Arch. Rational Mech. Anal. 222 (2016) 317–342
Digital Object Identifier (DOI) 10.1007/s00205-016-1002-2

Global Weak Solutions for Kolmogorov–Vicsek Type Equations with Orientational Interactions

Irene M. Gamba & Moon-Jin Kang

Communicated by D. Kinderlehrer

Abstract

We study the global existence and uniqueness of weak solutions to kinetic Kolmogorov–Vicsek models that can be considered as non-local, non-linear, Fokker–Planck type equations describing the dynamics of individuals with orientational interactions. This model is derived from the discrete Couzin–Vicsek algorithm as mean-field limit (Bolley et al., Appl Math Lett, 25:339–343, 2012; Degond et al., Math Models Methods Appl Sci 18:1193–1215, 2008), which governs the interactions of stochastic agents moving with a velocity of constant magnitude, that is, the corresponding velocity space for these types of Kolmogorov–Vicsek models is the unit sphere. Our analysis for L^p estimates and compactness properties take advantage of the orientational interaction property, meaning that the velocity space is a compact manifold.

Contents

1. Introduction

Recently, a variety of mathematical models capturing the emergent phenomena of self-driven agents have received extensive attention. In particular, the discrete Couzin–Vicsek algorithm (CVA) has been proposed as a model describing the interactions of agents moving with velocity of constant magnitude, and with angles measured from a reference direction (see [1,4,18,26]).

This manuscript focuses on analytical issues for the kinetic (mesoscopic) description associated with the discrete Couzin–Vicsek algorithm with stochastic dynamics corresponding to Brownian motion on a sphere. More precisely, we consider the corresponding kinetic Kolmogorov–Vicsek model describing stochastic particles with orientational interaction,

$$
\begin{aligned}
\partial_t f + \omega \cdot \nabla_x f &= -\nabla_\omega \cdot (f F_o) + \mu \Delta_\omega f, \\
F_o(x,\omega,t) &= \nu(\omega \cdot \Omega(f))(Id - \omega \otimes \omega)\Omega(f), \\
\Omega(J)(x,t) &= \frac{J(f)(x,t)}{|J(f)(x,t)|}, \\
J(f)(x,t) &= \int_{U\times\mathbb{S}^{d-1}} K(|x-y|)\omega f(y,\omega,t)\,\mathrm{d}y\,\mathrm{d}\omega, \\
f(x,\omega,0) &= f_0(x,\omega), \quad x \in U,\ \omega \in \mathbb{S}^{d-1},\ t > 0,
\end{aligned}
\tag{1.1}
$$

where $f = f(x,\omega,t)$ is the one-particle distribution function at position $x \in U$, velocity direction $\omega \in \mathbb{S}^{d-1}$ and time t. The spatial domain U denotes either $\mathbb{R}^d$ or $\mathbb{T}^d$. The operators ∇_ω and Δ_ω denote, respectively, the gradient and the Laplace–Beltrami operator on the sphere $\mathbb{S}^{d-1}$, and $\mu > 0$ is a diffusion coefficient. The term $F_o(x,\omega,t)$ is the mean-field force that governs the orientational interaction of self-driven particles by aligning them with the direction $\Omega(x,t) \in \mathbb{S}^{d-1}$ that depends on the flux $J(x,t)$.

This mean-field force is also proportional to the interaction frequency ν. Its reciprocal, ν^{-1}, represents the typical time-interval between two successive changes in the trajectory of the orientational swarm particle to accommodate the presence of other particles in the neighborhood. The function K is an isotropic observation kernel around each particle and it is assumed to be integrable in $\mathbb{R}$.

Following DEGOND and MOTSCH in [10], the interaction frequency function ν is taken to be a positive function of $\cos\theta$, where θ is the angle between ω and Ω. Such dependence of ν with respect to the angle θ represents different turning transition rates at different angles. Hence, the constitutive form of such an interaction frequency $\nu(\theta)$ is inherent to species being modeled by orientational interactions. As in [10], we assume that $\nu(\theta)$ is a smooth and bounded function of its argument.

The kinetic Kolmogorov–Fokker–Planck type model with orientational interactions (1.1) was formally derived in [10] as a mean-field limit of the discrete Couzin–Vicsek algorithm (CVA) with stochastic dynamics. There, the authors mainly focused on the model (1.1) with the local momentum $\tilde{J}$ instead of J:

$$
\Omega(\tilde{J})(x,t) = \frac{\tilde{J}(f)(x,t)}{|\tilde{J}(f)(x,t)|}, \quad \tilde{J}(f)(x,t) = \int_{\mathbb{S}^{d-1}} \omega f(x,\omega,t)\,\mathrm{d}\omega, \tag{1.2}
$$

where $\tilde{J}$ was derived from J in (1.1) by rescaling the kernel K in time and spatial variables. Such scaling describes dynamics for solutions to (1.1) at large time and length scales compared with the scales of individuals.

In the current manuscript, we focus on the existence and uniqueness properties of solutions to both models, with $J(f)$ as defined in (1.1) and with $\tilde{J}(f)$ as defined in (1.2).

In fact, since J with the kernel $K = \delta_0$(Dirac mass) is exactly $\tilde{J}$, it is enough to show the global existence and uniqueness of weak solutions to models (1.1) in an appropriate space, to be specified in Section 2. These results are easily applied to $\tilde{J}$, as in (1.2).

The classical Vicsek model has received extensive attention in the last few years, especially regarding its mean-field land hydrodynamic limits and its phase transition development. More specifically, Bolley, Cañizo and Carrillo have rigorously justified a mean-field limit in [2] when the force term acting on the particles is not normalized, that is, $\nu\Omega(x,t)$ is replaced by just $J(x,t)$ in force term F_o. This modification leads to the appearance of phase transitions from disordered states at low density to aligned (ordered) states at high densities. Such a phase transition problem has been studied in [1,4,8,9,16,18]. In addition, issues regarding hydrodynamic descriptions of the classical Vicsek model have been discussed in [8–12,15]. We also refer to [3,7,19] for related issues.

To date, there have been few results on the existence theory of true kinetic descriptions. Frouvelle and Liu [16] have shown the well-posedness in the space-homogeneous case of (1.2) with the regular force field $(Id - \omega \otimes \omega)\tilde{J}$ instead of $(Id - \omega \otimes \omega)\Omega(\tilde{J})$. There, they provided the convergence rates towards equilibria by using the Onsager free energy functional and Lasalle's invariance principle, and their results have been applied in [8]. Very recently, Figalli et al. [14] have shown the well-posedness in the space-homogeneous case of (1.2), and the convergence of solutions towards steady states, based on the gradient flow approach (see for example [13,21]).

On the other hand, the authors in [2] have shown the existence of weak solutions for the space-inhomogeneous equation for a force field F_o given by the difference between spatial convolutions of mass and momentum with bounded Lipschitz kernels K, namely $\omega K *_x \rho - K *_x J$, instead of $\nu\Omega$ as considered in this manuscript. Such a choice of force field has a regularizing effect for the spatial variable compared to our case $\nu\Omega$, which deals with stronger non-linearities.

This manuscript is mainly devoted to showing the existence and uniqueness properties of weak solutions to the kinetic Kolmogorov–Vicsek type model (1.1). A difficulty in our analysis arises from the fact that $\Omega(J)$ in the alignment force term of (1.1) is undefined as $J(f)$ becomes 0. Thus we restrict the problem of finding global weak solutions to (1.1) to a subclass of solutions with the non-zero local momentum, that is, $J(f) \neq 0$.

In the next section, we briefly present some known results for kinetic models with orientational interactions, (1.1) and (1.2), which give a heuristic justification for the *a priori* non-zero assumption on $J(f)$, which is to be stated in our main result. Section 3 presents *a priori* estimates and the compactness lemma, which play crucial roles in the main proof of the existence of a weak solution in the next section. Section 4 deals with the construction of weak solutions to (1.1) by means of introducing an ε-regularized problem, for an arbitrary parameter $\varepsilon > 0$, modifying the alignment force $\Omega(J)$ uniformly bounded in ε. We then solve the ε-regularized problem of (1.1) by constructing a sequence of functions $\{f_{n,\varepsilon}\}_{n\geqq 1}$ that converges to the solution f_ε. Finally we show that, within the class of solutions satisfying $J(f) \geqq 0$, there is a subsequence f_{ε_k} converging to f, solving (1.1). Section 5 is devoted to the proof of the uniqueness of weak solutions in a periodic spatial domain $U = \mathbb{T}^d$ under the additional constraint $J(f) \geqq \alpha > 0$.

2. Preliminaries and Main Results

In this section, we briefly review how the kinetic Kolmogorov–Viscek equations (1.1) and (1.2), can be formally derived from the discrete Couzin–Vicsek algorithm model [10] with stochastic dynamics. We then provide our main result and useful formulations.

2.1. Kinetic Kolmogorov–Vicsek Models

Following [10], the kinetic Kolmogorov–Vicsek model considered in (1.1) is derived from the classical discrete Vicsek formulation modeling Brownian motion of the sphere $\mathbb{S}^{d-1}$ given by the following stochastic differential equations for $1 \leqq i \leqq N$:

$$
\begin{aligned}
&dX_i = \omega_i \,\mathrm{d}t, \\
&d\omega_i = (Id - \omega_i \otimes \omega_i)\, \nu\left(\omega_i \cdot \bar{\Omega}_i\right) \bar{\Omega}_i \,\mathrm{d}t + \sqrt{2\mu}\,(Id - \omega_i \otimes \omega_i) \circ dB_t^i, \\
&\bar{\Omega}_i = \frac{\bar{J}_i}{|\bar{J}_i|}, \quad \bar{J}_i = \sum_{j,\ |X_j - X_i| \leqq R} \omega_j.
\end{aligned}
\tag{2.1}
$$

Here, the neighborhood of the i-th particle is the ball centered at $X_i \in \mathbb{R}^d$ with radius $R > 0$. The velocity director $\omega_i \in \mathbb{S}^{d-1}$ of the i-th particle tends to be aligned with the director Ω_i of the average velocity of the neighboring particles with noise B_t^i standing for N independent standard Brownian motions on $\mathbb{R}^d$ with intensity $\sqrt{2\mu}$. Then, its projection $(Id - \omega_i \otimes \omega_i) \circ dB_t^i$ represents the contribution of Brownian motion to the sphere $\mathbb{S}^{d-1}$, which should be understood in the Stratonovich sense. We refer to [20] for a detailed description of Brownian motions on Riemannian manifolds. We note that the first term in $d\omega_i$ is the sum of smooth binary interactions with identical speeds, whereas there is no constraint on the velocity in the Cucker–Smale model [5]. In addition, the interaction frequency (weight) function $\nu(\omega_i \cdot \Omega_i)$ depends on the angle between ω_i and Ω_i, parametrized by $\cos\theta_i = \omega_i \cdot \Omega_i$.

From the individual-based model (2.1), the corresponding kinetic mean-field limit (1.1) was proposed in [2,10] as the number of particles, N, tends to infinity. Notice that μ in (1.1) corresponds to the diffusive coefficient associated with the Brownian motion on the sphere $\mathbb{S}^{d-1}$.

The reduced model (1.1) with the modified definition of setting $J = \tilde{J}$, as in (1.2), was proposed in [10] by the following scaling argument. Considering the system dynamics at large times and length scales compared with the scales of individuals by the dimensionless rescaled variables $\tilde{x} = \varepsilon x, \tilde{t} = \varepsilon t$ with $\varepsilon \ll 1$, it makes the interactions become local and aligns the particle velocity with the direction of the local particle flux. This interaction term is balanced at leading order ε by the diffusion term.

Notice that $\Omega(f)$ in (1.1) is undefined when $J(f)$ becomes 0. Because of this issue, we study in this manuscript the existence of weak solutions to (1.1) for the subclass of solutions with non-zero local momentum, that is, $J(f) \neq 0$. As shown in [10], since ω is not a collisional invariant of operator Q, the momentum is not conserved. Thus, it is not straightforward to get $J(f)(x,t) \neq 0$ for all (x,t) by imposing non-zero initial momentum, that is, $J(f)(x,0) \neq 0$ for all x. Moreover, there is no canonical entropy for the type of the kinetic equations found in (1.1). Due to these analytical difficulties, we heuristically justify our constraint $J(f) \neq 0$ by observing equilibria of (1.2) in the three dimensional case, which has been studied in [10].

For the classification of equilibria in the $d = 3$ dimensional case, we recall the Fisher–von Mises distribution, given by

$$M_{\Omega}(\omega) = \frac{1}{\int_{\mathbb{S}^2} \exp(\frac{\sigma(\omega\cdot\Omega)}{\mu})\, \mathrm{d}\omega} \exp\left(\frac{\sigma(\omega\cdot\Omega)}{\mu}\right)$$

for a given unit vector $\Omega \in \mathbb{S}^2$, where σ denotes an antiderivative of ν, that is, $\frac{d\sigma}{d\tau}(\tau) = \nu(\tau)$. Since ν is positive, σ is an increasing function and then M_{Ω} is maximal at $\omega \cdot \Omega = 1$, that is, for ω pointing in the direction of Ω. Therefore, Ω plays the same role as the averaged velocity in the classical Maxwellian equilibria for classical kinetic models of rarefied gas dynamics with velocities defined in all space. The diffusion constant μ corresponds to the temperature strength, which measures the spreading of the equilibrium state about the average direction Ω. The present model has a constant diffusion μ that is in contrast with the classical gas dynamics where the temperature is a thermodynamical variable whose evolution is determined by the energy balance equation.

Using the Fisher–von Mises distribution, the operator Q and equilibria of (1.2) are expressed as follows:

Lemma 2.1. [10] (i) *The operator $Q(f)$ can be written as*

$$Q(f) = \mu \nabla_{\omega} \cdot \left[M_{\Omega(f)} \nabla_{\omega} \left(\frac{f}{M_{\Omega(f)}} \right) \right].$$

(ii) *The equilibria, that is, solutions $f(\omega)$ satisfying Q(f) = 0, form a three dimensional manifold $\mathcal{E}$ given by*

$$\mathcal{E} = \{\rho M_{\Omega}(\omega) \mid \rho > 0,\ \Omega \in \mathbb{S}^2\},$$

where ρ is the total mass and Ω is the flux director of $\rho M_\Omega(\omega)$, that is,

$$\rho = \int_{\mathbb{S}^2} \rho M_\Omega(\omega)\,\mathrm{d}\omega, \quad \Omega = \frac{\tilde{J}(\rho M_\Omega)}{|\tilde{J}(\rho M_\Omega)|},$$
$$\tilde{J}(\rho M_\Omega) := \int_{\mathbb{S}^2} \rho M_\Omega(\omega)\omega\,\mathrm{d}\omega = \rho c(\mu)\Omega,$$

with

$$c(\mu) = \frac{\int_0^\pi \cos\theta \exp\left(\frac{\sigma(\cos\theta)}{\mu}\right)\sin\theta\,\mathrm{d}\theta}{\int_0^\pi \exp\left(\frac{\sigma(\cos\theta)}{\mu}\right)\sin\theta\,\mathrm{d}\theta}.$$

We note that $c(\mu) \to 1$ as $\mu \to 0$, and $c(\mu) \to 0$ as $\mu \to \infty$. This means that the local momentum $\tilde{J}(\rho M_\Omega)$ of the equilibrium solution $f = \rho M_\Omega$ is not zero as long as the diffusion strength μ is not sufficiently large compared to orientational interaction. Consequently, it is expected that moderate values of μ would yield non-zero, local momentum $\tilde{J}(f)$ for solutions f near the Von Mises equilibria.

2.2. Main Result

We state now the main results for the global existence of weak solutions to equations (1.1).

We first introduce the following notations for simplification:

- **Notation** : We denote by $D := U \times \mathbb{S}^{d-1}$, and by $\mathbb{P}_{\omega^\perp} := Id - \omega \otimes \omega$, as the mapping $v \mapsto (Id - \omega \otimes \omega)v$ is the projection of the vector v onto the normal plane to ω.
- **Hypotheses ($\mathcal{H}$)** : As stated earlier, we assume that $\nu(\cdot)$ is a smooth and bounded function of its argument, and that $K(|\cdot|) \in L^1(U)$. Moreover, in order to avoid $\Omega(f)$ being undefined, we impose *a priori* assumptions stating that the weak solutions f of (1.1) belong to an admissible class

$$\mathcal{A} := \{f \mid J(f)(x,t) \neq 0, \quad \forall x \in U,\ t > 0\}. \tag{2.2}$$

Theorem 2.1. (Existence for spatial domains U, being either $\mathbb{R}^d$ or $\mathbb{T}^d$) *Assume ($\mathcal{H}$), and that f_0 satisfies*

$$f_0 \in (L^1 \cap L^\infty)(D) \quad \textit{and} \quad f_0 \geqq 0. \tag{2.3}$$

Then, for a given $T > 0$, the equation (1.1) has a weak solution f, which satisfies

$$\begin{aligned} f &\geqq 0, \\ f &\in C(0,T; L^1(D)) \cap L^\infty(D \times (0,T)), \\ \nabla_\omega f &\in L^2(D \times (0,T)) \end{aligned} \tag{2.4}$$

and the following weak formulation: for any $\phi \in C_c^\infty(D \times [0,T))$*,*

$$\begin{aligned}
&\int_0^t \int_D f\partial_t\phi + f\omega\cdot\nabla_x\phi + fF_o\cdot\nabla_\omega\phi - \mu\nabla_\omega f\cdot\nabla_\omega\phi \,\mathrm{d}x\,\mathrm{d}\omega\,\mathrm{d}s \\
&\quad + \int_D f_0\phi(0,\cdot)\,\mathrm{d}x\,\mathrm{d}\omega = 0, \\
&F_o(x,\omega,t) = \nu(\omega\cdot\Omega(f))\mathbb{P}_{\omega^\perp}\Omega(f).
\end{aligned} \tag{2.5}$$

Moreover, the weak solution f *satisfies the estimate*

$$\|f\|_{L^\infty(0,T;L^p(D))} + \frac{2\mu(p-1)}{p}\|\nabla_\omega f^{\frac{p}{2}}\|^{\frac{2}{p}}_{L^2(D\times(0,T))} \leqq e^{CT\frac{p}{p-1}}\|f_0\|_{L^p(D)}, \tag{2.6}$$

for any $1 \leqq p < \infty$*, and*

$$\|f\|_{L^\infty(D\times(0,T))} \leqq e^{CT}\|f_0\|_{L^\infty(D)}. \tag{2.7}$$

Remark 2.1. The proof of Theorem 2.1 is based on energy methods, where the diffusion term $\mu\Delta_\omega f$ plays a crucial role, yet the strength of $\mu > 0$ does not essentially affect the proof of existence. Therefore, without loss of generality, from now on we set $\mu = 1$.

We next present the uniqueness of weak solutions being constructed in Theorem 2.1, only for periodic domains $U = \mathbb{T}^d$, together with the following subclass:

$$\mathcal{A}_\alpha := \left\{ f \mid \exists\,\alpha > 0 \; s.t. \; |J(f)(x,t)| > \alpha, \; \forall (x,t) \in \mathbb{T}^d \times (0,T) \right\},$$

which is more restrictive than (2.2). Indeed this class corresponds to the subclass of weak solutions to the initial value problem (1.1), with uniformly bounded speed when solved in a spatial torus domain.

Theorem 2.2. (Uniqueness for periodic spatial domains $\mathbb{T}^d$) *Assume* ($\mathcal{H}$) *and* (2.3). *Then, for a given* $T > 0$*, the periodic boundary problem of* (1.1) *has a unique weak solution* f *in the subclass* $\mathcal{A}_\alpha$*.*

Remark 2.2. Our proof for uniqueness takes advantage of a uniformly positive lower bound α of $J(f)$ in order to control $\Omega(f)$, and consequently restricted to the periodic domain $\mathbb{T}^d$. Indeed, imposing that $J(f) \geqq \alpha > 0$ for all $x \in \mathbb{R}^d$ results in an infinite mass $\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} f\,\mathrm{d}x\,\mathrm{d}\omega = \infty$, due to

$$\begin{aligned}
\infty = \int_{\mathbb{R}^d} J(f)\,\mathrm{d}x &\leqq \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} |K *_x f|\,\mathrm{d}x\,\mathrm{d}\omega \\
&\leqq \int_{\mathbb{S}^{d-1}} \|K *_x f\|_{L^1(\mathbb{R}^d)}\,\mathrm{d}\omega \leqq \|K\|_{L^1}\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} f\,\mathrm{d}x\,\mathrm{d}\omega.
\end{aligned}$$

2.3. Formulas for Calculus on Sphere

We start recalling some useful formulas on the sphere $\mathbb{S}^{d-1}$ which are extensively used in this paper.
Let F be a vector-valued function and f be a scalar-valued function. The following formula, analogous to the integration by parts, holds:

$$\int_{\mathbb{S}^{d-1}} f\nabla_\omega \cdot F \,\mathrm{d}\omega = -\int_{\mathbb{S}^{d-1}} F \cdot (\nabla_\omega f - 2\omega f)\,\mathrm{d}\omega. \tag{2.8}$$

By the definition of the projection operator $\mathbb{P}_{\omega^\perp}$, it follows that

$$\begin{aligned} \mathbb{P}_{\omega^\perp}\omega = 0, \quad \mathbb{P}_{\omega^\perp}\nabla_\omega f = \nabla_\omega f, \\ \mathbb{P}_{\omega^\perp} u \cdot v = \mathbb{P}_{\omega^\perp} v \cdot u, \end{aligned} \tag{2.9}$$

for any scalar-valued function f, and vectors u and v.

In addition, for any constant vector $v \in \mathbb{R}^d$, we have

$$\begin{aligned} \nabla_\omega(\omega \cdot v) = \mathbb{P}_{\omega^\perp} v, \\ \nabla_\omega \cdot (\mathbb{P}_{\omega^\perp} v) = -(d-1)\omega \cdot v. \end{aligned} \tag{2.10}$$

These formulas can be easily derived classical calculus on spherical coordinates (see [16,24]).

3. A Priori Estimates and the Compactness Lemma

The following Lemma provides a priori estimates in $L^\infty(0,T;L^p(U))$, $1 \leqq p \leqq \infty$ for solutions to the initial value problem for the kinetic equation below. The subsequent Lemma 3.2 provides a compactness tool needed for the existence result proof of Theorem 2.1.

Lemma 3.1. *Assume that f_0 satisfies (2.3), and that f is a smooth solution to the equation*

$$\begin{aligned} &\partial_t f + \omega \cdot \nabla_x f = -\nabla_\omega \cdot (f\nu(\omega\cdot\Omega)\mathbb{P}_{\omega^\perp}\Omega) + \Delta_\omega f, \\ &f(x,\omega,0) = f_0(x,\omega), \end{aligned} \tag{3.1}$$

where $\Omega : U \times \mathbb{R}_+ \to \mathbb{R}^d$ is a bounded vector-valued function of (x,t).
Then, for any $1 \leqq p < \infty$,

$$\|f\|_{L^\infty(0,T;L^p(D))} + \frac{2(p-1)}{p}\|\nabla_\omega f^{\frac{p}{2}}\|^{\frac{2}{p}}_{L^2(D\times(0,T))} \leqq e^{CT\frac{p}{p-1}}\|f_0\|_{L^p(D)}. \tag{3.2}$$

In particular, if $p = \infty$, then

$$\|f\|_{L^\infty(D\times(0,T))} \leqq e^{CT}\|f_0\|_{L^\infty(D)}. \tag{3.3}$$

Proof. First of all, for any $1 \leqq p < \infty$, it follows from (1.1) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_D f^p \,\mathrm{d}x\,\mathrm{d}\omega &= -p\int_D f^{p-1}\nabla_\omega\cdot(f\nu(\omega\cdot\Omega)\mathbb{P}_{\omega^\perp}\Omega)\,\mathrm{d}x\,\mathrm{d}\omega \\ &\quad + p\int_D f^{p-1}\Delta_\omega f\,\mathrm{d}x\,\mathrm{d}\omega \\ &=: I_1 + I_2.\end{aligned} \tag{3.4}$$

Using formula (2.8) and $\omega\cdot\nabla_\omega f = 0$, we have

$$\begin{aligned} I_2 &= -p(p-1)\int_D f^{p-2}\nabla_\omega f\cdot\nabla_\omega f\,\mathrm{d}x\,\mathrm{d}\omega + 2p\int_D f^{p-1}\omega\cdot\nabla_\omega f\,\mathrm{d}x\,\mathrm{d}\omega \\ &= -\frac{4(p-1)}{p}\int_D |\nabla_\omega f^{\frac{p}{2}}|^2\,\mathrm{d}x\,\mathrm{d}\omega.\end{aligned}$$

Next, by formula (2.10), the term I_1 from (3.4) is estimated as follows:

$$\begin{aligned} I_1 &= -p\int_D f^{p-1}\Big(\nu(\omega\cdot\Omega)\nabla_\omega f\cdot\mathbb{P}_{\omega^\perp}\Omega + f\nu'(\omega\cdot\Omega)|\mathbb{P}_{\omega^\perp}\Omega|^2 \\ &\qquad - (d-1)f\nu(\omega\cdot\Omega)\omega\cdot\Omega\Big)\,\mathrm{d}x\,\mathrm{d}\omega \\ &\leqq p\|\nu(\omega\cdot\Omega)\|_{L^\infty}\int_D f^{p-1}|\nabla_\omega f|\,\mathrm{d}x\,\mathrm{d}\omega + p\|\nu'(\omega\cdot\Omega)\|_{L^\infty}\int_D f^p\,\mathrm{d}x\,\mathrm{d}\omega \\ &\quad + p(d-1)\|\nu(\omega\cdot\Omega)\|_{L^\infty}\int_D f^p\,\mathrm{d}x\,\mathrm{d}\omega.\end{aligned}$$

In addition, using Hölder's inequality, the first integral in the right hand side above can be estimated by

$$\begin{aligned}\int_D f^{p-1}|\nabla_\omega f|\,\mathrm{d}x\,\mathrm{d}\omega &\leqq \left(\int_D f^p\,\mathrm{d}x\,\mathrm{d}\omega\right)^{1/2}\left(\int_D f^{p-2}|\nabla_\omega f|^2\,\mathrm{d}x\,\mathrm{d}\omega\right)^{1/2} \\ &= \frac{2}{p}\left(\int_D f^p\,\mathrm{d}x\,\mathrm{d}\omega\right)^{1/2}\left(\int_D |\nabla_\omega f^{\frac{p}{2}}|^2\,\mathrm{d}x\,\mathrm{d}\omega\right)^{1/2}.\end{aligned}$$

Then, we have

$$I_1 \leqq \frac{2(p-1)}{p}\int_D |\nabla_\omega f^{\frac{p}{2}}|^2\,\mathrm{d}x\,\mathrm{d}\omega + C\left(\frac{p}{p-1}+p\right)\int_D f^p\,\mathrm{d}x\,\mathrm{d}\omega.$$

Finally, combining the estimates above for both I_1 and I_2, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_D f^p\,\mathrm{d}x\,\mathrm{d}\omega + \frac{2(p-1)}{p}\int_D |\nabla_\omega f^{\frac{p}{2}}|^2\,\mathrm{d}x\,\mathrm{d}\omega \leqq C\left(\frac{p}{p-1}+p\right)\int_D f^p\,\mathrm{d}x\,\mathrm{d}\omega,$$

which yields a Gronwall type inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f\|_{L^p(D)} \leqq C\frac{p}{p-1}\|f\|_{L^p(D)}.$$

Therefore,

$$\|f\|_{L^\infty(0,T;L^p(D))} \leqq e^{CT\frac{p}{p-1}}\|f_0\|_{L^p(D)},$$

which implies the L^p estimate in (3.2). Hence, taking $p \to \infty$ yields the L^∞ bound (3.3).

Remark 3.1. The boundedness of the alignment vector Ω is essential for the proof of Lemma 3.1, and the a priori estimates (3.2) and (3.3) still hold for $\Omega = \Omega(f)$ bounded for any f.

The following lemma provides the compactness property that ensures the strong L^p convergence of solutions to the initial value problem associated with linear equation (3.5). Such a strong compactness property relies on the boundedness of both the force term and velocity space (notice that the velocity variable would be unbounded; we would have to use the celebrated velocity averaging lemma [22,25]). As mentioned earlier, the compactness property obtained from the next lemma is crucial for the existence proof of Theorem 2.1.

Lemma 3.2. *Assume that f_0 satisfies (2.3), and that f_n is a smooth solution to*

$$\begin{aligned}&\partial_t f_n + \omega\cdot\nabla_x f_n = -\nabla_\omega\cdot\big(f_n\nu(\omega\cdot F_n)\mathbb{P}_{\omega^\perp}F_n\big) + \Delta_\omega f_n,\\ &f_n(x,\omega,0) = f_0(x,\omega),\end{aligned} \tag{3.5}$$

where $F_n : U\times\mathbb{R}_+ \to \mathbb{R}^d$ is a given function of (t,x).
If the sequence (F_n) is bounded in $L^\infty(U\times(0,T))$, then there exists a limit function f such that, up to a subsequence,

$$f_n \to f \quad \text{as } n\to\infty \text{ in } L^p(D\times(0,T))\cap L^2(U\times(0,T);H^1(\mathbb{S}^{d-1}))',\quad 1\leqq p<\infty.$$

Moreover, the associated sequence

$$J_n := \int_D K(|x-y|)\omega f_n(y,\omega,t)\,\mathrm{d}y\,\mathrm{d}\omega$$

strongly converges to the corresponding limit J in $L^p(U\times(0,T))$, where

$$J := \int_D K(|x-y|)\omega f(y,\omega,t)\,\mathrm{d}y\,\mathrm{d}\omega.$$

Proof. Since the sequence (F_n) is bounded in $L^\infty(U\times(0,T))$, there exists $F\in L^\infty(U\times(0,T))$ such that, up to a subsequence,

$$F_n \rightharpoonup F \quad \text{weakly} - * \text{ in } L^\infty(U\times(0,T)). \tag{3.6}$$

Let f be a solution of (3.5) corresponding to the limiting F. Then, the following identity holds:

$$\begin{aligned}\partial_t(f_n-f)+\omega\cdot\nabla_x(f_n-f) &= -\nabla_\omega\cdot\big((f_n-f)\nu(\omega\cdot F_n)\mathbb{P}_{\omega^\perp}F_n\big)\\ &\quad -\nabla_\omega\cdot\big(f(\nu(\omega\cdot F_n)-\nu(\omega\cdot F))\mathbb{P}_{\omega^\perp}F_n\big)\\ &\quad -\nabla_\omega\cdot\big(f\nu(\omega\cdot F)\mathbb{P}_{\omega^\perp}(F_n-F)\big)+\Delta_\omega(f_n-f).\end{aligned} \tag{3.7}$$

Next, for any fixed $p \in [1, \infty)$, multiplying the above equation by $p(f_n - f)^{p-1}$ and integrating over D yields the identity

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_D (f_n - f)^p \,\mathrm{d}x\,\mathrm{d}\omega \\
&\quad = -p\int_D (f_n - f)^{p-1}\nabla_\omega\cdot\big((f_n - f)\nu(\omega\cdot F_n)\mathbb{P}_{\omega^\perp}F_n\big)\,\mathrm{d}x\,\mathrm{d}\omega \\
&\quad\quad - p\int_D (f_n - f)^{p-1}\nabla_\omega\cdot\big(f(\nu(\omega\cdot F_n) - \nu(\omega\cdot F))\mathbb{P}_{\omega^\perp}F_n\big)\,\mathrm{d}x\,\mathrm{d}\omega \\
&\quad\quad - p\int_D (f_n - f)^{p-1}\nabla_\omega\cdot\big(f\nu(\omega\cdot F)\mathbb{P}_{\omega^\perp}(F_n - F)\big)\,\mathrm{d}x\,\mathrm{d}\omega \\
&\quad\quad + p\int_D (f_n - f)^{p-1}\Delta_\omega(f_n - f)\,\mathrm{d}x\,\mathrm{d}\omega \\
&\quad =: \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4.
\end{aligned} \tag{3.8}$$

We first estimate the term $\mathcal{J}_1$ using the same arguments as the ones used in Lemma 3.1 in order to estimate I_1. Indeed,

$$\begin{aligned}
\mathcal{J}_1 &= -p\int_D (f_n - f)^{p-1}\big(\nu(\omega\cdot F_n)\nabla_\omega(f_n - f)\cdot\mathbb{P}_{\omega^\perp}F_n \\
&\quad + (f_n - f)\nu'(\omega\cdot F_n)|\mathbb{P}_{\omega^\perp}F_n|^2 \\
&\quad -(d-1)(f_n - f)\nu(\omega\cdot F_n)\omega\cdot F_n\big)\,\mathrm{d}x\,\mathrm{d}\omega \\
&\leqq p\|\nu(\omega\cdot F_n)\|_{L^\infty}\|F_n\|_{L^\infty}\int_D (f_n - f)^{p-1}|\nabla_\omega(f_n - f)|\,\mathrm{d}x\,\mathrm{d}\omega \\
&\quad + p\|\nu'(\omega\cdot F_n)\|_{L^\infty}\|F_n\|_{L^\infty}^2\int_D (f_n - f)^p\,\mathrm{d}x\,\mathrm{d}\omega \\
&\quad + p(d-1)\|\nu(\omega\cdot F_n)\|_{L^\infty}\|F_n\|_{L^\infty}\int_D (f_n - f)^p\,\mathrm{d}x\,\mathrm{d}\omega. \\
&\leqq \frac{2(p-1)}{p}\int_D |\nabla_\omega(f_n - f)^{\frac{p}{2}}|^2\,\mathrm{d}x\,\mathrm{d}\omega + \frac{Cp^2}{p-1}\int_D (f_n - f)^p\,\mathrm{d}x\,\mathrm{d}\omega.
\end{aligned}$$

Similarly, $\mathcal{J}_4$ is also estimated as was done for I_2 in the proof of Lemma 3.1:

$$\mathcal{J}_4 = -\frac{4(p-1)}{p}\int_D |\nabla_\omega(f_n - f)^{\frac{p}{2}}|^2\,\mathrm{d}x\,\mathrm{d}\omega.$$

Hence, gathering these two last estimates, identity (3.8) yields the estimate

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_D (f_n - f)^p\,\mathrm{d}x\,\mathrm{d}\omega \\
&\quad \leqq C\int_D (f_n - f)^p\,\mathrm{d}x\,\mathrm{d}\omega - \frac{2(p-1)}{p}\int_D |\nabla_\omega(f_n - f)^{\frac{p}{2}}|^2\,\mathrm{d}x\,\mathrm{d}\omega + \mathcal{J}_2 + \mathcal{J}_3.
\end{aligned}$$

Next, since $f_n = f$ at $t = 0$, applying Gronwall's inequality to the above inequality it holds that for any $0 < t \leqq T$,

$$\begin{aligned}&\int_D (f_n - f)^p \,\mathrm{d}x\,\mathrm{d}\omega + \frac{2(p-1)}{p}\int_0^t\int_D |\nabla_\omega (f_n - f)^{\frac{p}{2}}|^2 \,\mathrm{d}x\,\mathrm{d}\omega\,\mathrm{d}s\\ &\leqq e^{CT}\int_0^t (\mathcal{J}_2 + \mathcal{J}_3)(s)\,\mathrm{d}s.\end{aligned}$$

The terms $\mathcal{J}_2$ and $\mathcal{J}_3$ can be rewritten using the calculus on the sphere formulas (2.10) as follows. First, note that the term $\mathcal{J}_2$ satisfies the identity

$$\begin{aligned}\mathcal{J}_2 = -p\int_D (f_n - f)^{p-1}\Big[&(\nu(\omega\cdot F_n) - \nu(\omega\cdot F))\nabla_\omega f\cdot \mathbb{P}_{\omega^\perp} F_n\\ &+ f(\nu'(\omega\cdot F_n)F_n - \nu'(\omega\cdot F)F)\cdot \mathbb{P}_{\omega^\perp} F_n\\ &- (d-1) f(\nu(\omega\cdot F_n) - \nu(\omega\cdot F))\omega\cdot F_n\Big]\,\mathrm{d}x\,\mathrm{d}\omega\\ = -p\int_D (f_n - f)^{p-1}\Big[&\nu'(\omega\cdot F_n^*)\omega\cdot(F_n - F)\nabla_\omega f\cdot \mathbb{P}_{\omega^\perp} F_n\\ &+ f\Big(\nu'(\omega\cdot F_n)(F_n - F) + \nu''(\omega\cdot F_n^{**})\omega\cdot(F_n - F)F\Big)\cdot \mathbb{P}_{\omega^\perp} F_n\\ &- (d-1) f\nu'(\omega\cdot F_n^*)\omega\cdot(F_n - F)\omega\cdot F_n\Big]\,\mathrm{d}x\,\mathrm{d}\omega\\ = -p\int_D (f_n - f)^{p-1}\Big[&\nu'(\omega\cdot F_n^*)\nabla_\omega f\cdot \mathbb{P}_{\omega^\perp} F_n\omega\\ &+ f\nu'(\omega\cdot F_n)\mathbb{P}_{\omega^\perp} F_n + f\nu''(\omega\cdot F_n^{**})F\cdot \mathbb{P}_{\omega^\perp} F_n\omega\\ &- (d-1) f\nu'(\omega\cdot F_n^*)\omega\cdot F_n\omega\Big]\cdot(F_n - F)\,\mathrm{d}x\,\mathrm{d}\omega,\end{aligned}$$

where F_n^* and F_n^{**} are some bounded functions due to the mean value theorem property, depending solely on the known bounded functions $F_n(x,t)$ and the limit F defined in (3.6).
Similarly, also by the identities in (2.10), the term $\mathcal{J}_3$ satisfies the identity

$$\begin{aligned}\mathcal{J}_3 = -p\int_D (f_n - f)^{p-1}\Big[&\nu(\omega\cdot F)\nabla_\omega f\cdot \mathbb{P}_{\omega^\perp}(F_n - F)\\ &+ f\nu'(\omega\cdot F)\mathbb{P}_{\omega^\perp} F\cdot \mathbb{P}_{\omega^\perp}(F_n - F)\\ &- (d-1) f\nu(\omega\cdot F)\omega\cdot(F_n - F)\Big]\,\mathrm{d}x\,\mathrm{d}\omega\\ = -p\int_D (f_n - f)^{p-1}\Big[&\nu(\omega\cdot F)\nabla_\omega f + f\nu'(\omega\cdot F)\mathbb{P}_{\omega^\perp} F\\ &- (d-1) f\nu(\omega\cdot F)\omega\Big]\cdot(F_n - F)\,\mathrm{d}x\,\mathrm{d}\omega.\end{aligned}$$

Thus, we get the *weighted* estimate

$$\begin{aligned}\|f_n - f\|^p_{L^p(D)} + \frac{4(p-1)}{p}\int_0^T \int_D |\nabla_\omega (f_n - f)^{\frac{p}{2}}|^2 \,\mathrm{d}x\,\mathrm{d}\omega\,\mathrm{d}s \\ \leqq e^{CT}\int_0^T \int_D \Phi(x, w, s) \cdot (F_n - F)\,\mathrm{d}x\,\mathrm{d}\omega\,\mathrm{d}s,\end{aligned} \tag{3.9}$$

where the weight function, given by

$$\begin{aligned}\Phi(x, w, s) = -p(f_n - f)^{p-1}\Big[&\nu'(\omega \cdot F_n^*)\nabla_\omega f \cdot \mathbb{P}_{\omega^\perp} F_n\omega + f\nu'(\omega \cdot F_n)\mathbb{P}_{\omega^\perp} F_n \\ &+ f\nu''(\omega \cdot F_n^{**})F \cdot \mathbb{P}_{\omega^\perp} F_n\omega - (d-1)f\nu'(\omega \cdot F_n^*)\omega \cdot F_n\omega \\ &+ \nu(\omega \cdot F)\nabla_\omega f + f\nu'(\omega \cdot F)\mathbb{P}_{\omega^\perp} F - (d-1)f\nu(\omega \cdot F)\omega\Big],\end{aligned}$$

is shown to satisfy $\Phi \in L^1(D \times (0, T))$.
In order to show this assertion, first we show the uniform control property of both f_n and f, and their gradients. Indeed, by the uniform boundedness of (F_n), applying the same estimates as in Lemma 3.1 for both $g = f_n$ and f, respectively, we obtain

$$\begin{aligned}&\|g\|_{L^\infty(0,T;L^p(D))} \leqq C\|f_0\|_{L^p(D)}, \quad 1 \leqq p \leqq \infty, \\ &\|\nabla_\omega g^{\frac{p}{2}}\|_{L^2(D\times(0,T))} \leqq C\|f_0\|^{p/2}_{L^p(D)}, \quad 1 \leqq p < \infty,\end{aligned}$$

where the positive constant C only depends on p and T.

Next, by Hölder's inequality it follows that

$$\begin{aligned}&\int_0^T \int_D (f_n - f)^{p-1}\nabla_\omega f \,\mathrm{d}x\,\mathrm{d}\omega\,\mathrm{d}s \\ &\leqq \left(\int_D (f_n - f)^p \,\mathrm{d}x\,\mathrm{d}\omega\right)^{1/2}\left(\int_D (f_n - f)^{p-2}|\nabla_\omega f|^2 \,\mathrm{d}x\,\mathrm{d}\omega\right)^{1/2} \\ &\leqq C\left(\int_D (f_n^p + f^p)\,\mathrm{d}x\,\mathrm{d}\omega\right)^{1/2}\left(\int_D |\nabla_\omega f^{\frac{p}{2}}|^2 \,\mathrm{d}x\,\mathrm{d}\omega\right)^{1/2} \\ &\leqq C_0,\end{aligned}$$

and

$$\begin{aligned}\int_0^T \int_D (f_n - f)^{p-1} f \,\mathrm{d}x\,\mathrm{d}\omega\,\mathrm{d}s &\leqq \left(\int_D (f_n - f)^p \,\mathrm{d}x\,\mathrm{d}\omega\right)^{\frac{p-1}{p}}\left(\int_D f^p \,\mathrm{d}x\,\mathrm{d}\omega\right)^{\frac{1}{p}} \\ &\leqq C\left(\int_D (f_n^p + f^p)\,\mathrm{d}x\,\mathrm{d}\omega\right)^{\frac{p-1}{p}}\left(\int_D f^p \,\mathrm{d}x\,\mathrm{d}\omega\right)^{\frac{1}{p}} \\ &\leqq C_0,\end{aligned}$$

where the positive constant C_0 depends only on $\|f_0\|_{L^p(D)}$.

Therefore, the weight function $\Phi(x, w, t)$ can be estimated by

$$\begin{aligned}\|\Phi\|_{L^1(D\times(0,T))} &\leqq C_*\Big(\|(f_n - f)^{p-1}\nabla_\omega f\|_{L^1(D\times(0,T))}\\ &\qquad + \|(f_n - f)^{p-1} f\|_{L^1(D\times(0,T))}\Big)\\ &\leqq C_* C_0,\end{aligned}$$

where the positive constant C_* is given by

$$\begin{aligned}C_* = pd\Bigg[&\Bigg(\bigg(\|\nu'\Big(\omega\cdot F_n^*\Big)\|_{L^\infty} + \|\nu'\Big(\omega\cdot F_n\Big)\|_{L^\infty}\bigg)\\ &+ \|\nu''\Big(\omega\cdot F_n^{**}\Big)\|_{L^\infty}\|F\|_{L^\infty}\Bigg)\|F_n\|_{L^\infty}\\ &+ \|\nu(\omega\cdot F)\|_{L^\infty} + \|\nu'(\omega\cdot F)\|_{L^\infty}\|F\|_{L^\infty}\Bigg],\end{aligned}$$

which does not depend on n thanks to the uniform boundedness of the sequence F_n.

Hence, applying (3.6) to (3.9), it follows that

$$\begin{aligned}&f_n \to f \quad \text{in } L^p(D\times(0,T)),\\ &\nabla_\omega f_n \to \nabla_\omega f \quad \text{in } L^2(D\times(0,T)).\end{aligned} \tag{3.10}$$

Finally, in order to complete the proof of Lemma 3.2, it remains to show that (3.10) implies the strong convergence of the associated sequence $(J_n) = (J(f_n))$ towards $J(f)$. Indeed, Minkowski inequality, Hölder's inequality and Young's inequality yield

$$\begin{aligned}\|J_n - J\|_{L^p(U\times(0,T))} &= \left(\int_0^T\int_U\left|\int_{\mathbb{S}^{d-1}} K *_x (f_n - f)\omega\,\mathrm{d}\omega\right|^p \mathrm{d}x\,\mathrm{d}s\right)^{\frac{1}{p}}\\ &\leqq \int_{\mathbb{S}^{d-1}}\left(\int_0^T\int_U |K *_x (f_n - f)|^p\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\omega\\ &\leqq C\left(\int_0^T\int_{\mathbb{S}^{d-1}}\|K *_x (f_n - f)\|_{L^p(U)}^p\,\mathrm{d}\omega\,\mathrm{d}s\right)^{\frac{1}{p}}\\ &\leqq C\|K\|_{L^1(\mathbb{R}^d)}\|f_n - f\|_{L^p(D\times(0,T))},\end{aligned} \tag{3.11}$$

which completes the proof.

4. Proof of Existence: Theorem 2.1

The proof of Theorem 2.1 entices the construction of an iteration scheme that generates a sequence (f_n), where f_n is a solution to the linear equation (3.5) at n-th step, with $F_n := \Omega(f_{n-1})$ evaluated at the $(n-1)$-th solution f_{n-1} obtained in the previous $(n-1)$-th step.

This first, intuitive approach confronts a difficulty, since the n-iteration scheme generating the sequence f_n does not secure the non-zero momentum $|J(f_n)| > 0$, even if $|J(f_{n-1})| > 0$. In fact, if that were the case, the term $\Omega(f_n)$ would be undefined and therefore we could not secure that it is bounded. In particular, since the compactness properties of Lemmas 3.1 and 3.2 require a bounded force term (in (3.1) and (3.5) respectively), then, at least with the tools developed in this paper, it would not be possible to secure the existence of a solution f_{n+1} for the next n-iterative step.

A way to avoid this difficulty is to use an ε-regularization approach by adding an arbitrary $\varepsilon > 0$ parameter to the denominator of $\Omega(f_n)$, for all $n \in \mathbb{N}$. Such regularization generates a double parameter (ε, n) sequence of solutions $f_{\varepsilon,n}$ that it is shown to satisfy the property $|J(f_{\varepsilon,n})| > 0$ for all $n \in \mathbb{N}$, uniformly in $\varepsilon > 0$.

4.1. The ε-Regularized Equation

The ε-regularization approach consists of solving the non-linear problem (1.1) by adding $\varepsilon > 0$ to the denominator of $\Omega(f)$, that is

$$
\begin{aligned}
\partial_t f_\varepsilon + \omega \cdot \nabla_x f_\varepsilon &= -\nabla_\omega \cdot \Big(f_\varepsilon \nu(\omega \cdot \Omega_\varepsilon) \mathbb{P}_{\omega^\perp} \Omega_\varepsilon \Big) + \Delta_\omega f_\varepsilon, \\
\Omega_\varepsilon(f_\varepsilon)(x,t) &:= \frac{J_\varepsilon(f_\varepsilon)(x,t)}{|J_\varepsilon(f_\varepsilon)(x,t)| + \varepsilon}, \\
J_\varepsilon(f_\varepsilon)(x,t) &= \int_{U \times \mathbb{S}^{d-1}} K(|x-y|) \omega f_\varepsilon(y,\omega,t)\, \mathrm{d}y\, \mathrm{d}\omega \\
f_\varepsilon(x,\omega,0) &= f_0(x,\omega), \quad x \in U,\ \omega \in \mathbb{S}^{d-1},\ t > 0.
\end{aligned} \tag{4.1}
$$

This new non-linear ε-problem is then solved by generating a sequence of solutions $f_{\varepsilon,n}$ to (3.5) with a bounded $F_{\varepsilon,n} := \Omega_\varepsilon(f_{\varepsilon,n-1})$ for the previous iterated solution $f_{\varepsilon,n-1}$.

In the sequel, we show first that it is possible to construct a sequence of solutions $f_{\varepsilon,n}$ converging to f_ε in $L^p(D \times (0,T)) \cap L^2(U \times (0,T); H^1(\mathbb{S}^{d-1}))$, $1 \leqq p \leqq \infty$ for any $\varepsilon > 0$, so that the results remain true in the $\varepsilon \to 0$ limit.

The details of this procedure are as follows.

4.2. Construction of Approximate Solutions

The construction of an (ε, n)-sequence of approximate solutions $f_{\varepsilon,n}$ to the non-linear ε-regularized Eq. (4.1) is now done by the following iteration scheme. For any fixed $\varepsilon > 0$, set $f_{\varepsilon,0}(x,\omega,t) := f_0(x,\omega)$ to be the initial state associated with (1.1). Then, define $f_{\varepsilon,1}$ as the solution of the following linear initial value problem:

$$\begin{aligned}
\partial_t f_{\varepsilon,1} + \omega \cdot \nabla_x f_{\varepsilon,1} &= -\nabla_\omega \cdot \left(f_{\varepsilon,1} \nu(\omega \cdot \Omega_{\varepsilon,0}) \mathbb{P}_{\omega^\perp} \Omega_{\varepsilon,0} \right) + \Delta_\omega f_{\varepsilon,1}, \\
\Omega_{\varepsilon,0}(x,t) &= \frac{J_{\varepsilon,0}(x,t)}{|J_{\varepsilon,0}(x,t)| + \varepsilon}, \\
J_{\varepsilon,0}(x,t) &= \int_{U \times \mathbb{S}^{d-1}} K(|x-y|) \omega f_{\varepsilon,0}(y,\omega,t)\, \mathrm{d}y\, \mathrm{d}\omega \\
f_{\varepsilon,1}(x,\omega,0) &= f_0(x,\omega).
\end{aligned}$$

Inductively, each $f_{\varepsilon,n+1}$ is defined to be the solution of the following linear initial value problem:

$$\begin{aligned}
\partial_t f_{\varepsilon,n+1} + \omega \cdot \nabla_x f_{\varepsilon,n+1} &= -\nabla_\omega \cdot \left(f_{\varepsilon,n+1} \nu(\omega \cdot \Omega_{\varepsilon,n}) \mathbb{P}_{\omega^\perp} \Omega_{\varepsilon,n} \right) + \Delta_\omega f_{\varepsilon,n+1}, \\
\Omega_{\varepsilon,n}(x,t) &= \frac{J_{\varepsilon,n}(x,t)}{|J_{\varepsilon,n}(x,t)| + \varepsilon}, \\
J_{\varepsilon,n}(x,t) &= \int_{U \times \mathbb{S}^{d-1}} K(|x-y|) \omega f_{\varepsilon,n}(y,\omega,t)\, \mathrm{d}y\, \mathrm{d}\omega \\
f_{\varepsilon,n+1}(x,\omega,0) &= f_0(x,\omega).
\end{aligned} \tag{4.2}$$

The justification for the unique solvability of the (ε, n)-approximate initial value problem (4.2), for $n \geqq 1$, follows from the next lemma.

Lemma 4.1. *For any $T > 0$, $\varepsilon > 0$, $n \geqq 1$, assume that $f_{\varepsilon,n}$ is a given integrable function and that f_0 satisfies (2.3). Then, there exists a unique solution $f_{\varepsilon,n+1} \geqq 0$ to the Eq. (4.2) satisfying the L^p-estimates: for any $1 \leqq p < \infty$,*

$$\|f_{\varepsilon,n+1}\|_{L^\infty(0,T;L^p(D))} + \frac{2(p-1)}{p} \|\nabla_\omega f_{\varepsilon,n+1}^{\frac{p}{2}}\|_{L^2(D\times(0,T))}^{\frac{2}{p}} \leqq e^{CT\frac{p}{p-1}} \|f_0\|_{L^p(D)}, \tag{4.3}$$

and

$$\|f_{\varepsilon,n+1}\|_{L^\infty(D\times(0,T))} \leqq e^{CT} \|f_0\|_{L^\infty(D)}. \tag{4.4}$$

The proof of Lemma 4.1 follows the same argument as Degond's proof in [6]. We include its proof in the Appendix for the reader's convenience.

4.3. *Passing to the Limit as $n \to \infty$*

The convergence of $f_{\varepsilon,n}$ towards some limit function f_ε, which solves the regularized Eq. (4.1), is secured by the following proposition.

Proposition 4.1. *For a given $T > 0$ and arbitrary $\varepsilon > 0$, if f_0 satisfies (2.3), then there exists a weak solution $f_\varepsilon \geqq 0$ to Eq. (4.1) satisfying the L^p-estimates: for $1 \leqq p < \infty$,*

$$\|f_\varepsilon\|_{L^\infty(0,T;L^p(D))} + \frac{2(p-1)}{p}\|\nabla_\omega f_\varepsilon^{\frac{p}{2}}\|_{L^2(D\times(0,T))}^{\frac{2}{p}} \leqq e^{CT\frac{p}{p-1}}\|f_0\|_{L^p(D)}, \quad (4.5)$$

and

$$\|f_\varepsilon\|_{L^\infty(D\times(0,T))} \leqq e^{CT}\|f_0\|_{L^\infty(D)}. \quad (4.6)$$

Proof. Since the sequence $(\Omega_{\varepsilon,n})$ defined in (4.2) is bounded in $L^\infty(U\times(0,T))$, we use Lemma 3.2 with $F_{\varepsilon,n} = \Omega_{\varepsilon,n}$. Thus, there exists a limit function f_ε such that, up to a subsequence,

$$\begin{aligned}
&f_{\varepsilon,n} \to f_\varepsilon \quad \text{as } n\to\infty \text{ in } L^p(D\times(0,T)) \cap L^2\left(U\times(0,T); H^1\left(\mathbb{S}^{d-1}\right)\right),\\
&J_{\varepsilon,n} \to J_\varepsilon \quad \text{as } n\to\infty \text{ in } L^p(U\times(0,T)),
\end{aligned}$$

which yields

$$\Omega_{\varepsilon,n} \;\to\; \Omega_\varepsilon := \frac{J_\varepsilon}{|J_\varepsilon|+\varepsilon} \quad \text{as } n\to\infty \;\text{ in } L^\infty(0,T;L^p(D)).$$

Indeed,

$$\begin{aligned}
&\int_U |\Omega_{\varepsilon,n}-\Omega_\varepsilon|^p\,\mathrm{d}x\\
&\quad= \int_U \left|\frac{\varepsilon(J_{\varepsilon,n}-J_\varepsilon)+|J_\varepsilon|(J_{\varepsilon,n}-J_\varepsilon)+J_\varepsilon(|J_\varepsilon|-|J_{\varepsilon,n}|)}{(|J_{\varepsilon,n}|+\varepsilon)(|J_\varepsilon|+\varepsilon)}\right|^p \mathrm{d}x\\
&\quad\leqq \frac{1}{\varepsilon^p}\int_U \left|\frac{\varepsilon(J_{\varepsilon,n}-J_\varepsilon)+|J_\varepsilon|(J_{\varepsilon,n}-J_\varepsilon)+J_\varepsilon(|J_\varepsilon|-|J_{\varepsilon,n}|)}{|J_\varepsilon|+\varepsilon}\right|^p \mathrm{d}x\\
&\quad\leqq C(\varepsilon)\int_U \left(|J_{\varepsilon,n}-J_\varepsilon|^p+|J_{\varepsilon,n}-J_\varepsilon|^p+||J_{\varepsilon,n}|-|J_\varepsilon||^p\right)\mathrm{d}x\\
&\quad\leqq C(\varepsilon)\int_U |J_{\varepsilon,n}-J_\varepsilon|^p\,\mathrm{d}x.
\end{aligned}$$

Therefore, the limit f_ε satisfies the following weak formulation of (4.1): for all $\phi \in C_c^\infty(D\times[0,T))$,

$$\begin{aligned}
&\int_0^t\int_D f_\varepsilon\partial_t\phi + f_\varepsilon\omega\cdot\nabla_x\phi + f_\varepsilon F_\varepsilon\cdot\nabla_\omega\phi - \nabla_\omega f_\varepsilon\cdot\nabla_\omega\phi\,\mathrm{d}x\,\mathrm{d}\omega\,\mathrm{d}s\\
&\quad+\int_D f_0\phi(0,\cdot)\,\mathrm{d}x\,\mathrm{d}\omega = 0,\\
&F_\varepsilon = \nu(\omega\cdot\Omega_\varepsilon)\mathbb{P}_{\omega^\perp}\Omega_\varepsilon, \quad \Omega_\varepsilon(x,t) = \frac{J_\varepsilon(x,t)}{|J_\varepsilon(x,t)|+\varepsilon}.
\end{aligned}$$

In addition, using Lemma 3.1 together with the boundedness of Ω_ε, above, the L^p estimates from (4.5) and (4.6) follow.

4.4. Passing to the Limit as $\varepsilon \to 0$

The proof of Theorem 2.1 is completed after showing the convergence from (4.1) to (1.1) as $0 < \varepsilon \to 0$, in the weak sense. In fact, it is enough to show such a limit for any convergent sequence $0 < \varepsilon_k \to 0$.

First, consider a sequence

$$F_k := \frac{J_{\varepsilon_k}}{|J_{\varepsilon_k}| + \varepsilon_k}, \quad J_{\varepsilon_k} = \int_{\mathbb{S}^{d-1}} \omega f_{\varepsilon_k} \, \mathrm{d}\omega.$$

Since such a sequence is bounded in $L^\infty(U \times (0,T))$ uniformly in ε_k, Lemma 3.2 can be applied, so there exists a limit function f such that, up to a subsequence,

$$\begin{aligned} & f_{\varepsilon_k} \to f \quad \text{as } k \to \infty \text{ in } L^p(D \times (0,T)) \cap L^2(U \times (0,T); H^1(\mathbb{S}^{d-1})), \\ & J_{\varepsilon_k} \to J \quad \text{as } k \to \infty \text{ in } L^p(U \times (0,T)). \end{aligned} \tag{4.7}$$

Next, in order to see that f is the weak solution to (1.1), it is enough to show that f satisfies the weak formulation (2.5) as a limit of the following formulation for (4.1):

$$\begin{aligned} \int_0^t \int_D & f_{\varepsilon_k} \partial_t \phi + f_{\varepsilon_k} \omega \cdot \nabla_x \phi + f_{\varepsilon_k} \nu \left(\frac{\omega \cdot J_{\varepsilon_k}}{|J_{\varepsilon_k}| + \varepsilon_k} \right) \mathbb{P}_{\omega^\perp} \frac{J_{\varepsilon_k}}{|J_{\varepsilon_k}| + \varepsilon_k} \cdot \nabla_\omega \phi \\ & - \nabla_\omega f_{\varepsilon_k} \cdot \nabla_\omega \phi \, \mathrm{d}x \, \mathrm{d}\omega \, \mathrm{d}s \\ & + \int_D f_0 \phi(0, \cdot) \, \mathrm{d}x \, \mathrm{d}\omega = 0, \end{aligned}$$

for any $\phi \in C_c^\infty(D \times [0,T))$.

By the convergence of f_{ε_k} in (4.7), clearly all linear terms in the above formulation converge to their corresponding terms in (2.5). On the other hand, the convergence of the nonlinear term requires further justification, which is provided in the following Lemma.

Lemma 4.2. *Assume that* $|J(x,t)| > 0$, *as in* (2.2). *Then, as* $k \to \infty$,

$$\begin{aligned} & \int_0^t \int_D f_{\varepsilon_k} \nu \left(\frac{\omega \cdot J_{\varepsilon_k}}{|J_{\varepsilon_k}| + \varepsilon_k} \right) \mathbb{P}_{\omega^\perp} \frac{J_{\varepsilon_k}}{|J_{\varepsilon_k}| + \varepsilon_k} \cdot \nabla_\omega \phi \, \mathrm{d}x \, \mathrm{d}\omega \, \mathrm{d}s \\ & \quad \longrightarrow \int_0^t \int_D f \nu \left(\frac{\omega \cdot J}{|J|} \right) \mathbb{P}_{\omega^\perp} \frac{J}{|J|} \cdot \nabla_\omega \phi \, \mathrm{d}x \, \mathrm{d}\omega \, \mathrm{d}s. \end{aligned} \tag{4.8}$$

Proof. By the properties (2.9) of calculus on the sphere applied to the projection operator, $\mathbb{P}_{\omega^\perp}$ is the identity operator acting on gradient functions of the sphere $\mathbb{S}^{d-1}$, that is, $\mathbb{P}_{\omega^\perp} \cdot \nabla_\omega \Phi = \nabla_\omega \Phi$ holds for any test function Φ of $w \in \mathbb{S}^{d-1}$. Then, the limit as $k \to \infty$ in (4.8) is identical to showing the analogous limit for the formulation without the projection operator. That is, for $k \to \infty$,

$$\begin{aligned} & \int_0^t \int_D f_{\varepsilon_k} \nu \left(\frac{\omega \cdot J_{\varepsilon_k}}{|J_{\varepsilon_k}| + \varepsilon_k} \right) \frac{J_{\varepsilon_k}}{|J_{\varepsilon_k}| + \varepsilon_k} \cdot \nabla_\omega \phi \, \mathrm{d}x \, \mathrm{d}\omega \, \mathrm{d}s \\ & \quad \to \int_0^t \int_D f \nu \left(\frac{\omega \cdot J}{|J|} \right) \frac{J}{|J|} \cdot \nabla_\omega \phi \, \mathrm{d}x \, \mathrm{d}\omega \, \mathrm{d}s. \end{aligned} \tag{4.9}$$

We first control the integrand in (4.9) using the estimates (4.6) and the boundedness of ν, so that there is a uniform constant C such that

$$\left\| f_{\varepsilon_k} \nu\left(\frac{\omega \cdot J_{\varepsilon_k}}{|J_{\varepsilon_k}| + \varepsilon_k}\right) \frac{J_{\varepsilon_k}}{|J_{\varepsilon_k}| + \varepsilon_k} \right\|_{L^\infty(D\times(0,T))} \leqq \| f_{\varepsilon_k} \|_{L^\infty(D\times(0,T))} \|\nu\|_{L^\infty} \leqq C,$$

which implies, for some F, that

$$f_{\varepsilon_k} \nu\left(\frac{\omega \cdot J_{\varepsilon_k}}{|J_{\varepsilon_k}| + \varepsilon_k}\right) \frac{J_{\varepsilon_k}}{|J_{\varepsilon_k}| + \varepsilon_k} \rightharpoonup F \quad \text{weakly} - * \text{ in } L^\infty(D \times (0, T)).$$

Then, it remains to show that

$$F = f\nu\left(\frac{\omega \cdot J}{|J|}\right) \frac{J}{|J|}, \quad \text{on } \left\{(t, x, \omega) \in (0, T] \times U \times \mathbb{S}^{d-1} \mid |J(x, t)| > 0\right\}.$$

In order to obtain this last identity, we consider the bounded set

$$X_{R,\delta} := \left\{(t, x, \omega) \in (0, T] \times B_R(0) \times \mathbb{S}^{d-1} \mid |J(x, t)| > \delta\right\},$$

where R and δ are any positive constants, and $B_R(0)$ denotes the ball in U, with radius R, centered at 0.

Since $f_{\varepsilon_k} \to f$ and $J_{\varepsilon_k} \to J$ almost everywhere on $X_{R,\delta}$ by (4.7), then by Egorov's theorem, for any $\eta > 0$, there exists a $Y_\eta \subset X_{R,\delta}$ such that $|X_{R,\delta} \backslash Y_\eta| < \eta$ and

$$f_{\varepsilon_k} \to f, \quad J_{\varepsilon_k} \to J \quad \text{in } L^\infty(Y_\eta),$$

and so, for sufficiently large k,

$$|J_{\varepsilon_k}(x, t)| > \frac{\delta}{2} \quad \text{for } (x, t) \in Y_\eta.$$

Therefore, the $L^\infty(Y_\eta)$ ε-convergence follows from

$$\begin{aligned}
\left\| \frac{J_{\varepsilon_k}}{|J_{\varepsilon_k}| + \varepsilon_k} - \frac{J}{|J|} \right\|_{L^\infty(Y_\eta)} &= \left\| \frac{|J|(J_{\varepsilon_k} - J) + J(|J| - |J_{\varepsilon_k}|) - \varepsilon_k J}{(|J_{\varepsilon_k}| + \varepsilon_k)|J|} \right\|_{L^\infty(Y_\eta)} \\
&\leqq \frac{2}{\delta} \left\| \frac{|J|(J_{\varepsilon_k} - J) + J(|J| - |J_{\varepsilon_k}|) - \varepsilon_k J}{|J|} \right\|_{L^\infty(Y_\eta)} \\
&\leqq \frac{2}{\delta} \left(\|J_{\varepsilon_k} - J\|_{L^\infty(Y_\eta)} + \||J_{\varepsilon_k}| - |J|\|_{L^\infty(Y_\eta)} - \varepsilon_k \right) \\
&\to 0,
\end{aligned}$$

which yields

$$
\begin{aligned}
&\left\| f_{\varepsilon_k}\nu\left(\frac{\omega\cdot J_{\varepsilon_k}}{|J_{\varepsilon_k}|+\varepsilon_k}\right)\frac{J_{\varepsilon_k}}{|J_{\varepsilon_k}|+\varepsilon_k} - f\nu\left(\frac{\omega\cdot J}{|J|}\right)\frac{J}{|J|}\right\|_{L^\infty(Y_\eta)}\\
&\quad = \left\| f_{\varepsilon_k}\left[\nu\left(\frac{\omega\cdot J_{\varepsilon_k}}{|J_{\varepsilon_k}|+\varepsilon_k}\right) - \nu\left(\frac{\omega\cdot J}{|J|}\right)\right]\frac{J_{\varepsilon_k}}{|J_{\varepsilon_k}|+\varepsilon_k}\right\|_{L^\infty(Y_\eta)}\\
&\qquad + \left\| f_{\varepsilon_k}\nu\left(\frac{\omega\cdot J}{|J|}\right)\left(\frac{J_{\varepsilon_k}}{|J_{\varepsilon_k}|+\varepsilon_k} - \frac{J}{|J|}\right)\right\|_{L^\infty(Y_\eta)}\\
&\qquad + \left\| (f_{\varepsilon_k}-f)\nu\left(\frac{\omega\cdot J_{\varepsilon_k}}{|J_{\varepsilon_k}|+\varepsilon_k}\right)\frac{J_{\varepsilon_k}}{|J_{\varepsilon_k}|+\varepsilon_k}\right\|_{L^\infty(Y_\eta)}\\
&\quad \leqq C\|f_{\varepsilon_k}\|_{L^\infty}(\|\nu'\|_{L^\infty}+\|\nu\|_{L^\infty})\left\|\frac{J_{\varepsilon_k}}{|J_{\varepsilon_k}|+\varepsilon_k} - \frac{J}{|J|}\right\|_{L^\infty(Y_\eta)}\\
&\qquad + C\|f_{\varepsilon_k}-f\|_{L^\infty(Y_\eta)}\|\nu\|_{L^\infty} \to 0.
\end{aligned}
$$

Hence, the following identity holds:

$$
F = f\nu\left(\frac{\omega\cdot J}{|J|}\right)\frac{J}{|J|} \quad \text{on } Y_\eta,
$$

and, since η, R and δ are arbitrary, taking $\eta, \delta \to 0$ and $R \to \infty$, it follows that

$$
F = f\nu\left(\frac{\omega\cdot J}{|J|}\right)\frac{J}{|J|} \quad \text{on } \left\{(t,x,\omega)\in(0,T]\times U\times\mathbb{S}^{d-1} \mid |J(x,t)|>0\right\},
$$

which completes the proof of Lemma 4.2.

Finally, thanks to lemma 4.2 and (4.7), it follows that f satisfies the weak formulation (2.5). In addition, estimates (2.6) and (2.7) follow directly from (4.5) and (4.6), respectively. Therefore, the proof of Theorem 2.1 is now completed.

5. Proof of Uniqueness: Theorem 2.2

The uniqueness argument is considered in the subclass $\mathcal{A}_\alpha$ of weak solutions constructed in Theorem 2.1. Let f and g be any weak solutions to the initial value problem (1.1) in $\mathcal{A}_\alpha$. A straightforward computation yields that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d\times\mathbb{S}^{d-1}}|f-g|^2\,\mathrm{d}x\,\mathrm{d}\omega + \int_{\mathbb{T}^d\times\mathbb{S}^{d-1}}|\nabla_\omega(f-g)|^2\,\mathrm{d}x\,\mathrm{d}\omega\\
&\quad = -\int_{\mathbb{T}^d\times\mathbb{S}^{d-1}}(f-g)\nabla_\omega\cdot\Big((f-g)\nu(\omega\cdot\Omega(f))\mathbb{P}_{\omega^\perp}\Omega(f)\Big)\,\mathrm{d}x\,\mathrm{d}\omega\\
&\qquad - \int_{\mathbb{T}^d\times\mathbb{S}^{d-1}}(f-g)\nabla_\omega\cdot\Big(g(\nu(\omega\cdot\Omega(f))-\nu(\omega\cdot\Omega(g)))\mathbb{P}_{\omega^\perp}\Omega(f)\Big)\,\mathrm{d}x\,\mathrm{d}\omega\\
&\qquad - \int_{\mathbb{T}^d\times\mathbb{S}^{d-1}}(f-g)\nabla_\omega\cdot\Big(g\nu(\omega\cdot\Omega(g))\mathbb{P}_{\omega^\perp}(\Omega(f)-\Omega(g))\Big)\,\mathrm{d}x\,\mathrm{d}\omega\\
&\quad =: J_1+J_2+J_3. \qquad (5.10)
\end{aligned}
$$

Using the same estimates applied to $\mathcal{J}_1$ in the proof of Lemma 3.2, we can also estimate

$$J_1 \leqq \frac{1}{4}\int_{\mathbb{T}^d\times\mathbb{S}^{d-1}} |\nabla_\omega(f-g)|^2\,\mathrm{d}x\,\mathrm{d}\omega + C\int_{\mathbb{T}^d\times\mathbb{S}^{d-1}} |f-g|^2\,\mathrm{d}x\,\mathrm{d}\omega.$$

Next, J_2 and J_3 can also be estimated by the same approach from (3.11) in Lemma 3.2 to get

$$\|J(f)-J(g)\|_{L^2(\mathbb{T}^d)} \leqq C\|K\|_{L^1(\mathbb{T}^d)}\|f-g\|_{L^2(\mathbb{T}^d\times\mathbb{S}^{d-1})}.$$

Moreover, since $|J(f)|\geqq\alpha$ in the set $\mathcal{A}_\alpha$, then the difference of alignment forces for any two weak solutions is controlled by

$$\begin{aligned}|\Omega(f)-\Omega(g)| &\leqq \frac{\Big||J(g)|(J(f)-J(g))-J(g)(|J(f)|-|J(g)|)\Big|}{\alpha|J(g)|}\\ &\leqq \frac{2}{\alpha}|J(f)-J(g)|,\end{aligned}$$

which yields

$$\|\Omega(f)-\Omega(g)\|_{L^2(\mathbb{T}^d)} \leqq C\|f-g\|_{L^2(\mathbb{T}^d\times\mathbb{S}^{d-1})}.$$

Therefore, by property (2.4) for any weak solution, the control of term J_2 in (5.10) follows from

$$\begin{aligned}J_2 &= \int_{\mathbb{T}^d\times\mathbb{S}^{d-1}} \nabla_\omega(f-g)\cdot\mathbb{P}_{\omega^\perp}\Omega(f)g(\nu(\omega\cdot\Omega(f))-\nu(\omega\cdot\Omega(g)))\,\mathrm{d}x\,\mathrm{d}\omega\\ &\leqq \|g\|_{L^\infty}\|\nu'\|_{L^\infty}\int_{\mathbb{T}^d\times\mathbb{S}^{d-1}} |\nabla_\omega(f-g)||\Omega(f)-\Omega(g)|\,\mathrm{d}x\,\mathrm{d}\omega\\ &\leqq \frac{1}{4}\int_{\mathbb{T}^d\times\mathbb{S}^{d-1}} |\nabla_\omega(f-g)|^2\,\mathrm{d}x\,\mathrm{d}\omega + C\int_{\mathbb{T}^d\times\mathbb{S}^{d-1}} |f-g|^2\,\mathrm{d}x\,\mathrm{d}\omega.\end{aligned}$$

Likewise, the control of the last term J_3 in (5.10) follows, since

$$J_3 \leqq \frac{1}{4}\int_{\mathbb{T}^d\times\mathbb{S}^{d-1}} |\nabla_\omega(f-g)|^2\,\mathrm{d}x\,\mathrm{d}\omega + C\int_{\mathbb{T}^d\times\mathbb{S}^{d-1}} |f-g|^2\,\mathrm{d}x\,\mathrm{d}\omega.$$

Hence, gathering the above estimates and using Gronwall's inequality, we have

$$\int_{\mathbb{T}^d\times\mathbb{S}^{d-1}} |f-g|^2\,\mathrm{d}x\,\mathrm{d}\omega \leqq e^{CT}\int_{\mathbb{T}^d\times\mathbb{S}^{d-1}} |f_0-g_0|^2\,\mathrm{d}x\,\mathrm{d}\omega,$$

which implies the uniqueness of weak solutions to the initial value problem (1.1) in $\mathcal{A}_\alpha$.

6. Conclusion

We have shown the existence of global weak solutions to problem (1.1) (as well for J, defined as in (1.2)) in a subclass of solutions with the non-zero local momentum. These solutions are unique on the subclass of solutions in the d dimensional torus whose mean speed is uniformly bounded below by a strictly positive constant.

An important future work would be to remove our assumption on the non-zero local momentum. The main difficulty is due to the lack of momentum conservation for solutions to problem (1.1), and the canonical entropy associated with the equation in (1.1). Thus, at this point, we have neither suitable functional spaces nor distances to study the behavior of solutions whose momentum may vanish locally. This difficulty is related to the issue of the stability of solutions to (1.1). Another future work is to extend the uniqueness result to the whole spatial domain $\mathbb{R}^d$.

Acknowledgments. IRENE M. GAMBA is supported by the NSF under Grants DMS-1109625, and NSF RNMS (KI-Net) Grant DMS11-07465, and MOON-JIN KANG is supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2013R1A6A3A03020506). The support from the Institute of Computational Engineering and Sciences at the University of Texas Austin is gratefully acknowledged.

Appendix A. Proof of Lemma 4.1

For notational simplicity, we omit the subindex $n+1$ in (4.2). Our goal is to prove the existence of solutions f to the linear equation

$$\begin{aligned}
&\partial_t f + \omega\cdot\nabla_x f = -\nabla_\omega\cdot\Big(f\nu(\omega\cdot\bar{\Omega})\mathbb{P}_{\omega^\perp}\bar{\Omega}\Big) + \Delta_\omega f,\\
&\bar{\Omega} = \frac{\bar{J}(x,t)}{|\bar{J}(x,t)|+\varepsilon}, \quad \bar{J}(x,t) = \int_D K(|x-y|)\omega g(y,\omega,t)\,\mathrm{d}y\,\mathrm{d}\omega,\\
&f(x,\omega,0) = f_0(x,\omega),
\end{aligned} \tag{7.1}$$

where g is just a given integrable function.
We begin by rewriting (7.1) as

$$\begin{aligned}
&\partial_t f + \omega\cdot\nabla_x f + \nu(\omega\cdot\bar{\Omega})\mathbb{P}_{\omega^\perp}\bar{\Omega}\cdot\nabla_\omega f\\
&\qquad + f\nu'(\omega\cdot\bar{\Omega})|\mathbb{P}_{\omega^\perp}\bar{\Omega}|^2 - (d-1)f\nu(\omega\cdot\bar{\Omega})\omega\cdot\bar{\Omega} - \Delta_\omega f = 0,\\
&f(x,\omega,0) = f_0(x,\omega),
\end{aligned} \tag{7.2}$$

where the formulas of (2.10) on projections and on calculus on the sphere were used.
Next, taking $\bar{f}(x,\omega,t) := e^{-\lambda t}f(x,\omega,t)$ for a given $\lambda > 0$ leads to the modified initial value problem

$$\begin{aligned}
&\partial_t\bar{f} + \omega\cdot\nabla_x\bar{f} + \psi_1\cdot\nabla_\omega\bar{f} + \Big(\lambda+\psi_2+\psi_3\Big)\bar{f} - \Delta_\omega\bar{f} = 0,\\
&\bar{f}(x,\omega,0) = f_0(x,\omega),
\end{aligned} \tag{7.3}$$

where the functions ψ_1, ψ_2 and ψ_3 are given by

$$\psi_1(x,\omega,t) = \nu(\omega\cdot\bar{\Omega})\mathbb{P}_{\omega^\perp}\bar{\Omega},$$
$$\psi_2(x,\omega,t) = \nu'(\omega\cdot\bar{\Omega})|\mathbb{P}_{\omega^\perp}\bar{\Omega}|^2,$$
$$\psi_3(x,\omega,t) = -(d-1)\nu(\omega\cdot\bar{\Omega})\omega\cdot\bar{\Omega},$$

respectively. Now, since $|\bar{\Omega}| \leqq 1$, and the smooth function ν is bounded, then ψ_1, ψ_2 and ψ_3 are also bounded. Therefore, by J. L. Lions' existence theorem in [23], the existence of a solution for (7.3) follows from the same argument given by Degond in [6]. That means equation (7.3) has a solution $\bar{f}$ in the space

$$Y := \left\{ f \in L^2\left([0,T]\times U; H^1\left(\mathbb{S}^{d-1}\right)\right) \mid \partial_t f + \omega\cdot\nabla_x f \in L^2\left([0,T]\times U; H^{-1}\left(\mathbb{S}^{d-1}\right)\right)\right\}.$$

Furthermore, by Green's formula, used in [6], the following identity holds: for any $f \in Y$,

$$\langle \partial_t f + \omega\cdot\nabla_x f, f\rangle = \frac{1}{2}\int_D (|f(x,\omega,T)|^2 - |f(x,\omega,0)|^2)\,\mathrm{d}x\,\mathrm{d}\omega, \tag{7.4}$$

where $\langle\cdot,\cdot\rangle$ denotes the pairing of $L^2([0,T]\times U; H^{-1}(\mathbb{S}^{d-1}))$ and $L^2([0,T]\times U; H^1(\mathbb{S}^{d-1}))$.

This identity (7.4) is needed below to show the uniqueness of solutions f in Y as follows.

Let $\bar{f} \in Y$ be a solution to (7.3) with initial data $f_0 = 0$. Then, by (7.4), it follows that

$$\begin{aligned}
0 &= \langle \partial_t \bar{f} + \omega\cdot\nabla_x \bar{f} + \psi_1\cdot\nabla_\omega \bar{f} + (\lambda+\psi_2+\psi_3)\bar{f} - \Delta_\omega \bar{f}, \bar{f}\rangle \\
&= \frac{1}{2}\int_D |\bar{f}(x,\omega,T)|^2\,\mathrm{d}x\,\mathrm{d}\omega - \frac{1}{2}\int_D \nabla_\omega\cdot\psi_1|\bar{f}|^2\,\mathrm{d}x\,\mathrm{d}\omega \\
&\quad + \int_D (\lambda+\psi_2+\psi_3)|\bar{f}|^2\,\mathrm{d}x\,\mathrm{d}\omega + \int_D |\nabla_\omega \bar{f}|^2\,\mathrm{d}x\,\mathrm{d}\omega \\
&\geqq \Bigg(\lambda - \frac{1}{2}\|\nabla_\omega\cdot\psi_1\|_{L^\infty([0,T]\times D)} - \|\psi_2\|_{L^\infty([0,T]\times D)} \\
&\quad - \|\psi_3\|_{L^\infty([0,T]\times D)}\Bigg)\int_D |\bar{f}|^2\,\mathrm{d}x\,\mathrm{d}\omega.
\end{aligned} \tag{7.5}$$

Next, since

$$\begin{aligned}
\nabla_\omega\cdot\psi_1 &= \nu'(\omega\cdot\bar{\Omega})\nabla_\omega(\omega\cdot\bar{\Omega})\cdot\mathbb{P}_{\omega^\perp}\bar{\Omega} + \nu(\omega\cdot\bar{\Omega})\nabla_\omega\cdot\mathbb{P}_{\omega^\perp}\bar{\Omega} \\
&= \nu'(\omega\cdot\bar{\Omega})|\mathbb{P}_{\omega^\perp}\bar{\Omega}|^2 - (d-1)\nu(\omega\cdot\bar{\Omega})\omega\cdot\bar{\Omega},
\end{aligned}$$

the term $\nabla_\omega\cdot\psi_1$ is bounded. Thus, choosing λ such that

$$\lambda > \frac{1}{2}\|\nabla_\omega\cdot\psi_1\|_{L^\infty([0,T]\times D)} + \|\psi_2\|_{L^\infty([0,T]\times D)} + \|\psi_3\|_{L^\infty([0,T]\times D)}, \tag{7.6}$$

estimate (7.5) yields $\bar{f} = 0$, which proves the uniqueness of the linear equation (7.3). Therefore, (7.3) has a unique solution $\bar{f} \in L^2([0,T] \times U; H^1(\mathbb{S}^{d-1}))$. Furthermore, since $f_0 \geqq 0$ and $f_0 \in L^\infty(D)$, by an argument similar to that in (7.5),

$$\bar{f} \geqq 0 \quad \text{and} \quad \bar{f} \in L^\infty([0,T] \times D).$$

Indeed, using the identity from [6] on any $f \in Y$, with $f_- := \max(-f, 0)$:

$$\langle \partial_t f + \omega \cdot \nabla_x f, f_- \rangle = \frac{1}{2} \int_D (|f_-(x,\omega,0)|^2 - |f_-(x,\omega,T)|^2)\, \mathrm{d}x\, \mathrm{d}\omega.$$

Then, since $f_-(x,\omega,0) = 0$ when $f_0 \geqq 0$, it follows that

$$\begin{aligned}
0 &= \langle \partial_t \bar{f} + \omega \cdot \nabla_x \bar{f} + \psi_1 \cdot \nabla_\omega \bar{f} + (\lambda + \psi_2 + \psi_3)\bar{f} - \Delta_\omega \bar{f}, \bar{f}_- \rangle \\
&= -\frac{1}{2} \int_D |\bar{f}_-(x,\omega,T)|^2 \,\mathrm{d}x\,\mathrm{d}\omega + \frac{1}{2} \int_D \nabla_\omega \cdot \psi_1 |\bar{f}_-|^2 \,\mathrm{d}x\,\mathrm{d}\omega \\
&\quad - \int_D (\lambda + \psi_2 + \psi_3)|\bar{f}_-|^2 \,\mathrm{d}x\,\mathrm{d}\omega - \int_D |\nabla_\omega \bar{f}_-|^2 \,\mathrm{d}x\,\mathrm{d}\omega \\
&\leqq -\Big(\lambda - \frac{1}{2}\|\nabla_\omega \cdot \psi_1\|_{L^\infty([0,T]\times D)} - \|\psi_2\|_{L^\infty([0,T]\times D)} \\
&\quad - \|\psi_3\|_{L^\infty([0,T]\times D)}\Big) \int_D |\bar{f}_-|^2 \,\mathrm{d}x\,\mathrm{d}\omega.
\end{aligned}$$

Using the same λ as in (7.6) yields $\bar{f}_- = 0$, which proves $\bar{f} \geqq 0$. The same argument also deduces that

$$\|\bar{f}\|_{L^\infty([0,T]\times D)} \leqq \|f_0\|_{L^\infty(D)}.$$

Finally, using the transformation $f(x,\omega,t) = e^{\lambda t} \bar{f}(x,\omega,t)$, the results hold for solutions of (7.2) as well. In addition, since the $\bar{f}$ properties are invariant under such a transformation, then the proof of existence is completed, and estimates (4.3) and (4.4) follow directly from Lemma 3.1 together with the boundedness of $\bar{\Omega}$.

References

1. ALDANA, M., HUEPE, C.: Phase transitions in self-driven many-particle systems and related non-equilibrium models: a network approach. *J. Stat. Phys*, **112**, 135–153 (2003)
2. BOLLEY, F., CAÑIZO, J.A., Carrillo, J.A.: Mean-field limit for the stochastic Vicsek model. *Appl. Math. Lett*, **25**, 339–343 (2012)
3. BOSTAN, M., CARRILLO, J.A.: Asymptotic fixed-speed reduced dynamics for kinetic equations in swarming. *Math. Models Methods Appl. Sci.* **23**, 2353–2393 (2013)
4. COUZIN, I.D., KRAUSE, J., JAMES, R., RUXTON, G.D., FRANKS, N.R.: Collective memory and spatial sorting in animal groups. *J. Theor. Biol.* **218**, 1–11 (2002)
5. CUCKER, F., SMALE, S.: Emergent behavior in flocks. *IEEE Trans. Autom. Control*, **52**, 852–862 (2007)
6. DEGOND, P.: Global existence of smooth solutions for the Vlasov–Fokker–Planck equation in 1 and 2 space dimensions. *Ann. Sci. Ecole Norm. Sup. (4)* **19**, 519–542 (1986)

7. Degond, P., Dimarco, G., Mac, T.B.N.: Hydrodynamics of the Kuramoto–Vicsek model of rotating self-propelled particles. *Math. Models Methods Appl. Sci.* **24**, 277–325 (2014)
8. Degond, P., Frouvelle, A., Liu, J.-G.: Phase transitions, hysteresis, and hyperbolicity for self-organized alignment dynamics. *Arch. Rat. Mech. Anal.* **216**, 63–115 (2015)
9. Degond, P., Frouvelle, A., Liu, J-G.: Macroscopic limits and phase transition in a system of self-propelled particles. *J. Nonlinear Sci.* **23**, 427–456 (2012)
10. Degond, P., Motsch, S.: Continuum limit of self-driven particles with orientation interaction. *Math. Models Methods Appl. Sci.* **18**, 1193–1215 (2008)
11. Degond, P., Motsch, S.: Macroscopic limit of self-driven particles with orientation interaction. *C. R. Acad. Sci. Paris Ser I.* **345**, 555–560 (2007)
12. Degond, P., Yang, T.: Diffusion in a continuum model of self-propelled particles with alignment interaction. *Math. Models Methods Appl. Sci.* **20**, 1459–1490 (2010)
13. Figalli, A., Gigli, N.: A new transportation distance between non-negative measures, with applications to gradients flows with Dirichlet boundary conditions. *J. Math. Pures Appl.* **94**, 107–130 (2010)
14. Figalli, A., Kang, M.-J., Morales, J.: *Global well-posedness of spatially homogeneous Kolmogorov–Vicsek model as a gradient flow*. http://arxiv.org/pdf/1509.02599 **(preprint)**
15. Frouvelle, A.: A continuum model for alignment of self-propelled particles with anisotropy and density dependent parameters. *Math. Mod. Meth. Appl. Sci.* **22**, 1250011 (2012)
16. Frouvelle, A., Liu, J.-G.: Dynamics in a kinetic model of oriented particles with phase transition. *SIAM J. Math. Anal.* **44**, 791–826 (2012)
17. Gamba, I.M., Haack, J.R., Motsch, S.: Spectral method for a kinetic swarming model. *J. Comput. Phys.* (2015). **(To appear)**
18. Grégoire, G., Chaté, H.: Onset of collective and cohesive motion. *Phys. Rev. Lett.* **92**, 025702 (2004)
19. Ha, S.-Y., Jeong, E., Kang, M.-J.: Emergent behaviour of a generalized Viscek-type flocking model. *Nonlinearity*, **23**, 3139–3156 (2010)
20. Hsu, E.P.: Stochastic analysis on manifolds. In: Graduate Series in Mathematics. Am. Math. Soc., Providence (2002)
21. Jordan, R., Kinderlehrer, D., Otto, F.: The variational formulation of the Fokker–Planck equation. *SIAM J. Math. Anal.* **29**, 1–17 (1998)
22. Kapper, T.K., Mellet, A., Trivisa, K.: Existence of weak solutions to kinetic flocking models. *SIAM J. Math. Anal.* **45**, 215–243 (2013)
23. Lions, J.L.: *Equations différentielles opérationelles et problèmes aux limites*. Springer, Berlin, 1961
24. Otto, F., Tzavaras, A.: Continuity of velocity gradients in suspensions of rod-like molecules. *Commun. Math. Phys.* **277**, 729–758 (2008)
25. Perthame, B., Souganidis, P.E.: A limiting case for velocity averaging. *Ann. Sci. École Norm. Sup.* **31**, 591–598 (1998)
26. Vicsek, T., Czirók, A., Ben-Jacob, E., Cohen, I., Shochet, O.: Novel type of phase transition in a system of self-driven particles. *Phys. Rev. Lett.* **75**, 1226–1229 (1995)

Department of Mathematics and ICES,
The University of Texas at Austin,
Austin,
TX78712,
USA.
e-mail: gamba@math.utexas.edu
e-mail: moonjinkang@math.utexas.edu

(*Received February 15, 2015 / Accepted April 8, 2016*)
Published online April 21, 2016 –