SOLUTIONS OF THE LINEAR BOLTZMANN EQUATION AND SOME DIRICHLET SERIES

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Abstract. It is shown that a broad class of generalized Dirichlet series (including the polylogarithm, related to the Riemann zeta-function) can be presented as a class of solutions of the Fourier transformed spatially homogeneous linear Boltzmann equation with a special Maxwell-type collision kernel. The result is based on an explicit integral representation of solutions to the Cauchy problem for the Boltzmann equation. Possible applications to the theory of Dirichlet series are briefly discussed.

1. Introduction

Classical special functions are used in different fields of mathematics and its applications. Most of such functions (spherical and cylindrical functions, Hermite and Laguerre polynomials and many others) appear as solutions of some ODEs or PDEs.

There is, however, another big class of functions which have no connections with differential equations. The well known examples are the Gamma and Beta functions, the Riemann Zeta-function $\zeta(z)$, obtained as an analytic continuation of the Dirichlet series [6]

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}\ , \qquad \mathrm{Re}\, z > 1\ , \tag{1.1}$$

and some similar functions.

One more example is a generalization of $\zeta(z)$, the so called polylogarithm

$$Li_z(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^z}\,, \qquad |x| < 1\ , \tag{1.2}$$

also known as de Longniére's function [3].

Hilbert conjectured in the beginning of last century that $\zeta(z)$ and other functions of the same type do not satisfy algebraic differential relations [4]. His conjectures were proved in [7, 8]. Other references and new results on differential independence of zeta-type functions can also be found, for example, in the book [6].

Therefore, it is clear that a possible connection of the functions $Li_z(x)$ and $\zeta(z) = Li_z(1)$ with solutions of certain equation would be an interesting fact. The main goal of this paper is to show that such connection does exist and that the corresponding equation is the linear Boltzmann equation for Maxwell-type interaction with a special "scattering cross-section" The equation is presented below in explicit form. This is just an observation and at the present time we do not know of any application of this representation to the theory of the Riemann zeta-function. On the other hand, this is a new example of a a formal connection between the zeta-function and some problem of physics (kinetic theory in our case).

Several other examples are discussed in detail in the recent book [5], which contains an impressive collection of references.

The paper is organized as follows. We first consider the linear spatially homogeneous Boltzmann equation for Maxwell-type collisions in Section 2 and show in Section 3 how to solve it via Fourier-transform. It is sufficient for our goals to consider isotropic solutions (see [2] for anisotropic case).

Then, in Section 4, we choose a special "cross-section", (basically a kernel of the Boltzmann collision operator) and a special initial data. Then we prove that the Fourier-transformed solution of the corresponding Cauchy problem coincides with the function $-|k|^{-2} Li_t\ (-|k|^2)$, where $k \in \mathbb{R}^3$ is the Fourier-variable and t is the time-variable.

The main result of the paper is stated in Proposition 4.1 of Section 4. A connection with moments Bose-Einstein and Fermi-Dirac distributions is also briefly discussed.

2. Linear Boltzmann Equation and the Fourier Transform

We consider the initial value problem for the linear spatially homogeneous Boltzmann equation for a distribution function $f(v,t)$ of test particles given by

$$
\begin{aligned}
&f_t = Q(f,M) = \int_{\mathbb{R}^3\times S^2} \left(f('v)M_T('w) - f(v)M_T(w)\right) g\left(\frac{u}{|u|}\cdot\omega\right) dw\, d\omega \\
&f(v,0) = f_0(v)
\end{aligned}
\tag{2.1}
$$

for $v \in \mathbb{R}^3$ and $t \in \mathbb{R}^+$, denoting velocity and time variables respectively, where

$$
\begin{aligned}
&u = v - w\ , \qquad \omega \in S^2 \\
&'v = \frac{1}{2}(v + w + |u|\omega) \qquad 'w = \frac{1}{2}(v + w - |u|\omega) \\
&M_T(w) = (2\pi T)^{-3/2} \exp[-|v|^2/2T]\ .
\end{aligned}
\tag{2.2}
$$

This model has been studied in detailed in [1, 2]. The solution to the initial value problem (2.1) describes the time evolution of the distribution function of test particles interacting with a gas in equilibrium with temperature T, under the assumption of Maxwellian-type collisions with scattering indicatrix $g(\cos\theta)$, $\theta \in [0,\pi]$ (i.e., the collision frequency is independent of the relative speed $|u|$). Formally, the function $g(cos\theta)$ is equal to a differential cross-section of scattering at the angle θ multiplied by the relative speed.

Via Fourier Transform in the velocity space, we introduce the characteristic function

$$
\varphi(k,t) = \int_{\mathbb{R}^3} f(v,t) e^{-ik\cdot v}\, dv\ , \qquad k \in \mathbb{R}^3\ ,\ t \in \mathbb{R}^+ \tag{2.3}
$$

and obtain the following equation for $\varphi(k,t)$ (see [1])

$$\begin{aligned}\varphi_t &= \int_{\mathbb{R}^3\times\mathbb{R}^3} dv\,dw\,f(v)M_T(w)\int_{S^2}(e^{-ik\cdot v'}-e^{-ik\cdot v})g\left(\frac{u}{|u|}\cdot\omega\right)d\omega \\ &= \int_{S^2}\Big(\varphi(k_+)\widehat{M}_T(k_-)-\varphi(k)\widehat{M}_T(0)\Big)\,g\left(\frac{k}{|k|}\cdot\omega\right)d\omega\ ,\end{aligned} \tag{2.4}$$

where

$$k_\pm = \frac{k\pm|k|\omega}{2}\ ,\qquad \widehat{M}_T(k)=\exp\left(-\frac{T|k|^2}{2}\right)\ .$$

The initial condition reads

$$\varphi(k,0)=\hat{f}_0(k)=\int_{\mathbb{R}^3} f_0(z)e^{-ik\cdot v}\,dv\ . \tag{2.5}$$

Notice that the general case for arbitrary constant $T>0$ can be reduced to the case $T=0$ by the transformation

$$\varphi(k,t)=\psi(k,t)\widehat{M}_T(k)\ . \tag{2.6}$$

Then, the transformed characteristic function $\psi(k,t)$ satisfies the initial value problem

$$\psi_t = \int_{S^2}\left(\psi\Big(\frac{k+|k|\omega}{2}\Big)-\psi(k)\right)g\left(\frac{k}{|k|}\cdot\omega\right)d\omega \tag{2.7}$$

$$\psi(k,0)=\hat{f}_0(k)\widehat{M}_T^{-1}(k)\ .$$

We note that, if $T=0$ in equations (2.1) and (2.2) (slow down process) then $\varphi\equiv\psi$ and equation (2.7) is obtained as a straightforward Fourier Transform of (2.1).

If one looks for isotropic solutions $f(|v|,t)$ of equation (2.1), then, clearly, $\psi=\psi(|k|^2,t)$ in equation (2.7). We denote

$$x=|k|^2\ ,\qquad s=\frac{1}{2}\left(1+\frac{k}{|k|}\cdot\omega\right)\ ,\qquad G(s)=4\pi g(2s-1)\ , \tag{2.8}$$

then $\psi(x,t)$ satisfies the initial value problem

$$\begin{aligned}\psi_t(x,t) &= L\psi = \int_0^1 G(s)(\psi(sx)-\psi(x))\,ds \\ \psi(x,0) &= \psi_0(x)\ ,\qquad x\ge 0\ .\end{aligned} \tag{2.9}$$

It is assumed below that

$$G(s)\ge 0\quad\text{and}\quad \int_0^1 G(s)(1-s)<\infty\ . \tag{2.10}$$

Remark 1. Property (2.10) holds for true Maxwell molecules, i.e., for particles interacting via repulsive potentials $V(r)\sim r^{-4}$, where $r>0$ is a distance between two particles.

3. General solution to the problem

For the sake of simplicity we assume below that $T = 0$ in equation (2.1) and consider isotropic solutions. Then the transformation (2.3) reads

$$\psi(x,t) = \frac{4\pi}{\sqrt{x}} \int_0^\infty f(r,t) \sin r\sqrt{x}\, r\, dr = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!} m_n(t) x^n \tag{3.1}$$

where

$$m_n(t) = 4\pi \int_0^\infty f(r,t) r^{2(n+1)}\, dr\ , \qquad x = |k|^2\ . \tag{3.2}$$

Under the assumption that

i) all moments $m_n(t)$ are finite, and

ii) the above series in (3.1) has a non-zero finite radius of convergence

it is sufficient to study solutions to the problem (2.9) in the form of the Taylor series

$$\psi(x,t) = \sum_{n=0}^\infty a_n(t) x^n\ . \tag{3.3}$$

We note that the linear operator L, defined in (2.9), satisfies

$$L\,1 = 0\ , \quad Lx^p = -\lambda(p)x^p\ , \quad \lambda(p) = \int_0^1 G(s)(1-s^p)\, ds \qquad p \geq 1\ . \tag{3.4}$$

Therefore

$$a_n(t) = a_n(0)e^{-\lambda(n)t}\ , \qquad \lambda(0) = 0\ ,\ n = 0, 1, \ldots \tag{3.5}$$

in (3.3). In other words, the initial value problem (2.9) can be solved explicitly in Taylor expansion form, provided the initial state $\psi(x,0)$ is given in a similar form, i.e.

$$\psi(x,0) = \sum_{n=0}^\infty a_n(0)\, x^n\ . \tag{3.6}$$

If, however, the initial state does not have this form, it is still possible to use a more general integral representation for $\psi(x,t)$.

In order to construct such a representation we rewrite equation (2.9) in the modified form. Setting $s = e^{-\tau}$ in the integration term of (2.9), yields

$$\psi_t = L\psi = \int_0^\infty \left[\psi(xe^{-\tau}, t) - \psi(x,t)\right] K(\tau)\, d\tau\ , \tag{3.7}$$

with

$$K(\tau) = e^{-\tau} G(e^{-\tau}) \ \text{ for } \ \tau \geq 0\ .$$

Then, we assume that

$$\psi(x,t) = \int_0^\infty \mathcal{R}(t,\tau)\psi_0(xe^{-\tau})\, d\tau\ , \qquad \lim_{t\to 0} \mathcal{R}(t,\tau) = \delta(\tau)\ . \tag{3.8}$$

If such representation (3.8) of the solution exits then, formally, $\psi(x,t)$ must satisfy

$$\psi_t = \int_0^\infty \mathcal{R}_t(t,\tau)\psi_0(xe^{-\tau})\,d\tau\ . \tag{3.9}$$

On the other hand,

$$\begin{aligned} L\psi &= \int_0^\infty \left[\int_0^\infty \mathcal{R}(t,\gamma)\psi_0(xe^{-\tau-\gamma})\,d\gamma - \int_0^\infty \mathcal{R}(t,\tau')\psi_0(xe^{-\tau'})\,d\tau'\right] K(\tau)\,d\tau \\ &= \int_0^\infty \left[\int_0^{\tau'} \mathcal{R}(t,\tau'-\tau)K(\tau)\,d\tau - \mathcal{R}(t,\tau')\int_0^\infty K(\tau)\,d\tau\right]\psi_0(xe^{-\tau'})\,d\tau' \qquad (3.10) \\ &= \int_0^\infty \mathcal{A}(t,\tau)\psi_0(xe^{-\tau})\,d\tau\,, \end{aligned}$$

where

$$\mathcal{A}(t,\tau) = \int_0^\tau \mathcal{R}(t,\tau-\tau')K(\tau')\,d\tau' - \mathcal{R}(t,\tau)\int_0^\infty K(\tau')\,d\tau'\,, \tag{3.11}$$

provided

$$\int_0^\infty K(\tau)\,d\tau < \infty\ .$$

Then, equating (3.9) to (3.10)–(3.11) one obtains the following initial value problem for $\mathcal{R}(t,\tau)$:

$$\begin{cases} \mathcal{R}_t(t,\tau) = \mathcal{A}(t,\tau) \\ \mathcal{R}(0,\tau) = \delta(\tau)\ . \end{cases} \tag{3.12}$$

We say that a solution $\mathcal{R}(t,\tau)$ of the initial value problem (3.12), with the right hand side given by (3.11), is a fundamental solution (Green's function) associated to problem (3.7) [2].

The Laplace transform is a natural tool to solve problem (3.12). Indeed, we denote

$$r(t;p) = \mathcal{L}(\mathcal{R}) := \int_0^\infty \mathcal{R}(t,\tau)e^{-p\tau}\,d\tau\ , \qquad k(p) = \mathcal{L}(K) := \int_0^\infty K(\tau)e^{-p\tau}\,d\tau \tag{3.13}$$

then $\mathcal{L}(\mathcal{A}) = r(t,p)(k(p)-k(0))$ and (3.12) becomes an evolution equation for $r(t,p)$

$$\begin{cases} r_t(t,p) = r(t,p)(k(p)-k(0)) \\ r(0,p) = 1 \quad . \end{cases}$$

Therefore

$$r(t,p) = \exp(-t\lambda(p))$$

where

$$\lambda(p) = k(0) - k(p) = \int_0^\infty K(\tau)(1-e^{-p\tau})\,d\tau \tag{3.14}$$

coincides with the spectral function $\lambda(p)$ defined in (3.4).

Hence, the representation formula for the solution of problem (3.7) is given by the kernel (fundamental solution of (3.12))

$$\mathcal{R}(t,\tau) = \mathcal{L}^{-1}(e^{-t\lambda(p)}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\tau p - t\lambda(p)}\, dp \tag{3.15}$$

under the condition that

$$K(\tau) \geq 0 \quad \text{and} \quad \lambda(p) = \int_0^\infty K(\tau)(1-e^{-\tau p})\, d\tau < \infty\ . \tag{3.16}$$

If $\int_0^\infty K(\tau)\, d\tau < \infty$, then the operator L given in (3.7) is bounded in $C(\mathbb{R}_+)$, i.e., on such $\psi(x)$ that

$$\|\psi\| = \sup_{x\geq 0} |\psi(x)| < \infty\ . \tag{3.17}$$

The initial value problem associated to (3.7), or its equivalent associated to (2.9), can be expressed in terms of analytic functions of differential operators as follows.

We denote

$$\widehat{\mathcal{D}} = x\frac{d}{dx} \quad \text{defined by} \quad e^{-\tau\widehat{\mathcal{D}}}u(x) = u(xe^{-\tau})\,, \tag{3.18}$$

then equation (3.7) can be rewritten as

$$\psi_t \ = \ -\lambda(\widehat{\mathcal{D}})\psi\,, \tag{3.19}$$

with

$$\lambda(p) = \int_0^\infty K(\tau)(1-e^{-p\tau})\, d\tau\ , \qquad p > 0\ .$$

Its formal solution reads

$$\psi(x,t) = e^{-t\lambda(\widehat{\mathcal{D}})}\psi(x,0)\ , \tag{3.20}$$

where

$$e^{-t\lambda(\widehat{\mathcal{D}})} = \int_0^\infty \mathcal{R}(t,\tau)e^{-\tau\widehat{\mathcal{D}}}\, d\tau\ , \qquad R(t,\tau) = \mathcal{L}^{-1}(e^{-t\lambda(p)})\ . \tag{3.21}$$

Formulas (3.20), (3.21) are, of course, equivalent to the more explicit representation (3.8), (3.12) of solution to the problem (3.7).

4. Applications to polylogarithm and the Riemann Zeta-function

The following interesting example follows from a special choice of initial state and the indicatrix function $g(2s-1)$.

Consider the initial value problem (2.1), (2.2) associated with the linear Boltzmann equation with $T \equiv 0$ and choose an initial condition

$$f(v,0) = \frac{1}{4\pi|v|}e^{-|v|}\ . \tag{4.1}$$

Let the function $g(\frac{u}{|u|}\cdot\omega) = g(\cos\theta)$ be given by

$$g(\cos\theta) = \left[4\pi\log\frac{1}{\cos^2\frac{\theta}{2}}\right]^{-1}\ . \tag{4.2}$$

Then, it is easy to verify that the function

$$\psi(|k|^2,t) = \int_{\mathbb{R}^3} f(|r|^2,t)e^{-ik\cdot v}\,dv$$

satisfies the initial value problem (3.7), where

$$K(\tau) = 4\pi e^{-\tau} g(2e^{-\tau}-1) = \frac{e^{-\tau}}{\tau}. \tag{4.3}$$

Hence, we obtain

$$\begin{cases} \psi_t = L\psi = \int_0^\infty \frac{e^{-\tau}}{\tau}\left(\psi(xe^{-\tau},t) - \psi(x,t)\right)\,d\tau \\ \psi(x,0) = \frac{1}{(1+x)} = \sum_{n=0}^\infty (-1)^n x^n\ . \end{cases} \tag{4.4}$$

A surprising fact is a connection between $\psi(x,t)$ solution of the initial value problem (4.4) with both a polylogarithm and the Riemann zeta-function.

For completion of this presentation, we recall the definition of a polylogarithm $Li_z(x)$ [3] as the function

$$Li_z(x) = \sum_{n=1}^\infty \frac{x^n}{n^z}\,. \tag{4.5}$$

defined for z in the complex plane over the open unit disk. Its definition on the whole complex plane then follows uniquely via analytic continuation.

The special case $x=1$ reduces to the Riemann zeta-function

$$\zeta(z) \;=\; \sum_{n=1}^\infty \frac{1}{n^z} \;=\; Li_z(1)\,. \tag{4.6}$$

In addition, it is well known that polylogarithms are related to integral moments of Fermi-Dirac and Bose-Einstein probability distributions through the integral formulas of the type [3]

$$Li_z(-x) \;=\; -\frac{1}{\Gamma(z)}\int_0^\infty \frac{\tau^{z-1}}{\frac{e^{-\tau}}{x}+1}\,d\tau\ , \qquad z>0\,. \tag{4.7}$$

We prove the following statement

Proposition 4.1. *The solution $\psi(x,t)$ of problem* (4.4) *is related to polylogarithm by equality*

$$Li_t(-x) = -x\psi(x,t)\ , \qquad x\ge 0\ ,\ t\ge 0\ ,$$

which can be extended to complex values of x and t by analytic continuation. The connection of $\psi(x,t)$ with zeta-function $\zeta(t)$ is given by the identity

$$\psi(1,t) = (1-2^{1-t})\zeta(t)\ ,$$

which can also be extended to complex t by analytic continuation.

Proof. We can compute explicitly the spectral function $\lambda(p)$, as defined in (3.14), for the specific $K(\tau)$ from (4.3). It is given by

$$\lambda(p) = \int_0^\infty \frac{e^{-\tau}}{\tau}(1 - e^{-\tau p})\, d\tau = \log(p+1) \tag{4.8}$$

so it is positive and finite for all $p > 0$.

Then, we can easily see that

$$\psi(x,t) = \sum_{n=0}^{\infty} a_n(0) e^{-\lambda(n)t} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(1+n)^t} \tag{4.9}$$

where the series converges for $|x| < 1$.

Then

$$-x\,\psi(x,t) \;=\; \sum_{n=1}^{\infty} \frac{(-x)^n}{n^t} \;=\; Li_t(-x) \;\text{ for any }\; x \geq 0\ . \tag{4.10}$$

On the other hand, we can present $\psi(x,t)$ in the integral form (3.8), (3.15). Indeed, since

$$\mathcal{R}(t,\tau) = \mathcal{L}^{-1}(e^{-t\lambda(p)}) = \mathcal{L}^{-1}\left[\frac{1}{(p+1)^t}\right] = \frac{e^{-\tau}\tau^{t-1}}{\Gamma(t)}\ , \tag{4.11}$$

with

$$\Gamma(t) = \int_0^\infty \tau^{t-1}\, e^{-\tau}\, d\tau\ ,$$

the corresponding integral representation reads

$$\psi(x,t) = \frac{1}{\Gamma(t)} \int_0^\infty \frac{\tau^{t-1} e^{-\tau}}{1 + x e^{-\tau}}\, d\tau\ , \qquad t > 0\ . \tag{4.12}$$

The right hand side of (4.12) can be recast as

$$\psi(x,t) = \frac{1}{x\Gamma(t)} \int_0^\infty \frac{\tau^{t-1}}{\frac{e^\tau}{x} + 1}\, d\tau \;=\; -\frac{1}{x} Li_t(-x)\,, \qquad t > 0\,, \tag{4.13}$$

by the use of identity (4.7), which recovers identity (4.10) by integral representations.

In particular, this formula can be used for analytic continuations of $\psi(x,z)$ to complex values of x and z respectively. Finally we discuss a connection of $\psi(x,t)$ with zeta-function.

We assume, for simplicity, that $t > 2$. Then it follows from (4.9) that

$$\psi(1,t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^t}\ . \tag{4.14}$$

On the other hand, one can write (4.14) as a difference of two positive series, namely

$$\begin{aligned} \psi(1,t) &= \sum_{k=0}^{\infty} \frac{1}{(2k+1)^t} - \sum_{k=1}^{\infty} \frac{1}{(2k)^t} = \\ &= \zeta(t) - 2\sum_{k=1}^{\infty} \frac{1}{(2k)^t} = (1-2^{1-t})\zeta(t) \end{aligned} \tag{4.15}$$

where $\zeta(t)$ is the zeta-function.

By analytic continuation, we conclude that the identity

$$\psi(1,t) = (1-2^{1-t})\zeta(t)$$

holds for all complex t. Hence the proof of Proposition 4.1 is completed. □

In addition the integral representation (4.12) for $\psi(x,t)$ yields the well known [10] integral representation of the zeta-function

$$\zeta(t) = \frac{1-2^{1-t}}{\Gamma(t)} \int_0^\infty \frac{\tau^{t-1} e^{-\tau}}{1+e^{-\tau}}\, d\tau \ . \tag{4.16}$$

Hence, both polylogarithm and zeta-function can be expressed through a solution of the Boltzmann-type integro-differential equation (4.4). Of course, this is just an observation, we do not know any reasonable applications of this property. Our observation shows that some classical special functions, which are not solutions of ODEs or PDEs, can be solutions of a more general class of equations.

For completeness, we also mention the well-known [9] connection of polylogarithm with power moments of the Bose-Einstein distribution

$$Li_{s+1}(z) = \frac{1}{\Gamma(s+1)} \int_0^\infty \frac{t^s}{\frac{e^t}{z} - 1}\, dt\,,$$

and the Fermi-Dirac distribution

$$-Li_{s+1}(-z) = \frac{1}{\Gamma(s+1)} \int_0^\infty \frac{t^s}{\frac{e^t}{z} + 1}\, dt\ .$$

Hence, the power moments of these quantum distributions can be also expressed through solutions of the problem (4.4).

5. Conclusions

The results of this paper can be formulated in a more general way. We consider a generalized Dirichlet series

$$\phi(s,x) = \sum_{n=1}^{\infty} \frac{a_n\, x^n}{n^s}\,, \qquad x \ge 0\,, \tag{5.1}$$

assuming its absolute convergence for $\mathrm{Re}\, s > \sigma_0$ and $|x| < r$, with some $\sigma_0 \ge 0$ and $r \ge 1$. Then $\phi(s,1)$ is the usual Dirichlet series [6]. It was shown above, that for real positive s (s plays a role of time t in (4.4)) the function (4.7) satisfies the equation

$$\frac{\partial}{\partial s}\phi(s,x) = L\phi = \int_0^\infty \frac{e^{-\tau}}{\tau} \left(\phi(s, xe^{-\tau}) - \phi(s,x)\right)\, d\tau\,.$$

It is clear that the same equation holds for the analytic continuation of $\phi(s,x)$ to complex values of s and x.

A typical problem for Dirichlet series $\phi_1(s), s = \sigma + it$, is to study its behavior along a "vertical" line $\sigma = \operatorname{Re} z =$ constant. Then $\phi_1(\sigma + it) = \phi(\sigma + it, 1)$ where $\phi(\sigma + it, x) = \phi^{(\sigma)}(t, x)$ is a solution of the equation

$$i\frac{\partial \phi^{(\sigma)}}{\partial t} = L\phi^{(\sigma)}\,, \qquad \sigma = \text{const.} \tag{5.2}$$

Hence, we obtain a linear Boltzmann-type equation with imaginary time. Its relation to the classical Boltzmann type equation is quite similar to the relation of the Schrödinger equation to the corresponding diffusion equation.

Perhaps a comprehensive study of this equation can help to answer some open questions in the theory of Dirichlet series. In particular the famous Lindelöf conjecture [6, 10] that states

$$\zeta(\frac{1}{2} + it) = O(|t|^{\varepsilon})$$

for any $\varepsilon > 0$, can be easily reformulated in terms of asymptotic estimates of a solution to equation (5.2) for large $t > 0$.

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