p_2|)R(p_2,p_1,p)\varphi(p) = \int \mathcal{K}(|p|,|p_1|,|p_2|)R(p,p_1,p_2)\varphi(p_2)\,. \tag{2.5}$$

Combining (2.3), (2.4), (2.5), we get the first equality in (2.1). Now, evaluate the Dirac in $p = p_1 + p_2$ (conservation of momentum) to obtain

$$\int_{\mathbb{R}^3} \mathrm{d}p\, \mathcal{Q}[f](p)\varphi(p) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathcal{K}(|p_1+p_2|,|p_1|,|p_2|)\delta(|p_1+p_2|-|p_1|-|p_2|) \Big[f(p_1)f(p_2) - f(p_1)f(p_1+p_2) - f(p_2)f(p_1+p_2) - f(p_1+p_2)\Big] \times \Big[\varphi(p_1+p_2) - \varphi(p_1) - \varphi(p_2)\Big]\mathrm{d}p_1\, \mathrm{d}p_2\,, \tag{2.6}$$

Now, observe that $|p_1+p_2|-|p_1|-|p_2|=0$ if and only if $\widehat{p_1}\cdot\widehat{p_2}=1$. Since,

$$|p_1+p_2|-|p_1|-|p_2| = \big(|p_1|^2+|p_2|^2+2|p_1||p_2|\widehat{p_1}\cdot\widehat{p_2}\big)^{1/2}-|p_1|-|p_2|\,,$$

it follows from a polar change of variable, taking $\widehat{p_1}$ as the zenith, that the following identity holds for any continuous function $F(p_2)$

$$\begin{aligned}
&\int_{\mathbb{R}^3}\mathrm{d}p_2\,F(p_2)\,\delta\big(|p_1+p_2|-|p_1|-|p_2|\big)\\
&\qquad=\int_{\mathbb{R}^+}|p_2|^2\mathrm{d}|p_2|\int_0^{2\pi}\mathrm{d}\phi\int_{-1}^{1}\mathrm{d}s\,F\big(p_2(s,\sin(\phi))\big)\delta(y(s))\\
&\qquad=2\pi\int_{\mathbb{R}^+}|p_2|^2\mathrm{d}|p_2|\,\frac{F(|p_2|\widehat{p_1})}{y'(1)}=2\pi\int_{\mathbb{R}^+}|p_2|^2\mathrm{d}|p_2|\,F(|p_2|\widehat{p_1})\frac{|p_1|+|p_2|}{|p_1||p_2|}\,,
\end{aligned}$$

where $y(s)=\big(|p_1|^2+|p_2|^2+2|p_1||p_2|s\big)^{1/2}-|p_1|-|p_2|$. In the second identity we used that $p_2(1,\sin(\phi))=|p_2|\widehat{p_1}$ and, for the latter, the fact that $y'(1)=\frac{|p_1|p_2|}{|p_1|+|p_2|}$. Using this identity in (2.6) proves the second equality in (2.1). Finally, for radially symmetric functions $f(p):=f(|p|)$ and $\varphi(p):=\varphi(|p|)$, one simply uses that $\big|p_1+|p_2|\widehat{p_1}\big|=|p_1|+|p_2|$ and polar coordinates in the p_1-integral to obtain (2.2) ■

Based on the weak formulation of the collision operator, we can deduce its strong formulation. The strong formulation will be important for finding L^∞-estimates to prove the BEC uniform stability. The nonlinear part of the operator will play an important role in the estimates, thus, in this context we write the operator as a quadratic part and a linear part

$$\mathcal{Q}[f](p)=\mathcal{Q}_q[f](p)+\mathcal{L}[f](p)\,,$$

and stress that this decomposition is different from that of gain and loss parts. Indeed, the linear part is only a piece of the loss operator which includes bilinear terms.

Corollary 2.1 (Strong Formulation) *Let f be a radially symmetric function. The strong formulation of the collision operator consists in 9 quadratic terms, namely,*

$$\begin{aligned}
\mathcal{Q}_q[f](|p|):=8\pi^2\bigg(&\int_0^{|p|}d|p_1|K(|p_1|,|p|-|p_1|)f(|p_1|)f(|p|-|p_1|)\\
&+\int_{|p|}^{\infty}d|p_1|\big(K(|p_1|-|p|,|p|)+K(|p|,|p_1|-|p|)\big)f(|p_1|)f(|p_1|-|p|)\bigg)
\end{aligned}$$

$$
\begin{aligned}
&+ 8\pi^2 f(|p|)\Big(\int_{|p|}^{\infty} d|p_1|\big(K(|p|, |p_1| - |p|) + K(|p_1| - |p|, |p|)\big) f(|p_1|) \\
&\qquad - \int_0^{|p|} d|p_1|\big(K(|p| - |p_1|, |p_1|) + K(|p_1|, |p| - |p_1|)\big) f(|p_1|) \\
&\qquad - \int_0^{\infty} d|p_1|\big(\mathcal{K}_0(|p|, |p_1|) + \mathcal{K}(|p_1|, |p|)\big) f(|p_1|)\Big) .
\end{aligned}
$$

The strong formulation of the linear operator reduces to 3 terms,

$$
\begin{aligned}
\mathcal{L}[f](|p|) = 8\pi^2\Big(&\int_{|p|}^{\infty} d|p_1|\big(K(|p|, |p_1| - |p|) + K(|p_1| - |p|, |p|)\big) f(|p_1|) \\
&- f(|p|) \int_0^{|p|} d|p_1| K(|p_1|, |p| - |p_1|)\Big) ,
\end{aligned}
$$

where the symmetric collision kernel is defined by

$$
K(|p_1|, |p_2|) := \mathcal{K}_0(|p_1| + |p_2|, |p_1|, |p_2|) = |p_1|^2 |p_2|^2 (|p_1| + |p_2|)^2 .
$$

In these expressions we included the polar Jacobian for notational simplicity.

Proof. The strong formulation follows by a simple, yet tedious, calculation involving change of variables. For instance, take the first term in the radial weak formulation (2.2)

$$
\begin{aligned}
&\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \mathrm{d}|p_1|\,\mathrm{d}|p_2|\, K\big(|p_1|, |p_2|\big)\, f(|p_1|) f(|p_2|) \varphi(|p_1| + |p_2|) = \\
&\qquad \int_{\mathbb{R}_+} \mathrm{d}|p| \varphi(|p|)\Big(\int_0^{|p|} \mathrm{d}|p_1|\, K\big(|p_1|, |p| - |p_1|\big)\, f(|p_1|) f(|p| - |p_1|)\Big) .
\end{aligned}
$$

Since this identity is valid for any suitable test function φ, one obtains the term

$$
\int_0^{|p|} \mathrm{d}|p_1|\, K\big(|p_1|, |p| - |p_1|\big)\, f(|p_1|) f(|p| - |p_1|)
$$

in the strong formulation. Other terms are left to the reader. ■

3 Conservation of laws and H-Theorem

The weak formulation presented in Proposition 2.1 implies the following conservation laws and a quantum version of the classical Boltzmann H-Theorem.

Corollary 3.1 (Conservation laws) *If (f, n_c) is a solution of the system* (1.1)*, it formally conserves mass, momentum and energy*

$$\int_{\mathbb{R}^3} dp\, f(t,p) \ + \ n_c(t) \ = \ \int_{\mathbb{R}^3} dp\, f_0(p) \ + \ n_c(0) \tag{3.1}$$

$$\int_{\mathbb{R}^3} dp\, f(t,p)\, p \ = \ \int_{\mathbb{R}^3} dp\, f_0(p)\, p\,, \tag{3.2}$$

$$\int_{\mathbb{R}^3} dp\, f(t,p)\, |p| \ = \ \int_{\mathbb{R}^3} dp\, f_0(p)\, |p|\,. \tag{3.3}$$

Remark 3.1 *Since f is the density related to the thermal cloud only, the mass is not conserved for f but for the total density $f + n_c\, \delta(p)$. Of course, particles enter and leave the condensate at all times.*

Corollary 3.2 (H-Theorem) *If $f(t,p)$ solves* (1.1)*, then*

$$\frac{d}{dt} \int_{\mathbb{R}^3} dp \Big[f(p) \log f(p) - \big(1 + f(p)\big) \log \big(1 + f(p)\big) \Big] \leq 0.$$

As a consequence, a radially symmetric equilibrium of the equation has the form

$$f_\infty(p) = \frac{1}{e^{\alpha\, \omega(p)} - 1}, \quad \textit{for some} \ \ \alpha > 0. \tag{3.4}$$

This distribution is usually referred as a Bose-Einstein distribution.

Remark 3.2 *The linearization of the equation* (1.1) *about Bose-Einstein states can be performed by setting*

$$f(t,p) = f_\infty(p) + f_\infty(p)\big(1 + f_\infty(p)\big)\Omega(t,p).$$

After plugging into the collision operator and neglecting the nonlinear terms, one has

$$f_\infty(p)\big(1 + f_\infty(p)\big) \frac{\partial \Omega}{\partial t}(t,p) = -M(p)\Omega(t,p) + \int_{\mathbb{R}^3} dp'\, \mathcal{U}(p,p')\Omega(t,p'),$$

for some explicit function $M(p)$ and measure $\mathcal{U}(p,p')$. We refer to [9, 14], for the study of this equation in this perturbative setting and further discussions on this direction.

Proof. We observe that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \mathrm{d}p \Big[f(p) \log f(p) - \big(1 + f(p)\big) \log \big(1 + f(p)\big) \Big] =$$

$$\int_{\mathbb{R}^3} \mathrm{d}p\, \partial_t f(p) \log\left(\frac{f(p)}{f(p)+1}\right).$$

In addition, we can rewrite

$$\begin{aligned}\int_{\mathbb{R}^3} \mathrm{d}p\, \mathcal{Q}[f](p)\varphi(p) = \int_{\mathbb{R}^9} & \mathcal{K}(|p|,|p_1|,|p_2|)\delta(p-p_1-p_2)\delta(|p|-|p_1|-|p_2|) \\ & \times \big(1+f(p)\big)\big(1+f(p_1)\big)\big(1+f(p_2)\big) \\ \times \left(\frac{f(p_1)}{f(p_1)+1}\frac{f(p_2)}{f(p_2)+1} - \frac{f(p)}{f(p)+1}\right) & \Big[\varphi(p)-\varphi(p_1)-\varphi(p_2)\Big]\mathrm{d}p\mathrm{d}p_1\mathrm{d}p_2.\end{aligned}$$

Choosing $\varphi(p) = \log\left(\frac{f(p)}{f(p)+1}\right)$ we obtain, in the case of equality, that

$$\frac{f(p_1)}{f(p_1)+1}\frac{f(p_2)}{f(p_2)+1} - \frac{f(p)}{f(p)+1} = 0,$$

or equivalently, putting $h(p) = \log\left(\frac{f(p)}{f(p)+1}\right)$, we get

$$h(p_1) + h(p_2) = h(p). \tag{3.5}$$

The fact that $h(\cdot)$ is radially symmetric yields $h(p) = -\alpha\,\omega(p)$, for all $p \in \mathbb{R}^3$ and some positive constant α. This proves the claim. ■

4 *A priori* estimates on a solution's moments

The aim of the following sections is to consider radially symmetric solutions of (1.1)-(1.7) that lie in $\mathcal{C}\big([0,\infty); L^1(\mathbb{R}^3, |p|^k \mathrm{d}p)\big)$ where

$$L^1(\mathbb{R}^3, |p|^k\mathrm{d}p) := \Big\{ f \text{ measurable} \mid \int_{\mathbb{R}^3} \mathrm{d}p\, |f(p)||p|^k < \infty,\ k \geq 0\Big\}.$$

That is, in the sections 4 and 5 the *a priori* estimates *assume* the existence of a radially symmetric solution $f(t,\cdot)$ enjoying time continuity in such Lebesgue spaces (thus, time continuity for such solution's moments), for k sufficiently large, say $0 \leq k \leq 5$. Define the solution's moment of order k as

$$\mathcal{M}_k\langle f\rangle(t) := \int_{\mathbb{R}^3} \mathrm{d}p f(t,p)|p|^k\,. \tag{4.1}$$

When f is as radially symmetric function $f(t,p) = f(t,|p|)$, one can use spherical coordinates to reduce the integral with respect to $\mathrm{d}p$ on $\mathbb{R}^3$ to an integral on $\mathbb{R}_+$ with respect to $\mathrm{d}|p|$. As a consequence,

$$\mathcal{M}_k\langle f\rangle(t) = \left|\mathbb{S}^2\right| \int_{\mathbb{R}^+} \mathrm{d}|p| f(t,|p|)|p|^{k+2}\,.$$

Thus, it will be convenient for notation purposes to introduce and work with what we call "line-moments"

$$m_k\langle f\rangle(t) := \int_0^\infty \mathrm{d}|p|\, f(t,|p|)|p|^k\,. \tag{4.2}$$

Observe that $\mathcal{M}_k\langle f\rangle = |\mathbb{S}^2| m_{k+2}\langle f\rangle$.

We are going to use the definition of moments in two contexts: In one hand, in sections 4, 5 and 7 we always consider the moment applied to a given *radial solution of the equation.* Thus, there is no harm to omit the function dependence and just write $\mathcal{M}_k(t)$, $\mathcal{M}_k$, $m_k(t)$ or m_k to denote moments and line-moments for simplicity. In the other hand, in section 6 we will use moments as norms of the spaces $L^1(\mathbb{R}^3,|p|^k\mathrm{d}p)$, as a consequence, the functional dependence will be important, so we write $m_k\langle f\rangle$. Note that according to the conservation law (3.2) and assuming initial energy finite, the following equivalent estimates hold

$$\mathcal{M}_1(t) = \mathcal{M}_1(0) < \infty\,, \qquad m_3(t) = m_3(0) < \infty\,.$$

Before entering into details, let us explain the necessity of considering radially symmetric solutions of the equation (1.1) in the following arguments. Choosing $\varphi(p) = |p|^k$ in the weak formulation Proposition 2.1, one is lead to estimate terms of the form

$$\int_{\mathbb{R}^3} \mathrm{d}p_1 f(t,p_1)|p_1|^i \int_{\mathbb{R}_+} \mathrm{d}|p_2| f(t,|p_2|\widehat{p_1})|p_2|^j\,, \qquad i,\, j \in \mathbb{N}.$$

These terms are not estimated by products of moments of f unless the function is radially symmetric. In such a case this particular term simply writes as a product of line-moments of f, namely $|\mathbb{S}^2| m_{i+2}\langle f\rangle\, m_j\langle f\rangle$. This technical issue will be central in finding closed *a priori* estimates in terms of line-moments of solutions.

Lemma 4.1 *For any suitable function $f \geq 0$, for $k \geq 0$, define the quantity*

$$\mathcal{J}_k = \int_{\mathbb{R}^3} dp\, \mathcal{Q}_q[f]|p|^k,$$

we have:

- *If* $k=0$, *then*

$$\mathcal{J}_0 \leq C_k m_2\langle f\rangle\, m_4\langle f\rangle. \tag{4.3}$$

- *If* $k\geq 1$, *then*

$$\mathcal{J}_k \leq C_k\big(m_{k+3}\langle f\rangle\, m_3\langle f\rangle + m_{k+1}\langle f\rangle\, m_5\langle f\rangle\big). \tag{4.4}$$

We only prove (4.4)*, the other inequality* (4.3) *can be proved by the same argument. The constant* $C_k>0$ *only depends on* k*. In addition, the linear part simply reads for all* $k\geq 0$

$$\int_{\mathbb{R}^3} dp\, \mathcal{L}[f]|p|^k = c_k\, m_{k+7}\langle f\rangle\,, \ \textit{with positive constant given by}$$

$$c_k = 8\pi^2\int_0^1 dz\, z^2(1-z)^2(1-z^k-(1-z)^k)\,.$$

Proof. Using the weak formulation (2.2), the pointwise inequality

$$0\leq (x+y)^k - x^k - y^k \leq C_k\big(y\, x^{k-1}+y^{k-1}x\big), \ \text{valid for any } \ k\geq 1\,,$$

and neglecting all the negative contributions, one concludes that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \mathrm{d}p\, \mathcal{Q}_q[f]\, |p|^k\\
&\leq C_k \int_0^\infty\int_0^\infty \mathrm{d}|p_1|\mathrm{d}|p_2| K(|p_1|,|p_2|) f(|p_1|) f(|p_2|)\big(|p_2|\,|p_1|^{k-1}+|p_2|^{k-1}|p_1|\big)\\
&\quad = 2\, C_k \int_0^\infty\int_0^\infty \mathrm{d}|p_1|\mathrm{d}|p_2| K(|p_1|,|p_2|) f(|p_1|) f(|p_2|)|p_2|\,|p_1|^{k-1}\\
&\qquad = 4\, C_k\left(m_{k+3}\langle f\rangle\, m_3\langle f\rangle + m_{k+1}\langle f\rangle\, m_5\langle f\rangle\right).
\end{aligned}$$

In the last inequality we used that $K(|p_1|,|p_2|)\leq 2|p_1|^2|p_2|^2(|p_1|^2+|p_2|^2)$.

Regarding the linear part, it follows from a direct computation that

$$\begin{aligned}
&\frac{1}{8\pi^2}\int_{\mathbb{R}^3} \mathrm{d}p\, \mathcal{L}[f]\, |p|^k\\
&= \int_0^\infty\int_0^\infty \mathrm{d}|p_1|\mathrm{d}|p_2| K(|p_1|,|p_2|) f(|p_1|+|p_2|)\Big(\big(|p_1|+|p_2|\big)^k - |p_1|^k - |p_2|^k\Big)\\
&= \int_0^\infty \mathrm{d}|p|\, f(|p|)|p|^{k+6}\int_0^{|p|} \mathrm{d}|p_1|\big(\tfrac{|p_1|}{|p|}\big)^2\big(1-\tfrac{|p_1|}{|p|}\big)^2\Big(1-\big(\tfrac{|p_1|}{|p|}\big)^k - \big(1-\tfrac{|p_1|}{|p|}\big)^k\Big)\,.
\end{aligned}$$

The result follows after the change of variables $z=|p_1|/|p|$ in the inner integral. ■

Theorem 4.1 (Propagation of polynomial moments) *Let $(f, n_c) \geq 0$ be a solution to the problem (1.11) with finite energy and initial k^{th} moment $m_k\langle f_0\rangle < \infty$, for fixed $k > 3$. Then, there exists a constant $C_k > 0$ that depends only on k such that*

$$\sup_{t\in[0,T]} m_k\langle f\rangle(t) \leq \max\left\{m_k\langle f_0\rangle, C_k\, m_3^{\frac{k+1}{4}}\right\}. \tag{4.5}$$

Here $T > 0$ is any time such that $n_c(t) > 0$ for $t \in [0, T]$.

Proof. Use the weak formulation for f with $\varphi(|p|) = |p|^k$, $k > 1$. Then, using Lemma 4.1

$$\frac{\mathrm{d}}{\mathrm{d}t} m_{k+2}(t) \leq \tfrac{\kappa_0}{n_c(t)}\Big(C_k\big(m_{k+3}(t)m_3(t) + m_{k+1}(t)m_5(t)\big) - c_k m_{k+7}(t)\Big).$$

Using the interpolations

$$m_{k+3} \leq m_3^{\frac{4}{k+4}}\, m_{k+7}^{\frac{k}{k+4}}, \quad m_{k+1} \leq m_3^{\frac{6}{k+4}} m_{k+7}^{\frac{k-2}{k+4}}, \quad \text{and} \quad m_5 \leq m_3^{\frac{k+2}{k+4}} m_{k+7}^{\frac{2}{k+4}},$$

one concludes that (we drop the time dependence for simplicity)

$$\frac{\mathrm{d}}{\mathrm{d}t} m_{k+2} \leq \tfrac{\kappa_0}{n_c(t)}\Big(C_k m_3^{\frac{k+8}{k+4}} m_{k+7}^{\frac{k}{k+4}} - c_k m_{k+7}\Big) \leq \tfrac{\kappa_0}{n_c(t)}\Big(C_k' m_3^{\frac{k+8}{4}} - \tfrac{c_k}{2} m_{k+7}\Big).$$

Now, interpolating again

$$m_{k+7} \geq m_3^{-\frac{5}{k-1}} m_{k+2}^{\frac{k+4}{k-1}}$$

and simplifying, one finally concludes that

$$\frac{\mathrm{d}}{\mathrm{d}t} m_{k+2} \leq \tfrac{\kappa_0}{n_c(t)} m_3^{-\frac{5}{k-1}}\left(\tilde{C}_k\, m_3^{\frac{(k+4)(k+3)}{4(k-1)}} - \tilde{c}_k\, m_{k+2}^{\frac{k+4}{k-1}}\right), \tag{4.6}$$

for some positive constants $\tilde{C}_k$ and $\tilde{c}_k$ depending only on $k > 1$. The result follows directly from (4.6) after observing that

$$Y(t) := \max\left\{m_{k+2}(0), \big(\tilde{C}_k/\tilde{c}_k\big)^{\frac{k-1}{k+4}} m_3^{\frac{k+3}{4}}\right\},$$

is a super-solution of (4.6), thus, $Y(t) \geq m_{k+2}(t)$. ■

5 L^∞-estimate and BEC stability

In this section we find natural conditions on the initial condition for global existence of solutions. Although global solutions are not expected to exists for arbitrary (f_0, n_0), we essentially prove that if $n_0 > 0$ is sufficiently large relatively to the amount of quasi-particles near zero temperature, the BEC will remain formed.

Lemma 5.1 *For any suitable $f \geq 0$, the quadratic operator can be estimated as*

$$\mathcal{Q}_q[f](|p|) \leq 2\, m_3\, |p| \left\| f(|\cdot|)|\cdot|^2 \right\|_{L^\infty} - 4\, m_3\, |p| \left(f(|p|)|p|^2 \right).$$

In addition, the linear operator satisfies

$$\mathcal{L}[f](|p|) \leq 2\, m_4\, |p|^2 - c_0\, |p|^5 \left(f(|p|)|p|^2 \right), \quad c_0 := \int_0^1 z^2(1-z)^2 dz\,.$$

Proof. Recall the strong formulation of $\mathcal{Q}_q[f]$ given in Corollary 2.1

$$\begin{aligned}
\mathcal{Q}_q[f](|p|) &= \int_0^{|p|} \mathrm{d}|p_1| K(|p_1|, |p| - |p_1|) f(|p_1|) f(|p| - |p_1|) \\
&+ \int_{|p|}^{\infty} \mathrm{d}|p_1| \Big(K(|p|, |p_1| - |p|) + K(|p_1| - |p|, |p|) \Big) f(|p_1|) f(|p_1| - |p|) \\
&+ f(|p|) \bigg(\int_{|p|}^{\infty} \mathrm{d}|p_1| K(|p|, |p_1| - |p|) f(|p_1|) - \int_0^{\infty} \mathrm{d}|p_1| K(|p|, |p_1|) f(|p_1|) \\
&\qquad - \int_0^{|p|} \mathrm{d}|p_1| K(|p| - |p_1|, |p_1|) f(|p_1|) \bigg) \\
&+ f(|p|) \bigg(\int_{|p|}^{\infty} \mathrm{d}|p_1| K(|p_1| - |p|, |p|) f(|p_1|) - \int_0^{\infty} \mathrm{d}|p_1| K(|p_1|, |p|) f(|p_1|) \\
&\qquad - \int_0^{|p|} \mathrm{d}|p_1| K(|p_1|, |p| - |p_1|) f(|p_1|) \bigg) =: \sum_{i=1}^{9} B_i[f](|p|)\,.
\end{aligned} \tag{5.1}$$

For the first term $B_1[f](|p|)$ use

$$\begin{aligned}
K(|p_1|, |p| - |p_1|) = |p_1|^2 |p|^2 (|p| - |p_1|)^2 &= |p||p_1|^2 (|p| - |p_1|)^2 (|p| - |p_1| + |p_1|) \\
&= |p||p_1|^3 (|p| - |p_1|)^2 + |p||p_1|^2 (|p| - |p_1|)^3\,.
\end{aligned}$$

For the second term $B_2[f](|p|)$, use that in the set $\{|p_1| \geq |p|\}$

$$K(|p|, |p_1| - |p|) = |p|^2 |p_1|^2 (|p_1| - |p|)^2 \leq |p||p_1|^3 (|p_1| - |p|)^2\,,$$

and with an identical estimate for $B_3[f](|p|)$. We obtain, after a change of variables, that

$$
\begin{aligned}
B_1[f](|p|)+&B_2[f](|p|) + B_3[f](|p|) \\
&\leq 2|p| \int_0^\infty |p_1|^3 f(|p_1|) \big||p|-|p_1|\big|^2 f\big(\big||p|-|p_1|\big|\big) \mathrm{d}|p_1| \\
&\qquad\qquad \leq 2\,|p|\,\big\|f(\cdot)|\cdot|^2\big\|_{L^\infty}\, m_3\,.
\end{aligned}
$$

Now, the sum of the terms $4^{th}, 5^{th}$ and 6^{th} can be rewritten as

$$
\begin{aligned}
&B_4[f](|p|) + B_5[f](|p|) + B_6[f](|p|) = \\
&\quad f(|p|)\bigg(\int_{|p|}^\infty \big(K(|p|,|p_1|-|p|) - K(|p|,|p_1|)\big) f(|p_1|)\mathrm{d}|p_1| \\
&\qquad\qquad - \int_0^{|p|} \big(K(|p|,|p_1|) + K(|p|-|p_1|,|p_1|)\big) f(|p_1|)\mathrm{d}|p_1| \bigg)\,.
\end{aligned}
$$

Note that an explicit calculation gives

$$
K(|p|,|p_1|-|p|) - K(|p|,|p_1|) = -4|p|^3|p_1|^3\,.
$$

Also, in the set $\{|p_1| \leq |p|\}$ it follows

$$
K(|p|,|p_1|) + K(|p|-|p_1|,|p_1|) = 2|p|^2|p_1|^2\big(|p|^2+|p_1|^2\big) \geq 2|p|^3|p_1|^3\,.
$$

Therefore, this sum can be estimated as

$$
\begin{aligned}
&B_4[f](|p|) + B_5[f](|p|) + B_6[f](|p|) \\
&\leq -2\,|p|^3\, f(|p|)\bigg(2\int_{|p|}^\infty |p_1|^3 f(|p_1|)\mathrm{d}|p_1| + \int_0^{|p|} |p_1|^3 f(|p_1|)\mathrm{d}|p_1| \bigg) \\
&\leq -2\,|p|^3\, f(|p|)\, m_3\,.
\end{aligned}
$$

Now, by symmetry $K(|p|,|p_1|) = K(|p_1|,|p|)$, one has the identity $B_4[f](|p|)+B_5[f](|p|)+B_6[f](|p|) = B_7[f](|p|)+B_8[f](|p|)+B_9[f](|p|)$, and consequently

$$
\mathcal{Q}_q[f](|p|) \leq 2\,m_3\,|p|\,\big\|f(|\cdot|)|\cdot|^2\big\|_{L^\infty} - 4\,m_3\,|p|\,\big(f(|p|)|p|^2\big)\,.
$$

Now, the strong formulation of the linear operator reads

$$
\begin{aligned}
&\mathcal{L}[f](|p|) = L_1[f](|p|) + L_2[f](|p|) + L_3[f](|p|) := \\
&\quad \int_{|p|}^\infty \big(K(|p|,|p_1|-|p|) + K(|p_1|-|p|,|p|)\big) f(|p_1|)\mathrm{d}|p_1| \\
&\qquad\qquad - f(|p|)\int_0^{|p|} K(|p_1|,|p|-|p_1|)\mathrm{d}|p_1|\,.
\end{aligned}
\tag{5.2}
$$

Note that $K(|p|,|p_1|-|p|) = |p|^2|p_1|^2(|p_1|-|p|)^2 \le |p|^2|p_1|^4$ in the set $\{|p| \le |p_1|\}$, thus,

$$L_1[f](|p|) + L_2[f](|p|) \le 2\,|p|^2\, m_4\,.$$

Finally, an elementary calculation gives for the current kernel $K(|p|,|p_1|) = |p|^2(|p|+|p_1|)^2|p_1|^2$

$$\begin{aligned} L_3[f](|p|) &= f(|p|)\int_0^{|p|} K(|p_1|,|p|-|p_1|)\mathrm{d}|p_1| \\ &= \int_0^1 z^2(1-z)^2\mathrm{d}z\, f(|p|)\,|p|^7 =: c_0\, f(|p|)\,|p|^7\,. \end{aligned}$$

■

Proposition 5.1 (L^∞-estimate) *Let $(f,n_c)\ge 0$ be a solution of* (1.11) *with finite energy and 4^{th} moment. Also, assume that $n_c(\cdot)$ is absolutely continuous and that $\|f_0(|\cdot|)|\cdot|^2\|_{L^\infty}<\infty$. Then,*

$$\sup_{0\le s\le T}\left\|f(s,\cdot)|\cdot|^2\right\|_{L^\infty} \le \max\left\{\left\|f_0(\cdot)|\cdot|^2\right\|_{L^\infty}, \frac{3\sup_{0\le s\le T} m_4(s)}{2\,c_0^{1/4} m_3^{3/4}}\right\}. \tag{5.3}$$

Here $T>0$ is any time such that $n_c(t)>\delta$ for $t\in[0,T]$ and for some fixed constant $\delta>0$.

Proof. The weak formulation leads to the strong representation

$$\partial_t f(t,|p|)|p|^2 = \tfrac{\kappa_0}{n_c(t)}\Big(\mathcal{Q}_q[f(t)](|p|) + \mathcal{L}[f(t)](|p|)\Big)\,,\ t\ge 0\,,\ |p|\ge 0\,.$$

Since $n_c(\cdot)>0$ is absolutely continuous in $[0,T]$, it is possible to solve uniquely the nonlinear ode

$$\alpha'(t) = \frac{1}{n_c(\alpha(t))}\,, \qquad \alpha(0)=0\,, \tag{5.4}$$

in the region $0\le\alpha(t)\le T$. The function α is strictly increasing.

Observe that

$$\int_{\mathbb{R}^3} dp\, f(t,p) \ + \ n_c(t) = \int_{\mathbb{R}^3} dp\, f_0(p) \ + \ n_c(0) = C(f_0,n_c(0));$$

hence, n_c is uniformly bounded in time by $C(f_0,n_c(0))$, then

$$\frac{1}{n_c(\alpha(t))} \ge \frac{1}{C(f_0,n_c(0))} > 0.$$

Thus the function α is strictly increasing and $\lim_{t\to\infty}\alpha(t)=\infty$. Let $\tilde T$ be the unique time such that $\alpha(\tilde T)=T$ and define the time scaled function

$$F(t,|p|):=f\big(\alpha(t),|p|\big)\,,\quad t\in[0,\tilde T]\,.$$

It follows that

$$\partial_t F(t,|p|)|p|^2=\kappa_0\Big(\mathcal{Q}_q[F(t)](|p|)+\mathcal{L}[F(t)](|p|)\Big)\,,\quad |p|\ge 0\,,$$

valid in the interval $t\in[0,\tilde T]$. Clearly, $m_3\langle F(t)\rangle=m_3\langle F(0)\rangle=m_3\langle f_0\rangle=:$ m_3.

Define, for simplicity, $g(t,|p|):=F(t,|p|)|p|^2$ and use the weak formulation and Lemma 5.1 to obtain

$$\begin{aligned}\partial_t g(t,|p|)\le{}& 2\,m_3\,|p|\,\|g(t,|p|)\|_\infty\\ &-4\,m_3\,|p|\,g(t,|p|)+2\,m_4\,|p|^2-c_0\,|p|^5\,g(t,|p|)\,.\end{aligned}$$

Integrating in time this differential inequality, and taking all supremum in $s\in[0,\tilde T]$, yields

$$\begin{aligned}g(t,|p|)\le{}& g(0,|p|)e^{-|p|(4m_3+c_0|p|^4)t}\\ &+2|p|\int_0^t e^{-|p|(4m_3+c_0|p|^4)(t-s)}\Big(m_3\big\|g(s,\cdot)\big\|_{L^\infty}+m_4(s)|p|\Big)\mathrm{d}s\\ &\le\ \max\left\{\|g(0,|p|)\|_\infty\,,\,\frac{2\,m_3\sup_s\big\|g(s,\cdot)\big\|_{L^\infty}+2\sup_s m_4(s)\,|p|}{4m_3+c_0\,|p|^4}\right\}\\ &\le\ \max\left\{\|g(0,|p|)\|_\infty\,,\,\tfrac12\sup_s\big\|g(s,\cdot)\big\|_{L^\infty}+\left\|\tfrac{2|\cdot|}{4m_3+c_0\,|\cdot|^4}\right\|_\infty\sup_s m_4(s)\right\}.\end{aligned}$$

Since by interpolation, we can estimate

$$\frac{2|p|}{4m_3+c_0\,|p|^4}\le\frac{3^{3/4}}{2^{5/2}c_0^{1/4}m_3^{3/4}}\,,$$

it follows, after taking supremum in $|p|\ge 0$ and then in $t\ge 0$, that

$$\sup_s\big\|g(s,\cdot)\big\|_{L^\infty}\le\max\left\{\big\|g(0,\cdot)\big\|_{L^\infty},\,\frac{3^{3/4}\sup_s m_4(s)}{2^{3/2}\,c_0^{1/4}m_3^{3/4}}\right\}.$$

Therefore, estimate (5.3) follows since

$$\sup_{s\in[0,\tilde T]}\big\|g(s,\cdot)\big\|_{L^\infty}=\sup_{s\in[0,T]}\big\|f(s,\cdot)|\cdot|^2\big\|_{L^\infty}\,,$$

and

$$\sup_{s\in[0,\tilde{T}]} m_4\langle F(s)\rangle = \sup_{s\in[0,T]} m_4\langle f(s)\rangle\,.$$

■

We are now in conditions to show the Bose-Einstein Condensation result for the Quantum Boltzmann condensation system.

Theorem 5.1 (BEC stability) *Let $(f, n_c) \geq 0$ be a solution of (1.11) with finite energy and 4^{th} moment. Also, assume that $n_c(\cdot) > 0$ is absolutely continuous and that $\|f_0(|\cdot|)|\cdot|^2\|_{L^\infty} < \infty$. Then, there exists a threshold $\mathcal{C}(f_0) > 0$, that can be taken as in (5.7), such that for any initial BEC having mass*

$$n_c(0) \geq \mathcal{C}(f_0) - m_2(0) + \delta\,, \qquad \delta > 0\,, \tag{5.5}$$

then, the BEC remains uniformly formed,

$$\inf_{0\leq s\leq T} n_c(s) \geq \delta\,.$$

Here $T > 0$ is any time were the aforementioned assumptions hold.

Proof. First, observe the following estimate that controls $m_2^2\langle f\rangle(t)$ by the conserved energy m_3 multiplied by the $\big\|f(|\cdot|)|\cdot|^2\big\|_{L^\infty}$.

Indeed, for any $\varepsilon > 0$,

$$\begin{aligned} m_2\langle f\rangle(t) = \int_0^\infty \mathrm{d}|p| f(|p|)|p|^2 &= \int_0^\varepsilon \mathrm{d}|p| f(|p|)|p|^2 + \int_\varepsilon^\infty \mathrm{d}|p| f(|p|)|p|^2 \\ &\leq \varepsilon\big\|f(\cdot)|\cdot|^2\big\|_{L^\infty} + \tfrac{1}{\varepsilon} m_3 \leq 2\sqrt{m_3\,\big\|f(|\cdot|)|\cdot|^2\big\|_{L^\infty}}\,, \end{aligned} \tag{5.6}$$

uniformly in time, where the last inequality follows after minimization over $\varepsilon > 0$.

Hence, since any solution $(f(t,\cdot), n_c(t)) \geq 0$ of (1.11) with continuous moments and with $n_c(t) > 0$ in $[0, T]$, the pair $(f(t), n_c(t))$ satisfies the total conservation of mass $m_2(t) + n_c(t) = m_2(0) + n_c(0)$ in such interval, then, by (5.6)

$$\begin{aligned} n_c(t) = n_c(0) + m_2(0) - m_2(t) &\geq n_c(0) + m_2(0) - \sup_{0\leq s\leq T} m_2(s) \\ &\geq n_c(0) + m_2(0) - 2\sqrt{m_3 \sup_{0\leq s\leq T} \big\|f(s,|\cdot|)|\cdot|^2\big\|_{L^\infty}}\,. \end{aligned}$$

Moreover, using Proposition 5.1 and Theorem 4.1,

$$
\begin{aligned}
m_2\langle f\rangle(0) &\le 2\sqrt{m_3 \sup_{0\le s\le T} \big\| f(s,|\cdot|)|\cdot|^2\big\|_{L^\infty}} \\
&\le 2\sqrt{m_3 \max\left\{ \big\| f_0(\cdot)|\cdot|^2\big\|_{L^\infty}, \frac{3\sup_{0\le s\le T} m_4(s)}{2\,c_0^{1/4} m_3^{3/4}} \right\}} \\
&\le 2\sqrt{m_3 \max\left\{ \big\| f_0(\cdot)|\cdot|^2\big\|_{L^\infty}, \frac{3\max\left\{m_4\langle f_0\rangle, C\, m_3^{\frac{5}{4}}\right\}}{2\,c_0^{1/4} m_3^{3/4}} \right\}} =: \mathcal{C}(f_0)\,.
\end{aligned}
\tag{5.7}
$$

Thus, fixing $\delta > 0$, if

$$
n_c(0) \ge \mathcal{C}(f_0) - m_2(0) + \delta\,,
$$

we have $\inf_{0\le s\le T} n_c(s) \ge \delta$ which concludes the proof. ∎

6 The Cauchy Problem

This section is devoted to show existence and uniqueness of positive solutions of the initial value problem (1.11) with quantum interaction operator $\mathcal{Q}[f]$ defined in (1.8), (1.9) and (1.10), associated to a transition probability $|\mathcal{M}|^2 = \kappa|p||p_1||p_2|$ valid in the low temperature regime.

The first observation is that the system (1.11) can be reduced to a single equation after explicit integration of $n_c(t)$. Indeed,

$$
n_c[f](t) := n_c(t) = \sqrt{n_0^2 - 2\kappa_0 \int_0^t \mathrm{d}s \int_{\mathbb{R}^3} \mathrm{d}p\, \mathcal{Q}[f](s,p)}\,. \tag{6.1}
$$

As a consequence, system (1.11) is equivalent to the single equation

$$
\frac{\mathrm{d}f}{\mathrm{d}t} = \tfrac{\kappa_0}{n_c[f]}\mathcal{Q}[f]\,, \quad t > 0\,, \tag{6.2}
$$

complemented with the initial condition $f(0,\cdot) = f_0(\cdot)$. This equivalence is valid as long as $n_c(\cdot) > 0$. Note that equation (6.2) is an nonlinear equation with memory.

The approach we follow here is based on an abstract ODE framework in Banach spaces. The following theorem, proved in the Appendix 8, is valid for causal operators. Fix spaces $\mathcal{S}$ and E, time $T > 0$, and causal operator

$$
\mathcal{O} : \mathcal{C}\big([0,T];\mathcal{S}\big) \to \mathcal{C}\big([0,T];E\big). \tag{6.3}
$$

We recall that an operator $\mathcal{O}$ is causal, if for any $t \in [0,T]$ the operator at time t is defined only by the values of f in $[0,t]$, that is, $\mathcal{O}[f](t) = \mathcal{O}[f(\cdot)\mathbf{1}_{\{\cdot \leq t\}}](t)$.

Theorem 6.1 *Let $E := (E, \|\cdot\|)$ be a Banach space, $\mathcal{S}$ be a bounded, convex and closed subset of E, and $\mathcal{O} : \mathcal{C}\big([0,T];\mathcal{S}\big) \rightarrow \mathcal{C}\big([0,T];E\big)$ be a causal operator satisfying the following properties:*

- *Hölder continuity condition: For any functions $f, g \in \mathcal{C}\big([0,T];\mathcal{S}\big)$ and times $0 \leq t \leq s \in [0,T]$, there is $\beta \in (0,1)$ such that*

$$\begin{aligned}&\big\|\mathcal{O}[f](t) - \mathcal{O}[g](s)\big\| \\ &\quad \leq C\big(\sup_{\sigma\in[0,t]} \big\|f(\sigma) - g(\sigma)\big\|^{\beta} + \big\|f(t) - g(s)\big\|^{\beta} + |t-s|^{\beta}\big)\,,\end{aligned} \tag{6.4}$$

- *sub-tangent condition: For any $f \in \mathcal{C}\big([0,T];\mathcal{S}\big)$*

$$\liminf_{h\to 0^+} h^{-1} \sup_{t\in[0,T]} dist\big(f(t) + h\,\mathcal{O}[f](t),\, \mathcal{S}\big) = 0\,, \tag{6.5}$$

- *and, one-sided Lipschitz condition: For any $f, g \in \mathcal{C}\big([0,T];\mathcal{S}\big)$ and $t \in [0,T]$*

$$\int_0^t ds\big[\mathcal{O}[f](s) - \mathcal{O}[g](s), f(s) - g(s)\big] \leq L\int_0^t ds\big\|f(s) - g(s)\big\|\,, \tag{6.6}$$

  where $\big[\varphi, \phi\big] := \lim_{h\to 0^-} h^{-1}\big(\|\phi + h\varphi\| - \|\phi\|\big)$.

Then, the equation

$$\partial_t f = \mathcal{O}[f] \;\; on \;\; [0,T) \times E, \qquad f(0) = f_0 \in \mathcal{S} \tag{6.7}$$

has a unique solution in $\mathcal{C}^1\big([0,T);E\big) \cap \mathcal{C}\big([0,T);\mathcal{S}\big)$.

This theorem is an extension of Theorem A.1 proved in [7] by Bressan in the context of solving the elastic Boltzmann equation for hard spheres in 3 dimension. We point out that [7] does not properly show that (6.5) is satisfied in that case. For completeness of this manuscript we rewrite Bressan's unpublished proof in the Appendix. The Bressan's needed techiques can be found in [23]. Indeed, referring to the argument given in [1], using conditions (6.4) and (6.5) combined with [23, Theorem VI.2.2] one has that conditions (C1), (C2) and (C3) in [23, pg. 229] are satisfied and hence, together with

(6.6), all needed conditions for the existence and uniqueness theorem [23, Theorem VI.4.3] for ODEs in Banach spaces are fulfilled.

For our particular case, we need to identify a suitable Banach space and a corresponding bounded, convex and closed subset $\mathcal{S}$. Choosing $E = L^1\big(\mathbb{R}^3, \mathrm{d}p\big)$ as Banach space, the choice of the subspace $\mathcal{S}$, defined below in (6.8), depends on the *a priori* estimates discussed in previous two sections and the desired continuity properties needed for existence.

More specifically, such subset $\mathcal{S} \subset L^1\big(\mathbb{R}^3, \mathrm{d}p\big)$ is characterized by the Hölder continuity and sub-tangent conditions (6.4) and (6.5), respectively, (to be shown next in subsection 6.2), and it is defined as follows:

$$
\begin{aligned}
\mathcal{S} \; := \; \Big\{ f \in L^1\big(\mathbb{R}^3, \mathrm{d}p\big) \;\big|\; &\textbf{i}.\ f \text{ nonnegative \& radially symmetric}\,,\\
&\textbf{ii}.\ m_3\langle f\rangle \;=\; \int_{\mathbb{R}_+} d|p|\, f(|p|)|p|^3 \;=\; \mathfrak{h}_3\,,\\
&\textbf{iii}.\ m_8\langle f\rangle \;=\; \int_{\mathbb{R}_+} d|p|\, f(|p|)|p|^8 \;\le\; \mathfrak{h}_8\,,\\
&\textbf{iv}.\ \|f(\cdot)\,|\cdot|^2\,\|_\infty \le \mathfrak{h}_\infty < \infty \Big\},
\end{aligned}
\tag{6.8}
$$

where $\mathfrak{h}_3 > 0$ is an arbitrary initial energy. The specific $\mathfrak{h}_8 > 0$ is defined below in (6.18), and $\mathfrak{h}_\infty > 0$ will be taken sufficiently large depending only on $\mathfrak{h}_3$ and $\mathfrak{h}_8$. We are now in conditions to state and prove the global well-posedness theorem.

Theorem 6.2 (Global well-posedness) *Let $f_0(p) = f_0(|p|) \in \mathcal{S}$ and assume that $(f_0, n_0 = n_c(0))$ satisfies the threshold condition* (5.5) *for $\delta > 0$. Then, system* (1.11) *(equivalently, system* (6.1)*-*(6.2)*) has a unique conservative solution (f, n_c) such that*

$$
\begin{aligned}
&0 \le f(t,p) = f(t,|p|) \in \mathcal{C}\big([0,T];\mathcal{S}\big) \cap \mathcal{C}^1\big((0,T];L^1\big(\mathbb{R}^3,dp\big)\big)\,,\\
&\delta \le n_c(t) = n_c[f](t) \in \mathcal{C}\big([0,T]\big) \cap \mathcal{C}^1\big((0,T]\big)\,,
\end{aligned}
\tag{6.9}
$$

for any $T > 0$. Momentum and energy are conserved for $f(t,\cdot)$, and the total mass of the system is conserved as well

$$
m_2\langle f(t)\rangle + n_c[f](t) = m_2\langle f_0\rangle + n_0\,.
$$

Proof. The proof of this theorem consists of verifying the three conditions (6.4), (6.5), and (6.6) to apply Theorem 6.1, respectively for the nonlinear causal operator $\mathcal{O}[f] = \frac{\kappa_0}{n_c[f]}\mathcal{Q}[f]$.

In the following estimates we fix a time $T := T_\delta > 0$ such that

$$\inf_{0\le s\le T_\delta} n_c[f](s) \ge \delta\,.$$

This can be done in the space $\mathcal{C}\big([0,T];\mathcal{S}\big)$ since

$$\int_{\mathbb{R}^3} \mathrm{d}p\big|\mathcal{Q}[f(t)]\big| \le C\big(m_2\langle f(t)\rangle, m_7\langle f(t)\rangle\big) \le C(\mathfrak{h}_3,\mathfrak{h}_8,\mathfrak{h}_\infty)\,.$$

In the sequel, we write $C(\mathcal{S})$ for a constant depending only on the parameters defining the set $\mathcal{S}$, namely $\mathfrak{h}_3$, $\mathfrak{h}_8$, and $\mathfrak{h}_\infty$. Therefore, from the definition of $n_c[f]$ it suffices to take $T_\delta := \frac{n_0^2-\delta^2}{2\kappa_0 C(\mathcal{S})} > 0$ to satisfy such lower bound on the condensate mass. A posteriori, knowing the total conservation of mass, we use Theorem 5.1 to conclude that $T>0$ is, in fact, arbitrary.

6.1 Hölder Estimate

Recall the definition of $m_k\langle f\rangle$, the k^{th}-line-moment of a radially symmetric $f(p) := f(|p|)$

$$m_k\langle f\rangle := \int_{\mathbb{R}_+} \mathrm{d}p\, f(|p|)|p|^k\,, \qquad k\ge 0\,, \tag{6.10}$$

and observe that $m_2\langle|f|\rangle$ is equivalent to the usual norm for a radially symmetric functions in $L^1\big(\mathbb{R}^3,\mathrm{d}p\big)$.

Lemma 6.1 (Hölder continuity) *The collision operator*

$$\tfrac{\kappa_0}{n_c[\cdot]}\mathcal{Q}[\cdot] : \mathcal{C}\big([0,T];\mathcal{S}\big) \to \mathcal{C}\Big([0,T];L^1\big(\mathbb{R}^3,dp\big)\Big)$$

is Hölder continuous with estimate

$$\begin{aligned} & m_2\Big\langle\Big|\tfrac{\kappa_0}{n_c[f](t)}\mathcal{Q}[f(t)] - \tfrac{\kappa_0}{n_c[g](s)}\mathcal{Q}[g(s)]\Big|\Big\rangle \\ &\le C_{\delta,T}(\mathcal{S})\Big(\sup_{\sigma\in[0,t]} m_2\big\langle|f(\sigma)-g(\sigma)|\big\rangle^{\frac16} + \sup_{\sigma\in[0,t]} m_2\big\langle|f(\sigma)-g(\sigma)|\big\rangle\Big) \\ &\quad + C_\delta(\mathcal{S})\Big(m_2\big\langle|f(t)-g(s)|\big\rangle^{\frac16} + m_2\big\langle|f(t)-g(s)|\big\rangle + |t-s|\Big)\,, \end{aligned} \tag{6.11}$$

valid for all f, $g\in\mathcal{C}\big([0,T];\mathcal{S}\big)$ and $0\le t\le s\in[0,T]$.

Proof. Recall that the interaction operator can be written as a sum of a nonlinear part and a linear part $\mathcal{Q}[f] = \mathcal{Q}_q[f] + \mathcal{L}[f]$. Besides, the nonlinear part is the sum of nine terms $\mathcal{Q}_q[f] = \sum_{i=1}^9 B_i[f]$, as in (5.1), and the linear part is the sum of three terms $\mathcal{L}[f] = \sum_{i=1}^{3} L_i[f]$, as in (5.2). An elementary calculation shows that the nonlinear terms satisfy for $1 \le i \le 9$

$$\begin{aligned}
&\int_{\mathbb{R}^3} \mathrm{d}p \big| B_i[f] - B_i[g] \big| \\
&\le 2 \max \big\{ m_2\langle f\rangle, m_4\langle f\rangle, m_2\langle g\rangle, m_4\langle g\rangle \big\} \Big(m_2\langle |f-g|\rangle + m_4\langle |f-g|\rangle \Big) \\
&\le 2 \max \big\{ m_2\langle f\rangle, m_4\langle f\rangle, m_2\langle g\rangle, m_4\langle g\rangle \big\} \times \\
&\times \Big(m_2\langle |f-g|\rangle + \big(m_8\langle f\rangle + m_8\langle g\rangle \big)^{1/3} m_2^{2/3}\langle |f-g|\rangle \Big) .
\end{aligned}$$

As for the linear terms,

$$\int_{\mathbb{R}^3} \mathrm{d}p \big| L_i[f] - L_i[g] \big| \le m_7\langle |f-g|\rangle \le \big(m_8\langle f\rangle + m_8\langle g\rangle \big)^{5/6} m_2^{1/6}\langle |f-g|\rangle .$$

The conclusion is that

$$\int_{\mathbb{R}^3} \mathrm{d}p \big| \mathcal{Q}[f] - \mathcal{Q}[g] \big| \le C(\mathcal{S}) \Big(m_2\langle |f-g|\rangle + m_2^{1/6}\langle |f-g|\rangle \Big) . \tag{6.12}$$

Additionally, for any $0 \le t \le s \in [0,T]$

$$\begin{aligned}
&\Big| \frac{1}{n_c[f](t)} - \frac{1}{n_c[g](s)} \Big| = \frac{\big| n_c^2[f](t) - n_c^2[g](s) \big|}{(n_c[f](t) + n_c[g](s))\, n_c[f](t)\, n_c[g](s)} \\
&\quad \le 2\kappa_0 \, \frac{\int_0^t \mathrm{d}\sigma\, m_2\langle \big| \mathcal{Q}[f(\sigma)] - \mathcal{Q}[g(\sigma)] \big| \rangle + \int_t^s \mathrm{d}\sigma\, m_2\langle \big| \mathcal{Q}[g(\sigma)] \big| \rangle}{(n_c[f](t) + n_c[g](s))\, n_c[f](t)\, n_c[g](s)} \\
&\le \frac{C(\mathcal{S})}{\delta^3} \Big(\int_0^t \mathrm{d}\sigma\, m_2\big\langle \big| f(\sigma) - g(\sigma) \big| \big\rangle + m_2^{1/6}\big\langle \big| f(\sigma) - g(\sigma) \big| \big\rangle + |t-s| \Big) .
\end{aligned} \tag{6.13}$$

We used, in the last inequality, the fact that $\min\{n_c[f], n_c[g]\} \ge \delta$ for any $f, g \in \mathcal{C}\big([0,T];\mathcal{S}\big)$. The result follows after applying $m_2\langle \cdot \rangle$ to

$$\begin{aligned}
\Big| \tfrac{\kappa_0}{n_c[f](t)} \mathcal{Q}[f(t)] - \tfrac{\kappa_0}{n_c[g](s)} \mathcal{Q}[g(s)] \Big| &\le \kappa_0 \Big| \tfrac{1}{n_c[f](t)} - \tfrac{1}{n_c[g](s)} \Big| \mathcal{Q}[f(t)] \\
&\quad + \tfrac{\kappa_0}{n_c[g](s)} \Big| \mathcal{Q}[f(t)] - \mathcal{Q}[g(s)] \Big| .
\end{aligned}$$

and using (6.12) and (6.13) to estimate each term in the right side. ■

6.2 Sub-tangent condition

This condition characterizes the stability of the space $\mathcal{S}$ defined in (6.8) under the equation's dynamics. Recall that the collision operator $\mathcal{Q}[\cdot]$ can be split as the sum of a gain and a loss operators, as mentioned earlier in (1.8)

$$\mathcal{Q}[f] = \mathcal{Q}^{+}[f] - f\,\nu[f]\,,$$

with (refer to the strong formulation and recall the symmetry of $K(\cdot,\cdot)$)

$$\begin{aligned}
\nu[f](p) = 2\int_0^\infty \mathrm{d}|p_1|\,K(|p_1|,|p|)\,f(|p_1|) + 2\int_0^{|p|} \mathrm{d}|p_1|\,K(|p_1|,|p|-|p_1|)\,f(|p_1|)\\
+\int_0^{|p|} \mathrm{d}|p_1|\,K(|p_1|,|p|-|p_1|)\\
\le 4|p|^4 m_2\langle f\rangle + 4|p|^2 m_4\langle f\rangle + 4|p|^7 \le C(\mathcal{S})|p|^2(1+|p|^5)\,.
\end{aligned}\tag{6.14}$$

The sub-tangent condition (6.5) follows as a corollary of next Proposition 6.1.

Proposition 6.1 *Fix $f \in \mathcal{C}\big([0,T];\mathcal{S}\big)$. Then, for any $t>0$ and $\epsilon>0$, there exists $h_* := h_*(f,\epsilon) > 0$, such that the ball centered at $f(t) + h\frac{\kappa_0}{n_c[f](t)}\mathcal{Q}[f(t)]$ with radius $h\,\epsilon > 0$ intersects $\mathcal{S}$, that is,*

$$B\Big(f(t) + h\tfrac{\kappa_0}{n_c[f](t)}\mathcal{Q}[f(t)], h\,\epsilon\Big)\cap\mathcal{S},\ \textit{is non-empty for any}\ 0<h<h_*.$$

Proof. Set $\chi_R(p)$ the characteristic function of the ball of radius $R>0$ and introduce the truncated function $f_R(t,p) := \mathbf{1}_{\{|p|\le R\}}f(t,p)$, then set $w_R(t,p) := f(t,p) + h\frac{\kappa_0}{n_c[f](t)}\mathcal{Q}[f_R(t)](p)$.

Since $0 \le f_R(t,p) \le f(t,p)$, one has that

$$m_2\langle f_R(t)\rangle \le m_2\langle f(t)\rangle\,,\quad m_7\langle f_R(t)\rangle \le m_7\langle f(t)\rangle\,.$$

Then, $\frac{\kappa_0}{n_c[f](t)}\mathcal{Q}[f_R(t)] \in \mathcal{C}\big([0,T],L^1\big(\mathbb{R}^3,\mathrm{d}p\big)$ by Lemma 6.1. As a consequence, $w_R \in \mathcal{C}\big([0,T];L^1(\mathbb{R}^3,\mathrm{d}p)\big)$. Note that, since $\mathcal{Q}^+$ is a positive operator, for any $f(t)\in\mathcal{S}$

$$\begin{aligned}
w_R(t) &= f(t) + h\tfrac{\kappa_0}{n_c[f](t)}\Big(\mathcal{Q}^+[f_R(t)] - f_R(t)\,\nu[f_R(t)]\Big)\\
&\ge f(t) - h\tfrac{\kappa_0}{n_c[f](t)} f_R(t)\,\nu[f_R(t)]\\
&\ge\ f(t)\Big(1 - h\,\delta^{-1}\,C(\mathcal{S})R^2\big(1+R^5\big)\Big) \ge 0
\end{aligned}\tag{6.15}$$

for any $0 < h < \delta/C(\mathcal{S})R^2\big(1+R^5\big)$. Moreover, by conservation of energy $\int_{\mathbb{R}_+} \mathrm{d}|p|\, \mathcal{Q}[f_R(t)]|p|^3 = 0$, yielding

$$
\begin{aligned}
m_3\langle w_R(t)\rangle &= \int_{\mathbb{R}_+} \mathrm{d}|p|\, w_R(t,|p|)|p|^3 \\
&= \int_{\mathbb{R}_+} \mathrm{d}|p| \left(f(t,|p|) + h\tfrac{\kappa_0}{n_c[f](t)}\mathcal{Q}[f_R(t)]\right)|p|^3 = \int_{\mathbb{R}_+} \mathrm{d}|p|\, f(t,|p|)|p|^3 = \mathfrak{h}_3\,.
\end{aligned} \tag{6.16}
$$

In summary, w_R satisfies, properties **i.** and **ii.** in the characterization of the $\mathcal{S}$. Let us show that w_R also satisfies property **iii.** in the set $\mathcal{S}$. First, recall the *a priori* estimate (4.6) for the line-moment inequalities, namely

$$
\begin{aligned}
\int_{\mathbb{R}^3} \mathrm{d}p\, \tfrac{\kappa_0}{n_c[f](t)}\mathcal{Q}[f(t)]|p|^k &\le \mathcal{L}_k\big(t, m_k\langle f(t)\rangle\big) \\
&:= \tfrac{\kappa_0}{n_c[f](t)}\, m_3^{-\frac{5}{k-3}} \left(\tilde{C}_k\, m_3^{\frac{(k+2)(k+1)}{4(k-3)}} - \tilde{c}_k\, m_k\langle f(t)\rangle^{\frac{k+2}{k-3}}\right) \\
&= \tfrac{\kappa_0}{n_c[f](t)}\, \mathfrak{h}_3^{-\frac{5}{k-3}} \left(\tilde{C}_k\, \mathfrak{h}_3^{\frac{(k+2)(k+1)}{4(k-3)}} - \tilde{c}_k\, m_k\langle f(t)\rangle^{\frac{k+2}{k-3}}\right).
\end{aligned} \tag{6.17}
$$

This estimate holds for any $k > 3$ and $\tilde{C}_k, \tilde{c}_k$ only depending on k. Note that the map $\mathcal{L}_k(t,\cdot) : [0,\infty) \to \mathbb{R}$ is decreasing and has only one root $\mathfrak{h}_*^k := \frac{\tilde{C}_k}{\tilde{c}_k}\mathfrak{h}_3^{(k+1)/4}$, at which $\mathcal{L}_k$ changes from positive to negative for any $k > 3$. Note that this root only depends on $\mathfrak{h}_3$ and k, in particular, it is time independent. Thus, it is always the case that for any $f \in \mathcal{C}\big([0,T];\mathcal{S}\big)$

$$
\int_{\mathbb{R}^3} \mathrm{d}p\, \tfrac{\kappa_0}{n_c[f](t)}\mathcal{Q}[f]|p|^k \le \mathcal{L}_k\big(t, m_k\langle f\rangle\big) \le \mathcal{L}_k(t,0) \le \tfrac{\kappa_0}{\delta}\, \tilde{C}_k\, \mathfrak{h}_3^{\frac{(k+6)}{4}}\,.
$$

Fix $k = 8$ and define

$$
\mathfrak{h}_8 := 2\mathfrak{h}_*^8 + \tfrac{\kappa_0}{\delta}\, \tilde{C}_8\, \mathfrak{h}_3^{\frac{7}{2}}\,. \tag{6.18}
$$

For any $f \in \mathcal{C}\big([0,T];\mathcal{S}\big)$, we have two sets: $I_1 = \{t : m_8\langle f(t)\rangle \le 2\mathfrak{h}_*^8\}$ and $I_2 = \{t : m_8\langle f(t)\rangle > 2\mathfrak{h}_*^8\}$. For the former, it readily follows that

$$
\begin{aligned}
m_8\langle w_R(t)\rangle = \int_{\mathbb{R}^3} \mathrm{d}p\, w_R(t,|p|)|p|^8 &= \int_{\mathbb{R}^3} \mathrm{d}p \left(f(t) + h\tfrac{\kappa_0}{n_c[f](t)}\mathcal{Q}[f_R(t)]\right)|p|^8 \\
&\le 2\mathfrak{h}_*^8 + h\tfrac{\kappa_0}{\delta}\, \tilde{C}_8\, \mathfrak{h}_3^{\frac{7}{2}} \le \mathfrak{h}_8,
\end{aligned}
$$

where in the last inequality we have assumed $h \le 1$ without loss of generality.

For the latter, we can choose $R = R_1(f)$ sufficiently large such that $\inf_{t\in I_2} m_8\langle f_R(t)\rangle \geq \mathfrak{h}_*^8$, and therefore,

$$\int_{\mathbb{R}^3} \mathrm{d}p\, \tfrac{\kappa_0}{n_c[f](t)}\mathcal{Q}[f_R(t)]|p|^8 \leq \mathcal{L}_8\big(t, m_8\langle f_R(t)\rangle\big) \leq 0\,, \qquad t\in I_2\,.$$

As a consequence, for any $t \in I_2$

$$\begin{aligned} m_8\langle w_R(t)\rangle &= \int_{\mathbb{R}^3} \mathrm{d}p\, \Big(f(t) + h\tfrac{\kappa_0}{n_c[f](t)}\mathcal{Q}[f_R(t)]\Big)|p|^8 \\ &\leq \int_{\mathbb{R}^3} \mathrm{d}p\, f(t)|p|^8 \leq \mathfrak{h}_8\,. \end{aligned}$$

The conclusion is that for any $f \in \mathcal{C}\big([0,T];\mathcal{S}\big)$, it is always the case that

$$m_8\langle w_R(t)\rangle \leq \mathfrak{h}_8\,, \quad \text{as long as} \quad R \geq R_1(f) > 0\,, \tag{6.19}$$

which ensures that w_R satisfies property **iii.** of the set $\mathcal{S}$ in (6.8). Let us prove now that w_R satisfies property **iv.** To this end, consider the sets

$$\begin{aligned} O &= \big\{(t,p) : f(t,p)\,|p|^2 \geq 0.9\,\mathfrak{h}_\infty\big\}\,, \\ O_R &= \big\{(t,p) : f_R(t,p)\,|p|^2 \geq 0.9\,\mathfrak{h}_\infty\big\}\,. \end{aligned}$$

In addition, consider the set $W = \big\{(t,p) : w_R(t,p)\,|p|^2 > \mathfrak{h}_\infty\big\}$. Assume that W is of positive measure. Then,

$$\begin{aligned} \mathfrak{h}_\infty < w_R(t,p)\,|p|^2 &= f(t,p)\,|p|^2 + h\tfrac{\kappa_0}{n_c[f](t)}\mathcal{Q}[f_R(t)](p)\,|p|^2 \\ &\leq f(t,p)\,|p|^2 + h\tfrac{\kappa_0}{n_c[f](t)}\mathcal{Q}^+[f_R(t)](p)\,|p|^2\,, \quad (t,p)\in W\,. \end{aligned}$$

It is not difficult to check, using the strong formulation, that for any function $F(t) \in \mathcal{S}$

$$\|\mathcal{Q}^+[F(t)](p)\,|p|^2\|_\infty \leq 6\,\|F(t,\cdot)\,|\cdot|^2\|_\infty m_4\langle F(t)\rangle\, + 2\,m_6\langle F(t)\rangle \leq C(\mathcal{S})\,.$$

Thus,

$$\begin{aligned} f(t,p)\,|p|^2 &> \mathfrak{h}_\infty - h\tfrac{\kappa_0}{n_c[f](t)}\big\|\mathcal{Q}^+[f_R(t)](p)\,|p|^2\big\|_\infty \\ &\geq \mathfrak{h}_\infty - h\,\delta^{-1}\,C(\mathcal{S}) \geq 0.9\,\mathfrak{h}_\infty\,, \quad (t,p)\in W\,, \end{aligned}$$

where, for the last step, $0 < h \leq 0.1\,\delta\,\mathfrak{h}_\infty/C(\mathcal{S})$. As a consequence, $W \subset O$. Since $O_R \nearrow O$ as $R \to \infty$, there exists $R = R_2(f) > 0$ sufficiently large such that $W \cap O_R$ is of positive measure. Take (s,q) in such intersection, then by Lemma 5.1

$$
\begin{aligned}
w_R(s,q)\,|q|^2 &= f(s,q)\,|q|^2 + h\tfrac{\kappa_0}{n_c[f](s)}\,\mathcal{Q}[f_R(s)](q)\,|q|^2 \\
&\le f(s,q)\,|q|^2 + h\tfrac{\kappa_0}{n_c[f](s)}\Big(2\,m_3\langle f_R(s)\rangle\,|q|\,\big\|f_R(s,\cdot)\,|\cdot|^2\big\|_\infty \\
&\quad - 4\,m_3\langle f_R(s)\rangle\,|q|\,\big(f_R(s,|q|)|q|^2\big) + 2\,m_4\langle f_R(s)\rangle\,|q|^2 - c_0\,|q|^5\,\big(f_R(s,|q|)|q|^2\big)\Big) \\
&\le \mathfrak{h}_\infty + h\tfrac{\kappa_0}{n_c[f](s)}|q|\Big(- \tfrac{9}{5}\,m_3\langle f_R(s)\rangle\,\mathfrak{h}_\infty - 0.9\,c_0\,|q|^4\,\mathfrak{h}_\infty + 2\,m_4\langle f_R(s)\rangle\,|q|\Big)\,.
\end{aligned}
$$

Using that $m_4 \le m_3^{3/4} m_7^{1/4}$ one obtains that the last parenthesis is majorized by

$$
m_3\langle f_R(s)\rangle\big(- \tfrac{9}{5}\,\mathfrak{h}_\infty + \tfrac{3}{2}\big) + \big(\tfrac{1}{2}m_7\langle f(s)\rangle - 0.9\,c_0\,\mathfrak{h}_\infty\big)|q|^4 \le 0\,,
$$

where the non positivity follows by taking $\mathfrak{h}_\infty \ge C(\mathfrak{h}_3, \mathfrak{h}_8) > 0$ sufficiently large. Therefore, $w_R(s,q)\,|q|^2 \le \mathfrak{h}_\infty$. This contradicts the definition of W, thus, we conclude that W must be empty for this choice of parameters $\mathfrak{h}_\infty$, R, and h. Then, it is always the case that $\|w_R(t,\cdot)\,|\cdot|^2\|_\infty \le \mathfrak{h}_\infty$ which verifies property **iv**.

We infer due to previous discussion that for any $f \in \mathcal{C}\big([0,T];\mathcal{S}\big)$, there exists $R := R_3(f)$ sufficiently large and $h_* := h_*(f,\mathcal{S}) > 0$ sufficiently small such that that $w_R \in \mathcal{C}\big([0,T];\mathcal{S}\big)$ for any $0 < h < h_*$.

We conclude the proof using the Hölder estimate from Lemma 6.1 to obtain

$$
\begin{aligned}
h^{-1}\,m_2\big\langle\big|f(t) + h\tfrac{\kappa_0}{n_c[f](t)}\,\mathcal{Q}[f(t)] - w_R(t)\big|\big\rangle& \\
= m_2\big\langle\big|\tfrac{\kappa_0}{n_c[f](t)}\,&\mathcal{Q}[f(t)] - \tfrac{\kappa_0}{n_c[f_R](t)}\,\mathcal{Q}[f_R(t)]\big|\big\rangle \\
\le C_{\delta,T}(\mathcal{S})\Big(\sup_{t\in[0,T]} m_2\big\langle\big|f(t) - f_R(t)\big|\big\rangle^{\frac{1}{6}} &+ \sup_{t\in[0,T]} m_2\big\langle\big|f(t) - f_R(t)\big|\big\rangle\Big) \le \epsilon\,,
\end{aligned}
$$

where the last inequality is valid for for $R = R_4(f,\epsilon) > 0$ sufficiently large. Then, $w_R(t) \in B\Big(f(t) + h\tfrac{\kappa_0}{n_c[f](t)}\,\mathcal{Q}[f(t)], h\,\epsilon\Big)$ for all times provided this choice of R. Thus, choosing $R = \max\big\{R_3(f), R_4(f,\epsilon)\big\}$, one concludes that

$$
w_R(t) \in B\Big(f(t) + h\tfrac{\kappa_0}{n_c[f](t)}\,\mathcal{Q}[f(t)], h\,\epsilon\Big) \cap \mathcal{S}\,, \quad 0 < h < h_*,\ \ t \in [0,T]\,.
$$

Consequently,

$$
h^{-1}\sup_{t\in[0,T]} \operatorname{dist}\big(f(t) + h\tfrac{\kappa_0}{n_c[f](t)}\,\mathcal{Q}[f(t)], \mathcal{S}\big) \le \epsilon\,, \qquad \forall\, 0 < h < h_*\,.
$$

The proof of Proposition 6.1 is now complete and accounts for the sub-tangent condition. ■

6.3 One-side Lipschitz condition

Using dominate convergence theorem one can show that

$$\big[\varphi(t), \phi(t)\big] \leq \int_{\mathbb{R}^3} \mathrm{d}p\, \varphi(t,p)\,\mathrm{sign}(\phi(t,p))\,.$$

Thus, the one-side Lipschitz condition is met after proving the following lemma showing a Lipschitz condition for the interaction Boltzmann operator. The following proof, which yields a uniqueness results, is in the same spirit of the original Di Blassio [10] uniqueness proof for initial value problem to the homogeneous Boltzmann equation for hard spheres, using data with enough initial moments.

Lemma 6.2 (Lipschitz condition) *Assume* $f, g \in \mathcal{C}\big([0,T];\mathcal{S}\big)$. *Then, there exists constant* $C := C_{\delta,T}(\mathcal{S}) > 0$ *such that*

$$\begin{aligned}&\int_0^t ds \int_{\mathbb{R}^3} dp \left(\tfrac{\kappa_0}{n_c[f](s)}\,\mathcal{Q}[f(s)] - \tfrac{\kappa_0}{n_c[g](s)}\,\mathcal{Q}[g(s)]\right)\\ &\qquad \times sign\big(f(s)-g(s)\big)\big(1+|p|^2\big) \leq C\int_0^t ds\, m_2\big\langle|f(s)-g(s)|\big\rangle\,, \quad t\in[0,T]\,.\end{aligned}$$

Proof. Writing $\mathcal{Q}[f] = \mathcal{Q}_q[f] + \mathcal{L}[f]$, one has that

$$\begin{aligned}\int_{\mathbb{R}^3} &\mathrm{d}p\,\big(\mathcal{Q}[f](p) - \mathcal{Q}[g](p)\big)\big(1+|p|^2\big)\mathrm{sign}(f-g) = \\ &\int_{\mathbb{R}^3} \mathrm{d}p\,\big(\mathcal{Q}_q[f](p) - \mathcal{Q}_q[g](p)\big)\big(1+|p|^2\big)\mathrm{sign}(f-g)\\ &+ \int_{\mathbb{R}^3} \mathrm{d}p\,\big(\mathcal{L}[f](p) - \mathcal{L}[g](p)\big)\big(1+|p|^2\big)\mathrm{sign}(f-g)\,.\end{aligned}$$

For the quadratic part it follows, after a simple inspection of the weak formulation, that

$$\begin{aligned}\int_{\mathbb{R}^3} &\mathrm{d}p\,\big(\mathcal{Q}_q[f](p) - \mathcal{Q}_q[g](p)\big)\big(1+|p|^2\big)\mathrm{sign}(f-g)\\ &\leq C\max\big\{m_2\big\langle f+g\big\rangle, m_4\big\langle f+g\big\rangle, m_6\big\langle f+g\big\rangle\big\}\\ &\times\Big(m_2\big\langle|f-g|\big\rangle + m_4\big\langle|f-g|\big\rangle + m_6\big\langle|f-g|\big\rangle\Big)\,.\end{aligned}\tag{6.20}$$

For the linear part it follows, after explicit calculation of the weak formulation for test function $\varphi(p) = \big(1+|p|^2\big)\mathrm{sign}(f-g)(p)$, that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \mathrm{d}p\, \big(\mathcal{L}[f](p) - \mathcal{L}[g](p)\big)\varphi(p) \\
&\qquad = \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \mathrm{d}|p_1|\,\mathrm{d}|p_2|\,\mathcal{K}_0\big(|p_1|+|p_2|,|p_1|,|p_2|\big) \\
&\qquad\quad \times (f-g)(|p_1|+|p_2|)\Big[\varphi(|p_1|)+\varphi(|p_2|)-\varphi(|p_1|+|p_2|)\Big] \\
&\qquad \le \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} d|p_1|\,d|p_2|\,\mathcal{K}_0\big(|p_1|+|p_2|,|p_1|,|p_2|\big) \\
&\qquad\quad \times \Big|(f-g)(|p_1|+|p_2|)\Big|\,\Big[|p_1|^2+|p_2|^2-\big(|p_1|+|p_2|\big)^2+1\Big]
\end{aligned}$$

Therefore,

$$\int_{\mathbb{R}^3} \mathrm{d}p\, \big(\mathcal{L}[f](p) - \mathcal{L}[g](p)\big)\varphi(p) \le c_0\, m_7\big\langle|f-g|\big\rangle - c_2\, m_9\big\langle|f-g|\big\rangle\,. \tag{6.21}$$

As a consequence, using estimates (6.20) and (6.21), it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \mathrm{d}p\, \big(\mathcal{Q}[f](p) - \mathcal{Q}[g](p)\big)\big(1+|p|^2\big)\mathrm{sign}(f-g) \\
&\qquad\qquad \le C(\mathcal{S})\Big(m_2\big\langle|f-g|\big\rangle + m_7\big\langle|f-g|\big\rangle\Big) - c_2\, m_9\,.
\end{aligned}\tag{6.22}$$

Now, writing

$$\begin{aligned}
&\tfrac{1}{n_c[f](t)}\,\mathcal{Q}[f(t)] - \tfrac{1}{n_c[g](t)}\,\mathcal{Q}[g(t)] \\
&\qquad = \Big(\tfrac{1}{n_c[f](t)} - \tfrac{1}{n_c[g](t)}\Big)\mathcal{Q}[f(t)] + \tfrac{1}{n_c[g](t)}\Big(\mathcal{Q}[f(t)] - \mathcal{Q}[g(t)]\Big)\,,
\end{aligned}$$

and using that

$$\Big|\tfrac{1}{n_c[f](t)} - \tfrac{1}{n_c[g](t)}\Big| \le \frac{C(\mathcal{S})}{\delta^3}\int_0^t \mathrm{d}s\, m_2\big\langle|f(s)-g(s)|\big\rangle + m_7\big\langle|f(s)-g(s)|\big\rangle\,,$$

together with (6.22), we can derive the estimate

$$\begin{aligned}
&\int_{\mathbb{R}^3} \mathrm{d}p\,\Big(\tfrac{\kappa_0}{n_c[f](t)}\,\mathcal{Q}[f(t)] - \tfrac{\kappa_0}{n_c[g](t)}\,\mathcal{Q}[g(t)]\Big)\varphi(p) \\
&\qquad \le C_\delta(\mathcal{S})\bigg(m_2\big\langle|f(t)-g(t)|\big\rangle + m_7\big\langle|f(t)-g(t)|\big\rangle \\
&\qquad \int_0^t \mathrm{d}s\, m_2\big\langle|f(s)-g(s)|\big\rangle + \int_0^t \mathrm{d}s\, m_7\big\langle|f(s)-g(s)|\big\rangle\bigg) \\
&\qquad\qquad - c(\mathcal{S},n_0)\, m_9\big\langle|f(t)-g(t)|\big\rangle\,.
\end{aligned}\tag{6.23}$$

After integrating estimate (6.23) from $[0,t]$, it follows that

$$\begin{aligned}
&\int^t \mathrm{d}s \int_{\mathbb{R}^3} \mathrm{d}p \left(\tfrac{\kappa_0}{n_c[f](s)} \mathcal{Q}[f(s)] - \tfrac{\kappa_0}{n_c[g](s)} \mathcal{Q}[g(s)] \right) \varphi(p) \\
&\leq \int_0^t \mathrm{d}s \Big[C_\delta(\mathcal{S})(1+T) \Big(m_2 \big\langle |f(s)-g(s)| \big\rangle + m_7 \big\langle |f(s)-g(s)| \big\rangle \Big) \qquad (6.24) \\
&- c(\mathcal{S}, n_0)\, m_9 \big\langle |f(s)-g(s)| \big\rangle \Big] \leq C_{\delta,T}(\mathcal{S}) \int_0^t \mathrm{d}s\, m_2 \big\langle |f(s)-g(s)| \big\rangle \,.
\end{aligned}$$

For the last inequality we used that

$$C_\delta(\mathcal{S})(1+T)\big(|p|^2 + |p|^7\big) - c(\mathcal{S}, n_0)|p|^9 \leq C_{\delta,T}(\mathcal{S})|p|^2 \,.$$

This completes the proof the the one-side Lipschitz property. ■

Let us complete now the proof of Theorem 6.2. As an application of Theorem 6.7, where the three conditions (6.4), (6.5), and (6.6) have been verified in subsections 6.1, 6.2, and 6.3, respectively, it follows that the system (6.1)-(6.2) has a unique solution $f \in \mathcal{C}\big([0,T];\mathcal{S}\big)$ where T is any time such that $n_c[f](t) \geq \delta$, $t \in [0,T]$. Clearly, such solution $(f(t), n_c[f](t))$ satisfies total conservation of mass

$$m_2\langle f(t)\rangle + n_c[f](t) = m_2\langle f_0\rangle + n_0 \,,$$

and all conditions of Theorem 5.1 are satisfied. Therefore,

$$\inf_t n_c[f](t) \geq \delta > 0 \,.$$

As a consequence, $T > 0$ is arbitrary. This proves Theorem 6.2. ■

Proposition 6.2 (Creation of polynomial moments) *Let the pair $0 \leq (f, n_c) \in \mathcal{C}\big([0,\infty);\mathcal{S}\big) \times \mathcal{C}\big([0,\infty)\big)$ be the solution of the system* (1.11) *with initial datum $(f_0, n_0) > 0$ satisfying condition* (5.5) *for some $\delta > 0$. Then, there exists a constant $C_k > 0$ that depends only on $k > 3$ such that*

$$m_k\langle f\rangle(t) \leq \Big(\tfrac{1}{\delta(k-3)}\Big)^{\frac{k-3}{5}} \frac{m_3}{t^{\frac{k-3}{5}}} + C_k\, m_3^{\frac{k+1}{4}} \,, \qquad t > 0 \,.$$

Proof. Recall estimate (4.6)

$$\frac{\mathrm{d}}{\mathrm{d}t} m_{k+2}(t) \leq \tfrac{\kappa_0}{n_c[f](t)} m_3^{-\frac{5}{k-1}} \Big(\tilde{C}_k\, m_3^{\frac{(k+4)(k+3)}{4(k-1)}} - \tilde{c}_k\, m_{k+2}^{\frac{k+4}{k-1}}(t) \Big) \,,$$

for some constants $\tilde{C}_k$ and $\tilde{c}_k$ depending only on $k > 1$. Since $n_c[f](t) > 0$, for $t \in [0, \infty)$, is Lipschitz continuous, we can solve uniquely the nonlinear ode

$$\alpha'(t) = \tfrac{1}{n_c[f](\alpha(t))}\,, \quad t > 0, \quad \alpha(0) = 0\,.$$

The solution $\alpha(t)$ is strictly increasing. Thus, we can rescale estimate (4.6) by defining the function $y(t) = m_{k+2}(\alpha(t))$, so that

$$\frac{\mathrm{d}y}{\mathrm{d}t} \leq \kappa_o\, m_3^{-\frac{5}{k-1}} \Big(\tilde{C}_k\, m_3^{\frac{(k+4)(k+3)}{4(k-1)}} - \tilde{c}_k\, y^{\frac{k+4}{k-1}} \Big)\,.$$

It is not difficult to prove that a super solution for previous differential inequality is given by

$$\begin{aligned} Y(t) &= \frac{m_{k+2}(0)}{\Big(1 + \tfrac{k-1}{5} \Big(\tfrac{m_{k+2}(0)}{m_3}\Big)^{\frac{5}{k-1}} t\Big)^{\frac{k-1}{5}}} + C_k\, m_3^{\frac{k+3}{4}} \\ &\leq \Big(\tfrac{5}{k-1}\Big)^{\frac{k-1}{5}} \frac{m_3}{t^{\frac{k-1}{5}}} + C_k\, m_3^{\frac{k+3}{4}}\,. \end{aligned}$$

Hence $y(t) \leq Y(t)$ for all times. Observe that $\alpha'(t) \leq \frac{1}{\delta}$, this implies that $\delta\, t \leq \alpha^{-1}(t)$. As a consequence,

$$\begin{aligned} m_{k+2}(t) \leq Y(\alpha^{-1}(t)) &\leq \Big(\tfrac{5}{k-1}\Big)^{\frac{k-1}{5}} \frac{m_3}{\big(\alpha^{-1}(t)\big)^{\frac{k-1}{5}}} + C_k\, m_3^{\frac{k+3}{4}} \\ &\leq \Big(\tfrac{5}{\delta(k-1)}\Big)^{\frac{k-1}{5}} \frac{m_3}{t^{\frac{k-1}{5}}} + C_k\, m_3^{\frac{k+3}{4}}\,. \end{aligned}$$

■

7 Mittag-Leffler moments

7.1 Propagation of Mittag-Leffler tails

In this section we are interested in studying the propagation and creation of Mittag-Leffler moments of order $a \in [1, \infty)$ and rate $\alpha > 0$ for radially symmetric solutions built in section 5. This concept of Mittag-Leffler tails was introduced recently in [30] and it is a generalization of the classical exponential tails for hard potentials in Boltzmann equations. The creation of exponential tail in the solutions formalize, at least qualitatively, the notion of low temperature regime which is key in the derivation of the model. We perform the analysis using standard moments $\mathcal{M}_k$ stressing that same estimates are valid for line moments since $\mathcal{M}_k = |\mathbb{S}^2|\, m_{k+2}$ in the context of radially symmetric solutions. In terms of infinite sums, see [30], this is equivalent to control the integral

$$\int_{\mathbb{R}^3} \mathrm{d}p\, f(t,p)\mathcal{E}_a(\alpha^a|p|) = \sum_{k=1}^{\infty} \frac{\mathcal{M}_k(t)\alpha^{ak}}{\Gamma(ak+1)}\,, \tag{7.1}$$

where

$$\mathcal{E}_a(x) := \sum_{k=1}^{\infty} \frac{x^k}{\Gamma(ak+1)} \approx e^{x^{1/a}} - 1\,, \qquad x \gg 1\,. \tag{7.2}$$

For convenience define for any $\alpha > 0$ and $a \in [1,\infty)$ the partial sums

$$\mathcal{E}_a^n(\alpha,t) := \sum_{k=1}^{n} \frac{\mathcal{M}_k(t)\alpha^{ak}}{\Gamma(ak+1)} \quad \text{and} \quad \mathcal{I}_{a,\rho}^n(\alpha,t) := \sum_{k=1}^{n} \frac{\mathcal{M}_{k+\rho}(t)\alpha^{ak}}{\Gamma(ak+1)}\,, \quad \rho > 0\,.$$

This notation will be of good use throughout this section.

Theorem 7.1 (Propagation of Mittag-Leffler tails) *Consider the pair $0 \le (f, n_c) \in \mathcal{C}\big([0,\infty);\mathcal{S}\big) \times \mathcal{C}\big([0,\infty)\big)$ to be the solution of* (1.11) *associated to the initial condition $(f_0, n_0) > 0$ satisfying condition* (5.5) *for some $\delta > 0$. Take $a \in [1,\infty)$ and suppose that there exists positive α_0 such that*

$$\int_{\mathbb{R}^3} dp\, f_0(p)\, \mathcal{E}_a(\alpha_0^a|p|) \le 1\,.$$

Then, there exists positive constant $\alpha := \alpha(\mathcal{M}_1(0), \alpha_0, a)$ such that

$$\sup_{t\ge 0} \int_{\mathbb{R}^3} dp\, f(t,p)\, \mathcal{E}_a(\alpha^a|p|) \le 2\,. \tag{7.3}$$

Lemma 7.1 (From Ref. [30]) *Let $k \ge 3$, then for any $a \in [1,\infty)$, we have*

$$\sum_{i=1}^{\left[\frac{k+1}{2}\right]} \binom{k}{i} B\big(ai+1, a(k-i)+1\big) \le \frac{C_a}{(ak)^{1+a}},$$

where $B(\cdot,\cdot)$ is the beta function. The constant $C_a > 0$ depends only on a.

Lemma 7.2 *Let $\alpha > 0$, $a \in [1,\infty)$. Then, the following estimate holds*

$$\mathcal{J} := \sum_{k=k_0}^{n} \sum_{i=1}^{\left[\frac{k+1}{2}\right]} \binom{k}{i} \frac{\mathcal{M}_{i+2}\mathcal{M}_{k-i}\,\alpha^{ak}}{\Gamma(ak+1)} \le \frac{C_a}{(ak_0)^a}\, \mathcal{E}_a^n\, \mathcal{I}_{a,2}^n\,, \quad n \ge k_0 \ge 1\,, \tag{7.4}$$

with universal constant C_a depending only on a.

Proof. Using the following identities for the Beta and Gamma functions

$$\begin{aligned} &B(ai+1,\, a(k-i)+1) \\ &\quad = \frac{\Gamma(ai+1)\,\Gamma(a(k-i)+1)}{\Gamma(ai+1+a(k-i)+1)} = \frac{\Gamma(ai+1)\,\Gamma(a(k-i)+1)}{\Gamma(ak+2)}\,, \end{aligned}$$

and the identity $\alpha^{ak} = \alpha^{\alpha i}\alpha^{a(k-i)}$, we deduce that

$$\begin{aligned} \mathcal{J} = \sum_{k=k_0}^{n} (ak+1) \sum_{i=1}^{\left[\frac{k+1}{2}\right]} \binom{k}{i} \frac{\mathcal{M}_{i+2}\alpha^{ai}}{\Gamma(ai+1)} \frac{\mathcal{M}_{k-i}\alpha^{a(k-i)}}{\Gamma(a(k-i)+1)} \\ \times B(ai+1, a(k-i)+1)\,, \end{aligned} \tag{7.5}$$

where we used that $\Gamma(ak+2) = (ak+1)\Gamma(ak+1)$. In addition, each component in the inner sum on the right side of (7.5) can be bounded as

$$\begin{aligned} &\sum_{i=1}^{\left[\frac{k+1}{2}\right]} \binom{k}{i} \frac{\mathcal{M}_{i+2}\alpha^{ai}}{\Gamma(ai+1)} \frac{\mathcal{M}_{k-i}\alpha^{a(k-i)}}{\Gamma(a(k-i)+1)} B(ai+1, a(k-i)+1) \\ &\quad \le \sum_{i=1}^{\left[\frac{k+1}{2}\right]} \frac{\mathcal{M}_{i+2}\alpha^{ai}}{\Gamma(ai+1)} \frac{\mathcal{M}_{k-i}\alpha^{a(k-i)}}{\Gamma(a(k-i)+1)} \sum_{j=1}^{\left[\frac{k+1}{2}\right]} \binom{k}{j} B(aj+1, a(k-j)+1)\,, \end{aligned}$$

which implies, by Lemma 7.1, that

$$\begin{aligned} &\sum_{i=1}^{\left[\frac{k+1}{2}\right]} \binom{k}{i} \frac{\mathcal{M}_{i+2}\alpha^{ai}}{\Gamma(ai+1)} \frac{\mathcal{M}_{k-i}\alpha^{a(k-i)}}{\Gamma(a(k-i)+1)} B(ai+1, a(k-i)+1) \\ &\qquad \le \frac{C_a}{(ak)^{1+a}} \sum_{i=1}^{\left[\frac{k+1}{2}\right]} \frac{\mathcal{M}_{i+2}\alpha^{ai}}{\Gamma(ai+1)} \frac{\mathcal{M}_{k-i}\alpha^{a(k-i)}}{\Gamma(a(k-i)+1)}\,. \end{aligned} \tag{7.6}$$

Combining (7.5) and (7.6) yields the estimate on $\mathcal{J}$

$$\mathcal{J} \le C_a \sum_{k=k_0}^{n} \frac{ak+1}{(ak)^{1+a}} \sum_{i=1}^{\left[\frac{k+1}{2}\right]} \frac{\mathcal{M}_{i+2}\alpha^{ai}}{\Gamma(ai+1)} \frac{\mathcal{M}_{k-i}\alpha^{a(k-i)}}{\Gamma(a(k-i)+1)}\,. \tag{7.7}$$

Noticing that $\frac{ak+1}{(ak)^{1+a}} \le \frac{1+a}{a}\frac{1}{(ak_0)^a}$ for $k \ge k_0$, one concludes from (7.7) that

$$\begin{aligned} \mathcal{J} &\le \frac{C_a'}{(ak_0)^a} \sum_{k=k_0}^{n} \sum_{i=1}^{\left[\frac{k+1}{2}\right]} \frac{\mathcal{M}_{i+2}\alpha^{ai}}{\Gamma(ai+1)} \frac{\mathcal{M}_{k-i}\alpha^{a(k-i)}}{\Gamma(a(k-i)+1)} \\ &\le \frac{C_a'}{(ak_0)^a} \sum_{i=1}^{n} \frac{\mathcal{M}_{i+2}\alpha^{ai}}{\Gamma(ai+1)} \sum_{i=1}^{n} \frac{\mathcal{M}_{i}\alpha^{ai}}{\Gamma(ai+1)} \le \frac{C_a'}{(ak_0)^a} \mathcal{E}_a^n\, \mathcal{I}_{a,2}^n. \end{aligned} \tag{7.8}$$

■

Lemma 7.3 *The following control is valid for any $\alpha > 0$ and $a \in [1, \infty)$*

$$\mathcal{I}^n_{a,5}(\alpha, t) \geq \frac{1}{\alpha^{5/2}} \mathcal{E}^n_a(\alpha, t) - \frac{1}{\alpha^2} \mathcal{M}_1 \mathcal{E}_a(\alpha^{a-1/2})\,. \tag{7.9}$$

Proof. Observe that

$$\mathcal{I}^n_{a,5}(\alpha, t) = \sum_{k=1}^{n} \frac{\mathcal{M}_{k+5}(t)\alpha^{ak}}{\Gamma(ak+1)} \geq \sum_{k=1}^{n} \int_{\{|p| \geq \frac{1}{\sqrt{\alpha}}\}} \mathrm{d}p\, \frac{|p|^{k+5}\alpha^{ak}}{\Gamma(ak+1)} f(t,p)\,.$$

Note that in the set $\{|p| \geq \frac{1}{\sqrt{\alpha}}\}$ one has $|p|^{k+5} \geq \frac{|p|^k}{\alpha^{5/2}}$, therefore

$$\begin{aligned}
\mathcal{I}^n_{a,6}(\alpha, t) &\geq \frac{1}{\alpha^{5/2}} \sum_{k=1}^{n} \int_{\{|p| \geq \frac{1}{\sqrt{\alpha}}\}} \mathrm{d}p\, \frac{|p|^k \alpha^{ak}}{\Gamma(ak+1)} f(t,p) \\
&= \frac{1}{\alpha^{5/2}} \left(\sum_{k=1}^{n} \int_{\mathbb{R}^3} \mathrm{d}p\, \frac{|p|^k \alpha^{ak}}{\Gamma(ak+1)} f(t,p) - \sum_{k=1}^{n} \int_{\{|p| < \frac{1}{\sqrt{\alpha}}\}} \mathrm{d}p\, \frac{|p|^k \alpha^{ak}}{\Gamma(ak+1)} f(t,p) \right).
\end{aligned}$$

In the set $\{|p| < \frac{1}{\sqrt{\alpha}}\}$ one has $|p|^k < |p|\alpha^{-(k-1)/2}$, consequently

$$\begin{aligned}
\mathcal{I}^n_{a,5}(\alpha, t) &\geq \frac{1}{\alpha^{5/2}} \left(\mathcal{E}^n_a(t) - \sum_{k=1}^{n} \int_{\mathbb{R}^3} \mathrm{d}p\, \frac{\alpha^{-(k-1)/2}\alpha^{ak}}{\Gamma(ak+1)} f(t,p)|p| \right) \\
&= \frac{1}{\alpha^{5/2}} \mathcal{E}^n_a(t) - \frac{\mathcal{M}_1}{\alpha^2} \sum_{k=1}^{n} \frac{\alpha^{(a-1/2)k}}{\Gamma(ak+1)} \geq \frac{1}{\alpha^{5/2}} \mathcal{E}^n_a(t) - \frac{\mathcal{M}_1}{\alpha^2} \mathcal{E}_a(\alpha^{a-1/2})\,.
\end{aligned}$$

■

Proof. (of Theorem 7.1) The proof consists in showing that for any $a \in [1, \infty)$, there exists positive constant α such that

$$\mathcal{E}^n_a(\alpha, t) \leq 2, \quad \forall\, t \geq 0, \quad \forall\, n \in \mathbb{N} \backslash \{0\}. \tag{7.10}$$

For this purpose we define for sufficiently small $\alpha > 0$, chosen in the sequel, the sequence of times

$$T_n := \sup \left\{ t \geq 0 \,\middle|\, \mathcal{E}^n_a(\alpha, \tau) \leq 2, \forall\, \tau \in [0, t] \right\}$$

and prove that $T_n = +\infty$. This sequence of times is well-defined and positive. Indeed, for any $\alpha \leq \alpha_0$

$$\mathcal{E}^n_a(\alpha, 0) = \sum_{k=1}^{n} \frac{\mathcal{M}_k(0)\alpha^{ak}}{\Gamma(ak+1)} \leq \sum_{k=1}^{n} \frac{\mathcal{M}_k(0)\alpha_0^{ak}}{\Gamma(ak+1)} = \int_{\mathbb{R}^3} \mathrm{d}p\, f_0(p) \mathcal{E}_a(\alpha_0^a |p|) \leq 1\,.$$

Since each term $\mathcal{M}_k(t)$ is continuous in t, the partial sum $\mathcal{E}_a^n(\alpha,t)$ is also continuous in t. Therefore, $\mathcal{E}_a^n(\alpha,t) \leq 2$ in some nonempty interval $(0,t_n)$ and, thus, T_n is well-defined and positive for every $n \in \mathbb{N}$.

Now, let us establish a differential inequality for the partial sums that implies $T_n = +\infty$. Note that

$$\tfrac{n_c}{\kappa_0}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_k \leq 2\sum_{i=1}^{\left[\frac{k+1}{2}\right]}\binom{k}{i}\mathcal{M}_{i+2}\mathcal{M}_{k-i} - c_k\,\mathcal{M}_{k+5}\,.$$

Here $c_k > 0$ was defined in Lemma 4.1. Multiplying the above inequality by $\frac{\alpha^k}{\Gamma(ak+1)}$ and summing with respect to k in the interval $k_0 \leq k \leq n$, with $k_0 \geq 1$ to be chosen later on sufficiently large,

$$\begin{aligned}\tfrac{n_c}{\kappa_0}\frac{\mathrm{d}}{\mathrm{d}t}&\sum_{k=k_0}^{n}\frac{\mathcal{M}_k\,\alpha^k}{\Gamma(ak+1)}\\ &\leq 2\sum_{k=k_0}^{n}\sum_{i=1}^{\left[\frac{k+1}{2}\right]}\binom{k}{i}\frac{\mathcal{M}_{i+2}\mathcal{M}_{k-i}\,\alpha^k}{\Gamma(ak+1)} - c_{k_0}\sum_{k=k_0}^{n}\frac{\mathcal{M}_{k+5}\,\alpha^k}{\Gamma(ak+1)}\,.\end{aligned} \tag{7.11}$$

Here we used the fact that c_k increases in k. We observe that the sum on the left side of (7.11) will become $\frac{n_c}{\kappa_0}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_a^n(\alpha,t)$ after adding

$$\tfrac{n_c}{\kappa_0}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=1}^{k_0-1}\frac{\mathcal{M}_k\,\alpha^k}{\Gamma(ak+1)} \leq C(k_0,\alpha_0,a) < \infty \tag{7.12}$$

to this expression. The latter inequality holds due to the choice $\alpha \leq \alpha_0$ and the control of moments Theorem 4.1. Therefore, from (7.11) and (7.12), we obtain the differential inequality

$$\begin{aligned}\tfrac{n_c}{\kappa_0}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_a^n(\alpha,t) \leq 2&\sum_{k=k_0}^{n}\sum_{i=1}^{\left[\frac{k+1}{2}\right]}\binom{k}{i}\frac{\mathcal{M}_{i+2}\mathcal{M}_{k-i}\,\alpha^k}{\Gamma(ak+1)}\\ &- c_{k_0}\sum_{k=k_0}^{n}\frac{\mathcal{M}_{k+5}\,\alpha^k}{\Gamma(ak+1)} + C(k_0,\alpha_0,a).\end{aligned} \tag{7.13}$$

Let us now estimate the sum on the right side of (7.13). Again, we deduce from propagation of moments Theorem 4.1 that

$$\sum_{k=1}^{k_0}\frac{\mathcal{M}_{k+5}\,\alpha^k}{\Gamma(ak+1)} \leq \sum_{k=1}^{k_0}\frac{\mathcal{M}_{k+5}\,\alpha_0^k}{\Gamma(ak+1)} \leq C(k_0,\alpha_0,a)\,,$$

which leads to the following estimate for (7.13)

$$\begin{aligned}\tfrac{n_c}{\kappa_0}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_a^n(\alpha,t) \le 2\sum_{k=k_0}^{n}\sum_{i=1}^{\left[\frac{k+1}{2}\right]}\binom{k}{i}\frac{\mathcal{M}_{i+2}\mathcal{M}_{k-i}\,\alpha^k}{\Gamma(ak+1)} \\ - c_{k_0}\sum_{k=1}^{n}\frac{M_{k+5}\,\alpha^k}{\Gamma(ak+1)} + C(k_0,\alpha_0,a)\,.\end{aligned} \tag{7.14}$$

Therefore, as a consequence of the definition of $\mathcal{I}^n_{a,5}$ and Lemma 7.2

$$\begin{aligned}\tfrac{n_c}{\kappa_0}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_a^n(\alpha,t) &\le 2\sum_{k=k_0}^{n}\sum_{i=1}^{\left[\frac{k+1}{2}\right]}\binom{k}{i}\frac{\mathcal{M}_{i+2}\mathcal{M}_{k-i}\,\alpha^k}{\Gamma(ak+1)} - c_{k_0}\,\mathcal{I}^n_{a,5} + C(k_0,\alpha_0,a) \\ &\le \tfrac{2C_a}{(ak_0)^a}\mathcal{E}^n_a\,\mathcal{I}^n_{a,2} - c_{k_0}\,\mathcal{I}^n_{a,5} + C(k_0,\alpha_0,a)\,.\end{aligned} \tag{7.15}$$

We now estimate the right hand side of (7.15) starting with the term $\mathcal{I}^n_{a,2}$. Using Cauchy inequality $|p|^2 \le \frac{3}{5} + \frac{2}{5}|p|^5$, then

$$\mathcal{M}_{k+2} \le \tfrac{3}{5}\mathcal{M}_k + \tfrac{2}{5}\mathcal{M}_{k+5}\,, \qquad k \ge 0\,.$$

Multiplying this inequality with $\frac{\alpha^{ak}}{\Gamma(ak+1)}$ and summing with respect to k in the interval $0 \le k \le n$ yields

$$\mathcal{I}^n_{a,2} \le \tfrac{3}{5}\mathcal{E}^n_a + \tfrac{2}{5}\mathcal{I}^n_{a,5} \le \tfrac{6}{5} + \tfrac{2}{5}\mathcal{I}^n_{a,5}\,,$$

where the last inequality follows since we are considering $t \in [0, T_n]$ so that $\mathcal{E}^n_a \le 2$. Therefore,

$$\tfrac{n_c}{\kappa_0}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^n_a \le \tfrac{5\,C_a}{(ak_0)^a}\big(1 + \tfrac{1}{3}\mathcal{I}^n_{a,5}\big) - c_{k_0}\mathcal{I}^n_{a,5} + C(k_0,\alpha_0,a)\,. \tag{7.16}$$

Choosing $k_0 := k_0(a)$ sufficiently large, the term $\frac{5C_a}{3(ak_0)^a}\mathcal{I}^n_{a,5}$ is absorbed by $\frac{c_{k_0}}{2}\mathcal{I}^n_{a,5}$. Thus,

$$\tfrac{n_c}{\kappa_0}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^n_a \le -\tfrac{c_{k_0}}{2}\mathcal{I}^n_{a,5} + C(\mathcal{M}_1,\alpha_0,a)\,. \tag{7.17}$$

Estimating the right side of (7.17) in terms of $\mathcal{E}^n_a$ using Lemma 7.3, it is concluded that

$$\tfrac{n_c}{\kappa_0}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^n_a \le -\tfrac{c_{k_0}}{2\alpha^{5/2}}\mathcal{E}^n_a + \tfrac{c_{k_0}}{2\alpha^2}\,\mathcal{M}_1\,\mathcal{E}_a(\alpha^{a-1/2}) + C(\mathcal{M}_1,\alpha_0,a)\,.$$

Therefore, one has that for $t \in [0, T_n]$

$$\mathcal{E}_a^n \leq \max\left\{1, \tfrac{2\alpha^{5/2}}{c_{k_0}}\left(\tfrac{c_{k_0}}{2\alpha^2}\,\mathcal{M}_1\,\mathcal{E}_a(\alpha^{a-1/2}) + C(\mathcal{M}_1, \alpha_0, a)\right)\right\} < 2\,, \qquad (7.18)$$

provided that $\alpha := \alpha(\mathcal{M}_1, \alpha_0, a) > 0$ is sufficiently small, for instance such that

$$\tfrac{2\alpha^{5/2}}{c_{k_0}}\left(\tfrac{c_{k_0}}{2\alpha^2}\mathcal{M}_1\mathcal{E}_a(\alpha^{a-1/2}) + C(\mathcal{M}_1, \alpha_0, a)\right) < 2\,.$$

Given the continuity of $\mathcal{E}_a^n(\alpha, t)$ with respect to t, estimate (7.18) readily implies that $T_n = +\infty$. Therefore, $\mathcal{E}_a^n(\alpha, t) \leq 2$ for $t \geq 0$ and $n \in \mathbb{N}\backslash\{0\}$. Now taking the limit as $n \to \infty$ and using the definition of Mittag-Leffler moments of order $a \in [1, \infty)$ and rate $\alpha > 0$, as defined in (7.1), yields

$$\int_{\mathbb{R}^3} \mathrm{d}p\, f(t,p)\,\mathcal{E}_a(\alpha^a|p|) = \lim_{n\to\infty} \mathcal{E}_a^n(\alpha, t) \leq 2\,.$$

This concludes the argument. ■

7.2 Creation of exponential tails

Theorem 7.2 *Let the pair $0 \leq (f, n_c) \in \mathcal{C}\big([0,\infty); \mathcal{S}\big) \times \mathcal{C}\big([0,\infty)\big)$ be the solution of* (1.11). *Assume that $(f_0, n_0) > 0$ is such that condition* (5.5) *is satisfied for some $\delta > 0$. Then, there exists a constant $\alpha > 0$ depending on $m_2(0)$, m_3, n_0, and $\delta > 0$, such that*

$$\int_{\mathbb{R}^3} dp\, f(t,p)|p|e^{\alpha\,\min\{1, t^{\frac{1}{5}}\}|p|} \leq \frac{1}{2\alpha}, \quad \forall\, t > 0. \qquad (7.19)$$

Proof. Thanks to Corollary 6.2, the moments of $f(t)$ enjoy the estimate

$$m_k(t) \leq C_k(\delta, m_3)\Big(t^{-\frac{k-3}{5}} + 1\Big), \quad \forall\, k > 3\,.$$

This implies that for any $0 \leq t \leq 1$

$$\mathcal{E}_1^n(t^{\frac{1}{5}}\alpha, t) = \int_{\mathbb{R}^3} \mathrm{d}p\, f(t,p)\mathcal{E}_1^n\big(t^{\frac{1}{5}}\alpha|p|\big) \leq C_n(\alpha)\, t^{\frac{1}{5}}\,, \quad \alpha > 0\,. \qquad (7.20)$$

Fix parameters $\alpha, \vartheta \in (0,1]$ and define

$$T_n := \sup\left\{t \in (0,1] \big| \mathcal{E}_1^n(t^{\frac{1}{5}}\alpha, t) \leq t^{\frac{1-\vartheta}{5}}\right\}.$$

We proof that for sufficiently small $\alpha > 0$ depending only on the initial data (through $m_2(0)$, m_3, and n_0), it holds that $T_n = 1$ for all $n \in \mathbb{N}$ and

$\vartheta \in (0,1]$. One notices first that $T_n > 0$ for each n thanks to (7.20). Also, for $n \geq k_0 \geq 1$ we have that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=k_0}^{n}\mathcal{M}_k(t)\frac{(t^{\frac15}\alpha)^k}{k!}\\
&\quad=\sum_{k=k_0}^{n}\mathcal{M}_k'(t)\frac{(t^{\frac15}\alpha)^k}{k!}+\frac{\alpha}{5t^{\frac45}}\sum_{k=k_0}^{n}\mathcal{M}_k(t)\frac{(t^{\frac15}\alpha)^{k-1}}{(k-1)!}\,.
\end{aligned}\tag{7.21}$$

Observe that for the last term in the right side of (7.21)

$$\begin{aligned}
&\frac{\alpha}{5t^{\frac45}}\sum_{k=k_0}^{n}\mathcal{M}_k(t)\frac{(t^{\frac15}\alpha)^{k-1}}{(k-1)!}\\
&\qquad=\frac{\alpha}{5t^{\frac45}}\sum_{k=k_0+5}^{n}\mathcal{M}_k(t)\frac{(t^{\frac15}\alpha)^{k-1}}{(k-1)!}+\frac{\alpha}{5t^{\frac45}}\sum_{k=k_0}^{k_0+5}\mathcal{M}_k(t)\frac{(t^{\frac15}\alpha)^{k-1}}{(k-1)!}\\
&\qquad=\frac{\alpha^5}{5}\sum_{k=k_0}^{n-5}\mathcal{M}_{k+5}(t)\frac{(t^{\frac15}\alpha)^{k}}{(k+4)!}+\frac{\alpha}{5t^{\frac45}}\sum_{k=k_0}^{k_0+5}\mathcal{M}_k(t)\frac{(t^{\frac15}\alpha)^{k-1}}{(k-1)!}\\
&\qquad\leq\frac{\alpha^5}{5}\sum_{k=k_0}^{n}\mathcal{M}_{k+5}(t)\frac{(t^{\frac15}\alpha)^{k}}{k!}+\frac{\alpha^{k_0}}{t^{\frac45}}C(k_0,m_3)\,,\quad 0<\alpha\leq 1\,.
\end{aligned}$$

Thus, arguing as in (7.11)-(7.15) we conclude that for the quantities

$$\mathcal{E}_1^n := \mathcal{E}_1^n(t^{\frac15}\alpha,t)\,,\quad \mathcal{I}_{1,5}^n := \mathcal{I}_{1,5}^n(t^{\frac15}\alpha,t)\,,$$

it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_1^n \leq \tfrac{C\,\kappa_0}{k_0\,n_c[f](t)}\mathcal{E}_1^n\,\mathcal{I}_{1,2}^n-\big(\tfrac{\kappa_0}{n_c[f](t)}c_{k_0}-\tfrac{\alpha^5}{5}\big)\mathcal{I}_{1,5}^n+\frac{\alpha}{t^{\frac45}}C(k_0,m_3)\,.\tag{7.22}$$

Using that $\mathcal{I}_{1,2}^n \leq \mathcal{E}_1^n + \mathcal{I}_{1,5}^n \leq 1+\mathcal{I}_{1,5}^n$, recalling the definition of T_n, it follows from (7.22)

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_1^n \leq \tfrac{C\,\kappa_0}{k_0\,n_c[f](t)}\\
&-\Big(\tfrac{\kappa_0}{n_c[f](t)}\big(c_{k_0}-\tfrac{C}{k_0}\big)-\tfrac{\alpha^5}{5}\Big)\mathcal{I}_{1,5}^n+\frac{\alpha}{t^{\frac45}}C\big(k_0,m_3\big)\,,\quad 0<t\leq T_n\,.
\end{aligned}\tag{7.23}$$

Now, fix $k_0 \in \mathbb{N}$ sufficiently large and, then, $\alpha \in (0,1]$ sufficiently small such that

$$c_{k_0}-\tfrac{C}{k_0}\geq\tfrac{c_{k_0}}{2}\,,\qquad \tfrac{\alpha^5}{5}\leq\frac{\kappa_0\,c_{k_0}}{4(m_2(0)+n_0)}\leq\frac{\kappa_0\,c_{k_0}}{4n_c[f](t)}\,,$$

to conclude from (7.23) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_1^n \le \tfrac{C\,\kappa_0}{k_0\,\delta} - \frac{\kappa_0\, c_{k_0}}{4(m_2(0)+n_0)}\mathcal{I}_{1,5}^n + \frac{\alpha}{t^{\frac{4}{5}}}C(k_0,m_3)\,, \quad 0<t\le T_n\,. \tag{7.24}$$

Also, observe that

$$\begin{aligned}
\mathcal{I}_{1,5}^n &= \sum_{k=1}^{n}\mathcal{M}_{k+5}(t)\frac{(t^{\frac{1}{5}}\alpha)^k}{k!}\\
&= \frac{1}{t\alpha^5}\sum_{k=6}^{n+5}\mathcal{M}_k(t)\frac{(t^{\frac{1}{5}}\alpha)^k}{(k-5)!} \ge \frac{1}{t\alpha^5}\sum_{k=6}^{n}\mathcal{M}_k(t)\frac{(t^{\frac{1}{5}}\alpha)^k}{k!}\\
&= \frac{1}{t\alpha^5}\mathcal{E}_1^n - \frac{1}{t\alpha^5}\sum_{k=1}^{5}\mathcal{M}_k(t)\frac{(t^{\frac{1}{5}}\alpha)^k}{k!} \ge \frac{1}{t\alpha^5}\mathcal{E}_1^n - \frac{C(m_3)}{t^{\frac{4}{5}}\alpha^4}\,.
\end{aligned}$$

Together with (7.24), this leads finally to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_1^n \le \frac{C(m_2(0),n_0,m_3,\delta)}{t^{\frac{4}{5}}\alpha^4} - \frac{c(m_2(0),n_0,m_3)}{t\alpha^5}\mathcal{E}_1^n\,, \quad 0<t\le T_n\,.$$

A simple integration of this differential inequality shows that choosing $\alpha>0$ sufficiently small, say

$$\frac{\alpha}{c+\alpha^5/5}\frac{C}{c}<1,$$

implies that $\mathcal{E}_1^n < t^{\frac{1}{5}}$. That is,

$$\int_{\mathbb{R}^3}\mathrm{d}p\, f(t,p)\mathcal{E}_1^n(t^{\frac{1}{5}}\alpha|p|) < t^{\frac{1}{5}}\,, \quad 0\le t\le T_n\,.$$

Time continuity of $\mathcal{E}_1^n$ and the maximality of T_n imply that $T_n=1$ for all $n\ge 1$ and $\vartheta\in(0,1]$. In particular, sending $\vartheta\to 0$ and, then, $n\to\infty$ one arrives to

$$\int_{\mathbb{R}^3}\mathrm{d}p\, f(t,p)\mathcal{E}_1(t^{\frac{1}{5}}\alpha|p|) \le t^{\frac{1}{5}}\,, \qquad 0\le t\le 1\,.$$

The result follows after noticing that

$$\mathcal{E}_1(t^{\frac{1}{5}}\alpha|p|) \ge t^{\frac{1}{5}}\alpha|p|e^{t^{\frac{1}{5}}\frac{\alpha}{2}|p|}, \qquad 0\le t\le 1,$$

and recalling that, after creation, exponential tails will uniformly propagate thanks to Theorem 7.1. ■

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8 Appendix: Proof of Theorem 6.1

The proof follows the same lines of the argument of Bressan's proof of Theorem A.1 in [7] with suitable modifications to deal with causal operators. The proof is divided into three steps:

Step 1.(Extension) Take $u \in \mathcal{C}\big([0,t];\mathcal{S}\big)$, for any fixed $t \in [0,T)$. Using the fact that $\mathcal{S}$ is bounded, the causality of $Q(\cdot)$, and the uniform Hölder estimate

$$\sup_{s\in[0,t]} \big\|Q(u)(s)\big\| \leq C \sup_{s\in[0,t]} \big\|u(s)\big\|^{\beta} \leq C\, C_{\mathcal{S}}^{\beta}\,.$$

Thanks to the uniform sub-tangent condition, for any fixed $\varepsilon \in (0,1)$ there exists $h(u,\varepsilon) > 0$ such that

$$B\big(u(t) + h\,Q(u)(t), \varepsilon\big) \cap \mathcal{S}\backslash\big\{u(t) + hQ(u)(t)\big\} \neq \emptyset\,, \quad \forall\, h \in \big(0, h(u,\varepsilon)\big]\,.$$

We fix $h > 0$ as $h := \min\{1, (\varepsilon/2C)^{\frac{1}{\beta}}(C_S^{\beta}+2)^{-1}, h(u,\varepsilon)\}$. As a consequence, there exists w in such set satisfying

$$\big\|w - u(t) - h\,Q(u)(t)\big\| \leq \tfrac{\varepsilon h}{2}\,.$$

Consider now the linear map

$$s \mapsto \rho(s) = u(t) + \frac{(s-t)\big(w - u(t)\big)}{h}, \quad s \in [t, t+h]\,.$$

Give the fact that the set $\mathcal{S}$ is convex and closed, $\rho(s) \in \mathcal{S}$ for all $s \in [t, t+h]$. Moreover, since the right derivative is $\dot{\rho}(s) = \frac{w-u(t)}{h}$ in $[t, t+h)$, it follows that

$$\big\|\dot{\rho}(s) - Q(u)(t)\big\| \leq \tfrac{\varepsilon}{2}\,, \quad s \in [t, t+h)\,.$$

Also, we observe that

$$\begin{aligned}\big\|\rho(s)-u(t)\big\| = \Big\|\tfrac{(s-t)(w-u(t))}{h}\Big\| &\le \big\|w-u(t)\big\| \\ &\le h\big\|Q(u)(t)\big\| + \tfrac{\varepsilon h}{2} \le h\big(C_{\mathcal{S}}^{\beta}+1\big)\,. \end{aligned} \tag{8.1}$$

Define now the extension $u_e \in \mathcal{C}\big([0,t+h];\mathcal{S}\big)$ as

$$u_e(s) = \begin{cases} u(s) & \text{for } s\in[0,t)\,, \\ \rho(s) & \text{for } s\in[t,t+h]\,, \end{cases}$$

Then, for any $0\le t\le s\in[t,t+h)\subset[0,T]$, the uniform Hölder continuity property of Q and estimate (8.1) imply that

$$\begin{aligned} &\big\|Q(u_e)(s)-Q(u)(t)\big\| \\ &\le C\Big(\sup_{\sigma\in[0,t]}\big\|u_e(\sigma)-u(\sigma)\big\|^{\beta} + \big\|u_e(s)-u(t)\big\|^{\beta} + |s-t|^{\beta}\Big) \\ &\le C\Big(\big\|\rho(s)-u(t)\big\|^{\beta}+h^{\beta}\Big) \le C\big(C_{\mathcal{S}}^{\beta}+2\big)h^{\beta} \le \tfrac{\varepsilon}{2}\,. \end{aligned}$$

Therefore, for the extension u_e follows that in the interval $s\in[t,t+h)$

$$\begin{aligned}\big\|\dot u_e(s)-Q(u_e)(s)\big\| &= \big\|\dot\rho(s)-Q(u)(t)+Q(u)(t)-Q(u_e)(s)\big\| \\ &\le \big\|\dot\rho(s)-Q(u)(t)\big\| + \big\|Q(u)(t)-Q(u_e)(s)\big\| \le \varepsilon\,. \end{aligned} \tag{8.2}$$

And, as consequence of this fact

$$\begin{aligned}\sup_{s\in[t,t+h]}\big\|\dot u_e(s)\big\| &\le 1+\sup_{s\in[0,t+h]}\big\|Q(u_e)(s)\big\| \\ &\le 1 + C\sup_{s\in[0,t+h]}\big\|u_e(s)\big\|^{\beta} \le 1 + C\,C_{\mathcal{S}}^{\beta}\,. \end{aligned} \tag{8.3}$$

valid for any $\varepsilon\in(0,1)$.

Step 2.(Piecewise approximations) Fix $\varepsilon\in(0,1)$. Starting from $t=0$ we use the extension procedure of Step 1 to construct a piecewise linear function $\rho := \rho^{\varepsilon}\in\mathcal{C}\big([0,\tau);S\big)$ satisfying the estimates

$$\sup_{s\in[0,\tau)}\big\|\dot\rho(s)-Q(\rho)(s)\big\| \le \varepsilon\,, \qquad \sup_{s\in[0,\tau)}\big\|\dot\rho(s)\big\| \le C\,, \tag{8.4}$$

with initial condition $\rho^{\varepsilon}(0)=u_0$.

Suppose that ρ is constructed on a series of intervals $[0,\tau_1]$, $[\tau_1,\tau_2]$, $\cdots$, $[\tau_n,\tau_{n+1}]$, $\cdots$. Moreover, suppose the increasing sequence $\{\tau_n\}$ is bounded and, set

$$\tau = \lim_{n\to\infty} \tau_n \,.$$

Since $\dot{\rho}$ is uniformly bounded, the sequence $\{\rho(\tau_n)\}$ has a limit. Therefore, we can define $\rho(\tau)$ as

$$\rho(\tau) = \lim_{n\to\infty} \rho(\tau_n)\,.$$

This implies that ρ is, in fact, defined on $[0,\tau]$. It also implies, by the extension procedure of Step 1, that $\tau = T$.

Step 3.(Limit) Let us now consider two sequences of approximate solutions u^ε, w^ε, where ε tends to 0. From Step 1 and Step 2, one can see that the time interval $[0,T]$ can be decomposed into

$$\Big(\bigcup_\gamma I_\gamma\Big)\bigcup \mathfrak{N},$$

where I_γ are countably many open intervals where u^ε, w^ε are affine, and $\mathfrak{N}$ is of measure 0. Thus, we can take the derivative of the difference $\big\|u^\varepsilon(t) - w^\varepsilon(t)\big\|$ gives

$$\begin{aligned}\frac{d}{dt}\big\|u^\varepsilon(t){-}w^\varepsilon(t)\big\| &= \Big[u^\varepsilon(t) - w^\varepsilon(t), \dot{u}^\varepsilon(t) - \dot{w}^\varepsilon(t)\Big] \\ &\le \Big[u^\varepsilon - w^\varepsilon, Q(u^\varepsilon)(t) - Q(w^\varepsilon)(t)\Big] + 2\,C\,\varepsilon\end{aligned}$$

In the last inequality we used the first estimate in (8.4). Integrating and using the one-sided Lipschitz property

$$\big\|u^\varepsilon(t) - w^\varepsilon(t)\big\| \le L\int_0^t \mathrm{d}s\big\|u^\varepsilon(s) - w^\varepsilon(s)\big\| + 2\,C\,t\,\varepsilon,$$

which yields, by Gronwall's lemma, that

$$\big\|u^\varepsilon(t) - w^\varepsilon(t)\big\| \le \tfrac{2CT}{L}e^{LT}\varepsilon\,.$$

As a consequence, the sequence $\{u^\varepsilon\}$ is Cauchy and converges uniformly to a continuous limit $u \in \mathcal{C}\big([0,T];\mathcal{S}\big)$. Cleary, the function u is *the* solution of our equation.

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