ON THE SELF-SIMILAR ASYMPTOTICS FOR GENERALIZED NON-LINEAR KINETIC MAXWELL MODELS

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Abstract. Maxwell models for nonlinear kinetic equations have many applications in physics, dynamics of granular gases, economics, etc. In the present manuscript we consider such models from a very general point of view, including those with arbitrary polynomial non-linearities and in any dimension space. It is shown that the whole class of generalized Maxwell models satisfies properties one of which can be interpreted as an operator generalization of usual Lipschitz conditions. This property allows to describe in detail a behavior of solutions to the corresponding initial value problem. In particular, we prove in the most general case an existence of self similar solutions and study the convergence, in the sense of probability measures, of dynamically scaled solutions to the Cauchy problem to those self-similar solutions, as time goes to infinity. A new application of multi-linear models to economics and social dynamics is discussed.

Keywords: Kac N-particle models for multi-linear interactions. Boltzmann Transport Equation for Maxwell models. Non-conservative systems. Self similar solutions. Spectral characteristics. Power tail formation. Economic games.

Contents

1. Introduction

The classical elastic Boltzmann equation with the Maxwell-type interactions is well-studied in literature (see [4, 15] and references therein). This is a mathematical model of a rarefied gas with binary collisions where the collision frequency is independent of the velocities of colliding particles.

Maxwell models of granular gases were introduced relatively recently in [6] (see also [2] for the one dimensional case). Soon after, these models became very popular among people studying granular gases (see, for example, the book [14] and references therein). There are two obvious reasons for this to happen. One is due to the fact that the inelastic Maxwell-Boltzmann equation can be essentially simplified by the Fourier transform similarly to the elastic one [5, 6], and the other one is that solutions to the spatially homogeneous inelastic Maxwell-Boltzmann equation have a non-trivial self-similar asymptotics approaching a corresponding self-similar solution that has a power-like tail for large velocities. This latter property was conjectured in [17] and later proved in [8,10,11]. It is actually remarkable that such an asymptotics is absent in the elastic case. This is due to the fact that, loosely speaking, the elastic Boltzmann equation has too many conservation laws. On the other hand, this self-similar asymptotics was also proved in the elastic case for initial data with infinite energy [7] using other mathematical tools from those of [8]. In another development, exact self-similar solutions [11] for elastic Maxwell mixtures, in the asymptotic limit of heavy particles in equilibrium with a cold background interacting with light particles dissipating the total kinetic energy, also exhibit power-like tails definitely suggesting self-similar asymptotics for such dissipative systems. Finally we mention recent publications [1,16,24] , where one dimensional Maxwell-type models were introduced for applications to economics and again the self-similar asymptotics and power-like tail were found.

All discussed models describe qualitatively different processes in physics or economics, however their solutions have a lot in common from the mathematical point of view. It is also clear that some further generalizations are possible: models for multiple, not just binary, interactions still assuming the constant Maxwell-type rate of interactions. A natural question that arises is whether multi-linear models have similar properties. The answer to this question is affirmative, as we shall see below. It becomes clear that there must be some general mathematical properties of Maxwell models, which, in turn, can explain properties of any particular model. Essentially, there must be just one *main theorem*, from which one can deduce all the above discussed facts and their possible generalizations. The goal of this paper is to consider Maxwell models from a very general point of view and to establish their key properties that lead to the self-similar asymptotics.

The paper is organized as follows. In Section 2 we focus on the classical homogeneous Boltzmann equation for Maxwell-type pair interactions and the stochastic N-particle model, introduced by M. Kac [20], related to this equation. Then we consider a generalization of the N-particle model which includes multi-particle interactions. It is shown that certain natural assumptions formally lead to class of equations which can be considered as the most general Maxwell-type model. We confine ourselves to the case of isotropic solutions when the phase space variable (i.e. the velocity for the classical Boltzmann equation) is a d-dimensional vector with $d \geq 2$. Then the Fourier transform leads to equations, which are the main object of our study in this paper (see Section 3). The same equations can be obtain by Laplace transform if the phase variable is a non-negative real number (such case is important for applications to economics as considered in Section 10). The concept of generalized multi-linear Maxwell model in the Fourier space is introduced in Section 3. Such models and their generalizations are studied in detail in Sections 4-9. The concept of an *L-Lipschitz* non-linear operator, one of the most important for our approach, is explained in Section 3 (Definition 3.1). It is proved (Theorem 3.2) that all multi-linear Maxwell models satisfy the L-Lipschitz condition. This property of the models constitutes a basis for the general theory.

The existence and uniqueness of solutions to the initial value problem is proved in Section 4 (Theorem 4.2). Then we study in Section 5 the large time asymptotics under very general conditions that are fulfilled, in particular, for all our models. It is shown that the L-Lipschitz condition leads to self-similar asymptotics provided the corresponding self-similar solution does exist. The existence and uniqueness of self-similar solutions is proved in Section 6 (Theorem 6.1). This result can be considered, to some extent, as the *main theorem* for general Maxwell-type models. Then, in Section 7, we go back to the multi-linear models of Section 3 and study more specific properties of their self-similar solution. We explain in Section 8 how to use our theory for applications to any specific model: it is shown that the results can be expressed in terms of just one function $\mu(p), p > 0$, that depends on the spectral properties of the specific model. General properties of the self-similar solutions, such as positivity, power-like tails, and more, are studied in Section 9. This study also includes the case of one dimensional models, where the Laplace (instead of Fourier) transform is used.

A new application of multi-linear Maxwell models to economics or social dynamics is presented in Section 10. It is shown how the general theory can predict a time dependent distribution of wealth among participants of economic games with simple rules and with arbitrary number of players.

For brevity, we did not include in this paper more traditional applications to Boltzmann type equations. Such applications were included in the original version of this manuscript (posted in arXiv:math-ph/0608035v1) but are now discussed in [9].

2. Maxwell models of the Boltzmann equation and their generalizations

We consider a spatially homogeneous d-dimensional ($d = 2, 3, \dots$) rarefied gas of particles having a unit mass. Let $f(v, t)$, where $v \in \mathbb{R}^d$ and $t \in \mathbb{R}_+$ denote respectively the velocity and time variables, be a one-particle distribution function with the usual normalization

$$\int_{\mathbb{R}^d} f(v,t)\, dv \;= 1 \; . \tag{2.1}$$

We also assume that the collision frequency is independent of the velocities of colliding particles (Maxwell-type interactions) and that the total scattering cross section is finite. Hence, we can choose such units of time such that the corresponding classical Boltzmann equation reads

$$f_t = Q_+(f) - f \; , \tag{2.2}$$

where $Q_+(f)$ is the gain term of the collision integral (e.g. see [15]).

We shall need only some general properties of $Q_+(f)$: $Q_+(f) \geq 0$ if $f \geq 0$, and

$$\int_{\mathbb{R}^d} [Q_+(f)](v)\, dv = 1 \, , \tag{2.3}$$

for any f satisfying (2.1). Therefore, the operator Q_+ transforms f to another probability density. The structure of Eq. (2.2) leads to well-known probabilistic interpretation by M. Kac [20]. Accordingly, we can forget for a while about the rarefied gas and consider stochastic dynamics of N particles with phase coordinates (velocities) $v_i(t) \in \mathbb{R}^d$, $i = 1, \dots, N$.. A bit simplified Kac rules of the dynamics are: on each time-step $\Delta t = 2/N$ choose randomly a pair of integers $1 \leq i < l \leq N$ and perform a transformation $(v_i, v_l) \to (v_i', v_l')$ which corresponds to a collision of two particles with pre-collisional velocities v_i and v_l. It is easy to show (at least formally) that these rules lead, under additional assumption of molecular chaos [20], to Eq. (2.2) in the limit $N \to \infty$. A similar stochastic model, with slightly different same type transformation $(v_i, v_l) \to (v_i', v_l')$, leads to the inelastic version of Eq. (2.2) [6, 14].

It is clear that the same stochastic model admits other possible generalizations. For example we can also include multiple interactions and interactions with a background (thermostat). This type of model will formally correspond to a version of Eq. (2.2) for some $Q_+(f)$. For example, we take

$$Q_+(f) = \alpha_1 Q_+^{(1)}(f) + \alpha_2 Q_+^{(2)}(f) + \cdots + \alpha_N Q_+^{(M)}(f) \ , \tag{2.4}$$

where $Q_+^{(j)}$, $j = 1, \dots, M$, are j-linear positive operators describing interactions of $j \geq 1$ particles, $\alpha_j \geq 0$ are relative probabilities of such interactions. It is assumed that

$$\text{each} \quad \int_{\mathbb{R}^d} [Q_+^{(j)}(f)](v)\, dv = 1 \ ; \qquad \text{and that} \quad \sum_{j=1}^{M} \alpha_j = 1 \ , \tag{2.5}$$

so the condition (2.1) always holds.

Next, we focus on what properties of operators $Q_+^{(j)}$ are needed to make them consistent with the Maxwell-type interactions.

We postulate the main property of multi-particle systems with such interactions in the following way: *Temporal evolution of the system is invariant under scaling transformations of the phase space.*

That is, if S_t is the evolution operator of the above discussed N-particle system such that

$$S_t\{v_1(0), \dots, v_M(0)\} = \{v_1(t), \dots, v_M(t)\} \ , \qquad t \geq 0 \ ,$$

then

$$S_t\{\lambda v_1(0), \dots, \lambda v_M(0)\} = \{\lambda v_1(t), \dots, \lambda v_M(t)\} \tag{2.6}$$

for any constant $\lambda > 0$. It is easy to see that this assumption leads to the following property of $Q_+^{(j)}$ $(j = 1, 2, \dots, M)$:

$$Q_+^{(j)}(A_\lambda f) = A_\lambda Q_+^{(j)}(f) \ , \qquad A_\lambda f(v) = \lambda^d f(\lambda v) \ , \qquad \lambda > 0 \ . \tag{2.7}$$

Note that the transformation A_λ is consistent with the normalization (2.1).

This property shows that it is convenient to use the Fourier Transform

$$\hat{f}(k,t) = \mathcal{F}(f) = \int_{\mathbb{R}^d} f(v,t) e^{-ik\cdot v}\, dv \ , \qquad k \in \mathbb{R}^d \ , \tag{2.8}$$

since the resulting equation

$$\hat{f}_t = \hat{Q}_+(\hat{f}) - \hat{f} \ , \qquad \hat{Q}_+(\hat{f}) = \sum_{j=1}^{M} \alpha_j \hat{Q}_+^{(j)}(\hat{f}) \ , \tag{2.9}$$

is invariant under scaling transformations $k \to \lambda k$, $k \in \mathbb{R}^d$. Finally, it is natural to assume (a least in the case when $v \in \mathbb{R}^d$ is a velocity of particle) that all interactions are invariant under rotations in $\mathbb{R}^d$. Then the general problem can be simplified if we confine ourselves to a class of isotropic distribution functions $f(|v|, t)$.

In particular, denoting

$$u(x,t) = \int_{\mathbb{R}^d} dv\, f(|v|,t) e^{-ik\cdot v} \ , \qquad x = |k|^2 \ , \tag{2.10}$$

we consider a particular form of Eq. (2.9) for $u(x,t)$. The resulting equation (see Section 3) is the main mathematical object studied in this paper.

All above considerations remain valid for $d = 1$, the only differences are that, first, Eq. (2.2) should be considered as the one-dimensional Kac equation [20], and second, rotations in $\mathbb{R}^1 = \mathbb{R}$ should be replaced by reflections. An interesting one-dimensional system, presented in section 9, is based on the above discussed multi-particle stochastic model with non-negative phase variables $v = \mathbb{R}_+$, for which the Laplace transform

$$u(x,t) = \int_0^\infty f(v,t)e^{-xv}\,dv\ , \qquad x \geq 0\ , \tag{2.11}$$

leads to exactly the same class of equations for $u(x,t)$, described in Section 3.

3. Main equations and statement of the problem

We consider Eq. (2.9) for the case of isotropic solutions $\hat{f}(k,t) = u(|k|^2,t)$. The operator $\hat{Q}_+$ is a linear combination of n-linear operators $\hat{Q}_+^{(j)}$, $1 \leq j \leq M$, acting on the $x = |k|^2$ variable and invariant under dilations $\{x_\lambda = \lambda x,\ \lambda > 0\}$ in $\mathbb{R}_+$. A general class of such operators acting on $u(x)$ can be written in the form

$$\hat{Q}_+^{(j)}(u) = \int_0^\infty da_1 \ldots \int_0^\infty da_j Q_j(a_1,\ldots,a_j) \prod_{i=1}^{j} u(a_i x)\ ,$$

where $Q_j(a_1,\ldots,a_j)$ can be an generalized function of j non-negative variables. In our case both $u(x)$ and $\hat{Q}_+^{(j)}(u)$ are related by Fourier (or Laplace) transforms to some probability densities in $\mathbb{R}^d$ (or $\mathbb{R}_+$) (see Eqs. 2.1, (2.5), (2.10), (2.11)). Hence

$$u(0) = 1\ , \qquad \int_0^\infty \int_0^\infty Q_j(a_1,\ldots,a_j)\,da_1\ldots da_j \ = \ 1\ ,$$

moreover $u \in C(\mathbb{R}_+)$. Finally we note that the original (before Fourier/Laplace transforms) operators $Q_+^{(j)}$ were positive, i.e., $Q_+^{(j)}(f) \geq 0$ if $f \geq 0$. To satisfy this condition it is sufficient to assume that $Q_j(a_1,\ldots,a_j) \geq 0$ in above formulas. This follows directly from the fact that the product of two transforms is the transform of convolution of originals. These arguments explain our choice of equations below.

We slightly change the notation and consider the following equation for $u(x,t)$:

$$u_t + u = \Gamma(u)\ , \qquad x \geq 0\ , \quad t \geq 0 \tag{3.1}$$

where

$$\begin{aligned} &\Gamma(u) = \sum_{j=1}^{M} \alpha_j \Gamma^{(j)}(u)\ , \quad \sum_{j=1}^{M} \alpha_j = 1\ ,\ \alpha_j \geq 0\ , \\ &\Gamma^{(j)}(u) = \int_0^\infty \cdots \int_0^\infty A_j(a_1,\ldots,a_j) \prod_{k=1}^{j} u(a_k x)\,da_1\ldots da_j, \quad j = 1,\ldots,M\ . \end{aligned} \tag{3.2}$$

We assume that

$$A_j(a) = A_j(a_1,\ldots,a_j) \geq 0\ , \qquad \int_0^\infty da_1 \ldots \int_0^\infty da_j\ A(a_1,\ldots,a_j) = 1\ , \tag{3.3}$$

where $A_j(a) = A_j(a_1, \dots, a_j)$ is a generalized density of a probability measure in $\mathbb{R}_+^j$ for any $j = 1, \dots, M$. We also assume that all $A_j(a)$ have a compact support, i.e.,

$$A_j(a_1, \dots, a_j) \equiv 0 \ \text{ if } \ \sum_{k=1}^{j} a_k^2 > R^2 \ , \qquad j = 1, \dots, M \ , \tag{3.4}$$

for sufficiently large $0 < R < \infty$. In fact a much weaker assumption that

$$\int_0^\infty \dots \int_0^\infty A_j(a_1, \dots, a_j) \sum_{k=1}^{j} a_k^p \, da_1 \dots da_j < \infty, \qquad j = 1, \dots, M \, , \tag{3.5}$$

for all $p > 0$ is needed for most of our results (the assumption (3.4) is used only in section 10).

Classical models of elastic or inelastic particle interactions of Maxwell type are particular cases of Eq. (3.1) with, for example

$$\begin{aligned} &M = 2 \ , \quad \alpha_1 = \int_0^1 ds \, H(s) \ , \quad \alpha_2 = \int_0^1 ds \, G(s) \\ &A_1(a_1) = \frac{1}{\alpha_1} \int_0^1 ds \, H(s) \delta[a_1 - c(s)] \\ &A_2(a_1, a_2) = \frac{1}{\alpha_2} \int_0^1 ds \, G(s) \delta[a_1 - a(s)] \delta[a_2 - b(s)] \ , \end{aligned} \tag{3.6}$$

where the interaction law is determined by the functions $a(s), b(s)$ and $c(s)$.

Then, it is clear that Eq. (3.1) can be considered as a generalized Fourier transformed isotropic Maxwell model with multiple interactions provided $u(0, t) = 1$. The case $M = \infty$ in Eqs. (3.2) can be treated in the same way.

Therefore, the general problem we consider below can be formulated in the following way. We consider the initial value problem

$$u_t + u = \Gamma(u) \ , \quad u_{|t=0} = u_0(x) \ , \quad x \ge 0 \ , \quad t \ge 0 \ , \tag{3.7}$$

in the Banach space $B = C(\mathbb{R}_+)$ of continuous functions $u(x)$ with the norm

$$\|u\| = \sup_{x \ge 0} |u(x)| \ . \tag{3.8}$$

It is usually assumed that $\|u_0\| \le 1$ and that the operator Γ is given by Eqs. (3.2). On the other hand, there are just a few properties of $\Gamma(u)$ that are essential for existence, uniqueness and large time asymptotics of the solution $u(x, t)$ of the problem (3.7). Therefore, in many cases the results can be applied to more general classes of operators Γ in Eqs. (3.7) and more general functional space, for example $B = C(\mathbb{R}^d)$ (anisotropic models). That is why we study below the class (3.2) of operators Γ as the most important example, but simultaneously indicate which properties of Γ are relevant in each case. In particular, most of the results of Section 3-6 do not use a specific form (3.2) of Γ and, in fact, are valid for a more general class of operators.

Following this way of study, we first consider the problem (3.7) with Γ given by Eqs. (3.2) and point out the most important properties of Γ.

We simplify notations and omit in most of the cases below the argument x of the function $u(x, t)$. The notation $u(t)$ (instead of $u(x, t)$) means then the function of the real variable $t \ge 0$ with values in the space $B = C(\mathbb{R}_+)$.

Remark 1. We shall omit below the argument $x \in \mathbb{R}_+$ of functions $u(x)$, $v(x)$, etc., in some cases when this does not cause a misunderstanding. In particular, inequalities of the kind $|u| \le |v|$, for functions $u(x)$ and $v(x)$, should be understood as a point-wise control in absolute value, i.e. "$|u(x)| \le |v(x)|$ for any $x \ge 0$".

We first start by giving the following general definition for operators acting in a unit ball of a Banach space B denoted by

$$U = \{u \in B : \|u\| \le 1\} \tag{3.9}$$

Definition 3.1. *The operator $\Gamma = \Gamma(u)$ is called an L-Lipschitz operator if there exists a linear bounded operator $L : B \to B$ such that the inequality*

$$|\Gamma(u_1) - \Gamma(u_2)|(x) \le (L|u_1 - u_2|)(x), \quad x \ge 0; \tag{3.10}$$

holds for any pair of functions $u_{1,2}$ in U.

Remark 2. Note that the L-Lipschitz condition (3.10) holds, by definition, at any point $x \in \mathbb{R}_+$. Thus, condition (3.10) is much stronger than the classical Lipschitz condition

$$\|\Gamma(u_1) - \Gamma(u_2)\| < C\|u_1 - u_2\| \ \text{ if } \ u_{1,2} \in U \tag{3.11}$$

which obviously follows from (3.10) with the constant $C = \|L\|_B$, the norm of the operator L in the space of bounded operators acting in B. In other words, the terminology "L-Lipschitz condition" means the point-wise Lipschitz condition with respect to a specific linear operator L. A generalization to the case $B = C(\mathbb{R}^d)$ is obvious: we just need to change $x \ge 0$ to $x \in \mathbb{R}^d$ in Eqs. (3.7), (3.8) and (3.10).

The next lemma shows that the operator $\Gamma(u)$ defined in Eqs. (3.2), which satisfies $\Gamma(1) = 1$ (mass conservation) and maps U into itself, satisfies an L-Lipschitz condition, where the linear operator L is the one given by the linearization of Γ near the unity.

We assume without loss of generality that the kernels $A_j(a_1, \dots, a_j)$ in Eqs. (3.2) are symmetric with respect to any permutation of the arguments $(a_1, \dots, a_j)$, $j = 2, 3, \dots, M$.

Theorem 3.2. *The operator $\Gamma(u)$ defined in Eqs.* (3.2) *maps U into itself and satisfies the L-Lipschitz condition* (3.10)*, where the linear operator L is given by*

$$Lu(x) = \int_0^\infty da\, K(a) u(ax) , \tag{3.12}$$

with

$$K(a) = \sum_{j=1}^{M} j\alpha_j K_j(a) ,$$
$$\text{where} \quad K_j(a) = \int_0^\infty \dots \int_0^\infty A_j(a, a_2, \dots, a_j)\, da_2 \dots da_j \quad \text{and} \quad \sum_{j=1}^{M} \alpha_j = 1 . \tag{3.13}$$

for symmetric kernels $A_j(a_1, a_2, \dots, a_j), j = 2, \dots$.

Proof. First, the operator $\Gamma(u)$ in (3.2)-(3.4) maps B into itself and also satisfies

$$\|\Gamma(u)\| \le \sum_{j=1}^{M} \alpha_j \|u\|^j , \qquad \sum_{j=1}^{M} \alpha_j = 1 . \tag{3.14}$$

Hence,

$$\|\Gamma(u)\| \leq 1 \quad \text{if} \quad \|u\| \leq 1 \ , \tag{3.15}$$

and then $\Gamma(U) \subset U$, so its maps U into itself.

Since $\Gamma(1) = 1$, we introduce the linearized operator $L : B \to B$ such that formally

$$\Gamma(1 + \varepsilon u) = 1 + \varepsilon L u + O(\varepsilon^2) \ . \tag{3.16}$$

By using the symmetry of kernels $A_j(a)$, $j = 2, 3, \ldots, M$, one can easily check that L is given by equations (3.12) and (3.13).

In order to prove the L-Lipschitz property (3.10) for the operator Γ given in Eqs. (3.2), we make use of the multi-linear structure of the integrand associated with the definition of $\Gamma(u)$. Indeed, from the elementary identity

$$ab - cd = \frac{a+c}{2}(b-d) + \frac{b+d}{2}(a-c) \, ,$$

we estimate

$$\begin{aligned} &|u_1(a_1 x) u_1(a_2 x) - u_2(a_1 x) u_2(a_2 x)| \\ &\quad \leq |u_1(a_1 x) - u_2(a_1 x)| + |u_1(a_2 x) - u_2(a_2 x)| \ , \qquad x \geq 0 \ , \ a_{1,2} \geq 0 \ , \end{aligned}$$

provided $\|u_{1,2}\| \leq 1$. Then we obtain

$$|\Gamma^{(2)}(u_1) - \Gamma^{(2)}(u_2)| \leq 2 \int_0^\infty da \, K_2(a) |u_1(ax) - u_2(ax)|$$

in the notation of Eqs. (3.2), (3.13). It remains to prove that

$$|\Gamma^{(j)}(u_1) - \Gamma^{(j)}(u_2)| \leq j \int_0^\infty K_j(a) |u_1(ax) - u_2(ax)| \, da \tag{3.17}$$

for $3 \leq j \leq M$ (the case $j = 1$ is trivial). This problem can be obviously reduced to an elementary inequality

$$\Big| \prod_{k=1}^{j} x_k - \prod_{k=1}^{j} y_k \Big| \leq \sum_{k=1}^{j} |x_k - y_k| \ , \qquad j = 3, \ldots, \tag{3.18}$$

provided $|x_k| \leq 1$, $|y_k| \leq 1$, $k = 1, \ldots, j$. Since this is true for $j = 2$, we can use the induction. Let

$$a = x_{j+1} \ , \quad c = y_{j+1} \ , \quad b = \prod_{k=1}^{j} x_k \ , \quad d = \prod_{k=1}^{j} y_k \ ,$$

then

$$\begin{aligned} \Big| \prod_{k=1}^{j+1} x_k - \prod_{k=1}^{j+1} y_k \Big| &= |ab - cd| \leq |a - c| + |b - d| \leq \\ &\leq |x_{j+1} - y_{j+1}| + \sum_{k=1}^{j} |x_k - y_k| \ , \end{aligned}$$

and the inequality (3.18) is proved for any $j \geq 3$. Then we proceed exactly as in case $j = 2$ and prove the estimate (3.18) for arbitrary $j \geq 3$. Inequality (3.10) follows directly from the definition of operators Γ and L. □

Corollary. *The Lipschitz condition* (3.11) *is fulfilled for* $\Gamma(u)$ *given in Eqs.* (3.2) *with the constant*

$$C = \|L\| = \sum_{j=1}^{M} j\alpha_j \ , \qquad \sum_{j=1}^{M} \alpha_j = 1 \ , \tag{3.19}$$

where $\|L\|$ *is the norm of* L *in* B.

Proof. The proof follows directly from the inequality (3.10) and Eqs. (3.12), (3.13). □

It is also easy to prove that the L-Lipschitz condition holds in $B = C(\mathbb{R}^d)$ for "gain-operators" in the Fourier transformed Boltzmann equations for both elastic and inelastic Maxwell models.

4. Existence and uniqueness of solutions

The aim of this Section is to state and prove, with minimal requirements, the existence and uniqueness results associated with the initial value problem (3.7) in the space $B = C(\mathbb{R}_+)$. In fact, this existence and uniqueness result is an application of the classical Picard iteration scheme and holds for any operator Γ which satisfies the usual Lipschitz condition (3.11) and transforms the unit ball U into itself. We include its proof for the sake of completeness.

Lemma 4.1. *(Picard Iteration scheme) If the conditions in* (3.15) *and* (3.11) *are fulfilled then the initial value problem* (3.7) *with arbitrary* $u_0 \in U$ *has a unique solution* $u(t)$ *such that* $u(t) \in U$ *for any* $t \geq 0$.

Proof. Consider the integral form of Eq. (3.1)

$$u(t) = u_0 e^{-t} + \int_0^t e^{-(t-\tau)} \Gamma[u(\tau)]\, d\tau \tag{4.1}$$

and apply the standard Picard iteration scheme

$$u^{(n+1)}(t) = u_0 e^{-t} + \int_0^t e^{-(t-\tau)} \Gamma[u^{(n)}(\tau)]\, d\tau \,, \qquad u^{(0)} = u_0 \ . \tag{4.2}$$

Consider a finite interval $0 \leq t \leq T$ and denote

$$|\!|\!| u |\!|\!|_T = \sup_{0 \leq t \leq T} \|u(t)\| \ .$$

Then

$$\|u^{(n+1)}(t)\| \leq \|u_0\| e^{-t} + (1 - e^{-t}) |\!|\!| \Gamma(u^{(n)}) |\!|\!|_t$$

and therefore, by induction

$$\|u^{(n)}(t)\| \leq 1 \ \text{ for all } \ n = 1, 2, \ldots, \ \text{ and } \ t \in [0, T] \ ,$$

since $\|u_0\| \leq 1$ and $\Gamma(u)$ satisfies the inequality (3.15). If, in addition, $\Gamma(u)$ satisfies the Lipschitz condition (3.11), then it is easy to verify that

$$|\!|\!| u_{n+1} - u_n |\!|\!|_T \leq (1 - e^{-T}) C |\!|\!| u_n - u_{n-1} |\!|\!|_T \ , \qquad n = 1, 2, \ldots,$$

and therefore, the iteration scheme (4.2) converges uniformly for any $0 \leq t \leq T$ provided

$$C(1 - e^{-T}) < 1 \Longrightarrow T < \ln \frac{C}{C-1} \quad \text{if} \ \ C > 1 \tag{4.3}$$

(T can be taken arbitrary large if $C \le 1$). It is easy to verify that

$$u(t) = \lim_{n\to\infty} u^{(n)}(t) \; , \qquad 0 \le t \le T \; ,$$

is a solution of Eqs. (3.7), (4.1), satisfying the inequality

$$\|u(t)\| \le 1 \; , \qquad 0 \le t \le T \; . \tag{4.4}$$

The Lipschitz condition (3.11) is also sufficient to show that this solution is unique in a class of functions satisfying the inequality (4.4) on any interval $0 \le t \le \varepsilon$. Since the length T of the initial time-interval does not depend on the initial conditions (see Eq. (4.3)), then we can proceed by taking the next interval $T \le t \le 2T$ and so on. Thus we obtain the global in time solution $u(t) \in U$, where U is the closed unit ball in B, of the Cauchy problem (3.7). □

This proof of existence and uniqueness for the Cauchy problem (3.7)-(3.8) is quite standard (see any textbook on ODEs) and therefore we omit some details.

The next statement shows that L-Lipschitz condition (3.10) implies pointwise stability.

Theorem 4.2. *Consider the Cauchy problem* (3.7) *with* $\|u_0\| \le 1$ *and assume that the operator* $\Gamma : B \to B$

- **(a)** *maps the closed unit ball* $U \subset B$ *into itself, and*
- **(b)** *satisfies a* L*-Lipschitz condition* (3.10) *for some positive bounded linear operator* $L : B \to B$.

Then, for any $t \ge 0$

- **i)** *there exists a unique solution* $u(t)$ *of the problem* (3.7) *such that* $\|u(t)\| \le 1$*;*
- **ii)** *any two solutions* $u(t)$ *and* $w(t)$ *of problem* (3.7) *with initial data in the unit ball* U *satisfy the pointwise in* x *inequality*

$$|u(t) - w(t)| \le \exp\{t(L-1)\}(|u_0 - w_0|) \; . \tag{4.5}$$

Proof. The proof of **i)** follows directly from Lemma 4.1 (with the Lipschitz constant $C = \|L\|$). For the proof of **ii)**, let $u(t)$ and $w(t)$ be two solutions of this problem such that

$$u(0) = u_0 \; , \quad w(0) = w_0 \; , \quad \|u_0\| \le 1 \; , \quad \|w_0\| \le 1 \; . \tag{4.6}$$

Then the function $y(t) = u(t) - w(t)$ satisfies the equation

$$y_t + y = g(x,t) = \Gamma(u) - \Gamma(w) \; , \qquad y\big|_{t=0} = u_0 - w_0 = y_0 \; .$$

Hence,

$$y(t) = e^{-t} y_0 + \int_0^t e^{-(t-\tau)} g(x,\tau)\, d\tau \; ,$$

and, applying theorem 3.2 for the use of the L-Lipschitz condition (3.10) , we obtain

$$|y(t)| \le |y_0| e^{-t} + \int_0^t e^{-(t-\tau)} L |y(\tau)|\, d\tau \; . \tag{4.7}$$

Clearly, $|y(t)| \le y_*(t)$, where $y_*(t)$ satisfies the equation

$$y_*(t) = |y_0| e^{-t} + \int_0^t e^{-(t-\tau)} L y_*(\tau)\, d\tau \; . \tag{4.8}$$

Since the linear operator $L : B \to B$ is positive and bounded, then Eq. (4.8) has a unique solution

$$y_*(t) = e^{-t(1-L)}|y_0| = e^{-t}\sum_{n=0}^{\infty}\frac{t^n}{n!}L^n|y_0|\,,$$

so estimate (4.5) follows and the proof of the theorem is completed. □

Theorem 3.2 and the inequality (3.14) show that the operator Γ given in Eqs. (3.2) satisfies all conditions of the theorem.

Remark 3. The above consideration is, of course, a simple generalization of the proof of the usual Gronwall inequality for the scalar function $y(t)$. The essential difference is, however, that $y(t)$ is a "vector" $y(x,t)$ with values in the Banach space $B = C(\mathbb{R}_+)$ and, consequently, the estimate (4.5) for the functions $u(x,t)$ and $w(x,t)$ holds at any point $x \in \mathbb{R}_+$.

Remark 4. We stress that estimates (4.7)-(4.8) do not depend on specific properties of the operator L beyond that of being positive and bounded. The Banach space $B = C(\mathbb{R}_+)$ can be also replaced, for example, by $B = C(\mathbb{R}^d)$ (that is the case of non-isotropic models) provided some obvious changes in Eqs. (3.7), (3.8) and (3.10) are made.

At this point we remind the reader that the initial value problem (3.7) appeared as a generalization of the initial value problem (2.9) for a characteristic function $\varphi(x,t)$, i.e., for the Fourier transform of a probability measure see Eqs. (2.9), (2.10)). It is important therefore to show that the solution $u(x,t)$ of the problem (3.7) is a characteristic function for any $t > 0$ provided this is so for $t = 0$. The answer to such and similar questions is given in the following statement.

Lemma 4.3. *Let $U' \subset U \subset B$ be any closed convex subset of the unit ball U (i.e., $u = (1-\theta)u_1 + \theta u_2 \in U'$ for any $u_{1,2} \in U'$ and $\theta \in [0,1]$). If $u_0 \in U'$ in Eq. (3.7) and U is replaced by U' in the condition (**a**) of Theorem 4.2, the theorem holds and $u(t) \in U'$ for any $t \geq 0$.*

Proof. The only important point which should be changed in the proof is the consideration of Eqs. (4.2) in the proof of Lemma 4.1. We need to verify that, for any $u_0 \in U'$ and $v(\tau) \in U'$, $0 \leq \tau \leq t$,

$$\hat{u}(t) = u_0 e^{-t} + \int_0^t e^{-(t-\tau)}v(\tau)\,d\tau \quad \in U'\ . \tag{4.9}$$

In order to see that (4.9) holds, we note that $v(\tau) = \Gamma[u^{(n)}(\tau)]$ in Eqs. (4.2) is, by construction, a continuous function of $\tau \in [0,t]$ and that

$$e^{-t} + \int_0^t e^{-(t-\tau)}\,d\tau \ = 1\ .$$

Therefore we can approximate $\hat{u}(t)$ by an integral sum

$$\hat{u}_m(t) = u_0 e^{-t} + \sum_{k=1}^{m}\gamma_k(t)v(\tau_k)\ , \qquad \sum_{k=1}^{m}\gamma_k(t) = 1 - e^{-t}\ , \tag{4.10}$$

Then $u_m(t) \in U'$ as a convex linear combination of elements of U'. Taking the limit $m \to \infty$ to the sequence generated in (4.10), with U' is a closed subset of U, it follows that $\hat{u}(t) \in U'$ so (4.9) holds. Hence, the corresponding sequence as in Eqs. (4.2) also satisfies $u^{(n)}(t) \in U'$

for all $n \geq 0$. The rest of the proof continues as the one of Lemma 4.1, so that the result remains true for any closed convex subset $U' \subset U$ and so does Theorem 4.2. Thus the proof of Lemma 4.3 is completed. □

Remark 5. It is well-known (see, for example, the textbook [18]) that the set $U' \subset U$ of Fourier transforms of probability measures in $\mathbb{R}^d$ (Laplace transforms in the case of $\mathbb{R}_+$) is convex and closed with respect to uniform convergence. On the other hand, it is easy to verify that the inclusion $\Gamma(U') \subset U'$, where Γ is given in Eqs. (3.2), holds in both cases of Fourier and Laplace transforms. Hence, all results obtained for Eqs. (3.1), (3.2) can be interpreted in terms of "physical" (positive and satisfying the condition (2.1)) solutions of corresponding Boltzmann-like equations with multi-linear structure of any order.

We also note that all results of this Section remain valid for operators Γ satisfying conditions (3.2) with a more general condition such as

$$\sum_{j=1}^{M} \alpha_j \leq 1 , \qquad \alpha_j \geq 0 , \tag{4.11}$$

so that $\Gamma(1) < 1$ and so the mass may not be conserved. The only difference in this case is that the operator L satisfying conditions (3.12), (3.13) is not a linearization of $\Gamma(u)$ near the unity. Nevertheless Theorem 3.2 remains true. The inequality (4.11) is typical for Fourier (Laplace) transformed Smoluchowski-type equations where the total number of particles is decreasing in time (see [22, 23] for related work).

5. Large time asymptotics and self-similar solutions

In this Section we study in more detail the solutions to the initial value problem (3.7)-(3.8) constructed in Theorem 4.2 and, in particular, their long time behavior.

First of all, we note that the operator Γ given in Eqs. (3.2) has the followings properties.

Main properties of the operator Γ:

(a) Γ *maps the unit ball U of the Banach space $B = C(\mathbb{R}_+)$ into itself, that is*

$$\|\Gamma(u)\| \leq 1 \quad \text{for any} \quad u \in C(\mathbb{R}_+) \quad \text{such that} \quad \|u\| \leq 1.$$

(b) Γ *is a L-Lipschitz operator with L given by*

$$L\,u(x) = \int_0^\infty K(a)u(a\,x)\,da \;, \qquad x \geq 0, \tag{5.1}$$

where $K(a)$ ia a generalized density of a positive measure in $\mathbb{R}_+$ satisfying,

$$0 < \int_0^\infty K(a)a^p\,da < \infty, \qquad \text{for any } p \geq 0\,. \tag{5.2}$$

That means

$$|\Gamma(u_1) - \Gamma(u_2)|(x) \leq (L|u_1 - u_2|)(x) = \int_0^\infty K(a)|u_1(ax) - u_2(ax)|\,da\,, \tag{5.3}$$

for all $x \geq 0$ and for any two functions $u_{1,2} \in C(\mathbb{R}_+)$ such that $\|u_{1,2}\| \leq 1$.

(c) Γ *is invariant under dilations:*

$$e^{\tau\mathcal{D}}\Gamma(u) = \Gamma(e^{\tau\mathcal{D}}u) \;, \quad \mathcal{D} = x\frac{\partial}{\partial x} \;, \quad e^{\tau\mathcal{D}}u(x) = u(xe^{\tau}), \quad \tau \in \mathbb{R} \;. \tag{5.4}$$

No specific information about Γ beyond these three conditions will be used in Sections 5 and 6.

It was already shown in Theorem 4.2 that the conditions **(a)** and **(b)** guarantee existence and uniqueness of the solution $u(x,t)$ to the initial value problem (3.7)-(3.8). The property **(b)** yields the estimate (4.5) that is very important for large time asymptotics, as we shall see below. The property **(c)** suggests a special class of self-similar solutions to Eq. (3.7).

Note that the operator L in property **(b)** has a general form of linear positive operator invariant under dilations, i.e. its specific form is connected with property **(c)**

Next, we recall the usual meaning of the notation $y = O(x^p)$ (often used below): $y = O(x^p)$ if and only if there exists a positive constant C such that

$$|y(x)| \leq Cx^p \quad \text{for any} \quad x \geq 0. \tag{5.5}$$

In order to study long time stability properties to solutions whose initial data differs in terms of $O(x^p)$, we will need some spectral properties of the linear operator L.

The integral formula for L (see condition **(b)**) allows to extend L to a much wider functional classes than $B = C(\mathbb{R}_+)$. We assume below that this obvious extension is already done.

Definition 5.1. *Let L be the integral operator given in Eq.* (5.3)*,then*

$$Lx^p = \lambda(p)x^p \ , \qquad 0 < \lambda(p) = \int_0^\infty K(a)a^p \, da \ < \ \infty \ , \qquad p \geq 0 \, , \tag{5.6}$$

and the spectral function $\mu(p)$ is defined by

$$\mu(p) = \frac{\lambda(p) - 1}{p} \ . \tag{5.7}$$

An immediate consequence of properties **(a)** and **(b)**, as stated in (5.3), is that one can obtain a criterion for a point-wise in x estimate of the difference of two solutions to the initial value problem (3.7).

Lemma 5.2. *Let $u_{1,2}(x,t)$ be any two classical solutions of the problem* (3.7) *with Γ satisfying* **(a)** *and* **(b)***, and initial data satisfying the conditions*

$$|u_{1,2}(x,0)| \leq 1, \qquad |u_1(x,0) - u_2(x,0)| \leq C\, x^p \, , \quad x \geq 0 \tag{5.8}$$

for some positive constant C and p. Then

$$|u_1(x,t) - u_2(x,t)| \leq Cx^p \, e^{-t(1-\lambda(p))} \, , \qquad \textit{for all} \ \ t \geq 0 \tag{5.9}$$

Proof. The existence and uniqueness of $u_{1,2}(x,t)$ follow from Theorem 4.2. Estimate (4.5) (a consequence from the L-Lipschitz condition!) yields

$$|u_1(x,t) - u_2(x,t)| \leq e^{-t} e^{Lt} w(x) \, , \quad \text{with} \ \ w(x) = |u_1(x,0) - u_2(x,0)| \ . \tag{5.10}$$

The operator L from (5.3) is positive, and therefore monotone. Hence we obtain

$$e^{tL} w(x) = \sum_j \frac{t^j}{j!} L^j \, w(x) \leq C \, e^{tL} \, x^p = Ce^{\lambda(p)\, t} x^p,$$

that completes the proof. □

Corollary 1. *The minimal constant C for which condition* (5.8) *is satisfied is*

$$C_0 = \sup_{x \geq 0} \frac{|u_1(x,0) - u_2(x,0)|}{x^p} = \left\| \frac{u_1(x,0) - u_2(x,0)|}{x^p} \right\| , \tag{5.11}$$

and the following estimate holds

$$\left\| \frac{u_1(x,t) - u_2(x,t)}{x^p} \right\| \leq e^{-t(1-\lambda(p))} \left\| \frac{u_1(x,0) - u_2(x,0)}{x^p} \right\| \tag{5.12}$$

for any $p > 0$.

Proof. It follows directly from Lemma 5.2. □

We note that a result similar to Lemma 5.2 was first obtained in [8] for the inelastic Boltzmann equation. Its corollary in the form similar to (5.12) for equation (2.10) was stated later in [10] and was interpreted there as "the contraction property of the Boltzmann operator" (note that the left hand side of Eq.(5.12) can be understood as a non-expansive distance between any two solutions). However, independently of the terminology, the key reason for estimates (5.9)-(5.12) is the L-Lipschitz property of the operator Γ as defined in (5.4). It is remarkable that the large time asymptotics of $u(x,t)$, satisfying the problem (3.7) with such Γ, can be explicitly expressed through spectral characteristics of the linear operator L.

Hence, in order to study the large time asymptotics of $u(x,t)$ in more detail, we distinguish two different kinds of asymptotic behavior:

1) convergence to stationary solutions,
2) convergence to self-similar solutions provided the condition (**c**), of the main properties on Γ, is satisfied.

The case **1)** is relatively simple. Any stationary solution $\bar{u}(x)$ of the problem (3.7) satisfies the equation

$$\Gamma(\bar{u}) = \bar{u} \,, \qquad \bar{u} \in C(\mathbb{R}_+) \,, \quad \|\bar{u}\| \leq 1 \,. \tag{5.13}$$

If the stationary solution $\bar{u}(x)$ does exists (note, for example, that $\Gamma(0) = 0$ and $\Gamma(1) = 1$ for Γ given in Eqs. (3.2)) then the large time asymptotics of some classes of initial data $u_0(x)$ in (3.7) can be studied directly on the basis of Lemma 5.2. It is enough to assume that $|u_0(x) - \bar{u}(x)|$ satisfies (5.8) with p such that $\lambda(p) < 1$. Then $u(x,t) \to \bar{u}(x)$ as $t \to \infty$, for any $x \geq 0$.

This simple consideration, however, does not answer at least two questions:

A) What happens with $u(x,t)$ if the inequality (5.8) for $|u_0(x) - \bar{u}(x)|$ is satisfied with such p that $\lambda(p) > 1$?
B) What happens with $u(x,t)$ for large x ? (note that the estimate (5.9) becomes trivial if $x \to \infty$).

In order to address these questions we consider a special class of solutions of Eq. (3.7), the so-called self-similar solutions. Indeed the property **(c)** of Γ shows that Eq. (3.7) admits a class of formal solutions $u_s(x,t) = w(x\, e^{\mu_* t})$ with some real μ_*. It is convenient for our goals to use a terminology that slightly differs from the usual one.

Definition 5.3. *The function $w(x)$ is called a self-similar solution associated with the initial value problem* (3.7) *if it satisfies the problem*

$$\mu_* \mathcal{D}w + w = \Gamma(w) \ , \quad \mathcal{D} = x\frac{\partial}{\partial x} \ , \qquad \|w\| \leq 1 \ . \tag{5.14}$$

Note that the convergence of solutions $u(x,t)$ of the initial value problem (3.7) to a stationary solution $\bar{u}(x)$ can be considered as a special case of the self-similar asymptotics with $\mu_* = 0$.

Under the assumption that self-similar solutions exist (the existence is proved in the next Section), we prove a fundamental result on the convergence of solutions $u(x,t)$ of the initial value problem (3.7) to self-similar ones (sometimes called in the literature *self-similar stability*).

Lemma 5.4. *We assume that*

- **i)** *for some $\mu_* \in \mathbb{R}$, there exists a classical (continuously differentiable if $\mu_* \neq 0$) solution $w(x)$ of the problem* (5.14)*;*
- **ii)** *the initial data $u(x,0) = u_0$ in the problem* (3.7) *satisfies*

$$u_0 = w + O(x^p) \ , \qquad \|u_0\| \leq 1 \ , \ \textit{for} \ \ p > 0 \ \ \textit{such that} \ \mu(p) < \mu_*, \tag{5.15}$$

where $\mu(p)$ defined in (5.7) *is the spectral function associated to the operator L.*

Then

$$|u(xe^{-\mu_* t}, t) - w(x)| = O(x^p)e^{-pt(\mu_* - \mu(p))} \tag{5.16}$$

and therefore

$$\lim_{t\to\infty} u(xe^{-\mu_* t}, t) = w(x) \ , \qquad x \geq 0 \ . \tag{5.17}$$

Proof. By assumption, the function $u_2(x,t) = w(x\, e^{\mu_* t})$ satisfies Eq. (3.7). Let $u_1(x,t)$ be a solution of the problem (3.7) such that $u_1(x,0) = u_0(x)$. Then, by Lemma 5.2 and by assumption **ii)** we obtain

$$|u_1(x,t) - w(xe^{\mu_* t})| \leq C\, x^p e^{-(1-\lambda(p))\, t} \ ,$$

for some constant $C > 0$ and all $x \geq 0\,, t \geq 0$.

We can change in this inequality x to $\tilde{x}\, e^{-\mu_* t}$, then

$$|u_1(xe^{-\mu_* t}, t) - w(x)| \leq C\, x^p e^{-(p\mu_* + 1 - \lambda(p))\, t} \ ,$$

where the tildes are omitted. Note that $u_1(x,t) = u(x,t)$ in the formulation of the lemma and that $p\mu_* + 1 - \lambda(p) = p(\mu_* - \mu(p))$ in the notation (5.7).

Hence, the estimate (5.16) is proved. Eq. (5.17) follows from (5.16) since $\mu_* < \mu(p)$. So the proof is completed. □

Remark 6. Lemma 5.4 shows how to find a part of the domain of attraction of any self-similar solution provided the self-similar solution is itself known. It is remarkable that this part of the domain of attraction can be expressed in terms of just the *spectral function* $\mu(p)$, $p > 0$, defined in (5.7). Generally speaking, the equality (5.17) can be also fulfilled for some other values of p with $\mu(p) > \mu_*$ in Eq. (5.15), but, at least, it always holds if $\mu(p) < \mu_*$.

We shall need some properties of the *spectral function* $\mu(p),\ p > 0$. Note that

$$\mu(p) \approx \frac{\lambda(0) - 1}{p} \ , \qquad p \to 0^+ \ , \tag{5.18}$$

i.e. $\mu(p) > 0$ ($\mu(p) < 0$), for small $p > 0$, if $\lambda(0) > 1$ ($\lambda(0) < 1$).

In the particular case of Γ given by (3.2) and L by (3.12) and (3.13), we obtain

$$\lambda(0) = \int_0^\infty K(a)\, da = \sum_{j=1}^M \alpha_j \ , \ j \geq 1 \ , \qquad \sum_{j=1}^M \alpha_j = 1 \ , \ \alpha_j \geq 0 \, . \tag{5.19}$$

Note that, for this case $\lambda(0) = 1$ if and only if $M = 1$, i.e. for a linear operator Γ (3.2). It is easy to see that the problem (5.18) with a linear operator Γ given by (3.2) has no solutions (the condition $\|w\| \leq 1$ is important!), except for the trivial ones $w = 0, 1$.

Having in mind applications to non-linear operators Γ from (3.2), we assume below that $\lambda(0) > 1$.

Lemma 5.5. *The spectral function $\mu(p)$ given in Eqs.* (5.7) *and* (5.6), *with any $K(a)$ from the property (***b***), is analytic for $\mathcal{R}e\, p > 0$. It also has the following properties for real valued $p > 0$, provided $\lambda(0) > 1$:*

- **i)** *$\mu(p)$ is positive and unbounded as $p \to 0^+$, with asymptotic behavior given by* (5.18)*;*
- **ii)** *there is not more than one point $0 < p_0 < \infty$, where the spectral function $\mu(p)$ achieves its minimum.*

Proof. The regularity (analyticity) of $\mu(p)$ in the half-space $\mathcal{R}e\, p > 0$ follows from standard considerations of Laplace transforms in $\mathbb{R}_+$.

Next, **i)** is obvious from (5.18).

The statement **ii)** follows, first, from the convexity of $\lambda(p)$ since

$$\lambda''(p) = \int_0^\infty K(a) a^p (\ln a)^2\, da \ \geq 0 \ , \tag{5.20}$$

and from the identity

$$\mu'(p) = \frac{\psi(p)}{p^2} \ , \qquad \psi(p) = p\lambda'(p) - \lambda(p) + 1 \ . \tag{5.21}$$

We note that $\psi(p)$ in (5.21) is a monotone increasing function of p,($\psi' = p\lambda'' \geq 0$) and therefore it has not more than one zero, at say, $p = p_0 > 0$. Then $p = p_0$ is also a minimum point for $\mu(p)$ since from Eq. (5.18) $\mu(p) \to +\infty$ as $p \to 0$ and thus $\mu'(p) < 0$ for $p \to 0$. This completes the proof. □

The following corollaries are readily obtained from Lemma 5.5 under its assumptions.

Corollary 2. *The spectral function $\mu(p)$ is always monotone decreasing in the interval $(0, p_0)$, and $\mu(p) \geq \mu(p_0)$ for $0 < p < p_0$. This implies that there exists a unique inverse function* $\mathrm{p}(\mu) : (\mu(p_0), +\infty) \to (0, p_0)$, *monotone decreasing in its domain of definition.*

Proof. It follows immediately from Lemma 5.5, part **ii)** and its proof. □

Corollary 3. *There are precisely four different kinds of qualitative behavior of $\mu(p)$ shown on Fig.1 (the intermediate case with $\mu(p_0) = 0$ is considered as a coincidence of Fig.1 (c) and Fig.1(d)).*

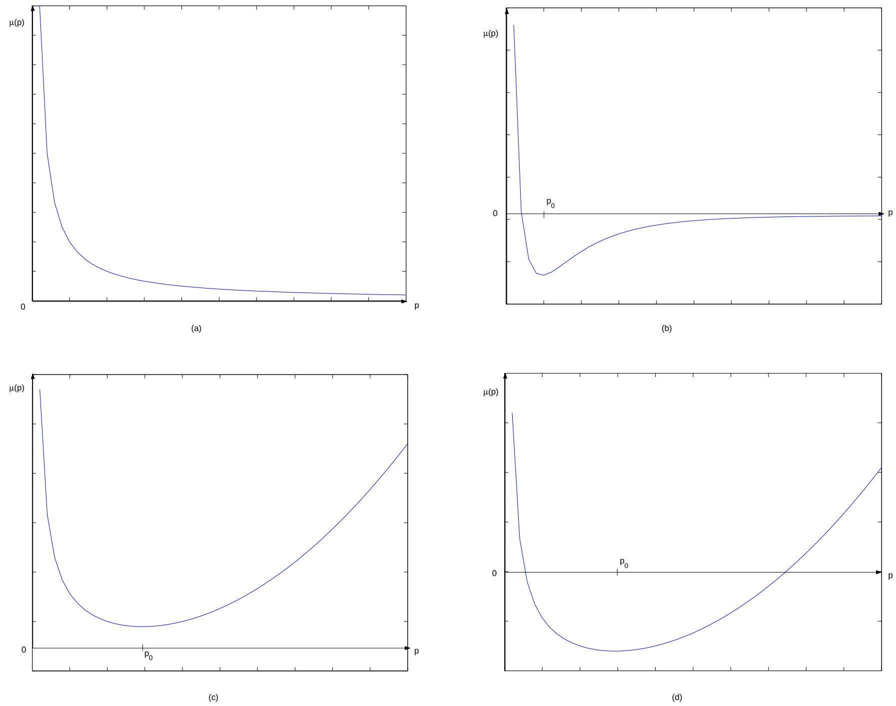

FIGURE 1. Possible profiles of the spectral function $\mu(p)$

Proof. There are two options: $\mu(p)$ is a monotone decreasing function (Fig.1 (a)) or $\mu(p)$ has a minimum at $p = p_0$ (Fig.1 (b,c,d)). In the first case $\mu(p) > 0$ for all $p > 0$ since $\mu(p) > -1/p$ and therefore $\lim_{p\to\infty} \mu(p) \geq 0$. The asymptotics of $\lambda(p)$ (5.6) is clear:

$$\textbf{(1)} \qquad \lambda(p) \xrightarrow[p\to\infty]{} \lambda_\infty \in \mathbb{R}_+ \ \text{ if } \int_{1+}^{\infty} K(a)\, da = 0 \ ; \tag{5.22}$$

$$\textbf{(2)} \qquad \lambda(p) \xrightarrow[p\to\infty]{} \infty \text{ if } \int_{1+}^{\infty} K(a)\, da > 0 \ . \tag{5.23}$$

In the case **(1)** when $\mu(p) \to \infty$ as $p \to 0$, two possible pictures (with and without minimum) are shown on Fig.1 (b) and Fig.1 (a) respectively. In case **(2)**, from Eq. (5.6) it is clear that $\lambda(p)$ grows exponentially for large p, therefore $\mu(p) \to \infty$ as $p \to \infty$. Then the minimum always exists and we can distinguish two cases: $\mu(p_0) < 0$ (Fig.1 (d) and $\mu(p_0) > 0$ (Fig.1 (c)) □

Fig.1 gives a clear graphic representation of the domains of attraction of self-similar solutions (Lemma 5.4): it is sufficient to draw the line $\mu(p) = \mu_* = constant$, and to consider a p such that the graph of $\mu(p)$ lies below this line. Therefore, the following corollary follows directly from the properties of the spectral function $\mu(p)$, as characterized by the behaviors in Fig.1, where we assume that $\mu(p_0) = 0$ for $p_0 = \infty$, for the case shown on Fig.1 (a).

Corollary 4. *Any self-similar solution $u_s(x,t) = w(xe^{\mu_* t})$ with $\mu(p_0) < \mu_* < \infty$ has a non-empty domain of attraction, where p_0 is the unique (minimum) critical point of the spectral function $\mu(p)$.*

Proof. We use Lemma 5.4 part **ii)** on any initial state $u_0 = w + O(x^p)$ with $p > 0$ such that $\mu(p_0) \le \mu(p) < \mu_*$. In particular, Eqs. (5.16) and (5.17) show that the domain of attraction of $w(xe^{\mu_* t})$ contains any solution to the initial value problem (3.7) with the initial state as above. □

The inequalities of the kind $u_1 - u_2 = O(x^p)$ for any $p > 0$ such that $\mu(p) < \mu_*$, for any fixed $\mu_* \ge \mu(p_0)$ play an important role for the self-similar stability. We can use specific properties of $\mu(p)$ in order to express such inequalities in a more convenient form.

Lemma 5.6. *Let the conditions of lemma 5.5 be satisfied. Then, for any given* $\mu_* \in (\mu(p_0), \infty)$ *and* $u_{1,2}(x)$ *such that* $\|u_{1,2}\| < \infty$*, the following two statements are equivalent:*

i) *there exists* $p > 0$ *such that*

$$u_1 - u_2 = O(x^p) \ , \qquad \text{with} \ \ \mu(p) < \mu_* \, . \tag{5.24}$$

ii) *There exists* $\varepsilon > 0$ *such that*

$$u_1 - u_2 = O(x^{\mathrm{p}(\mu_*)+\varepsilon}) \ , \tag{5.25}$$

where $\mathrm{p}(\mu)$ *is the inverse to* $\mu(p)$ *function, as defined in* **Corollary 2***.*

Proof. Let property **i)** hold, then recall from **Corollary 2** that $\mu(p)$ is monotone on the interval $0 < p \le p_0$, so its inverse function $\mathrm{p}(\mu)$ is defined uniquely.

It is clear, as it can be seen in from Fig.1, that the condition $\mu(p) < \mu_*$ are satisfied only for some $p > \mathrm{p}(\mu_*)$, therefore inequality (5.25) with $\varepsilon = p - \mathrm{p}(\mu_*)$ follows directly from (5.24).

Conversely, if **ii)** holds, first note that for any pair of uniformly bounded functions (note that $\|u_{1,2}\| < \infty$ by assumption) which satisfy inequality

$$|u_1(x) - u_2(x)| < C\, x^q \ , \qquad C = const. \ ,$$

for some $q > 0$, then the same inequality holds with any p such that $0 < p < q$ and perhaps another constant. Therefore, if the condition (5.25) is satisfied, then one can always find a sufficiently small $0 < \varepsilon_1 \le \varepsilon$ such that taking $p = \mathrm{p}(\mu_*) + \varepsilon_1$ condition (5.24) is fulfilled. This completes the proof. □

Finally, to conclude this Section, we show a general property of the initial value problem (3.7) for any non-linear Γ operator satisfying conditions **(a)** and **(b)** given in the beginning of this Section. This property gives the control to the point-wise difference of any two rescaled solutions to (3.7) in the unit sphere of B , whose initial states differ by $O(x^p)$. It is formulated as follows.

Lemma 5.7. *Consider the problem* (3.7)*, where* Γ *satisfies the conditions* **(a)** *and* **(b)***. Let* $u_{1,2}(x,t)$ *be two solutions satisfying the initial conditions* $u_{1,2}(x,0) = u_0^{1,2}(x)$ *such that*

$$\|u_0^{1,2}\| \le 1 \ , \quad u_0^1 - u_0^2 = O(x^p) \ , \quad p > 0 \ . \tag{5.26}$$

then, for any real μ_**,*

$$\Delta_{\mu_*}(x,t) = u_1(xe^{-\mu_* t}, t) - u_2(xe^{-\mu_* t}, t) = O(x^p) e^{-pt[\mu_* - \mu(p)]} \tag{5.27}$$

and therefore

$$\lim_{t\to\infty} \Delta_{\mu_*}(x,t) = 0 \ , \qquad x \ge 0 \ , \tag{5.28}$$

for any $\mu_* > \mu(p)$*.*

Proof. The proof is a repetition of the arguments that led to Eq. (5.16). In particular, we obtain from Eqs. (4.5), (5.9) the estimate

$$|u_1(x,t) - u_2(x,t)| = x^p\, e^{t[\lambda(p)-1]}\, \| \frac{u_0^1 - u_0^2}{x^p} \|$$

and then change x to $xe^{-\mu_* t}$. This leads to Eq. (5.27) in the notation (5.7) and this completes the proof. □

Remark 7. There is an important point to understand here: Lemmas 5.2 and 5.7 hold for any operator Γ that satisfies just the two properties **(a)** and **(b)** stated at the beginning of this Section. It says that, in some sense, a distance between any two solutions with initial conditions satisfying Eqs. (5.26) tends to zero as $t \to \infty$, i.e. *non-expansive distance*. Such terminology and corresponding distances were introduced in [19] for the elastic Maxwell-Boltzmann with finite initial energy, and used in specific forms of inelastic binary Maxwell-Boltzmann models in [3, 24]. It should be pointed out, however, that this *contraction property* may not say much about large time asymptotics of $u(x,t)$, unless the corresponding self-similar solutions are known.

Therefore one must study the problem of existence of self-similar solutions, which is considered in the next Section.

6. Existence of self-similar solutions

A goal is to study problem (5.14) Theorem 6.1 below shows a criterion for existence and uniqueness of self-similar solutions, in the sense of Definition 5.3 for any operator Γ that satisfies just conditions **(a)** and **(b)**. Then Theorem 6.2 follows, showing a general criterion to self-similar asymptotics of the problem (3.7) for any operator Γ satisfying conditions **(a)**, **(b)** and **(c)**.

We consider Eq. (5.14) written in the form

$$\mu_* x w'(x) + w(x) = g(x)\ , \qquad g = \Gamma(w)\ ,\ \mu_* \in \mathbb{R}\ , \tag{6.1}$$

and, assuming that $\|w\| < \infty$, transform this equation to the following integral form. Multiplying the equation by the integrating factor ${\mu_*}^{-1} x^{\frac{1-\mu_*}{\mu_*}}$, integrating and changing coordinates in the resulting right hand side integral, it is easy to verify that the resulting integral equation reads

$$w(x) = \int_0^1 g(x\tau^{\mu_*})\, d\tau\ . \tag{6.2}$$

We prove the following result formulated in terms of the spectral function $\mu(p)$ from (5.7).

Theorem 6.1. *Consider Eq.* (6.1) *with arbitrary* $\mu_* \in \mathbb{R}$ *and the operator* Γ *satisfying the conditions* **(a)** *and* **(b)** *from Section 5. Assume that there exists a continuous function* $w_0(x)$, $x \geq 0$, *such that*

i) $\|w_0\| \leq 1$ *and*

ii)
$$\int_0^1 g_0(x\tau^{\mu_*})\, d\tau = w_0(x) + O(x^p)\ , \qquad g_0 = \Gamma(w_0)\ , \tag{6.3}$$

with some $p > 0$ *satisfying the inequality* $\mu(p) < \mu_*$.

Then, there exists a classical solution $w(x)$ of Eq. (6.1) *such that*

$$\|w\| \le 1, \;\; w(x) = w_0(x) + O(x^p) \,, \qquad \textit{with the same } p > 0. \tag{6.4}$$

The solution is unique in the class of continuous functions satisfying conditions

$$\|w\| < \infty \;, \qquad w(x) = w_0(x) + O(x^{p_1}) \;, \tag{6.5}$$

with any positive p_1 such that $\mu(p_1) < \mu_*$.

Proof. The existence is proven by the following iteration procedure. We choose an initial approximation $w_0 \in U$ such that $\|w_0\| \le 1$ and consider the iteration scheme

$$w_{n+1}(x) = \int_0^1 g_n(x\tau^{\mu_*})\, d\tau \;, \qquad g_n = \Gamma(w_n) \;, \; n = 0, 1, \dots \;. \tag{6.6}$$

Then, because of property **(a)** of Γ, $\|w_n\| \le 1$ for all $n \ge 1$ and

$$|w_{n+1}(x) - w_n(x)| \le \int_0^1 |g_n(x\tau^{\mu_*}) - g_{n-1}(x\tau^{\mu_*})|, d\tau \;, \qquad n \ge 1\,.$$

By using the inequality (5.3) (i.e. property **(b)** of Γ), we control the right hand side of the above inequality by

$$\begin{aligned} |w_{n+1}(x) - w_n(x)| &\le \int_0^1 L(|w_n - w_{n-1}|)(x\tau^{\mu_*})\, d\tau \\ &= \int_0^1 d\tau \int_0^\infty K(a)|w_n - w_{n-1}|(ax\tau^{\mu_*})\, da \;. \end{aligned} \tag{6.7}$$

Next, by assumption ii), initially

$$|w_1(x) - w_0(x)| \le Cx^p \;, \qquad x \ge 0 \;, \; C = \textit{constant}\,. \tag{6.8}$$

Then, recalling the definition for $\lambda(p)$ given in (5.6), we can control the right hand side of (6.7) by

$$x^p \int_0^\infty K(a)a^p\, da \int_0^1 \tau^{p\mu_*} d\tau = x^p \frac{\lambda(p)}{1 + p\mu_*} \,, \qquad p\mu_* > p\mu(p) > -1\,.$$

Therefore, we estimate the left hand side of (6.7) by

$$|w_{n+1}(x) - w_n(x)| \le C\gamma^n(p, \mu_*)\, x^p \;, \qquad \gamma(p, \mu_*) = \frac{\lambda(p)}{1 + p\mu_*} \;. \tag{6.9}$$

Then, $\lambda(p) > 0$ and $\mu(p) = \frac{\lambda(p)-1}{p} < \mu_*$ imply $0 < \gamma(p, \mu_*) < 1$. Therefore, there exists a point-wise limit

$$w(x) = \lim_{n\to\infty} w_n(x) \tag{6.10}$$

satisfying the inequality

$$|w(x) - w_0(x)| \le \sum_{n=0}^{\infty} |w_{n+1} - w_n(x)| \le \frac{C}{1 - \gamma(p, \mu_*)}\; x^p \;. \tag{6.11}$$

Estimate (6.9) with $\gamma < 1$ shows that the convergence $w_n(x) \to w(x)$ is uniform on any interval $0 \le x \le R$, for any $R > 0$. Therefore $w(x)$ is a continuous function, moreover $\|w\| \le 1$ since $\|w_n\| \le 1$ for all $n = 0, 1, \dots$.

The next step is to prove that the limit function $w(x)$ from (6.10) satisfies Eqs. (6.1), or equivalently, (6.2). We note that

$$|g_{n+1}(x) - g_n(x)| \le \int_0^\infty K(a)|w_{n+1}(ax) - w_n(ax)|\, da \le \\ \le C\,\lambda(p)\gamma^n(p,\mu_*)\,x^p\ . \tag{6.12}$$

Therefore $g_n(x) \to g(x)$, where $g(x) \in C(\mathbb{R}_+)$ and $\|g\| \le 1$, since $\|g_n\| \le 1$ for all n. In addition, from the continuity of the operator $\Gamma(u)$ for $\|u\| \le 1$ follows that $g = \Gamma(w)$, and the transition to the limit in the right hand side of Eq. (6.2) is justified since $\|g_n\| \le 1$. Hence, $w(x)$ satisfies Eq. (6.2).

When $\mu_* \neq 0$ one also needs to check that Eq. (6.1) is satisfied. We note that, for any continuous and bounded $w_0(x)$, all functions $w_n(x)$, $n \ge 1$, are differentiable for $x > 0$ and their derivatives $w_n'(x)$ satisfy the equations (see Eqs. (6.1), (6.6))

$$\mu_*\, x\, w_n'(x) = w_n(x) + g_{n-1}(x)\ , \qquad n = 1, 2, \dots\ .$$

Hence,

$$\mu_*\, x\, (w_{n+1}' - w_n') = (w_n - w_{n+1}) + (g_n - g_{n-1})\ ,$$

and, by using inequalities (6.9), (6.12), we obtain

$$|\mu_*|\, |w_{n+1}' - w_n'| \le C\,\gamma^{p-1}(p,\mu_*)\,(\gamma(p,\mu_*) + \lambda(p))\,x^p\ , \qquad n \ge 1\ .$$

Therefore the sequence of derivatives $\{w_n'(x),\, n = 1, \dots\}$ converges uniformly on any interval $\varepsilon \le x \le R$. Hence, the limit function $w(x)$ from (6.10) is differentiable for $x > 0$,

$$w'(x) = O(x^{p-1})\ , \qquad p > 0\ , \tag{6.13}$$

and the equality (6.1) is also satisfied for $\mu_* \neq 0$.

Finally we note that the condition of convergence $0 < \gamma(p,\mu_*) < 1$ is equivalent to the condition $\mu(p) < \mu_*$ (see Eq. (5.7)) that has already appeared in Lemma 5.4.

It remains to prove the statement concerning the uniqueness of the solution to (6.1). If there are two solutions $w^{1,2}$ satisfying (6.5), with $p_1 = p^{1,2}$ respectively, then the integral equation (6.2) yields

$$|w^1(x) - w^2(x)| \le \int_0^1 |g^1(x\tau^{\mu_*}) - g^2(x\tau^{\mu_*})|\ d\tau\ , \qquad g^{1,2} = \Gamma(w^{1,2})\ . \tag{6.14}$$

Since $\|w^{1,2}\| \le 1$, we obtain

$$|w^1(x) - w^2(x)| \le C\,x^q\ , \qquad q = \min(p^1, p^2)\ ,$$

where obviously $\mu(q) < \mu_*$. Then we again apply the inequality (5.3) to the integral in Eq. (6.2) and get the new estimate

$$|w^1(x) - w^2(x)| \le C_1 x^q\ , \qquad C_1 = \gamma(q,\mu_*)C < C\ .$$

By repeating the same considerations as in the existence argument, it follows that

$$|w^1(x) - w^2(x)| \le C\gamma^n(q,\mu_*)x^q\ , \qquad \text{with} \quad \gamma(q,\mu_*) < 1\ ,$$

for any integer $n \ge 0$. Therefore $w^1(x) \equiv w^2(x)$ and the proof is completed. □

Now we can combine Lemma 4.1 with Theorem 6.1 and prove the general statement related to the self-similar asymptotics for the problem (3.7).

Having in mind applications to operators Γ from (3.2), we assume below that $\lambda(0) > 1$ in Eqs. (5.6), (5.7). Then Lemma 5.5 and its corollaries show that the spectral function $\mu(p)$ has a qualitative behavior shown on Fig. 1. In particular, there exists a $p_0 > 0$ such that $\mu(p) > \mu(p_0) = \inf_{p>0} \mu(p)$, including formally the case $p_0 = \infty$ (Fig.1a).

Theorem 6.2. *Let $u(x,t)$ be a solution of the problem* (3.7) *with initial data $\|u_0\| \le 1$ and Γ satisfying conditions* **(a)**, **(b)** *with $\lambda(0) > 1$ in the notation of Eqs.* (5.6), *and* **(c)** *from Section 5. We assume that*

1) *There exists a continuous function $y(x), x \ge 0$, such that*

$$\|y\| \le 1 \qquad \int_0^1 g(x\tau^{\mu(p)}) = y(x) + O(x^{p+\varepsilon_1})\, d\tau \ , \qquad g = \Gamma(y), \tag{6.15}$$

for some $p \in (0, p_0)$ and $\varepsilon_1 > 0$;

2) *The initial state satisfies*

$$u_0(x) = y(x) + O(x^{p+\varepsilon_2}) \tag{6.16}$$

for the same p and some $\varepsilon_2 > 0$.

Then,

i) *there exists a self-similar solution $u_p(x,t) = w(x\, e^{\mu(p)t})$ of equation* (3.7) *such that*

$$\|w\| \le 1 \ , \qquad w(x) = y(x) + O(x^{p+\varepsilon_1})\, ; \tag{6.17}$$

ii)

$$\lim_{t\to\infty} u(x\, e^{-\mu(p)t}, t) = w(x) \ , \qquad x \ge 0 \ , \tag{6.18}$$

where the convergence is uniform on any bounded interval in $\mathbb{R}_+$ and

$$u(x\, e^{-\mu(p)t}, t) - w(x) = O(x^{p+\varepsilon} e^{-\beta(p,\varepsilon)t})\, , \tag{6.19}$$

with $\beta(p,\varepsilon) = (p+\varepsilon)(\,\mu(p) - \mu(p+\varepsilon)\,) > 0$ and $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$.

iii) *If the condition* **1)** *for $y(x)$ holds simultaneously with two different values $p_{1,2} \in (0, p_0)$, with $p_1 > p_2$, then $w(x) = const.$ (trivial self-similar solutions) for p_2.*

Proof. Since $\lambda(0) > 1$, we can apply Lemma 5.6 and see that the condition **1)** for $y(x)$ coincides with the conditions of Theorem 6.1 for $w_0(x)$ provided the obvious changes of notation is made in (6.3). The $w(x)$ in **i)** is the function constructed in Theorem 6.1 for $w_0 = y$ and $\mu_* = \mu(p)$. We note that, in accordance with Eqs. (6.16) and (6.17),

$$u_0(x) - w(x) = O(x^{p+\varepsilon})\, , \qquad \text{with} \;\; \varepsilon = \min\{\varepsilon_1, \varepsilon_2\}\, .$$

Then, statement **ii)** follows directly from Lemma 5.4 since $\mu(p)$ is monotone decreasing on $(0, p_0)$.

It remains to prove **iii)**. If there are two different values $p_1 > p_2$ for which Eq. (6.15) is satisfied, then there are two different self-similar solutions $w_i(x\, e^{\mu(p_i)t}),\ i = 1,2$. It follows from (6.17) that $\|w_{1,2}\| \le 1$, with $w_1(x) = w_2(x) + O(x^{p_2+\varepsilon})$, where $\varepsilon > 0$ can be made as small as we want.

Then, recalling that $w_{1,2}(x)$ are initial data for two different self-similar solutions, apply Eq. (5.16) with $w(x) = w_2(x)$, $u(x,t) = w_1(x\, e^{\mu(p_1)t})$, $\mu_* = \mu(p_2)$ and $p = p_2 + \varepsilon$ to obtain $w_1(x\, e^{(\mu(p_1)-\mu(p_2))t}) - w_2(x) = O(x^{p_2+\varepsilon}\, e^{-\beta(p_2,\varepsilon)t})$ in the notation of Eq. (6.19). Hence,

$w_2(x) = w_1(0) = constant$ since $w_1(x)$ is continuous at $x = 0$, and this completes the proof of Theorem 6.2. □

Thus we obtain a general criterion of self-similar asymptotics of $u(x,t)$ for a given initial condition $u_0(x)$. It is important that the case of non-trivial ($w \neq const.$ in (6.18)) asymptotics can appear only if $p \in (0, p_0)$ is the maximal number for which Eq. (6.15) is satisfied. For brevity, we do not consider the case with $\lambda(0) \le 1$ (a criterion for self-similar asymptotics in this case can be also obtained by a combination of Theorem 6.1 and Lemma 5.4), since we do not know any specific application of this case. On the other hand, it was already noted above that all conditions of Theorem 6.2 (in particular $\lambda(0) > 1$) for operators Γ from (3.2) are satisfied in the non-linear case ($N \ge 2$). Applications to these type of operators are considered in Sections 8 and 9.

7. Properties of self-similar solutions

We consider the integral equation (6.2) written as

$$w = \Gamma_{\mu_*}(w) = \int_0^1 g(x\tau^{\mu_*})\, d\tau\ s\ , \qquad g = \Gamma(w)\ ,\ \mu_* \in \mathbb{R}\ . \tag{7.1}$$

The following two properties of $w(x)$ that are independent of the specific form (3.2) of Γ.

Lemma 7.1.

- **i)** *If there exist a closed subset $U' \subset U$ of the unit ball U in B, as given in* (3.15)*, such that $\Gamma_{\mu_*}(U') \subset U'$ for any $\mu_* \in \mathbb{R}$, and for some function $w_0 \in U'$ the conditions of Theorem 6.1 are satisfied, then $w \in U'$, where w is constructed by the iterative scheme as defined in* (6.6)*.*
- **ii)** *If the conditions of Theorem 6.1 for Γ are satisfied and, in addition, $\Gamma(1) = 1$,*

then the solution $w_* = 1$ of Eq. (7.1) *is unique in the class of functions $w(x)$ satisfying conditions*

$$\|w\| < \infty \qquad w(x) = 1 + O(x^p)\ , \qquad \mu(p) < \mu_*\ . \tag{7.2}$$

Proof. The first statement follows from the iteration scheme (6.6) with $w_0 \in U'$. Then, by assumption **i)**, $w_0 \in U'$ for all integer $n \ge 1$, and $w_n \to w \in U'$. The second statement follows from the obvious fact that $w_* = 1$ satisfies Eq. (7.1) with any $\mu_* \in \mathbb{R}$ provided $\Gamma(1) = 1$ and from the uniqueness of $w(x)$ stated in Theorem 6.1. This completes the proof. □

The statement **ii)** can be interpreted as a necessary condition for existence of "nice" non-trivial ($w \neq$ const.) solutions of Eq. (7.1):
if there exists a non-trivial solution $w(x)$ of Eq. (7.1) *with* $\Gamma(1) = 1$ *and with some* $\mu_* \in \mathbb{R}$*, such that*

$$\|w\| = 1\ , \qquad w = 1 + O(x^p)\ , \quad p > 0\ , \qquad \textit{then} \qquad \mu_* \le \mu(p)\ . \tag{7.3}$$

Let us consider now the specific class (3.2)–(3.3) of operators Γ, with functions $u(x)$ satisfying the condition $u(0) = 1$. That is, $u(0,t) = 1$ for the solution $u(x,t)$ of the problem (3.7).

Since the operators (3.2) are invariant under dilation transformations (5.4) (property **(c)**, Section 5), the problem (3.7) with the initial condition $u_0(x)$ satisfying

$$u(0) = 1\ , \quad \|u_0\| = 1\ ; \qquad u_0(x) = 1 - \beta x^p + \cdots\ , \quad x \to 0\ , \tag{7.4}$$

can be always reduced to the case $\beta = 1$ by the transformation $x' = x\beta^{1/p}$.

Moreover, the whole class of operators (3.2), with different kernels $A_j(a_1, \dots, a_j)$, $j = 1, 2, \dots$, is invariant under transformations $\tilde{x} = x^p$, $p > 0$. The result of such transformation acting on Γ is another operator $\widetilde{\Gamma}$ of the same class (3.2) with kernels $\tilde{A}_j(a_1, \dots, a_j)$ and corresponding spectral function $\tilde{\mu}(p)$ (see the end of this Section).

Therefore, we fix the initial condition (7.4) with $\beta = 1$ and transform the function (7.4) and Eq. (3.7) to new variables $\tilde{x} = x^p$. Then, we omit the tildes and reduce the problem (3.7), with initial condition (7.4) to the case $\beta = 1$, $p = 1$. We study this case in detail and formulate afterwards the results in terms of initial variables.

Our goal now is to apply the general theory (in particular, Theorem 6.2) to the particular case where the initial data $u_0(x)$ satisfies,

$$\|u_0\| = 1\ , \quad u_0(x) = 1 - x + O(x^{1+\varepsilon})\ , \quad x \to 0\ , \tag{7.5}$$

with some $\varepsilon > 0$. We also assume that the spectral function $\mu(p)$, given by Eqs. (5.6), (5.7) and (3.13), corresponds to one of the four cases shown on Fig.1 with a unique minimum (infimum) achieved at $p_0 > 0$.

Let us consider a typical function $u_0 = e^{-x}$ satisfying (7.5) and investigate the criterion (6.15) from Theorem 6.2 for this particular function.

Theorem 7.2. *Whenever $\mu(1) > -1$, the estimate*

$$\Gamma_{\mu(p)}(e^{-x}) - e^{-x} = 0(x^{p+\varepsilon})\,, \qquad \varepsilon > 0\,, \tag{7.6}$$

holds for positive p if and only if $p \leq 1$.

Proof. In order to prove (7.6), we assume that

$$\mu(p) > -1\,, \tag{7.7}$$

and investigate the structure of $\Gamma_\mu(e^{-x})$ for any $\mu > -1$. Its explicit formula reads

$$\Gamma_\mu(e^{-x}) = \sum_{j=1}^{M} \alpha_j \int_{\mathbb{R}_+^j} A_j(a_1, \dots, a_j) I_\mu\Big[x \sum_{k=1}^{j} a_k\Big]\, da_1 \dots da_j\ , \tag{7.8}$$

with

$$I_\mu(y) = \int_0^1 e^{-y\tau^\mu}\, d\tau\ , \qquad \mu \in \mathbb{R}\,,\ y > 0\,, \qquad\qquad \sum_{j=1}^{M} \alpha_j = 1\,. \tag{7.9}$$

The following lemma holds.

Lemma 7.3. *If $\mu > -1$, $y \geq 0$, then $0 < I_\mu(y) \leq 1$, and*

$$I_\mu(y) = e^{-y}\left(1 + \frac{\mu y}{1+\mu}\right) + \frac{\mu^2}{1+\mu}\, r_\mu(y)\ , \tag{7.10}$$

where $0 \leq r_\mu(y) \leq B(y,\mu)$ *with*

$$B(y,\mu) = \begin{cases} y^2(2\mu+1)^{-1} & \text{if } \mu > -\frac{1}{2}\,, \\ 2y^2\left(-\ln y + y\right) & \text{if } \mu = -\frac{1}{2}\,; \\ \frac{\Gamma(2-|\mu|^{-1})}{|\mu|}\; y^{\frac{1}{|\mu|}} & \text{if } \mu \in \left(-1,-\frac{1}{2}\right)\,. \end{cases} \tag{7.11}$$

Proof. We consider the integral $I_\mu(y)$, integrate twice by parts and obtain Eq. (7.10) with

$$r_\mu(y) = y^2 \int_0^1 e^{-y\tau^\mu}\tau^{2\mu}\,d\tau\ .$$

If $\mu > -1/2$ then the estimate (7.11) is obvious. Otherwise we transform $r_y(y)$ into the form

$$r_\mu(y) = \frac{1}{|\mu|} y^{\frac{1}{|\mu|}} \int_y^\infty e^{-\tau}\tau^{1-\frac{1}{|\mu|}}\,d\tau\ , \qquad \mu \leq -\frac{1}{2}\ ,$$

so that the estimate (7.11) with $\mu \in (-1,-\frac{1}{2})$ is also clear.

Finally, in the case $\mu = -\frac{1}{2}$ we obtain

$$r_{-1/2}(y) = 2y^2E(y)\ , \qquad \text{with}\ \ E(y) = \int_y^\infty \frac{e^{-\tau}}{\tau}\,d\tau\ .$$

Hence, rewriting

$$\begin{aligned} E(y) &= \int_1^\infty \frac{e^{-\tau}}{\tau}\,d\tau + \int_y^1 \frac{d\tau}{\tau} + \int_y^1 (e^{-\tau}-1)\,\frac{d\tau}{\tau} \\ &= -\ln y + \int_0^y (1-e^{-\tau})\,\frac{d\tau}{\tau} + C\ , \end{aligned}$$

where C is the well-known integral

$$C = \int_1^\infty \frac{e^{-\tau}}{\tau}\,d\tau + \int_0^1 (e^{-\tau}-1)\,\frac{d\tau}{\tau} = \int_0^\infty e^{-\tau}\ln\tau\,d\tau = -\gamma\ ,$$

where $\gamma = \Gamma'(1) \simeq 0.577$ is the Euler constant. Therefore, since $C<0$ and $(1-e^{-\tau}) \leq \tau$, then the term $E(y)$ is estimated by

$$E(y) \leq -\ln y + y\ .$$

Hence, we obtain the estimate (7.11) for $\mu = -1/2$ and the proof is completed. □

We can characterize now possible values of $p > 0$ for which Eq. (7.6) holds.

From the previous lemma we obtain

$$I_\mu(y) = 1 - \frac{y}{1+\mu} + O(y^{1+\varepsilon})\ , \qquad \varepsilon > 0\ ,\ y \to 0\ ,$$

provided $\mu > -1$. Therefore

$$\Gamma_\mu(e^{-x}) - e^{-x} = \theta(\mu)x + O(x^{1+\varepsilon})\ ,$$

where

$$\theta(\mu) = 1 - \frac{1}{1+\mu}\sum_{j=1}^M \alpha_j \int_{\mathbb{R}_+^j} A_j(a_1,\dots,a_j)\sum_{k=1}^j a_k\,da_1\dots da_j\ .$$

We recall that kernels $A_j(a_1,\dots,a_j)$, $j=1,\dots,M$, are assumed to be symmetric functions of their arguments. Then

$$\theta(\mu) = 1 - \frac{1}{1+\mu}\lambda(1)\ , \qquad \lambda(p) = \int_0^\infty K(a)a^p\, da\ , \tag{7.12}$$

where $K(a)$ is given in Eqs. (3.13). Recalling the definition of $\mu(p)$ in (5.7), we obtain

$$\Gamma_{\mu(p)}(e^{-x}) - e^{-x} = \theta[\mu(p)]x + O(x^{1+\varepsilon})\ , \tag{7.13}$$

where

$$\theta[\mu(p)] = \frac{\mu(p)-\mu(1)}{1+\mu(p)}\ ,. \tag{7.14}$$

It follows from Eqs. (7.13), (7.14) that the condition (7.6) is fulfilled if and only if $p \le 1$. Thus, Theorem 7.2 is proved. □

Hence all conditions of Theorem 6.2 for $y = u_0 = e^{-x}$ are fulfilled for p satisfying

$$\mu(p) > -1 \quad p \le 1 \ \text{ and } 0 < p < p_0\,,$$

where $p_0 > 0$ is a unique critical point of $\mu(p)$ (Fig.1).

However, Theorem 6.2 states that only a maximal point p_{max} of this set can lead to a non-trivial self-similar solution $w(x) \neq const$. Such point $p_{max} = 1$ does exist only in the case when $p_0 > 1$. Then, automatically

$$\mu(p) > \mu(p_0) > -\frac{1}{p_0} > -1 \qquad \text{for all } \ p > 0\,.$$

Finally, it remains to show that the iteration scheme

$$w_{n+1} = \Gamma_{\mu_*}(w_n)\,, \qquad \mu_* = \mu(1)\,, \qquad w_0 = e^{-x}\,,\ n = 0,1,\dots, \tag{7.15}$$

leads to a non-trivial solution

$$w(x) = \lim_{n\to\infty} w_n(x) \neq const.\ , \qquad x \ge 0\,. \tag{7.16}$$

According to Theorem 6.1, $w(x)$ is a continuously differentiable function on $[0,\infty)$ and

$$w(x) = w_0(x) + O(x^{1+\varepsilon})\,. \tag{7.17}$$

Therefore $w'(0) = w_0'(0) = -1$ and so $w(x) \neq const$. We shall study $w(x)$ in more detail in this Section.

By Theorem 6.1, $w \in C_1(\mathbb{R}_+)$ and satisfies the equation

$$\mu_*\, x\, w'(x) + w(x) = \Gamma(x)\ , \qquad \mu_* = \mu(1)\ . \tag{7.18}$$

The differentiability of $w(x)$ was proved in Theorem 6.1 only for $\mu \neq 0$, but the proof can be easily extended to the case $\mu = 0$ since $w_0 = e^{-x}$ in Eqs. (7.15) has a bounded and continuous derivative.

From (7.15) and (7.17) it is clear that the limit function w satisfies

$$0 \le w(x) \le 1\ , \qquad w(0) = 1\ ,\ w'(0) = -1\ ; \tag{7.19}$$

and, by considering a sequence of derivatives in Eqs. (7.15), it is easy to see that

$$w'(x) \le 0\ , \qquad |w'(x)| \le 1\ . \tag{7.20}$$

Then, estimates from Theorem 6.1 and Lemma 7.3 yield that

$$w(x) = e^{-x} + O(x^{\pi(\mu_*)})\ , \tag{7.21}$$

where

$$\pi(\mu_*) = \begin{cases} 2 & \text{for } \mu_* > -\frac{1}{2} \ , \\ 2-\varepsilon \text{ with any } \varepsilon > 0 & \text{for } \mu_* = -\frac{1}{2} \ , \\ \frac{1}{|\mu_*|} & \text{for } -1 < \mu_* < -\frac{1}{2} \ . \end{cases} \tag{7.22}$$

Hence, we collect all essential properties of $w(x)$ in the following statement.

Theorem 7.4. *The limiting function $w(x)$ constructed in* (7.16) *satisfies Eq.* (7.1) *with $\mu = \mu(1)$ and Eqs.* (7.18)*, where Γ is given in Eqs.* (3.2)*, $\mu(p)$ is defined in Eqs.* (5.6), (5.7) *and* (3.13)*. The conditions* (7.19), (7.20), (7.21) *are fulfilled for $w(x)$. Moreover*

$$1 \geq w(x) \geq e^{-x} \ , \qquad \lim_{x\to\infty} w(x) = 0 \ , \tag{7.23}$$

and there exists a generalized non-negative function $R(\tau)$, $\tau \geq 0$, such that

$$w(x) = \int_0^\infty R(\tau) e^{-\tau x}\, d\tau \ , \qquad \int_0^\infty R(\tau)\, d\tau \ = \int_0^\infty R(\tau)\tau\, d\tau \ = \ 1 \ . \tag{7.24}$$

Proof. It remains to prove (7.23) and (7.24).

First we note that Eq. (7.1) is obtained as the integral form of Eq. (6.1). Then, the identity

$$\mu\, x\, v'(x) + v(x) = \Gamma(v) + \Delta(x) \, ,$$

where

$$\Delta(x) = \mu\, x\, v'(x) + v(x) - \Gamma(v) \, , \tag{7.25}$$

is fulfilled for any function $v(x)$, and the integral form of this identity reads

$$v(x) = \Gamma_\mu(v) + \int_0^1 \Delta(x\tau^\mu)\, d\tau \ .$$

Hence, if $\Delta(x) \leq 0$ then $v \leq \Gamma_\mu(v)$ and vice-versa.

We intend to prove that $\Delta(x) \leq 0$ for $v = e^{-x}$. If so, then $w_{n+1}(x) \geq w_n(x)$ at any $x \geq 0$ in the sequence (7.15) generated by the corresponding iteration scheme with $w_0 = e^{-x}$, and obviously $w(x) \geq e^{-x}$.

Indeed, by substituting $v = e^{-x}$ in Eqs. (7.25) we obtain, for $\mu = \mu(1)$,

$$\Delta(x) = \sum_{j=1}^{M} \alpha_j \Delta_j(x) \ ,$$

where using (7.9),

$$\Delta_j(x) = \int_{\mathbb{R}_+^j} A_j(a_1, \ldots, a_j) P\Big(x, \sum_{k=1}^{j} a_k\Big)\, da_1, \ldots, da_j \ , \qquad \text{with}$$

$$P(x,s) = e^{-x}[1-(s-1)x] - e^{-sx} \leq 0 \ .$$

We note that $P(x,s) \leq 0$ for any real s and x, since $e^y \leq 1+y$ for any real y. Then $\Delta_n(x) \leq 0$, and so also $\Delta(x) \leq 0$. Hence, the inequality in (7.23) is proved.

In order to prove the limiting identity (7.23) we denote

$$w_\infty = \lim_{x\to\infty} w(x) \ .$$

Such limit exists since $w(x)$ is a monotone function. From theorem 7.4, the nice properties of $w(x)$ allow to take the limit in both sides of Eq. (7.1). Then

$$w_\infty = \sum_{n=1}^{M} \alpha_n w_\infty^n \ , \qquad \sum_{n=1}^{M} \alpha_n = 1 \ , \quad \alpha_n \geq 0 \ ,$$

and therefore we obtain

$$\sum_{n=2}^{\infty} \alpha_n w_\infty (1 - w_\infty^{n-1}) = 0 \ .$$

This equation has just two non-negative roots: $w_\infty = 0$ and $w_\infty = 1$. The root $w_\infty = 1$ is possible only if $w(x) = 1$ for all real x. Since by (7.16) this is not the case, then $w_\infty = 0$.

It remains to prove the integral representation (7.24). In order to do this we denote by U' the set of Laplace transforms of probability measures in $\mathbb{R}_+$, i.e., $u \in U'$ if there exists a generalized function $F(\tau) \geq 0$ such that

$$u(x) = \int_0^\infty F(\tau) e^{-x\tau} \, d\tau \ , \qquad \int_0^\infty F(\tau) \, d\tau \ = \ 1 \ .$$

Then $e^{-x} \in U'$ (with $F = \delta(\tau - 1)$) and it is easy to check that $\Gamma_\mu(U') \subset U'$ for any real μ. The set U' is closed with respect to uniform convergence in $\mathbb{R}_+$ (see, for example, [18]). Thus, according to Lemma 7.1, **[i]**, $w \in U'$. On the other hand, it is already known from (7.19) that $w'(0) = -1$. Hence, the corresponding function $R(\tau)$ has a unit first moment [18]. This completes the proof of Theorem 7.4. □

The integral representation (7.24) is important for the properties of the corresponding distribution functions satisfying Boltzmann-type kinetic equations. Now it is easy to return to initial variables with u_0 given in Eq. (7.4) and to describe the complete picture of the self-similar relaxation for the problem (3.7).

In order to explain how this picture is obtained we formulate below a related result to this Section.

Lemma 7.5. *Let $u(x,t)$ be a solution of the problem* (3.7) *with a non-linear operator Γ from* (3.2) *($N \geq 2$) and initial condition $u_0(x)$ satisfying* (7.5). *Then $u(x,t)$ has a non-trivial self-similar asymptotics*

$$\lim_{t \to \infty} u(x \, e^{-\mu(1)t}, t) = w(x) \ , \qquad x \geq 0 \ , \tag{7.26}$$

provided $p_0 > 1$, where $p_0 > 0$ is a critical point of $\mu(p)$ given in Eqs. (5.6), (5.7) *and* (3.11). *The function $w(x)$ is described in Theorem 7.4.*

Proof. Since all conditions of Theorem 6.2 are satisfied for $y(x) = e^{-x}$, it remains to make sure that $w(x) \neq const$. This fact, in turn, follows from Theorem 7.4 and completes the proof. □

In order to generalize this statement to more general initial conditions (7.4) with fixed $\beta = 1$ and some $p = p_1 > 0$, we just need to pass from x to x^{p_1} in Eqs. (3.2) and (3.7). Then a new spectral function reads (see Eq. (3.13))

$$\tilde{\mu}(p) = \frac{1}{p} \mu(p_1 \, p) \, ,$$

where $\mu(p)$ is the original spectral function. Hence we can apply Lemma 7.5 with $\tilde{\mu}(p)$ and then reformulate the result. This leads to a scenario described in Section 8.

8. Main results for Fourier transformed Maxwell models with multiple interactions

We consider the Cauchy problem (3.7) with a fixed non-linear operator Γ (3.2) ($M \geq 2$) and study the time evolution of $u_0(x)$ satisfying the conditions

$$\|u_0\| = 1 \ ; \quad u_0 = 1 - x^p + O(x^{p+\varepsilon}) \ , \ x \to 0 \ , \tag{8.1}$$

with some positive p and ε. Then there exists a unique classical solution $u(x,t)$ of the problem (3.7), (8.1) such that, for all $t \geq 0$,

$$\|u(\cdot,t)\| = 1 \ ; \qquad u(x,t) = 1 + O(x^p) \ , \ x \to 0 \ . \tag{8.2}$$

We explain below the simplest way to analyze this solution, in particular in the case of self-similar asymptotics.

Step 1. Consider the linearized operator L given in Eqs. (3.12)–(3.13) and construct the spectral function $\mu(p)$ given in Eqs. (5.6)–(5.7). The resulting $\mu(p)$ will be of one of four kinds described qualitatively on Fig.1.

Step 2. Find the value $p_0 > 0$ where the minimum (infimum) of $\mu(p)$ is achieved. Note that $p_0 = \infty$ just for the case described on Fig.1 (a), otherwise $0 < p_0 < \infty$. Compare p_0 with the value p from Eqs. (8.1).

If $p < p_0$ then the problem (3.7), (8.1) has a self-similar asymptotics (see below).

The above consideration shows that two different cases are possible:

(1) $p \geq p_0$ provided $p_0 < \infty$;

(2) $0 < p < p_0$, that is $\mu(p)$ is monotone decreasing for all $0 < p < p_0$.

In case (1) a behavior of $u(x,t)$ for large t may depend strictly on initial conditions. The only general conclusion that can be drawn for the initial data (8.1) with $p \geq p_0$ is the following:

$$\lim_{t\to\infty} u(xe^{-\mu t}, t) = 1 \ , \qquad x \geq 0 \ , \tag{8.3}$$

for any $\mu > \mu(p_0)$. This follows from Lemma 5.7 with $u^{(1)} = u$, $u^{(2)} = 1$ and from the fact that any such function $u_0(x)$ satisfies the condition

$$u_0 = 1 + O(x^{p_0})$$

Case (2) with $0 < p < p_0$ in Eqs. (8.1) is more interesting. In this case (assume that $p \in (0, p_0)$ is fixed) there exists a unique self-similar solution

$$u_s(x,t) = \psi(xe^{\mu(p)t}) \tag{8.4}$$

satisfying Eqs. (7.1) at $t = 0$. We again use Lemma 5.7 with $u_1 = u$ and $u_2 = u_s$ and obtain for the solution $u(x,t)$ of the problem (3.7), (8.1):

$$\lim_{t\to\infty} u(xe^{-\mu t}, t) = \begin{cases} 1 & \text{if } \ \mu > \mu(p) \\ \psi(x) & \text{if } \ \mu = \mu(p) \\ 0 & \text{if } \ \mu(p) > \mu > \mu(p+\delta) \ , \end{cases} \tag{8.5}$$

with sufficiently small $\delta > 0$. The third equality follows from the fact that

$$u_0(x) - \psi(x) = O(x^{p+\varepsilon})$$

and from the equality (see Eqs. (7.23))

$$\lim_{x\to\infty} \psi(x) = 0 \ .$$

We note that $\psi(x) = w(x^p)$, where $w(x)$ has all properties described in Theorem 7.4. The equalities (8.5) explain the exact meaning of the approximate identity,

$$u(x,t) \approx \psi(xe^{\mu(p)t}) \ , \qquad t \to \infty \ , \ xe^{\mu(p)t} = \text{const.} \ , \tag{8.6}$$

that we call self-similar asymptotics. We collect the results in the following statement.

Proposition 8.1. *The solution* $u(x,t)$ *of the problem* (3.7), (8.1), *with* Γ *given in Eqs.* (3.2), *satisfies either one of the following limiting identities:*

(1) *Eq.* (8.3) *if* $p \geq p_0$ *for the initial data* (8.1) ,

(2) *Eqs.* (8.5) *provided* $0 < p < p_0$.

The convergence in Eqs. (8.3), (8.5) *is uniform on any bounded interval* $0 \leq x \leq R$, *and*

$$u(xe^{\mu(p)t}, t) - \psi(x) = O(x^{p+\varepsilon})e^{-\beta(p,\varepsilon)t} \ , \qquad \beta(p,\varepsilon) = (p+\varepsilon)\,(\mu(p) - \mu(p+\varepsilon)\,),$$

for $0 < p < p_0$ *and* $0 < \varepsilon < p_0 - p$.

Proof. It remains to prove the last statement. It follows in both cases from the estimate (5.27) for the remainder term in Lemma (5.7). This completes the proof. □

It is interesting that our considerations are the same for both positive and negative values of $\mu(p)$. There are, however, certain differences if we want to consider the "pure" large time asymptotics, i.e., the limits (8.3), (8.5) with $\mu = 0$. Then we can conclude that

(1) $\lim_{t\to\infty} u(x,t) = 1 \qquad$ if $\ p \geq p_0$ and $\mu(p_0) < 0$, or $0 < p < p_0$ and $\mu(p) < 0$;

(2) $\lim_{t\to\infty} u(x,t) = \psi(x) \quad$ if $\quad 0 < p < p_0$ and $\mu(p) = 0$.

It seems probable that $u(x,t) \to 0$ for large t in all other cases, but our results, obtained on the basis of Lemma 5.7, are not sufficient to prove this.

Remark 8. We mention that the self-similar asymptotics becomes more transparent in logarithmic variables

$$y = \ln x \ , \quad u(x,t) = \hat{u}(y,t) \ , \quad \psi(x,t) = \hat{\psi}(y,t)$$

Then Eq. (8.6) reads

$$\hat{u}(y,t) \approx \hat{\psi}(y + \mu(p)t) \ , \qquad t \to \infty \ , \quad y + \mu(p)t = \text{const.} \ , \tag{8.7}$$

i.e., the self-similar solutions are simply nonlinear waves (note that $\psi(-\infty) = 1$, $\psi(+\infty) = 0$) propagating with constant velocities $c_p = -\mu(p)$ to the right if $c_p > 0$ or to the left if $c_p < 0$. If $c_p > 0$ then the value $u(-\infty,t) = 1$ is transported to any given point $y \in \mathbb{R}$ when $t \to \infty$. If $c_p < 0$ then the profile of the wave looks more natural for the functions $\tilde{u} = 1 - \hat{u}$, $\tilde{\psi} = 1 - \psi$.

Thus, Eq. (3.7) can be considered in some sense as the equation for nonlinear waves. The self-similar asymptotics (8.7) means a formation of the traveling wave with a universal profile for a broad class of initial conditions. It is a purely non-linear phenomenon, it is easy to see that such asymptotics cannot occur in the particular case ($M = 1$ in Eqs. (3.2)) of the linear operator Γ.

9. Distribution functions, moments and power-like tails

We have described above the general picture of the behavior of the solutions $u(x,t)$ to the problem (3.7), (8.1). On the other hand, Eq. (3.7) (in particular, its special cases in [3-11]) was obtained as the Fourier transform of the kinetic equation. Therefore we need to study in more detail the corresponding distribution functions.

We assume in this Section that $u_0(x)$ in the problem (3.7) is an isotropic characteristic function of a probability measure in $\mathbb{R}^d$, i.e.,

$$u_0(x) = \mathcal{F}[f_0] = \int_{\mathbb{R}^d} f_0(|v|)e^{-ik\cdot v}\,dv\,, \qquad k \in \mathbb{R}^d\ ,\ x = |k|^2\ , \tag{9.1}$$

where f_0 is a generalized positive function normalized such that $u_0(0) = 1$ (distribution function). Let U be a closed unit ball in the $B = C(\mathbb{R}_+)$ as defined in (3.9). Then, as was already mentioned at the end of Section 4, the set $U' \subset U$ of isotropic characteristic functions is a closed convex subset of U. Moreover, $\Gamma(U') \subset U'$ if Γ is defined in Eqs. (3.2). Hence, we can apply Lemma 4.3 and conclude that there exists a distribution function $f(v,t)$, $v \in \mathbb{R}^d$, satisfying Eq. (2.1), such that

$$u(x,t) = \mathcal{F}[f(\cdot,t)]\ , \qquad x = |k|^2\ , \tag{9.2}$$

for any $t \geq 0$.

A similar conclusion can be obtain if we assume the Laplace (instead of Fourier) transform in Eqs. (9.1). Then there exists a distribution function $f(v,t)$, $v > 0$, such that

$$u(x,t) = \mathcal{L}[f(\cdot,t)] = \int_0^\infty f(v,t)e^{-xv}\,dv\ , \qquad u(0,t) = 1\ ,\ \ x \geq 0\ ,\ \ t \geq 0\ , \tag{9.3}$$

where $u(x,t)$ is the solution of the problem (3.7) constructed in Theorem 4.2 and Lemma 4.3.

We remind the reader that the point-wise convergence $u_n(x) \to u(x)$, $x \geq 0$, where $\{u_n,\, n = 1,2,\dots\}$ and u are characteristic functions (or Laplace transforms) is sufficient for the weak convergence of the corresponding probability measures [18]. Hence, all results of pointwise convergence related to self-similar asymptotics can be easily re-formulated in corresponding terms for the related distribution functions (or, equivalently, probability measures).

The approximate equation (8.6) in terms of distribution functions (9.2) reads

$$f(|v|,t) \simeq e^{-\frac{d}{2}\mu(p)\,t}F_p(|v|e^{-\frac{1}{2}\mu(p)\,t})\ , \quad t \to \infty\ ,\ |v|e^{-\frac{1}{2}\mu(p)\,t} = \text{const.}\ , \tag{9.4}$$

where $F_p(|v|)$ is a distribution function such that for $x = |k|^2$

$$\psi_p(x) = \mathcal{F}[F_p]\ \ , \tag{9.5}$$

with ψ_p given in Eq. (8.4) (the notation ψ_p is used in order to stress that ψ defined in (8.4), depends on p). The factor $1/2$ in Eqs. (9.4) is due to the notation $x = |k|^2$. Similarly, for the Laplace transform (9.3), we obtain

$$f(v,t) \simeq e^{-\mu(p)t}\Phi_p(ve^{-\mu(p)t})\ , \qquad t \to \infty\ ,\ ve^{-\mu(p)t} = \text{const.}\ , \tag{9.6}$$

where

$$\psi_p(x) = \mathcal{L}[\Phi_p]\ . \tag{9.7}$$

In the space of distributions, the approximate relation $\simeq$ is weak in the sense of distributions, i.e. the classical approximation concept of real valued expression obtained after integrating by test functions.

The positivity and some other properties of $F_p(|v|)$ follow from the fact that $\psi_p(x) = w_p(x^p)$, where $w_p(x)$ satisfies Theorem 7.4. Hence

$$\psi_p(x) = \int_0^\infty R_p(\tau) e^{-\tau x^p}\, d\tau \ , \qquad \int_0^\infty d\tau\, R_p(\tau)\, d\tau \ = \int_0^\infty R_p(\tau)\tau\, d\tau = 1 \ , \tag{9.8}$$

where $R_p(\tau)$, $\tau \geq 0$, is a non-negative generalized function (of course, both ψ_p and R_p depend on p).

We stress here that if $u_0(x)$ is a characteristic function given in Eq. (9.1), and condition (8.1) is fulfilled, then $p \leq 1$ in Eqs. (8.1). In addition, the case $p > 1$ is impossible for the non-negative initial distribution f_0 in Eqs. (9.1) since

$$u'(0) = -C_d \int_{\mathbb{R}^d} f_0(|v|)|v|^2\, dv\,, \tag{9.9}$$

where $C_d > 0$ is a constant factor that depends only on the space dimension for the problem.

Hence, the self-similar asymptotics (9.4) for any initial data $f_0 \geq 0$ occurs if $p_0 > 1$ (see Step 2 at the beginning of Section 8). Otherwise it occurs for $p \in (0, p_0) \subset (0, 1)$. Therefore, for any spectral function $\mu(p)$ (Fig.1), the approximate relation (9.4) holds for sufficiently small $0 < p \leq 1$. It follows from Eq. (9.9) that, if (8.1) holds, then

$$m_2 = \int_{\mathbb{R}^d} f_0(|v|)|v|^2\, dv \ < \infty$$

for $p = 1$, and $m_2 = \infty$ for $p < 1$.

Similar conclusions can be made for the Laplace transforms (9.3) since in that case

$$u'(0,t) = -\int_0^\infty f(v,t) v\, dv\,,$$

therefore the first moment of f plays the same role as the second moment in case of Fourier transforms.

The positivity of $F(|v|)$ in Eqs. (9.5) and (9.7) follows from the integral representation (9.8) with $p \leq 1$. It is well-known that

$$\mathcal{F}^{-1}(e^{-|k|^{2p}}) > 0 \ , \qquad \mathcal{L}^{-1}(e^{-x^{2p}}) > 0$$

for any $0 < p \leq 1$ (the so-called stable distributions [18]). Thus, Eqs. (9.8) explains the connection of the self-similar solutions of generalized Maxwell models with stable distributions. We can use standard formulas for the inverse Fourier (Laplace) transforms and denote ($d = 1, 2, \ldots$ is fixed)

$$\begin{aligned} M_p(|v|) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-|k|^{2p} + ik\cdot v}\, dk \ , \\ N_p(v) &= \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{-x^p + xv}\, dx\,, \qquad 0 < p \leq 1 \ . \end{aligned} \tag{9.10}$$

Then the self-similar solutions F_p and Φ_p (distribution functions) given in the right hand sides of Eqs. (9.6) and (9.8) respectively, satisfy

$$\begin{aligned} F_p(|v|) &= \int_0^\infty R_p(\tau)\tau^{-\frac{d}{2p}} M_p(|v|\tau^{-\frac{1}{2p}}), d\tau\,, \\ \Phi_p(v) &= \int_0^\infty R_p(\tau)\tau^{-\frac{1}{p}} N_p(v\tau^{-\frac{1}{p}})\, d\tau \ , \qquad v \geq 0\ ,\ 0 < p \leq 1\ . \end{aligned} \tag{9.11}$$

That is, they admit an integral representation through stable distributions. Note that $M_1(|v|)$ is the standard Maxwellian in $\mathbb{R}^d$. The functions $N_p(v)$ (9.10) are studied in detail in the literature [18, 21].

Thus, for given $0 < p \leq 1$, the kernel $R_p(\tau)$, $\tau \geq 0$, is the only unknown function that is needed to describe the distribution functions $F_p(|v|)$ and $\Phi_p(v)$. Therefore we study $R_p(\tau)$ and its s-moments in more detail.

It was already noted in Section 7 that the general problem (7.1), (7.3), with given $0 < p < p_0$, can be reduced to the case $p = 1$ by the transformation of variables $\tilde{x} = x^p$. We assume therefore that such transformation is already made and consider the case $p = 1$. Then the equation for $R(\tau) = R_1(\tau)$ can be obtained (see Eqs. (7.24)) by applying the inverse Laplace transform to Eq. (7.18). Then we obtain, with $\mu_* = \mu(1)$,

$$- \mu(1)\frac{\partial}{\partial\tau}\tau R(\tau) + R(\tau) = Z(R) = \mathcal{L}^{-1}[\Gamma(w)] \ , \tag{9.12}$$

where (see Eqs. (3.2))

$$Z(R) = \sum_{j=1}^{M} \alpha_j Z_j(R) \ , \qquad \sum_{j=1}^{M} \alpha_j = 1 \ , \ \alpha_j \geq 0 \ ,$$

$$Z_j(R) = \int_{\mathbb{R}_+^j} \frac{A_j(a_1, \dots, a_j)}{a_1 a_2 \dots a_j} \prod_{k=1}^{j}{}^{*} R\Big(\frac{\tau}{a_k}\Big)\, da_1, \dots, da_j \ ,$$

$$\prod_{k=1}^{j}{}^{*} R_k(\tau) = R_1 * R_2 * \dots * R_j \ , \qquad R_1 * R_2 = \int_0^\tau R_1(\tau')R_2(\tau - \tau')\, d\tau' \ .$$

Let us denote the s-moment of R

$$m_s = \int_0^\infty R(\tau)\tau^s\, d\tau \ , \qquad s \geq 0 \ , \tag{9.13}$$

then, in order to obtain an identity for these m_s moments, we multiply Eq. (9.12) by τ^s and obtain, after integration in $\tau_k = {a_k}^{-1}\tau$, the following identity

$$\begin{aligned} &(\mu(1)s + 1)m_s = \\ &= \sum_{j=1}^{M} \alpha_j \int_{\mathbb{R}_+^j} A_j(a_1 \dots a_j)\Bigg(\int_{\mathbb{R}_+^j}\Big(\sum_{k=1}^{j} a_k\tau_k\Big)^s \prod_{k=1}^{j} R(\tau_k)d\tau_1 \dots d\tau_j\Bigg)da_1 \dots da_j. \end{aligned} \tag{9.14}$$

Next, recalling the notations from (3.13), (5.6), that is $\mu(p) = (\lambda(p) - 1)\, p^{-1}$, with (5.7)

$$\lambda(p) = \int_0^\infty K(a)a^p\, da \ = \sum_{j=1}^{M} \alpha_j \int_{\mathbb{R}_+^j} A_j(a_1, \dots, a_j)\Big(\sum_{k=1}^{j} a_k^p\Big)\, da_1 \dots da_j \ ,$$

then, subtracting $\lambda(p)\, m_s$ from both sides of identity (9.14), the s-moment equation associated to Eq. (9.12) can be written in the form

$$(s\mu(1) - \lambda(s) + 1)m_s = s\,(\mu(1) - \mu(p))\, m_s = \sum_{j=2}^{M} \alpha_j\, I_j(s) \ , \tag{9.15}$$

where

$$I_j(s) = \int_{\mathbb{R}^j_+} A(a_1, \dots, a_j) \left(\int_{\mathbb{R}^j_+} g_j^{(s)}(a_1\tau_1, \dots, a_j\tau_j) \prod_{k=1}^{j} R(\tau_k)\, d\tau_1 \dots d\tau_j \right) da_1 \dots da_j \,,$$
$$g_j^{(s)}(y_1, \dots, y_j) = \left(\sum_{k=1}^{j} y_k \right)^s - \sum_{k=1}^{j} y_k^s \,, \qquad j = 1, \dots, M. \tag{9.16}$$

We note that $g_1^{(s)} = 0$ for any $s \geq 0$ and that $m_0 = m_1 = 1$ (see Eqs. (9.8)).

Our aim is to study the moments m_s, $s > 1$, on the basis of Eq. (9.15). The approach is related to the one used in [24] for a simplified version of Eq. (9.15) with $M = 2$. The main results are formulated below in terms of the spectral function $\mu(p)$ (see Fig.1) under the assumption that $p_0 > 1$.

Proposition 9.1.

[i] *If the equation $\mu(s) = \mu(1)$ has the only solution $s = 1$, then $m_s < \infty$ for any $s > 0$.*
[ii] *If this equation has two solutions $s = 1$ and $s = s_* > 1$, then $m_s < \infty$ for $s < s_*$ and $m_s = \infty$ for $s > s_*$.*
[iii] *$m_{s_*} < \infty$ only if $I_j(s_*) = 0$ in Eq. (9.15) for all $j = 2, \dots, M$.*

Proof. The proof is based on the following inequality that controls, from above and below, the right hand side of Eq. (9.15)

$$0 \leq \sum_{j=2}^{M} \alpha_j I_j(s) \leq C_M(s) m_1 m_{s-1} \tag{9.17}$$

with $s > 1$ and some positive constant $C_M(s)$ to be determined. Then, as (9.17) holds, combining with identity (9.15), we obtain, for $m_s \geq 0$

$$m_s \leq \frac{C_M(s)}{s[\mu(1) - \mu(s)]}\; m_{s-1} \,, \qquad \text{with } m_0 = m_1 = 1 \,.$$

In the case [i] $\mu(1) > \mu(s)$ (see Fig.1) for all $s > 1$. The same is true in the case [ii] for $s < s_*$. It is clear from Eq. (9.13) that $m_s > 0$ since $R(\tau) \geq 0$ and $m_1 = 1$. This means that the inequality (9.17) cannot be satisfied for $s > s_*$, therefore moments of orders $s > s_*$ cannot exist. The statement [iii] follows directly from Eq. (9.15). □

We note that Proposition 9.1 relates to moments (9.13) of $R(\tau) = R_1(\tau)$ in Eqs. (9.11). A similar statement can be formulated for moments of $R_p(\tau)$, with any $p > 0$ (by using the change of variables $\tilde{x} = x^p$), but we shall need below just the case $p = 1$ as described in Proposition 9.1.

Hence, it remains to prove the inequality (9.17). The proof is based on the following elementary inequality.

Lemma 9.2. *In the notation of Eqs. (9.16),*

$$0 \leq g_j^{(s)}(y_1, \dots, y_j) \leq 2^{s-1} s \sum_{k=1}^{j-1} \gamma_k\, \psi(y_{k+1}, Y_k) \,, \qquad s > 1 \,, \tag{9.18}$$

where

$$\psi(y_1, y_2) = y_1^{s-1} y_2 + y_2^{s-1} y_1 \ , \qquad Y_k = \max(y_1, \dots, y_k) \ ,$$

$$\gamma_k = \max(k, k^{s-1}) \ , \qquad k = 1, \dots, j \ ; \ j = 2, 3, \dots$$

Proof. If $j = 2$, we assume without loss of generality that $y_1 \leq y_2$ and reduce the upper estimate problem to the inequality

$$\Delta(x) = (1+x)^s - 1 - x^s - 2^{s-1} s (x + x^{s-1}) \leq 0 \ , \qquad x = \frac{y_1}{y_2} \leq 1 \ .$$

Its proof is obvious since, $\Delta(0) = 0$, $\Delta'(x) \leq 0$. The lower estimate in Eqs. (9.18) is similarly reduced for $j = 2$ to the inequality

$$g(\theta) = 1 - \theta^s - (1-\theta)^s \geq 0 \ , \qquad \theta = \frac{x}{x+y} \leq 1 \ .$$

Then, its proof follows from the fact that $g(0) = g(1) = 0$, $g''(\theta) \leq 0$.

We proceed by induction. It is easy to see that

$$g_{j+1}^{(s)}(y_1, \dots, y_{j+1}) = g_j^{(s)}(y_1, \dots, y_j) + g_2^{(s)}\left(y_{j+1}, \sum_{k=1}^{j} y_k \right) \ , \qquad j = 3, \dots \ .$$

Then the lower estimate (9.18) becomes clear for any $j \geq 2$. By applying the upper estimate (9.18) for $g_2(s)$ we obtain

$$g_{j+1}^{(s)}(y_1, \dots, y_{j+1}) \leq g_j^{(s)} + 2^{s-1} s \, \psi\left(y_{j+1}, \sum_{k=1}^{j} y_k \right)$$

and note that $\psi(x, y)$ is an increasing function of y. Clearly,

$$\sum_{k=1}^{j} y_k \leq j Y_j \ , \qquad Y_j = \max(y_1, \dots, y_j) \ ,$$

and therefore

$$g_{j+1}^{(s)}(y_1, \dots, y_{j+1}) \leq g_j^{(s)}(y_1, \dots, y_j) + 2^{s-1} s \gamma_j \psi(y_{j+1}, Y_j) \ .$$

This is precisely what is needed to get the upper estimate (9.18) by induction. This completes the proof of Lemma 9.2. □

In order to complete the proof of Proposition 9.1, we just need to prove the inequality (9.17). First substitute the estimates (9.18) applied to $g_j^{(s)}(a_1\tau_1, \dots, a_j\tau_j)$ into the right hand side of Eq. (9.15). Then the lower estimate (9.17) becomes clear since the functions $g_j^{(s)}$ are non negative. The upper estimate (9.17) follows after we note that

$$\max(a_1\tau_1, \dots, a_j\tau_j) \leq \bar{a}_j \max(\tau_1, \dots, \tau_j) \ , \quad \bar{a}_j = \max(a_1, \dots, a_j) \ , \ j = 1, \dots, M \ ;$$

and we can estimate (9.15) as follows. First we show that, for $s > 1$,

$$g_j^{(s)}(a_1\tau_1, \dots, a_j\tau_j) \leq s \, 2^{s-1} \, \bar{a}_j^s \sum_{k=1}^{j-1} \gamma_k \, \psi(\tau_{k+1}, \max(\tau_1, \dots, \tau_k)) \ ,$$

as it follows from Eqs. (9.18). Then the internal integral in Eqs. (9.16) is controlled by a sum of integrals having the following structure

$$\int_{\mathbb{R}^j_+} \prod_{i=1}^{j} R(\tau_i)\, \psi(\tau_{k+1}, \max(\tau_1, \dots, \tau_k)\,)\; d\tau_1 \dots d\tau_j$$

$$= \; k! \int_{0 \le \tau_1 \le \dots \le \tau_k < \infty} [m_1 \tau_k^{s-1} + m_{s-1}\tau_k] \big(\prod_{i=1}^{k} R(\tau_i) \big)\; d\tau_1 \dots d\tau_k$$

$$\le \; 2k!\, m_1\, m_{s-1}\,, \qquad\qquad k = 1, \dots, j-1\,,$$

in the notation of Eqs. (9.13), with $m_0 = 1$. Hence, we obtain from (9.16)

$$I_j(s) \;\le\; m_1\, m_{s-1}\; 2^s\, s \left(\sum_{k=1}^{j-1} k!\, \gamma_k \right) \int_{\mathbb{R}^j_+} A(a_1, \dots, a_j)\, \bar{a}_j^s\; da_1 \dots da_j, \qquad j = 2, \dots, M,$$

and the upper estimate (9.17) follows from conditions (3.2)–(3.5). Thus, the proof of Proposition 9.1 is completed. □

Now we can draw some conclusions concerning the moments of the distribution functions (9.11). We denote

$$m_s(\Phi_p) = \int_0^\infty \Phi_p(v) v^s\, dv\ , \qquad m_s(R_p) = \int_0^\infty R_p(\tau)\tau^s\, d\tau\ ,$$

$$m_{2s}(F_p) = \int_{\mathbb{R}^d} F_p(|v|)|v|^{2s}\, dv\ , \qquad s > 0\ ,\ 0 < p \le 1\ ,$$

and use similar notations for $N_p(v)$ and $M_p(|v|)$ in Eqs. (9.11). Then, by formal integration of Eqs. (9.11), we obtain

$$m_s(\Phi_p) = m_s(N_p) m_{s/p}(R_p)$$

$$m_{2s}(F_p) = m_{2s}(M_p) m_{s/p}(R_p)\ ,$$

where M_p and N_p are given in Eqs. (9.10).

First we consider the case $0 < p < 1$. It follows from general properties of stable distributions that the moments $m_s(N_p)$ and $m_{2s}(M_p)$, $0 < p < 1$, are finite if and only if $s < p$ (see [18]). On the other hand, $m_0(R_p) = m_1(R_p) = 1$, therefore $m_s(R_p)$ is finite for any $0 \le s \le 1$. Hence, in this case $m_s(\Phi_p)$ and $m_{2s}(F_p)$ are finite only for $s < p$.

The remaining case $p = 1$ is less trivial since *all* moments of functions

$$M_1(|v|) = (4\pi)^{-d/2} \exp\Big[-\frac{|v|^2}{4} \Big]\ , \qquad v \in \mathbb{R}^d\ ;$$

$$N_1(v) = \delta(v-1)\ , \qquad v \in \mathbb{R}_+\ ,$$

are finite. Therefore everything depends on moments of R_1 in Eqs. (9.12) with $p = 1$. It remains to apply Proposition 9.1. Hence, the following statement is proved for the moments of the distribution functions (9.4), (9.6).

Proposition 9.3.

[i] *If* $0 < p < 1$*, then* $m_{2s}(F_p)$ *and* $m_s(\Phi_p)$ *are finite if and only if* $0 < s < p$*.*
[ii] *If* $p = 1$*, then Proposition 9.1 holds for* $m_s = m_{2s}(F_1)$ *and for* $m_s = m_s(\Phi_1)$*.*

Remark 9. Proposition 9.3 can be interpreted in other words: the distribution functions $F_p(|v|)$ and $\Phi_p(v)$, $0 < p \le 1$, can have finite moments of all orders in the only case when two conditions are satisfied

(1) $p = 1$, and
(2) the equation $\mu(s) = \mu(1)$ (see Fig.1) has the unique solution $s = 1$.

In all other cases, the maximal order s of finite moments $m_{2s}(F_p)$ and $m_s(\Phi_p)$ is bounded.

This fact means, roughly speaking, that the distribution functions F_p and Φ_p have power-like tails for large values of their arguments.

For the sake of reader's convenience, we end this section with some general comments on distribution functions related to generalized Maxwell models. Coming back to solutions $f(|v|, t)$ of Eqs. (2.2), (2.4) discussed at the beginning of this Section, we observe that their self-similar asymptotics is described by Eqs. (9.4) (Eqs. (9.6) in one-dimensional case), where the parameter $0 < p \le 1$ is related to initial data (9.1) in the following way

$$u_0(x) = 1 - \alpha x^p + O(x^{p+\varepsilon}), \qquad x \to 0, \qquad 0 < p \le 1. \tag{9.19}$$

We proved that the sufficient condition for such asymptotics is the inequality $p < p_0$, where p_0 is a minimum point of corresponding spectral function $\mu(p)$ defined in (5.7). The case $p < 1$ in (9.19) corresponds to infinite initial energy (the second moment of $f_0(|v|)$), for which the corresponding self-similar solution $F_p(|v|)$ from (9.11) has also infinite energy.

The most important case $p = 1$ (finite initial energy) leads to a self-similar solution $F_p(|v|)$, which has a power-like tail provided there exists a second root $s > 1$ of the equation $\mu(s) = \mu(1)$. The existence of such root depends on specific form of $\mu(p)$ (see Fig.1). For example, it does not exist in the case (a) and always exists in the cases (c) and (d).

A dissipative Boltzmann equation corresponds to the case (b) with $\mu(1) < 0$, therefore the second root exists and we have a power-like tail. The classical (elastic) Boltzmann Equation also relates to the case (b), however with $\mu(1) = 0$ (energy conservation). Therefore the second root does not exist in this case and the "self-similar" solution (usual stationary Maxwellian) has bounded moments of all orders.

More detailed information on self-similar solutions is contained in Eqs (9.11) and in Proposition 9.3 in this Section. The rate of convergence in Eqs. (9.4), (9.6) is characterized in terms of uniform convergence of the corresponding characteristic functions in Proposition 8.1 in the previous Section.

10. Applications

The main feature of multi-particle stochastic models discussed in Section2 was independence of two scales (in time and in phase variable respectively). Maxwell molecules, for which the frequency of pair collisions is independent of velocities of colliding particles, are the classical example of such system. Perhaps it is not very easy to find another example in physics, but at least one example of such system from "real life" is obvious. Imagine that the phase variable are *goods*, such as moneys or wealth, whereas the particles are *players* participating in various economical "games." Then, a realistic assumption is that a scaling transformation of the phase variable, such as a change of currency in this case, does not influence a behavior of players.

A phase state of $j \geq 1$ players (identical particles) is characterized by a vector $V_j = (v_1, \dots, v_j) \in \mathbb{R}^j_+$ of their individual capitals (velocities). The game of these j partners is understood as a random linear transformation (j-particle collision)

$$V'_j = \hat{G} V_j \ , \qquad V_j = (v_1, \dots, v_j) \ , \quad V'_j = (v'_1, \dots, v'_j) \ , \tag{10.1}$$

where $\hat{G}$ is a square $j \times j$ matrix with non-negative random elements, V_j and V'_j are understood as vector-columns. The matrix $\hat{G}$ must satisfy certain conditions to ensure that the model does not depend on numeration of identical particles. The simplest class of such matrices is a two-parameter family

$$\hat{G} = \{g_{ik} \ , \ i, k = 1, \dots, j\} \ , \qquad g_{ik} = \begin{cases} a & \text{if } i = k \\ b & \text{otherwise,} \end{cases} \tag{10.2}$$

with arbitrary non-negative a and b. The parameters (a, b) can be fixed or randomly distributed in $\mathbb{R}^2_+$ with some probability density $B_j(a, b)$. The corresponding transformation

$$v'_k = a v_k + b \sum_{i=1, (i \neq k)}^{j} v_i \ , \qquad k = 1, \dots, j \ , \tag{10.3}$$

can be easily interpreted in terms of economics, the interpretation is discussed below.

In order to derive a kinetic equation we can consider the stochastic model from Section2 with *large* phase vector $V_N(t) = (v_1(t), \dots, v_N(t)) \in \mathbb{R}^N_+$ (in terminology of "particles" and "collisions"). We use the standard approach of Kac [20] and assume that $V_N(t)$ undergoes random jumps caused by collisions. Intervals between two successive jumps have the Poisson distribution with the average $\Delta t_N = \Theta/N$, $\Theta =$ const.. Then we introduce N-particle distribution function $F(V_N, t)$ and consider a weak form of the Kac Master equation [20]

$$\frac{d}{dt} \int_{\mathbb{R}^N_+} F(V_N, t) \Phi(V_N) \, dV_N \ = \frac{N}{\Theta} \int_{\mathbb{R}^N_+} F(V_N, t) \langle \Phi(V'_N) - \Phi(V_N) \rangle \ dV_N \, , \tag{10.4}$$

where $\Phi(V_N)$ is a nice test function, the average $\langle \cdot \rangle$ is taken over all possible jumps.

We assume that

1) jumps are caused by interactions of $1 \leq j \leq M < N$ particles (the case $j = 1$ is understood as an interaction with background) and

2) relative probabilities of interactions which involve $1, 2, \dots, M$ particles are given respectively by non-negative real numbers $\beta_1, \beta_2, \dots, \beta_M$ such that

$$\beta_1 + \beta_2 + \dots + \beta_M = 1 \ . \tag{10.5}$$

Then

$$\langle \Phi(V'_N) \rangle = \sum_{j=1}^{M} \beta_j \langle \Phi(V'_N) \rangle_j \ , \tag{10.6}$$

where $\langle \dots \rangle_j$ is the average over all j-particle interactions, with $1 \leq j \leq M$. Assuming that the interactions are described by Eqs. (10.3) with various random parameters a and b (if

$j \geq 2$), we obtain for $j = 1, 2$

$$\langle \Phi(V_N') \rangle_1 = \int_0^\infty B_1(a) \frac{1}{N} \sum_{k=1}^{N} \Phi(v_1, \ldots, av_k, \ldots, v_N)\, da \ ,$$

$$\langle \Phi(V_N') \rangle_2 = \int_0^\infty \int_0^\infty B_2(a,b) \times \\ \times \frac{2}{N(N-1)} \sum_{1 \leq k < l \leq N} \Phi(v_1, \ldots, av_k + bv_l, \ldots, bv_k + av_l, \ldots, v_N)\, db\, da \ ,$$

$$\int_0^\infty B_1(a)\, da = \int_0^\infty \int_0^\infty B_2(a,b)\, db\, da = 1 \ .$$

Those terms with $j \geq 3$, where it is assumed that a and b in Eqs. (9.3) are distributed with probability density $B_j(a,b)$ in $\mathbb{R}_+^2$, are defined in similar way. Then we introduce a one-particle distribution function (setting $v_1 = v$)

$$f(v,t) = \int_0^\infty \int_0^\infty F(V_N, t)\ dv_2, \ldots dv_N \ , \qquad \int_0^\infty f(v,t)\, dv \ = 1, \tag{10.7}$$

and its Laplace transform

$$u(x,t) = \int_0^\infty f(v,t) e^{-xv}\, dv \ , \qquad x \geq 0 \ . \tag{10.8}$$

It remains to substitute $\Phi(V_N) = e^{-xv_1}$ into Eqs. (10.4), (10.6) and to assume formally that ("molecular chaos")

$$F(V_N, t) \approx \prod_{k=1}^{N} f(v_k, t) \ , \qquad N \to \infty \ . \tag{10.9}$$

This assumption is postulated below.

Then we obtain in the limit $N \to \infty$

$$\frac{\partial}{\partial t} u(x,t) + u(x,t) = \frac{1}{\Theta} \Big\{ \beta_1 \int_0^\infty B_1(a) u(ax,t)\ da + \\ + \sum_{j=2}^{M} j\beta_j \int_0^\infty \int_0^\infty B_2(a,b) u(ax) u^{j-1}(bx,t)\ db\, da \Big\} \ . \tag{10.10}$$

We assume without loss of generality that

$$\Theta = \sum_{j=1}^{M} j\beta_j \ . \tag{10.11}$$

Then Eq. (10.10) is a particular case of Eqs. (3.1)–(3.4) with

$$\alpha_j = \frac{j\beta_j}{\Theta} \ , \quad 1 \leq j \leq M \ ; \qquad A_1(a) = B_1(a) \ , \quad A_2(a_1, a_2) = B_2(a_1, a_2) \\ A_j(a_1, a_2, \ldots, a_j) = B_2(a_1, a_2) \prod_{k=3}^{j} \delta(a_k - a_2) \ , \quad 3 \leq j \leq M \ . \tag{10.12}$$

Hence, all our results can be applied to this system. The spectral function reads

$$\mu(p) = \frac{\lambda(p)-1}{p} \,, \qquad \lambda(p) = \int_0^\infty \left\{ \alpha_1 a^p B_1(a) + \sum_{j=2}^{M} \alpha_j \int_0^\infty B_2(a,b)[a^p + (j-1)b^p]\, db \right\} da \,, \tag{10.13}$$

and its knowledge is sufficient for our goals. A typical example of application in terms of particles-players and velocities-goods is discussed below. Assume a typical initial condition

$$f_{|t=0} = \delta(v-1) \quad \Longleftrightarrow \quad u_{|t=0} = e^{-x} \,, \tag{10.14}$$

in the notation of Eq. (10.8). Thus, each player has one unit of currency at $t = 0$ (full equality).

The game of $j \geq 1$ players is played in three steps:

(1) the participants collect all their goods and form a sum $\Sigma = v_1 + v_2 + \cdots + v_j$;
(2) the sum Σ is multiplied by a random number $\theta \geq 0$ distributed with given probability density $q(\theta)$ in $\mathbb{R}_+$;
(3) the resulting sum $\Sigma' = \theta\Sigma = v'_1 + \cdots + v'_j$ is given back to the players in accordance with the following rule: a part of it $\Sigma'_1 = (1-\gamma)\Sigma'$ is divided proportionally to initial contributions, whereas the rest $\Sigma'_2 = \gamma\Sigma'$ is divided among all players equally, with some fixed or random parameter $0 \leq \gamma \leq 1$.

Simple algebra shows that this "game" is equivalent to Eq. (10.3) with

$$a = \theta\left[1 - \frac{j-1}{j}\gamma\right] \,, \qquad b = \frac{\gamma\theta}{j} \,, \qquad j \geq 1 \qquad 0 \leq \gamma \leq 1\,. \tag{10.15}$$

The meaning of the parameter θ is clear: something was bought (or produced) for the value Σ and then sold for $\Sigma' = \theta\Sigma$ (with gain if $\theta > 1$ or loss if $\theta < 1$).

An interesting example arises from assuming the following probability density for θ:

$$q(\theta) = q\delta(\theta) + (1-q)\delta(\theta - s) \,, \qquad s > 1 \tag{10.16}$$

where $0 \leq q \leq 1$ characterizes a *risk of complete loss*.

The parameter γ can be interpreted as a fixed control parameter to give more chances to losers, and hence γ is introduced in the game in order to prevent large differences between *affluent* and *destitute* in the future. In particular, the theory we present here can explain exactly how these differences depend on the parameter γ.

In order to clarify this point we make one more simplification by assuming that only games with some fixed number $j \geq 2$ of players are allowed. Then Eq. (10.10) for such model reads

$$\begin{aligned} u_t + u &= \int_0^\infty q(\theta) u\left(\theta\left(1 - \frac{j-1}{j}\gamma\right)x\right) u^{j-1}\left(\frac{\gamma\theta}{j}x\right) d\theta = \\ &= q u^j(0) + (1-q) u\left[s\left(1 - \frac{j-1}{j}\gamma\right)x\right] u^{j-1}\left(\frac{\gamma s}{j}x\right) \,, \qquad u_{|t=0} = e^{-x} \,, \end{aligned} \tag{10.17}$$

provided Eqs. (10.14), (10.16) hold. The corresponding spectral function (10.13) reads

$$\mu(p) = \frac{1}{p}\left\{(1-q)s^p\left[\left(1 - \frac{j-1}{j}\gamma\right)^p + (j-1)\left(\frac{\gamma}{j}\right)^p\right] - 1\right\} \,. \tag{10.18}$$

Then, if we assume that $j(1-q) > 1$ (which it is always the case if $q < 1/2$), $\mu(p) > 0$ for small $p > 0$ and we have a typical example of functions shown on Fig.1. In this case

$$\begin{aligned} &\mu(1) = s(1-q) - 1\ , \qquad s > 1\ ; \\ &\mu'(1) = 1 + (1-q)s\left[\log s - 1 + \psi_j\left(\frac{j-1}{j}\gamma\right)\right]\ , \\ &\psi_j(x) = x\log\frac{x}{j-1} + (1-x)\log(1-x)\ . \end{aligned} \tag{10.19}$$

Next, we apply to Eq. (10.18) our criteria for self-similar asymptotics.

According to the analysis of the spectral function $\mu(p)$, for $p = 1$, presented in the previous sections, the condition $\mu'(1) < 0$ would mean that the large time asymptotics of $u(x,t)$ is described by the self-similar solution $w[x\exp(\mu(1)t)]$ studied in previous sections. It is easy to verify that $\mu'(1)$ is monotone decreasing function of $\gamma \in [0,1]$ and

$$\mu'(1) = 1 + (1-q)s(\log s - 1 - \log j) \quad \text{if} \quad \gamma = 1\ .$$

Then the inequality

$$j > s\exp\left[\frac{1}{(1-q)s} - 1\right]$$

guarantees that there exists a real number $\gamma_* > 0$ such that we do have self-similar asymptotics for all $\gamma_* < \gamma \le 1$. This number is a unique root of equality

$$\psi_j\left(\frac{j-1}{j}\gamma_*\right) + 1 + (1-q)s(\log s - 1) = 0$$

in the notation of Eq. (10.19).

There is also the second important number γ_{**}, which indicates an appearance of *power-like tails* for the probability distribution function $f(v,t)$ defined in (10.7) whose Laplace transform satisfies Eqs. (10.17). We note that “long” tails (i.e. values of $f(v,t)$ that decay no faster than negative powers v, for large v) correspond to relatively big differences between *affluent* and *destitute*.

We can expect existence of such tails for $\gamma_* < \gamma < \gamma_{**} < 1$, provided they disappear for γ close to $\gamma_{\max} = 1$. We already know that power-like tails do exist if the equation $\mu(p) = \mu(1)$ has the second root $p > 1$. Therefore, conditions of appearance of the second root can be easily understood from comparison of Fig.1 with Eq. (10.18). It is clear that

$$\mu(p) \xrightarrow[p\to\infty]{} \begin{cases} \infty & \text{if } s\left(1 - \frac{j-1}{j}\gamma\right) > 1 \\ 0 & \text{if } s\left(1 - \frac{j-1}{j}\gamma\right) \le 1. \end{cases}$$

Hence, the second root exists (see Fig.1c,d) if

$$\gamma_* < \gamma < \gamma_{**} = \frac{j}{j-1}\left(1 - \frac{1}{s}\right)\ , \qquad s > 1\ ,$$

independently of all other parameters. If $\gamma > \gamma_{**}$ then the second root still exists, provided $\mu(1) < 0$ (loss of total wealth), since $\mu(p)$ has a profile shown on Fig.1b. Note that $\gamma_{**} \le 1$ only if $s \le j$, i.e., the second root always exists if $s > j$.

Finally we note that Eq. (10.17) can be simplified in the limit $j = \infty$, i.e. for games with a very large number os participants. Assuming that

$$u(x,t) = 1 - \alpha(t)x + \mathcal{O}(x^{1+\varepsilon}), \qquad x \to 0, \quad t > 0\,,$$

we consider Eq. (10.17), where we formally obtain

$$u^j(0) = 1 \qquad u\left[s\left(1 - \frac{j-1}{j}\gamma\right)x\right] \underset{j\to\infty}{\longrightarrow} u\left[(1-\gamma)\,sx\right]$$

$$\text{and} \qquad u^{j-1}\left(\frac{\gamma s}{j}x\right) \approx \left(1 - \frac{\alpha\gamma s}{j}x\right)^{j-1} \underset{j\to\infty}{\longrightarrow} \exp(-\alpha\,\gamma s x)$$

where

$$\alpha = \alpha(t) = -u_x(0,t). \tag{10.20}$$

Hence, the formal limit of Eq. (10.18) at $j = \infty$ reads

$$u_t + u = q + (1-q)u[(1-\gamma)sx]e^{-\alpha(t)\gamma s x}, \tag{10.21}$$

with $\alpha(t)$ from Eq. (10.20). As always, we assume that $\alpha(0) = 1$ and find $\alpha(t)$ by considering Eq. (10.21) for small x. Then we obtain

$$\alpha_t + \alpha - (1-q)\left[(1-\gamma)\,s + \gamma s\right]\alpha = 0,$$

and therefore

$$\alpha(t) = \exp\{[(1-q)s - 1]\,t\}\,.$$

Thus, Eq. (10.21) becomes linear for a class of initial data satisfying the condition $\alpha(0) = 1$. For brevity we omit simple conclusions about this equation.

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