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UPPER MAXWELLIAN BOUNDS FOR THE BOLTZMANN EQUATION WITH PSEUDO-MAXWELL MOLECULES

Abstract. We consider solutions to the initial value problem for the spatially homogeneous Boltzmann equation for pseudo-Maxwell molecules and show uniform in time propagation of upper Maxwellians bounds if the initial distribution function is bounded by a given Maxwellian. First we prove the corresponding integral estimate and then transform it to the desired local estimate. We remark that propagation of such upper Maxwellian bounds were obtained by Gamba, Panferov and Villani for the case of hard spheres and hard potentials with angular cut-off. That manuscript introduced the main ideas and tools needed to prove such local estimates on the basis of similar integral estimates. The case of pseudo-Maxwell molecules needs, however, a special consideration performed in the present paper.

1. **Introduction.** The paper is related to a conjecture that looks very natural: if a solution of the spatially homogeneous Boltzmann equation is bounded by some Maxwellian at initial time, then there exists another Maxwellian which bounds the solution for any positive value of time. In fact this is true, at least for hard spheres and hard potentials with cut-off. This was proved in two steps. The first step was to prove similar statement at the level of moments for hard spheres in three dimensions [6]. Further results were obtained in [8] for hard potentials in any dimensions, including pointwise bounds, based on combination of the moment approach of [6] with some new ideas consisting in comparison principles for Boltzmann equations, which we discuss below. However, the results of [8] are proved just for hard potentials, not including the pseudo-Maxwell molecules case (Maxwell-type of collision kernel with bounded total cross-section). The goal of this paper is to fill this gap and to present the proof for the pseudo-Maxwell case. The first part of the proof is based on the complex version of the Fourier transform, not on moment inequalities used in [6], [8]. The second part includes a generalization of some estimates from [8]. For brevity we do not discuss many interesting results, which have only indirect connection with our subject (for example, non-Maxwellian bounds for solutions or bounds for derivatives of solutions [1]). A comprehensive review of literature can be found in [8], [1].

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The paper is organized as follows. In Sect. 2 the statement of the problem is discussed and main results are formulated (Theorem 1 and Theorem 2). Here we also show why the straightforward way of proof based on the Wild sum formula for solutions does not work. The proof of Theorem 2 is given in Sect. 3 where we use the complex version of the Fourier transform in order to prove an estimate of the norm in weighted L^1 space. In Section 4 we combine this estimate with the comparison principle from [8]. The proof of needed estimate for the gain term of the collision integral is given in Appendix. These estimates, combined also with results of Wennberg [10, 11] for the Maxwell case, allow to present the final proof of Theorem 1 in Sect. 4.

2. **Statement of the problem and main results.** We consider below the spatially homogeneous Boltzmann equation for the one-particle distribution function $f(v,t)$, where $v \in L^1(\mathbb{R}^3)$ and $t \in \mathbb{R}_+$. The equation reads

$$\frac{\partial f}{\partial t} = Q(f,f) = \int\limits_{\mathbb{R}^3\times S^2} dw d\omega\; g(|u|, \hat{u}\cdot\omega)\; [f(v')f(w') - f(v)f(w)]\,, \tag{1}$$

with $u = v - w$, $\hat{u} = u/|u|$, $\omega \in S^2$, and the precollisional velocities are given by the classical elastic interaction law, written in center of mass and relative velocity coordinates,

$$v' = \frac{1}{2}(v + w + |u|\,\omega), \quad w' = \frac{1}{2}(v + w - |u|\,\omega)\,.$$

We note that the function $\sigma(|u|,\mu) = |u|^{-1}\,g(|u|,\mu) \geq 0$ is referred as the differential cross-section of scattering at the angle $\theta = \arccos\mu, \theta \in [0,\pi]$. Such function $\sigma(|u|,\cos\theta)$ is defined for given intermolecular potential [9]. For example, $\sigma(|u|,\mu) = d^2/4$ for particles - hard spheres of diameter d. If we assume that the particles interact via power-like potential $U(r) = \alpha^2/r^n,\ n > 1$, then

$$g_n(|u|,\mu) = |u|^{\gamma} g_{\gamma}(\mu), \qquad \gamma = \gamma(n) = 1 - \frac{4}{n}\,. \tag{2}$$

Implicit formulas for $g_\gamma(\mu)$ can be found in standard textbooks in classical mechanics, e.g. [9]. These functions have a non-integrable singularity at $\mu = 1$, i.e. for scattering angle $\theta = 0$. Therefore a natural representation of the Boltzmann collision integral as a difference of two positive terms

$$Q(f,f) = Q^+(f,f) - Q^-(f,f) \tag{3}$$

is impossible for power-like potentials. For this reason many authors of rigorous mathematical and numerical work prefer to use some model cross-sections, for example, that $g_\gamma(\mu) \in L^1_+([-1,1])$. Moreover, the so-called VHS-models (Variable Hard Spheres) with $g_\gamma(\mu) = \text{const}$ are often used in applications to rarefied gas dynamics.

We consider below mainly the case $\gamma = 0$ with integrable function $g_0(\mu)$ (pseudo-Maxwellian molecules). Then we denote in (1)

$$f(v,0) = f_0(v) \geq 0, \quad g_{n=4}(|u|,\mu) = b(\mu) \in L^1_+([-1,1])\,, \tag{4}$$

and consider the corresponding Cauchy problem for $t > 0$. It is convenient to assume without loss of generality that

$$\int\limits_{\mathbb{R}^3} dv\, f_0(v) = 1, \quad \int\limits_{\mathbb{R}^3} dv\, f_0(v) v = 0, \quad 2\pi \int\limits_{-1}^{1} d\mu\, b(\mu) = 1\,. \tag{5}$$

We also assume that

$$0 \le f_0(v) \le A_0 \exp(-a_0\, |v|^2)\,, \tag{6}$$

for some positive constants A_0 and a_0 and almost all $v \in \mathbb{R}^3$. For brevity, the words 'almost all' for similar inequalities and equalities are usually omitted below. Our main goal is to prove following statement.

Theorem 1 (L^∞-Maxwellian weighted estimate)**.** *Let $f(v,t)$ be a solution of the problem (1), (4) under assumptions (5), (6) and an additional assumption that $b(\mu)$ is bounded on $[-1,1]$. Then there exist such constants $A \ge A_0$ and $0 < a \le a_0$ that*

$$0 \le f(v,t) \le A \exp(-a\, |v|^2) \tag{7}$$

for all $t \ge 0$. In particular, this inequality holds for any $0 \le a < a_1$, where

$$a_1 \;=\; \frac{a_0^{5/2}}{A_0\, \pi^{3/2}}\,. \tag{8}$$

The corresponding constant A depends on a, $f_0(v)$ and $b(\mu)$.

We will use the same strategy as in [8] for the proof of Theorem 1. In particular, we begin with the following integral estimate.

Theorem 2 (L^1-Maxwellian weighted estimate)**.** *Let $f(v,t)$ be a solution of the problem (1), (4) under assumptions (5), (6). Then the following estimate holds for any $0 \le a < a_1$ uniformly in time $t \ge 0$:*

$$\|f(v,t) e^{a|v|^2}\|_{L^1} = \int\limits_{\mathbb{R}^3} dv\, f(v,t)\; e^{a|v|^2} \;\le \left(1 - \frac{a}{a_1}\right)^{-3/2}, \tag{9}$$

where a_1 is given in (8).

This theorem is proven in Sect. 3.

Then we use a simplified version of the comparison theorem from [8] and prove Theorem 1 in Sect. 4. Thus, the proof of inequality (7), which looks quite natural to physicists, is rather long. We shall see below that a local in time estimate (7) with $a = a_0$ can be obtained in a simpler way.

Indeed we can use a straightforward approach based on the Wild sum representation under assumption (5). Then the formula for $f(v,t)$ reads (see e.g. [5])

$$f(v,t) = \sum_{n=0}^{\infty}\; e^{-t}\left(1 - e^{-t}\right)^n f_n(v)\;, \tag{10}$$

where

$$f_{n+1}(v) = \frac{1}{n+1} \sum_{k=0}^{n} Q^+(f_k, f_{n-k}), \quad n = 0, 1, ... \tag{11}$$

$$Q^+(f_1, f_2) = \int\limits_{\mathbb{R}^3 \times \mathbb{S}^2} dw d\omega \, b(\hat{u} \cdot \omega) f_1(v') f_2(w'),$$

in the notations of (1). Note that

$$\begin{aligned} Q^+ \left(e^{-c|v|^2}, e^{-c\,|v|^2} \right) =& e^{-c|v|^2} \int\limits_{\mathbb{R}^3} dw \; e^{-c\,|w|^2} = \\ =& \left(\frac{\pi}{c} \right)^{3/2} \; e^{-c\,|v|^2}, \quad c > 0. \end{aligned} \tag{12}$$

Hence, a direct application of assumption (6) to (10), (11) yields the estimates

$$\begin{aligned} &f_n(v) \le A_0 B^n e^{-a_0|v|^2}, \quad B = A_0(\pi/a_0)^{3/2}, \quad n = 0, 1, ..., \\ &f(v,t) \le A_0 K(t) e^{-a_0|v|^2}, \quad K(t) = \left[1 - B(1 - e^{-t})\right]^{-1}. \end{aligned} \tag{13}$$

This proves inequality (7) with $a = a_0$ for all $0 \le t < T(A_0, a_0)$, where

$$T = \log \frac{B}{B-1}, \quad B = A_0 \left(\frac{\pi}{a_0} \right)^{3/2}.$$

It is easy to see that $B > 1$ because

$$1 = \int\limits_{\mathbb{R}^3} dv \; f_0(v) \le A_0 \int\limits_{\mathbb{R}^3} dv \, e^{-a_0|v|^2} = B$$

and the equality $B = 1$ is possible only in the trivial case $f_0(v) = A_0 \exp(-a_0|v|^2)$.

Thus we need to look for other venues to obtain the proof of Theorem 1.

From here to the end of the paper, all identities and inequalities for functions of $v \in \mathbb{R}^3$ and $\mu \in [-1, 1]$ are assumed to be valid for almost all values of their arguments. We also assume below that all basic conservation laws

$$\int\limits_{\mathbb{R}^3} dv \, f(v,t) = const., \int\limits_{\mathbb{R}^3} dv \, f(v,t) \, v = const., \int\limits_{\mathbb{R}^3} dv \, f(v,t) |v|^2 = const.$$

are fulfilled for every $t > 0$.

3. **Complex Fourier transform and the integral estimate.** Our goal in this section is to prove Theorem 2. It is well known that all considerations for Maxwell molecules (Eq. (1) with $g(|u|, \mu) = b(\mu)$) can be simplified by the Fourier transform [3], [5]. The equation for

$$\hat{f}(k,t) = \mathfrak{F}(f) = \int\limits_{\mathbb{R}^3} dv \, f(v) e^{-ik \cdot v} \tag{14}$$

reads

$$\begin{aligned} &\hat{f}_t = \int\limits_{S^2} d\omega \, b(\hat{k} \cdot \omega)[\hat{f}(k_+)\hat{f}(k_-) \; - \; \hat{f}(0)\hat{f}(k)], \\ &k_+ = \frac{1}{2}(k \pm |k|\omega), \quad \omega \in S^2, \quad \hat{k} = \frac{k}{|k|}. \end{aligned} \tag{15}$$

It was shown in Section 2 that the condition $f_0(v) = f(v,0) \le A_0 \exp(-a_0|v|^2)$ implies similar inequality (13) for $t \in [0, T(A_0, a_0)\,)$. Hence, for such t we can define the complex Fourier transform (14) with $k = x + iy,\ x \in \mathbb{R}^3,\ y \in \mathbb{R}^3$. In particular, we can consider the real-valued integral transform

$$\phi(y,t) = \hat{f}(iy,t) = \int_{\mathbb{R}^3} dv\, f(v,t) e^{y\cdot v}\ , \quad y \in \mathbb{R}^3. \tag{16}$$

Note that Eq. (15) is invariant under scaling transformations $k \to \lambda k$, with λ constant. Therefore it is not surprising that the equation (16) for $\phi(y,t)$ coincides with Eq. (15). One can verify it directly by repeating the same steps as for the usual Fourier transformation (see also a remark on p. 208 of [5]).

Thus, under conditions (5) of integrability of the cross section, as well as conservation of mass and kinetic energy, both renormalized to unity, we obtain the following equation for $\phi(y,t)$

$$\begin{aligned} &\phi_t + \phi = \hat{Q}^+(\phi,\phi) = \int_{S^2} d\omega\, b(\hat{y}\cdot\omega)\phi(y_+)\phi(y_-), \\ &y_\pm = \frac{1}{2}(y \pm |y|\omega), \quad \hat{y} = \frac{y}{|y|}, \quad y \in \mathbb{R}^3, \quad t > 0. \end{aligned} \tag{17}$$

The initial condition reads

$$\phi(y,0) = \phi_0(y) = \int_{\mathbb{R}^3} dv\, f_0(v) e^{y\cdot v}\ , \quad \phi_0(0) = 1. \tag{18}$$

The next estimate of the solution $\phi(y,t)$ of the initial value problem (17) (18) is important for the proof of Theorem 2.

Proposition 3.1. *If the assumptions (5), (6) of* **Theorem** *2 hold, then,*

$$\phi(y,t) = \int_{\mathbb{R}^3} dv\, f(v,t) e^{y\cdot v}\ \le\ \exp[\gamma\,|y|^2], \quad \text{with} \ \ \gamma = \frac{A_0}{4}\,\frac{\pi^{3/2}}{a_0^{5/2}}\ , \tag{19}$$

uniformly in $t \ge 0$.

Proof. We start by seeing that this result holds on the initial data $\phi_0(y) = \int_{\mathbb{R}^3} dv\, f_0(v) e^{y\cdot v}$, where f_0 satisfies conditions (5). Indeed, an exact calculation yields

$$\phi_0(y) \le A_0 \int_{\mathbb{R}^3} dv\, e^{-a_0|v|^2 + y\cdot v}\ = A_0 \left(\frac{\pi}{a_0}\right)^{3/2} \exp\left[\frac{|y|^2}{4a_0}\right].$$

On the other hand,

$$\phi_0(0) = 1, \quad \nabla\phi_0|_{y=0} = \int_{\mathbb{R}^3} dv\, f_0(v) v = 0.$$

These two conditions allow us to obtain a better estimate for $\phi_0(y)$. Indeed,

$$\phi_0(y) = 1 + \int_{\mathbb{R}^3} dv\, f(v)\psi(y\cdot v),$$

where $\psi(z) = e^z - 1 - z \ge 0,\ z \in \mathbb{R}$.

Hence, by assumption (6) in **Theorem 1** , the above identity yields the estimate

$$\phi_0(y) \le 1 + A_0 \int_{\mathbb{R}^3} dv\, \psi(y\cdot v)\, e^{-a_0|v|^2} = 1 + B\left(e^{c|y|^2} - 1\right),$$

$$\text{with} \quad B = A_0 \left(\frac{\pi}{a_0}\right)^{3/2}, \quad \text{and} \quad c = (4a_0)^{-1}.$$

In addition, the elementary inequality

$$1 + B\,(e^z - 1) \le e^{Bz}, \quad B \ge 1, \quad z \ge 0,$$

leads to

$$\phi_0(y) \le \exp(\gamma|y|^2), \quad \gamma = \gamma(A_0, a_0) = \frac{A_0}{4}\,\frac{\pi^{3/2}}{a_0^{5/2}}\,,$$

so estimate (19) holds for $t = 0$.

Now it is easy to show that the same inequality holds for $\phi(y,t)$, $t \ge 0$.

We denote in Eq. (17)

$$\tau = 1 - e^{-t}, \quad \phi(y,t) = e^{-t}\,\psi(y,\tau).$$

Then the initial value problem for $\psi(y,\tau)$ reads

$$\psi_\tau = \hat{Q}(\psi,\psi), \quad 0 \le \tau < 1, \quad \psi(y,0) = \phi_0(y).$$

Functions $\phi_0(y)$ and $b(\mu)$ in Eqs. (17), (18) are non-negative.

In addition, the solution $\psi(y,\tau)$ can be constructed in the form of power series in τ (similarly to the Wild sum (10), (11) for the Boltzmann equation).

Moreover, the function $\psi(y,\tau)$ is monotone with respect to initial data. Indeed,

$$\psi^{(1)}(y,0) \ge \psi^{(2)}(y,0) \ge 0 \implies$$
$$\psi^{(1)}(y,\tau) \ge \psi^{(2)}(y,\tau) \ge 0, \quad \text{for all} \quad 0 \le \tau < 1\,.$$

Therefore, setting

$$\psi^{(1)}(y,0) = e^{\gamma|y|^2}\,, \quad \psi^{(2)}(y,0) = \phi_0(y)\,,$$

it is easy to check that

$$\psi^{(1)}(y,\tau) = \frac{1}{1-\tau} e^{\gamma|y|^2}$$

and therefore

$$\phi(y,\tau) \le (1-\tau)\,\psi^{(1)}(y,\tau) = e^{\gamma|y|^2},$$

which completes the proof of **Proposition 3.1**. □

We are now in conditions to prove Theorem 2, that is to show by means of Proposition 3.1 that estimate (9) holds.

Proof of **Theorem 2**. We denote

$$F(p,t) = \int_{\mathbb{R}^3} dv\, f(v,t) e^{p|v|^2}\,,\ p \in \mathbb{R},\ I(\lambda,t) = \int_{\mathbb{R}^3} dy\, \phi(y,t) e^{-\lambda|y|^2}\,, \tag{20}$$

$\lambda > \gamma$ with $f(v,t)$, $\phi(y,t)$ and $\gamma = A_0\pi^{3/2}/4\,a_0^{5/2}$ from **Proposition 3.1**.

Taking $I(\lambda,t)$ combined with Eq. (19) it follows after a change in order of integration that

$$I(\lambda,t) = \int_{\mathbb{R}^3} dv\, f(v,t) \int_{\mathbb{R}^3} dy \exp\left(y\cdot v - \lambda |y|^2\right) \;=\; \left(\frac{\pi}{\lambda}\right)^{3/2} F[(4\lambda)^{-1}, t]\,.$$

Hence,

$$F(p,t) = \| f(v,t)\, e^{p\,|v|^2} \|_{L^1} = (4\pi p)^{-3/2}\; I\left(\frac{1}{4p}, t\right)\,. \tag{21}$$

On the other hand, estimate (19) from **Proposition** 3.1 yields

$$I(\lambda,t) \le \left(\frac{\pi}{\lambda-\gamma}\right)^{3/2}\,.$$

Therefore, calculating estimate (21) in the above expression for $\gamma = (4p)^{-1}$ we finally obtain the L^1 Maxwellian weighted estimate for the solution of the Boltzmann equation for pseudo Maxwell molecules for any p satisfying $0 < p < (4\gamma)^{-1}$

$$\| f(v,t)\, e^{p\,|v|^2} \|_{L^1} \le (4\pi p)^{-3/2} \left(\frac{\pi}{(4p)^{-1}-\gamma}\right)^{3/2} = (1-4p\gamma)^{-3/2}\,, \tag{22}$$

uniformly in time t, where $\gamma = \gamma(A_0, a_0) = \frac{A_0 \pi^{3/2}}{4\, a_0^{5/2}}$ as given in Eq. (19).

Finally, setting $a = p < \frac{a_0^{5/2}}{A_0 \pi^{3/2}}$ into inequality (22) yields , under conditions of Theorem 1, the desired estimate (9). Hence, the proof of Theorem 2 is complete. $\square$

Remark 3.2. Theorem 2 is proved for integrable functions $b(\mu)$. However, its result does not depend on $b(\mu)$. The same result can be proved for more general class of angular cross-section, in particular, for true Maxwell molecules (particles, interacting via potential $U(r) \simeq \alpha/r^4$). Only some modifications in the proof Proposition 3.1 are needed. However, for brevity, we do not consider them here.

The next section is devoted to the proof of Theorem 1. The proof is partly based on Theorem 2.

4. **Local estimate.** Our aim is now to prove Theorem 1. This is in fact a local pointwise estimate in v-space, uniformly in time, for the solution $f(v,t)$ of the the initial value problem for the Boltzmann equation with initial data satisfying the conditions stated in Theorem 1.

In order to get these estimates, we briefly describe some ideas developed in [8]. Consider a solution $f(v,t)$ of the spatially homogeneous Boltzmann equation (1)

$$f_t = Q(f,f), \quad f_{t=0} = f_0(v) \ge 0\,, \tag{23}$$

under very general assumptions on $g(|u|,\mu)$. We assume that $f(v,t) \in L^\infty(\mathbb{R}^3)$ for all $t \ge 0$, such solutions were studied by Arkeryd [2] (for hard potentials with cut-off) and Wennberg [10],[11] (in particular, for pseudo-Maxwell molecules). Suppose that we want to prove that

$$\psi(v,t) = f(v,t) - h(v,t) \le 0, \quad t \ge 0\,, \tag{24}$$

where $h(v,t) \geq 0$ is given function. To this goal we introduce a bilinear operator

$$\tilde{Q}(f_1,f_2) = \tilde{Q}^{+}(f_1,f_2) - \tilde{Q}^{-}(f_1,f_2) = \tag{25}$$

$$= \int dw d\omega \; \tilde{g}(|u|, \hat{u}\cdot\omega) \; [f_2(v')f_1(w') - f_2(v)f_1(w)] \;, \tag{26}$$

such that $\tilde{Q}(f,f) = Q(f,f)$ (generally speaking, it does not mean that $g(|u|,\mu) = \tilde{g}(|u|,\mu)$). Then we consider the equation for $\psi(v,t)$:

$$\psi_t = \tilde{Q}(f,f) - h_t = \tilde{Q}(f,\psi) + \triangle(v,t), \tag{27}$$

$$\triangle(v,t) = \tilde{Q}(f,h) - h_t, \quad \psi|_{t=0} = \psi_0(v) = f_0(v) - h(v,0)\,. \tag{28}$$

Our aim is to prove that $\psi(v,t) \leq 0$ for all $t \geq 0$. Of course, we need to assume that $\psi_0(v) \leq 0$. What extra assumptions are needed? The answer in simplified form is given below (tildes are omitted). The following property of solutions to the linear equation for $\psi(v,t)$ is proved below in exactly the same way as Theorem 5 in [8].

Proposition 4.1. *Let $\psi(v,t)$ be a solution of the problem*

$$\psi_t = Q(f,\psi) + \triangle(v,t), \quad \psi|_{t=0} = \psi_0(v)\,, \tag{29}$$

in the notation of Eq. (25), *where*

$$f(v,t) \geq 0, \quad Q^{\pm}(f,\psi) \in L^1(\mathbb{R}^3), \quad \Delta(v,t) \in L^1(\mathbb{R}^3) \quad \textit{for all}\; t \geq 0.$$

We assume that

$$(a) \quad \psi_0(v) \leq 0, \quad v \in \mathbb{R}^3\,;$$

$$(b) \quad \psi(v,t) \leq 0 \quad \textit{for all} \quad t \geq 0 \quad \textit{and} \quad v \in H \subset \mathbb{R}^3,$$
$$\textit{where}\; H \;\textit{is a measurable subset of}\; \mathbb{R}^3\,;$$

$$(c) \quad \triangle(v,t) \leq 0 \quad \textit{for all} \;\; t \geq 0 \;\textit{and}\; v \in H^C = \mathbb{R}^3 \backslash H.$$

Then $\psi(v,t) \leq 0$ for all $t \geq 0$ and almost all $v \in \mathbb{R}^3$.

Proof. We assume that $\psi(v,t)$ has for $t > 0$ both positive and negative parts: $\psi(v,t) = \psi^+(v,t) + \psi^-(v,t)$. Then

$$\psi^+(v,t) = \frac{1}{2}\,(\,|\psi(v,t)| + \psi(v,t)\,)$$

and therefore

$$\partial_t \psi^+(v,t) = \frac{1}{2}\,(\,\mathrm{sign}\,(\psi(v,t)) + 1)\;\psi_t(v,t) =$$
$$= \eta\,(\psi(v,t))\;\psi_t(v,t) = \eta(\psi)\,(Q(f,\psi) + \triangle)\;,$$

where $\eta(x)$ is the unit step function defined by $\eta(x) = 1$ for $x > 0$ and $\eta(x) = 0$ for $x < 0$; or equivalently $\eta(x) = \frac{1}{2}\,(\mathrm{sign}(x) + 1)$. It is convenient to set $\eta(0) = 0$.

Then, we obtain (note that $\psi^+(v,0) = 0$)

$$\psi^+(v,t) = \int_0^t ds\, \eta\,(\psi(s))\,(\,Q(f(s),\psi(s)) + \triangle(s)\,)\,,$$

where the argument v of functions $f(v,s)$, $\psi(v,s)$ and $\triangle(v,s)$ is omitted in the integrand.

Integrating this identity over $\mathbb{R}^3$, we obtain

$$\int_{\mathbb{R}^3} dv\psi^+(v,t) = \int_0^t ds[I_1(s) + I_2(s)],$$

with

$$I_1(s) = \int_{\mathbb{R}^3} dv\, \eta\,(\psi(s))\; Q[f(s), \psi(s)], \quad I_2(s) = \int_{\mathbb{R}^3} dv\, \eta\,(\psi(s))\; \triangle(s)\,.$$

Fixing $s > 0$, we first consider $I_2(s)$ and use the conditions (b) and (c) that clearly yield

$$I_2(s) = \int_{H^C} dv \triangle(v,s) \le 0.$$

In order to estimate $I_1(s)$, we use the classical weak formulation for the collisional integral

$$I_1(s) = \int_{\mathbb{R}^3} dv\, \phi(v)\, Q(f,\psi) =$$

$$= \int_{\mathbb{R}^3} dw\, f(w) \int_{\mathbb{R}^3\times S^2} dv d\omega\, g(|u|, \hat{u}\cdot\omega)\, \psi(v)\; (\, \phi(v') - \phi(v)\,)\,,$$

with $\phi(v) = \eta\,(\psi(v))$ and omitting the argument s of $\psi(v,s)$ and $f(v,s)$.

Therefore, for any pair of real numbers x and x' we obtain

$$x\,(\eta(x') - \eta(x)) = \frac{1}{2}\,x\;(\mathrm{sign}(x') - \mathrm{sign}(x)) = \frac{1}{2}\,(x\,\mathrm{sign}(x') - |x|) \le 0.$$

Hence, $I_1(s) \le 0$ for any $s \ge 0$. Therefore for any $t \ge 0$

$$\int_{\mathbb{R}^3} dv\, \psi^+(v,t) = 0 \implies \psi^+(v,t) = 0$$

for almost all $v \in \mathbb{R}^3$. This completes the proof of Proposition 4.1. □

Now we can apply Proposition 4.1 to the proof of Theorem 1.

Proof of Theorem 1. Assuming all conditions of Theorem 1 are satisfied, consider Eqs. (24) - (27), where

$$\begin{aligned} h(v,t) &= M(v) = Ae^{-a|v|^2}, \\ \tilde{g}(|u|,\mu) &= \tilde{b}(\mu) = \,[b(\mu) + b(-\mu)]\eta(\mu), \end{aligned} \tag{30}$$

with $\eta(\mu)$ denoting a unit step function. A similar transformation of $g(|u|,\mu)$ was used in [8].

Our aim is to show that there exist such values $A > 0$ and $a > 0$ that all conditions of Proposition 4.1 with $\triangle(v,t) = \tilde{Q}(f,M)$ are satisfied. Then we can conclude that

$$\psi(v,t) = f(v,t) - M \le 0, \quad t \ge 0. \tag{31}$$

This inequality is exactly what is needed for Theorem 1.

Hence, it remains to check conditions (a), (b), (c) of Proposition 4.1. The first condition (a) is obviously satisfied for any pair (A,a) provided $A \ge A_0$, $0 \le a \le a_0$ in the notation of Eq. (6). Two other conditions (b) and (c) depend on the set

$H \in \mathbb{R}^3$. Following [8] we fix a number $R > 0$ and choose $H = B_R(0) = \{v \in \mathbb{R}^3,\ |v| \le R\}$. Then we can show that the condition (b) of Proposition 4.1 can be fulfilled for any $0 < R < \infty$. This follow from results of Wennberg [10, 11], who generalized earlier results of Arkeryd [2] to the case of pseudo-Maxwell molecules. One can conclude from [10, 11] that, under conditions of Theorem 1, the L^∞-norm of $f(v,t)$ is bounded uniformly in time:

$$\|f\|_\infty = \sup_{t\ge 0} \|f(\cdot,t)\|_{L^\infty} < \infty, \quad \|f(\cdot,t)\|_{L^\infty} = ess \sup_{v\in\mathbb{R}^3} |f(v,t)|. \tag{32}$$

Therefore the following estimate is valid for all $a \ge 0,\ R > 0$, and $t \ge 0$

$$0 \le f(v,t) \le \|f\|_\infty \ \exp[a(R^2 - |v|^2)], \quad |v| \le R\,.$$

Hence, the condition (b) for $H = B_R(0)$ holds with any $a \ge 0,\ R > 0$, and $A = \|f\|_\infty \exp(aR^2)$, because

$$\psi(v,t) = f(v,t) - Ae^{-a|v|^2} \le f(v,t) - \|f\|_\infty \le 0 \tag{33}$$

for all $v \in B_R(0)$, $t \ge 0$. It is easy to see that both conditions (a) and (b) of Proposition 4.1 hold for any $0 \le a \le a_0$, $R > 0$ and

$$A = \max\left(A_0,\ \|f\|_\infty\ e^{aR^2}\right)\,. \tag{34}$$

It remains to show that the third condition (c) is also satisfied for sufficiently large R. We note that, under assumption (5) of Theorem 1, the source term in Eq. (29) of Proposition 4.1 reads

$$\triangle(v,t) = \tilde{Q}(f,M) = A\left[\tilde{Q}^+\left(f, e^{-a|v|^2}\right) - e^{-a|v|^2}\right] \tag{35}$$

in the notation of Eqs. (25), (30). Hence, the condition (c) reduces to inequality

$$e^{a|v|^2}\tilde{Q}^+\left(f, e^{-a|v|^2}\right) \le 1, \quad |v| > R, \tag{36}$$

with free parameters $0 \le a \le a_0$ and $R > 0$. Its proof for sufficiently large R and small a is based on the following estimate.

Proposition 4.2. *We consider the bilinear form*

$$Q^+(f,F) = \int\limits_{\mathbb{R}^3\times S^2} dw d\omega \tilde{b}(\hat{u}\cdot\omega)\, F(v')f(w') \tag{37}$$

in notation of Eqs. (1), (30) and assume that (1) $b(\mu) \in L^\infty([-1,1])$ and (2) $\|f(v)\ \exp(c|v|^2)\|_{L^1} < \infty$ for some $c > 0$. Then, for any $0 < a < c$,

$$e^{a|v|^2}\left|Q^+(f, e^{-a|v|^2})\right| \le L\,\frac{\|f(v)\,e^{c|v|^2}\|_{L^1}}{1 + k(a,\lambda)|v|^2}\,, \tag{38}$$

where $\lambda = c - a$, $k(a,\lambda) = a\lambda(\sqrt{a} + \sqrt{\lambda})^{-2}$, $L = 16\pi\|b\|_{L^\infty}$.

The proof of Proposition 4.2 is given in the Appendix.

To complete the proof of Theorem 1 we fix $t > 0$ and substitute $f(v)$ by $f(v,t)$ from Theorem 1 into inequality (38). Then we use Theorem 2 and obtain from (38)

$$e^{a|v|^2}Q^+(f, e^{-a|v|^2}) \le \frac{L(1 - c/a_1)^{-3/2}}{1 + k(a,\lambda)|v|^2},$$

where a_1 is given in Theorem 2. This estimate holds for any $0 < a < c < a_1$, $v \in \mathbb{R}^3$. Its right hand side is obviously less than one if

$$|v|^2 \geq \frac{L}{k(a,\lambda)(1-c/a_1)^{3/2}}, \quad \lambda = c - a.$$

To satisfy inequality (36) (equivalent to condition (c) of Proposition 4.1) for given $0 < a < a_1$ it is sufficient to choose any $0 < \lambda < a_1 - a$ and to set in (36)

$$R^2 = R^2(a,\lambda) = \frac{L}{k(a,\lambda)} \left(1 - \frac{a+\lambda}{a_1}\right)^{-3/2} .$$

In particular, we can choose

$$R = R_{min}(a) = \min R(a,\lambda), \quad 0 < \lambda < a_1 - a. \tag{39}$$

The existence of $R_{min}(a)$ follows from elementary considerations since $R(a,\lambda)$ is a positive continuous function of λ on the open interval $0 < \lambda < a_1 - a$, and $R(a,\lambda)$ tends to infinity at both ends of the interval.

Finally, setting the constant A given in Eqs. (34), with R from (39) completes the proof **Theorem 1**. □

Appendix A. **Inequality for the gain term** $Q^+(f,h)$**.** There are many equivalent forms of the integral

$$Q^+(f,h) \;=\; \int\limits_{\mathbb{R}^3\times S^2} dw\, d\omega g(|u|, \hat{u}\cdot\omega)\, f(w')h(v') \tag{A.1}$$

in the notation of Eqs.(1)-(3). It is convenient for our goals to choose the form of Q^+, which was first introduced in [4] for derivation of the Landau equation. Then we obtain

$$\begin{aligned} Q^+(f,h) = \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} dwdk\; \delta\left(k\cdot u + \frac{|k|^2}{2}\right)\, \sigma\left(|u|, 1 - \frac{|k|^2}{2|u|^2}\right) \times \\ \times\, f\big(w - \frac{k}{2}\big)h\big(v + \frac{k}{2}\big), \\ u = v - w, \quad g(|u|,\mu) = |u|\; \sigma(|u|,\mu)\,. \end{aligned} \tag{A.2}$$

By substitution $\{w = \tilde{w} + \tilde{k},\ k = 2\tilde{k}\}$ we obtain (tildes are omitted)

$$\begin{aligned} Q^+(f,h) \;=\; 4\int\limits_{\mathbb{R}^3} dw\, dk\; \delta(k\cdot u)\; \sigma\left[\sqrt{|u|^2+|k|^2},\, \frac{|u|^2-|k|^2}{|u|^2+|k|^2}\right] \times \\ \times\, f(w)\; h(v+k)\,. \end{aligned} \tag{A.3}$$

It was shown in Section 4 that for our goals it is enough to use the symmetrized collision kernel $\tilde{g}(|u|,\mu)$ (30) or, equivalently, the function

$$\tilde{\sigma}(|u|,\mu) = [\sigma(|u|,\mu) + \sigma(|u|,-\mu)]\eta(\mu), \tag{A.4}$$

where $\eta(\mu)$ denotes the unit step function. The corresponding bilinear form $Q^+(f,h)$ reads

$$Q^+(f,h) \;=\; 8\int\limits_{\mathbb{R}^3} dw\, f(w)\, \tilde{R}_h(v,w)\,, \tag{A.5}$$

$$\tilde{R}_h(v,w) = \frac{1}{2}\int\limits_{|k|^2\le|u|^2} dk\, \delta(k\cdot u)\; h(v+k)\tilde{\sigma}\left[\sqrt{|u|^2+|k|^2},\, \frac{|u|^2-|k|^2}{|u|^2+|k|^2}\right]\,,$$

$u = v-w$. A similar formula was used in [7] for some local estimates of $Q^+(f,f)$ for power-like potentials with cut-off. It is equivalent to the Carleman representation used in [8].

Proof of Proposition 4.2 . Assuming in (A5) that

$$f(v)\ge 0,\quad \sigma(|u|,\mu) = b(\mu)/|u|,\quad 0\le b(\mu)\le \|b\|_{L^\infty}\,, \tag{A.6}$$

we need to estimate the integral

$$e^{a|v|^2}\, Q^+(f, e^{-a|v|^2}) = 8\int\limits_{\mathbb{R}^3} dw f(w) P_a(v,w), \tag{A.7}$$

$$P_a(v,w) \;= \frac{1}{4}\int\limits_{|k|\le|u|} dk\;\delta(k\cdot u)\;\big(|u|^2+|k|^2\big)^{-1/2}\;\left[b\left(\frac{|u|^2-|k|^2}{|u|^2+|k|^2}\right)+\right.$$
$$\left. +b\left(\frac{|k|^2-|u|^2}{|u|^2+|k|^2}\right)\right]\;\exp\big[-a(k+v)^2 + a|v|^2\big]\,.$$

Note that

$$(k+v)^2 - v^2 = (k+w)^2 - w^2$$

under the integral sign, since $k\cdot v = k\cdot w$.

Thus we obtain

$$P_a(v,w)\;\le\;\frac{\|b\|}{|u|}\;e^{a|w|^2}\;I_a(u,w)\;,\quad \|b\| = \|b\|_{L^\infty}, \tag{A.8}$$

where

$$I_a(u,w) = \int\limits_{|k|\le|u|} dk\;\delta(k\cdot u)\;\exp\big[-a\,(k+w)^2\big]\;=$$
$$=\frac{1}{|u|}\int\limits_{|k_\perp|\le|u|} dk_\perp\;\exp\big[-a(k_\perp+w)^2\big]\,.$$

where the integration is done over a disc in the plane $k_\perp\cdot u = 0$. First, one can easily check that

$$I_a(u,w)\;\le\;\frac{1}{|u|}\int\limits_{|k_\perp|\le|u|} dk_\perp\;=\;\pi\,|u|\,.$$

Next, integrating over the whole plane for large $|u|$, it follows the estimate

$$I_a(u,w)\le\frac{1}{|u|}\int\limits_{k_\perp\cdot u=0} dk_\perp\, e^{-a(k_\perp+|w|)^2}\le\frac{1}{|u|}\int\limits_{k_\perp\cdot u=0} dk_\perp\; e^{-a|k_\perp|^2}\;=\frac{\pi}{a\,|u|}\,.$$

Hence, we obtain

$$P_a(v,w) \le \pi\, \|b\|\, e^{a|w|^2}\, \mathrm{Min}\Big(1,\, \frac{1}{a\,|u|^2}\Big) \;\le\; \frac{2\pi \|b\| e^{a|w|^2}}{1+a|u|^2}$$

and therefore, combining Eqs. (A.7) and (A.8) yields the following estimate

$$e^{a|v|^2}\, Q^+\big(f, e^{-a|v|^2}\big) \;\le\; 16\pi\, \|b\| \int\limits_{\mathbb{R}^3} \frac{dw\; f(w) e^{a|w|^2}}{1+a|v-w|^2}\,. \tag{A.9}$$

The final step is to estimate this integral under assumption that

$$\|f(v)\; e^{c|v|^2}\|_{L^1} \;<\; \infty$$

for some $c > a$. Then we denote $\lambda = c - a \;>\; 0$,

$$\phi_\lambda(v,w) = e^{-\lambda |w|^2}\, \frac{1}{1+a|v-w|^2} \;=\; e^{-\psi_\lambda(v,w)}\,, \tag{A.10}$$
$$\psi_\lambda(v,w) = \lambda |w|^2 + \log[1+a|v-w|^2]\,.$$

Note that $|v-w|^2 \ge (|v|-|w|)^2$. If $x = |v|, y = |w|$, then for any $0 < \theta \le 1$

$$\begin{aligned}\psi_\lambda(v,w) &\ge \lambda y^2 \eta(y-\theta x) + \eta(\theta x - y)\log[1+a(y-x)^2] \ge\\ &\ge \lambda \theta^2 x^2 \eta(y-\theta x) + \eta(\theta x - y)\log[1+a(1-\theta)^2 x^2] \ge\\ &\ge \eta(y-\theta x)\log(1+\lambda\theta^2 x^2) + \eta(\theta x - y)\log[1+a(1-\theta)^2x^2]\,,\end{aligned}$$

where $\eta(y)$ is the unit step function and the last line follows from elementary estimate $x \ge \log(1+x)$, $x > 0$.

Thus, choosing $\theta = \theta_0$ such that

$$a(1-\theta_0)^2 = \lambda\theta_0^2 \implies \theta_0 = (1+\sqrt{\lambda/a})^{-1}\,.$$

We obtain

$$\psi_\lambda(v,w) \ge \log[1+k(\lambda,a)|v|^2], \quad k(\lambda,a) = \frac{\lambda\, a}{(\sqrt{\lambda}+\sqrt{a})^2}\,. \tag{A.11}$$

It remains to substitute this estimate into Eqs.(A.9),(A.10). Then, the proof of Proposition 4.2 is completed.

□

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