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003

SIMULATION OF THE TRANSIENT BEHAVIOR OF A ONE-DIMENSIONAL SEMICONDUCTOR DEVICE II*

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Abstract. A numerical method based on treating the potential by a mixed finite-element method and the electron and hole density equations by a finite-element version of a modification of the method of characteristics is introduced to simulate the transient behavior of a semiconductor device, for which the dependence of the coefficients of the conductivity equations on the electric field is considered and the Einstein relations are not assumed. L^2-error estimates that are independent of L^∞-error estimates for the approximate electric field are derived for a single space variable model.

Key words. semiconductor simulation, mixed finite elements, modified method of characteristics

AMS(MOS) subject classification. 65

0. Introduction. We assume the reader is familiar with the work of Douglas, Martínez, and Squeff [4]. General references to the literature were given in the bibliography of [4] and they are given again herein. We state briefly the problem in this section.

We consider [1], [8]-[11], [13] the nonlinear parabolic/elliptic system of equations that describe the transient behavior of a semiconductor device in a closed interval of R^1 ($\partial_x = \partial/\partial x$, etc.):

$$\partial_x q = -\partial_x(\varepsilon \partial_x \psi) = -Q(e-p-c), \tag{0.1a}$$

$$Q\partial_t e - \partial_x(J_e) = QR(e,p), \tag{0.1b}$$

$$Q\partial_t p + \partial_x(J_p) = QR(e,p), \tag{0.1c}$$

where the electric field q and the carrier densities e and p are related through the current densities

$$J_e = \varepsilon^{-1}Q\mu_e(q)eq + QD_e(q)\partial_x e, \qquad J_p = \varepsilon^{-1}Q\mu_p(q)pq - QD_p(q)\partial_x p,$$

with ε and Q being positive constants. We assume Dirichlet boundary conditions

$$\psi(0,t) = r_0(t), \qquad \psi(1,t) = r_1(t), \tag{0.2a}$$

$$e(0,t) = f_0(t), \qquad e(1,t) = f_1(t), \tag{0.2b}$$

$$p(0,t) = g_0(t), \qquad p(1,t) = g_1(t), \tag{0.2c}$$

and the initial conditions

$$e(x,0) = e^0(x), \qquad p(x,0) = p^0(x). \tag{0.2d}$$

If μ_e, D_e, μ_p, D_p are assumed to be positive constants, the equations (0.1) are quasilinear and have been treated by Douglas, Martínez, and Squeff [4]. In this paper we generalize their method to the nonlinear system that results from assuming that μ_e, D_e, μ_p, and D_p are functions of the electric field q. Actually, these assumptions on the coefficients are more realistic, since the mobilities μ_e and μ_p are the proportionality factors of the drift velocities to the electric field:

$$v_e^d = -\mu_e q, \qquad v_p^d = \mu_p q.$$

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Also, they allow us to improve the estimates of [4]. Here we derive L^2-norm error estimates for the carrier densities that are independent of the L^∞-norm of the approximation q_h to the electric field q. Appropriate models for the mobilities are given in Selberherr [13], and they suggest that μ_α and D_α for $\alpha = e$ or p can be assumed to satisfy the following conditions.

There exist positive constants D^*, M_1, L_1, L_2, and K_D such that

$$D^* \leqq \inf\{D_i(q)\colon q \in R, i = e, p\}, \tag{0.3a}$$

$$|\mu_\alpha(q)q| \leqq M_1, \qquad q \in R, \tag{0.3b}$$

$$|\mu_\alpha(q_1)q_1 - \mu_\alpha(q_2)q_2| \leqq L_1|q_1 - q_2|, \qquad |\mu_\alpha(q_1) - \mu_\alpha(q_2)| \leqq L_2|q_1 - q_2|, \tag{0.3c}$$

$$|D_\alpha(q_1) - D_\alpha(q_2)| \leqq K_D|q_1 - q_2|, \tag{0.3d}$$

for $\alpha = e$ or p. Let us also assume, for this case of single space variable problem, that

$$|(\partial_q\mu_\alpha(q))q| \leqq M_2, \qquad |(\partial_q\mu_\alpha(q_1))q_1 - (\partial_q\mu_\alpha(q_2))q_2| \leqq L_3|q_1 - q_2|, \qquad \text{for } \alpha = e \text{ or } p, \tag{0.3e}$$

where M_2 and L_3 are positive constants. If μ_α is a Lipschitz function as, for example, in Markowich [10],

$$\mu_e = \frac{\mu_e^s}{1 + (\mu_e^s|q|/v_s^e)}, \qquad \mu_p = \frac{\mu_p^s}{1 + (\mu_p^s|q|/v_s^p)},$$

where v_s^e and v_s^p are the saturation velocities and μ_e^s and μ_p^s stand for one of the field independent, scattering mobility models, then assumption (0.3e) can be verified provided that q and its derivatives are bounded. In the single space variable problem q and its derivatives are bounded. So, in particular, it follows from the first inequality of (0.3e) that

$$|\partial_x(\mu_\alpha(q)q)| \leqq M_3, \qquad \alpha = e \text{ or } p, \tag{0.3f}$$

where M_3 is a positive constant depending on M_2, the bound for the right-hand side of the potential equation, and the bound for $\mu_s(q)$. Without loss of generality we can take $L_3 \leqq M_3$.

We will not assume the Einstein relations for the mobility and diffusion coefficients.

In § 1 we describe the proposed numerical procedure. In § 2 we derive L^2-norm error estimates for the approximate electric field q_h. Finally, in § 3 we obtain L^2-norm error estimates for the approximations e_h and p_h to their respective carrier densities e and p.

1. Description of the numerical procedure. If (0.1) are scaled as in [10], [11], [15], then

$$\partial_x q = -\partial_{xx}\psi = -z(e - p - c), \tag{1.1a}$$

$$\partial_t e - U_T\mu_e(q)q\partial_x e - \partial_x(D_e(q)\partial_x e) - qe\partial_x(U_T\mu_e(q)) + U_T z\mu_e(q)e(e - p - c) = R(e, p), \tag{1.1b}$$

$$\partial_t p + U_T\mu_p(q)q\partial_x p - \partial_x(D_p(q)\partial_x p) + qp\partial_x(U_T\mu_p(q)) - U_T z\mu_p(q)p(e - p - c) = R(e, p), \tag{1.1c}$$

where z is the inverse square of the normed characteristic Debye length of the device [10].

As in the previous work [4] we will use a mixed finite-element method to approximate q and ψ simultaneously and a modified method of characteristics [3], [5], [6], [12] to approximate the densities e and p.

To introduce the modified method of characteristics for e and p let $\tau_e = \tau_e(x, t)$ be the unit vector in the direction $(-U_T\mu_e(q)q, 1)$ and τ_p the unit vector in the direction $(U_T\mu_p(q)q, 1)$. Set $\varphi_\alpha = [1+(U_T\mu_\alpha(q)q)^2]^{1/2}$ for $\alpha = e$ or p. Then the derivatives in the τ_α directions are given by

$$\varphi_e\partial/\partial\tau_e = \partial_t - U_T\mu_e(q)q\partial_x, \qquad \varphi_p\partial/\partial\tau_p = \partial_t + U_T\mu_p(q)q\partial_x,$$

so that equations (1.1b) and (1.1c) can be written in the following form:

$$\varphi_e\partial e/\partial\tau_e - \partial_x(D_e(q)\partial_x e) - qe\partial_x(U_T\mu_e(q)) + U_T\mu_e(q)ze(e-p-c) = R(e, p), \tag{1.2a}$$

$$\varphi_p\partial p/\partial\tau_p - \partial_x(D_p(q)\partial_x p) + qp\partial_x(U_T\mu_p(q)) - U_T\mu_p(q)zp(e-p-c) = R(e, p). \tag{1.2b}$$

Now, for $\alpha = e$ or p,

$$\partial_x\mu_\alpha(q) = \partial_q\mu_\alpha(q)\cdot\partial_x q = -(\partial_q\mu_\alpha(q))z(e-p-c),$$

so that the weak formulation for (1.2a) and (1.2b) is given as the determination of maps e and p of the time interval $J = [0, T]$ into $H^1(\Omega)$ such that

$$\begin{aligned}(\varphi_e\partial e/\partial\tau_e, \zeta) + (D_e(q)\partial_x e, \partial_x\zeta) + U_Tz([\mu_e(q) + q\partial_q(\mu_e(q))]e(e-p-c), \zeta)\\ = (R, \zeta),\end{aligned} \tag{1.3a}$$

$$\begin{aligned}(\varphi_p\partial p/\partial\tau_p, \zeta) + (D_p(q)\partial_x p, \partial_x\zeta) - U_Tz([\mu_p(q) + q\partial_q(\mu_p(q))]p(e-p-c), \zeta)\\ = (R, \zeta),\end{aligned} \tag{1.3b}$$

for $\zeta \in H_0^1(\Omega)$, and such that the boundary and initial conditions (0.2a)-(0.2d) are satisfied.

First, consider a partition of J into subintervals $[t^{m-1}, t^m]$, $t^m = m\Delta t$, $\Delta t = T/N$, and a partition of Ω into subintervals $[x_{i-1}, x_i]$, $0 = x_0 < x_1 < \cdots < x_k = 1$, with $\max(x_i - x_{i-1}) = h_d$. Let

$$Z_h = \{\zeta \in C^0(\Omega) : \zeta|_{[x_{j-1}, x_j]} \in P_1([x_{i-1}, x_i])\}.$$

We will seek approximations e_h^m and p_h^m in Z_h, $0 \leqq m \leqq N$, to $e^m = e(\cdot, t^m)$ and $p^m = p(\cdot, t^m)$, respectively. We denote by q_h^m and ψ_h^m the corresponding approximations to q^m and ψ^m; they lie in different spaces to be discussed later in this section.

Next, we approximate $\varphi_e\partial e/\partial\tau_e$ via backward differencing along the tangent to the τ_e-characteristics at (x, t^m):

$$\varphi_e\partial e/\partial\tau_e \approx [e(x, t^m) - e(\tilde{x}_e^m, t^m - \tilde{\Delta}t_e^m)]/\tilde{\Delta}t_e^m, \tag{1.4}$$

where $\tilde{x}_e^m = \tilde{x}_e^m(x) = x + U_T\mu_e(q^m)q^m\tilde{\Delta}t_e^m$, with

$$\tilde{\Delta}t_e^m = \tilde{\Delta}t_e^m(x) = \begin{cases} -x/U_T\mu_e(q^m)q^m & \text{if } x + U_T\mu_e(q^m)q^m\Delta t < 0, \\ (1-x)/U_T\mu_e(q^m)q^m & \text{if } x + U_T\mu_e^m(q^m)q^m\Delta t > 1, \\ \Delta t & \text{otherwise.} \end{cases}$$

Note that, if $\tilde{\Delta}t_e^m < \Delta t$, then $\tilde{x}_e^m \in \partial\Omega$ and $e(\tilde{x}_e^m, t^m - \tilde{\Delta}t_e^m)$ should be evaluated using the boundary value specification.

Similarly,

$$\varphi_p\partial p/\partial\tau_p \approx [p(x, t^m) - p(\tilde{x}_p^m, t^m - \tilde{\Delta}t_p^m)]/\tilde{\Delta}t_p^m, \tag{1.5}$$

where $\tilde{x}_p^m = \tilde{x}_p^m(x) = x - U_T\mu_p(q^m)q^m\tilde{\Delta}t_p^m$, with

$$\tilde{\Delta}t_p^m = \tilde{\Delta}t_p^m(x) = \begin{cases} x/U_T\mu_p(q^m)q^m & \text{if } x - U_T\mu_p(q^m)q^m\Delta t < 0, \\ (x-1)/U_T\mu_p(q^m)q^m & \text{if } x - U_T\mu_p(q^m)q^m\Delta t > 1, \\ \Delta t & \text{otherwise.} \end{cases}$$

Also, note that $\tilde{x}_e^m$ and $\tilde{x}_p^m$ cannot be evaluated exactly. So, let $\hat{x}_e^m$, $\hat{\Delta}t_e^m$, $\hat{x}_p^m$, and $\hat{\Delta}t_p^m$ be defined by the corresponding relations when $q^m = q(x, t^m)$ is replaced by q_h^{m-1}.

Let e_h^0 and p_h^0 lie in Z_h and approximate $e(x, 0)$ and $p(x, 0)$, respectively. Then, for $m \geqq 1$, define $\hat{e}_h^{m-1} = e_h(\hat{x}_e^m(x),\ t^m - \hat{\Delta} t_e^m)$, and $\hat{p}_h^{m-1} = p_h(\hat{x}_p^m(x),\ t^m - \hat{\Delta} t_p^m)$. We remark here that, if $\hat{\Delta} t_e^m = \Delta t$ then $\hat{e}_h^{m-1} = e_h(\hat{x}_e^m(x), t^{m-1})$. Now, we define e_h^m and p_h^m as the unique solution of the following (algebraically linear) equations:

$$
\begin{aligned}
&((e_h^m - \hat{e}_h^{m-1})/\hat{\Delta} t_e^m, \zeta) + (D_e(q_h^{m-1})\partial_x e_h^m, \partial_x \zeta) \\
&\quad + U_T z([\mu_e(q_h^{m-1}) + q_h^{m-1}\partial_q \mu_e(q_h^{m-1})] e_h^m (\hat{e}_h^{m-1} - \hat{p}_h^{m-1} - c^m), \zeta) \\
&= (\hat{R}^{m-1}, \zeta), \qquad \zeta \in Z_h,
\end{aligned} \tag{1.6a}
$$

$$
\begin{aligned}
&((p_h^m - \hat{p}_h^{m-1})/\hat{\Delta} t_p^m, \zeta) + (D_p(q_h^{m-1})\partial_x p_h^m, \partial_x \zeta) \\
&\quad - U_T z([\mu_p(q_h^{m-1}) + q_h^{m-1}\partial_q \mu_p(q_h^{m-1})] p_h^m (\hat{e}_h^{m-1} - \hat{p}_h^{m-1} - c^m), \zeta) \\
&= (\hat{R}^{m-1}, \zeta), \qquad \zeta \in Z_h,
\end{aligned} \tag{1.6b}
$$

where $\hat{R}^{m-1} = R(\hat{e}_h^{m-1}, \hat{p}_h^{m-1})$.

For later convenience, we define approximations to the derivative along the approximate characteristics by

$$\varphi_{e,h}\partial e^m/\partial \tau_{e,h} = \partial_t e^m - U_T \mu_e(q_h^{m-1}) q_h^{m-1} \partial_x e^m,$$

$$\varphi_{p,h}\partial p^m/\partial \tau_{p,h} = \partial_t p^m + U_T \mu_p(q_h^{m-1}) q_h^{m-1} \partial_x p^m.$$

Finally, we describe the mixed finite-element method to approximate q and ψ simultaneously, defined as follows. First write the potential equation (1.1a) in the following form:

$$q + \partial_x \psi = 0, \quad x \in \Omega, \quad t \in J, \tag{1.7a}$$

$$\partial_x q = -z(e - p - c), \quad x \in \Omega, \quad t \in J, \tag{1.7b}$$

$$\psi = r, \quad x \in \partial\Omega, \quad t \in J. \tag{1.7c}$$

Then, if (1.7a) is tested against a function in $H^1(\Omega)$ and (1.7b) against one in $L^2(\Omega)$, we find the mixed weak form:

$$(q, v) - (dv/dx, \psi) = rv(1) - rv(0), \qquad v \in H^1(\Omega), \tag{1.8a}$$

$$(\partial_x q, w) = (-z(e - p - c), w), \qquad w \in L^2(\Omega). \tag{1.8b}$$

Let Ω be partitioned into subintervals $[y_{i-1}, y_i]$, $0 = y_0 < y_1 < \cdots < y_L = 1$, with $\max(y_i - y_{i-1}) = h_q$. Let

$$V_h = \{v \in C^0(\Omega) \colon v|_{[y_{i-1}, y_i]} \in P_1([y_{i-1}, y_i])\},$$

$$W_h = \{w \colon w|_{[y_{i-1}, y_i]} \in P_0([y_{i-1}, y_i])\},$$

where $P_j(E)$ denotes the class of restrictions of polynomials of degree not greater than j to the set E. Then, for $m = 0, \cdots, N$, find $\{q_h^m, \psi_h^m\} \in V_h \times W_h$ such that

$$(q_h^m, v) - (dv/dx, \psi_h^m) = r_0^m v(0) - r_1^m v(1), \qquad v \in V_h, \tag{1.9a}$$

$$(\partial_x q_h^m, w) = (-z(e_h^m - p_h^m - c^m), w), \qquad w \in W_h. \tag{1.9b}$$

Let us note that our computational algorithm is now complete. First, let e_h^0 and p_h^0 be the piecewise-linear interpolants of e and p, respectively. Then, given $\{e_h^m, p_h^m\}$, (1.9) can be used to evaluate $\{q_h^m, \psi_h^m\}$. Finally, (1.6) can be used to advance e_h and p_h to the time level t^{m+1}.

2. Error estimates for the electric field. In this section the following error estimates will be derived:

(2.1a) $$\|q^m - q_h^m\|_0 \leqq 2z\{\|e^m - e_h^m\|_0 + \|p^m - p_h^m\|_0\} + M\|(e-p-c)^m\|_1 h_q^2,$$

(2.1b) $$\|\partial_x(q^m - q_h^m)\|_0 \leqq z\{\|e^m - e_h^m\|_0 + \|p^m - p_h^m\|_0\} + M\|(e-p-c)^m\|_1 h_q.$$

The error in the approximation of q and ψ can be considered to come from two sources. Let $\{Q_h^m, \Psi_h^m\} \in V_h x W_h$ be such that

(2.2a) $$\begin{aligned}(Q_h^m, v) - (dv/dx, \Psi_h^m) &= (q^m, v) - (dv/dx, \psi) \\ &= r_0^m v(0) - r_1^m v(1), \qquad v \in V_h,\end{aligned}$$

(2.2b) $$(\partial_x Q_h^m, w) = (\partial_x q^m, w), \qquad w \in W_h;$$

i.e., $\{Q_h^m, \Psi_h^m\}$ is a mixed method approximation to $\{q^m, \psi^m\}$. It was noted in [4] that

(2.3a) $$\|q^m - Q_h^m\|_0 \leqq M\|q^m\|_2 h_q^2,$$

(2.3b) $$\|\partial_x(q^m - Q_h^m)\|_0 \leqq M\|q^m\|_2 h_q.$$

Now we estimate the error $q_h^m - Q_h^m$. Subtracting (1.9) from (2.2) and using (1.8), we see that

(2.4a) $$(q_h^m - Q_h^m, v) - (dv/dx, \psi_h^m - \Psi_h^m) = 0, \qquad v \in V_h,$$

(2.4b) $$(\partial_x(q_h^m - Q_h^m), w) = (z[(e^m - e_h^m) - (p^m - p_h^m)], w), \qquad w \in W_h.$$

First, if we let $w = \partial_x(q_h^m - Q_h^m)$ and $v = q_h^m - Q_h^m$ in (2.4), then we see that

(2.5) $$\|\partial_x(q_h^m - Q_h^m)\|_0 \leqq z[\|e^m - e_h^m\|_0 + \|p^m - p_h^m\|_0],$$

and

(2.6) $$\|q_h^m - Q_h^m\|_0^2 \leqq .5\{z[\|e^m - e_h^m\|_0 + \|p^m - p_h^m\|_0]\}^2 + .5\|\psi_h^m - \Psi_h^m\|_0^2.$$

To estimate the error $\psi_h^m - \Psi_h^m$ a duality argument will be used. Let $g \in L^2(\Omega)$ and $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$ be such that $-\partial_{xx}\varphi = g$. Then, if $\Pi_h : V \to V_h$ denotes piecewise linear interpolation, by dropping momentarily the superscript m, we have

$$(\psi_h - \Psi_h, g) = (\psi_h - \Psi_h, -\partial_{xx}\varphi) = (\psi_h - \Psi_h, -\partial_x(\Pi_h \partial_x \varphi)).$$

Thus,

$$\begin{aligned}(\psi_h - \Psi_h, g) &= (q_h - Q_h, -\Pi_h(\partial_x\varphi)) \\ &= (q_h - Q_h, \partial_x\varphi - \Pi_h(\partial_x\varphi)) + (\partial_x(q_h - Q_h), \varphi),\end{aligned}$$

and

$$\begin{aligned}|(\psi_h - \Psi_h, g)| &\leqq \|q_h - Q_h\|_0 \|\partial_x\varphi - \Pi_h(\partial_x\varphi)\|_0 + \|\partial_x(q_h - Q_h)\|_0 \|\varphi\|_0 \\ &\leqq K\|q_h - Q_h\|_0 \|\varphi\|_2 h_q + \|\partial_x(q_h - Q_h)\|_0 \|\varphi\|_2.\end{aligned}$$

Hence,

(2.7) $$\|\psi_h - \Psi_h\|_0 \leqq K\|q_h - Q_h\|_0 h_q + \|\partial_x(q_h - Q_h)\|_0.$$

Therefore, replacing (2.7) in (2.6) and using (2.5), we have that

$$\|q_h^m - Q_h^m\|_0 \leqq \sqrt{2}z[\|e^m - e_h^m\|_0 + \|p^m - p_h^m\|_0] + K\|q_h^m - Q_h^m\|_0 h_q.$$

Then, for h_q sufficiently small, it follows that

(2.8) $$\|q_h^m - Q_h^m\|_0 \leqq z^2[\|e^m - e_h^m\|_0 + \|p^m - p_h^m\|_0].$$

Since $\|q\|_2 \leqq z\|e-p-c\|_1$, the desired estimates (2.1) now follow from (2.3), (2.5), and (2.8).

3. L^2 error estimates for the densities. Consider the projection $E \times P: J \to Z_h \times Z_h$ of the solution (e, p) defined by

$$(D_e(q)\partial_x(e-E), \partial_x\zeta) = 0, \qquad \zeta \in Z_h, \tag{3.1a}$$

$$(D_p(q)\partial_x(p-P), \partial_x\zeta) = 0, \qquad \zeta \in Z_h, \tag{3.1b}$$

and such that the boundary conditions (0.2b) and (0.2c) are satisfied.

Wheeler [14] obtained the following bounds:

$$\left\|\frac{\partial^k \eta_\alpha}{\partial t^k}\right\|_{0,s} + h_d \left\|\frac{\partial^k \eta_\alpha}{\partial t^k}\right\|_{1,s} \leqq K \left\|\frac{\partial^k \alpha}{\partial t^k}\right\|_{2,s} h_d^2, \tag{3.2}$$

$t \in J$, $\alpha = e$ or p, $k \geqq 0$, $1 \leqq s \leqq \infty$, and

$$\eta_e = e - E, \qquad \eta_p = p - P.$$

Also, set

$$\sigma_e = e_h - E, \qquad \sigma_p = p_h - P.$$

Then,

$$e - e_h = \eta_e - \sigma_e, \qquad p - p_h = \eta_p - \sigma_p,$$

and it follows from (3.2) that it suffices to estimate σ_α, $\alpha = e$, and p. The argument for handling σ_p is quite similar to the one for σ_e, and therefore we will concentrate on the error equations for σ_e. Combining (1.3a), (1.6a), and (3.1a), we have

$$\begin{aligned}
&((\sigma_e^m - \hat{\sigma}_e^{m-1})/\hat{\Delta} t_e^m, \zeta) + (D_e(q_h^{m-1})\partial_x e_h^m - D_e(q^m)\partial_x E^m, \partial_x \zeta) \\
&\quad = (\varphi_e \partial e^m/\partial \tau_e - \varphi_{e,h}\partial e^m/\partial \tau_{e,h}, \zeta) \\
&\qquad + (\varphi_{e,h}\partial e^m/\partial \tau_{e,h} - (e^m - \hat{e}^{m-1})/\hat{\Delta} t_e^m, \zeta) + ((\eta_e^m - \hat{\eta}_e^{m-1})/\hat{\Delta} t_e^m, \zeta) \\
&\qquad - U_T z([\mu_e(q_h^{m-1}) + q_h^{m-1}\partial_q \mu_e(q_h^{m-1})] e_h^m (\hat{e}_h^{m-1} - \hat{p}_h^{m-1} - c^m), \zeta) \\
&\qquad + U_T z([\mu_e(q^m) + q^m \partial_q \mu_e(q^m)] e^m (e^m - p^m - c^m), \zeta) - (R^m - \hat{R}^{m-1}, \zeta),
\end{aligned} \tag{3.3}$$

for every $\zeta \in Z_h$.

Set

$$\Omega_1^m = \{x \in \Omega: \hat{\Delta} t_e^m = \Delta t\}, \qquad \Omega_2^m = \Omega \backslash \Omega_1^m.$$

Then, since σ_e vanishes on $\partial\Omega$, $\hat{\sigma}_e^{m-1} = 0$ in Ω_2^m, and

$$\begin{aligned}
&((\sigma_e^m - \hat{\sigma}_e^{m-1})/\hat{\Delta} t_e^m, \zeta) = ((\sigma_e^m - \sigma_e^{m-1})/\Delta t, \zeta)_{\Omega_1^m} \\
&\qquad + ((1/\hat{\Delta} t_e^m)\sigma_e^m, \zeta)_{\Omega_2^m} - ((\hat{\sigma}_e^{m-1} - \sigma_e^{m-1})/\Delta t, \zeta)_{\Omega_1^m}, \qquad \zeta \in Z_h.
\end{aligned} \tag{3.4}$$

Next, write

$$D_e(q_h^{m-1})\partial_x e_h^m - D_e(q^m)\partial_x E^m = D_e(q_h^{m-1})\partial_x \sigma_e^m + [D_e(q_h^{m-1}) - D_e(q^m)]\partial_x E^m. \tag{3.5}$$

Hence, combining (3.3)-(3.5), we obtain the following equality:

$$
\begin{aligned}
&((\sigma_e^m-\sigma_e^{m-1})/\Delta t, \zeta)_{\Omega_1^m}+((1+\hat{\Delta} t_e^m)\sigma_e^m, \zeta)_{\Omega_2^m}+(D_e(q_h^{m-1})\partial_x\sigma_e^m, \zeta)\\
&\quad=((\hat{\sigma}_e^{m-1}-\sigma_e^{m-1})/\Delta t, \zeta)_{\Omega_1^m}+([D_e(q^m)-D_e(q_h^{m-1})]\partial_x E^m, \partial_x\zeta)\\
&\qquad+(\varphi_e\partial e^m/\partial\tau_e-\varphi_{e,h}\partial e^m/\partial\tau_{e,h}, \zeta)\\
&\qquad+(\varphi_{e,h}\partial e^m/\partial\tau_{e,h}-(e^m-\hat{e}^{m-1})/\hat{\Delta} t_e^m, \zeta)\\
&\qquad+((\eta_e^m-\hat{\eta}_e^{m-1})/\hat{\Delta} t_e^m, \zeta)\\
&\qquad-U_Tz([\mu_e(q_h^{m-1})+q_h^{m-1}\partial_q\mu_e(q_h^{m-1})]e_h^m(\hat{e}_h^{m-1}-\hat{p}_h^{m-1}-c^m), \zeta)\\
&\qquad+U_Tz([\mu_e(q^m)+q^m\partial_q\mu_e(q^m)]e^m(e^m-p^m-c^m), \zeta)\\
&\qquad-(R^m-\hat{R}^{m-1}, \zeta)\\
&\quad=T_1+T_2+T_3+T_4+T_5+T_6+T_7+T_8=\sum_{i=1}^{8} T_i(\zeta), \qquad \zeta\in Z_h.
\end{aligned}
$$

For an L^2-estimate choose as test function $\zeta=\sigma_e^m$. Note that

$$(2\Delta t)^{-1}[\|\sigma_e^m\|_0^2-\|\sigma_e^{m-1}\|_0^2]\leqq((\sigma_e^m-\sigma_e^{m-1})/\Delta t, \sigma_e^m)_{\Omega_1^m}+((1/\hat{\Delta} t_e^m)\sigma_e^m, \sigma_e^m)_{\Omega_2^m},$$

and by (0.3a)

$$D^*\|\partial_x\sigma_e^m\|_0^2\leqq(D_e(q_h^{m-1})\partial_x\sigma_e^m, \partial_x\sigma_e^m).$$

Therefore, replacing ζ by σ_e^m in each T_i, we get the following inequality:

$$(3.6)\qquad (2\Delta t)^{-1}[\|\sigma_e^m\|_0^2-\|\sigma_e^{m-1}\|_0^2]+D^*\|\partial_x\sigma_e^m\|_0^2\leqq\sum_{i=1}^{8} T_i(\sigma_e^m).$$

Consider first T_1. Since

$$\sigma_e^{m-1}-\hat{\sigma}_e^{m-1}=\int_{x+U_T\mu_e(q_h^{m-1})q_h^{m-1}\Delta t}^{x}\partial_x(\sigma_e^{m-1})\,dy,$$

it follows from (0.3b) that

$$(3.7)\qquad \|\sigma_e^{m-1}-\hat{\sigma}_e^{m-1}\|_{0,\Omega_1}\leqq U_TM_1\Delta t\|\sigma_e^{m-1}\|_1.$$

Hence,

$$(3.8)\qquad |T_1|=|((\sigma_e^{m-1}-\hat{\sigma}_e^{m-1})/\Delta t, \sigma_e^m)_{\Omega_1^m}|\leqq K\|\sigma_e^{m-1}\|_1\|\sigma_e^m\|_0$$

where $K=K(M_1, U_T)$.

Next, by (0.3d),

$$|D_e(q^m)-D_e(q_h^{m-1})|\leqq K_D|q^m-q_h^{m-1}|,$$

so that

$$
(3.9)\qquad
\begin{aligned}
|T_2|&=|([D_e(q^m)-D_e(q_h^{m-1})]\partial_xE^m, \partial_x\sigma_e^m)|\\
&\leqq K\|E^m\|_{1,\infty}(\|\partial_t q\|_{L^\infty(J;L^2(\Omega))}\Delta t+\|q^{m-1}-q_h^{m-1}\|_0)\|\sigma_e^m\|_1,
\end{aligned}
$$

where $K=K(K_D, \Omega)$.

To estimate T_3 note that

$$
\begin{aligned}
\varphi_e\partial e^m/\partial\tau_e-\varphi_{e,h}\partial e^m/\partial\tau_{e,h}&=U_T[\mu_e(q^m)q^m-\mu_e(q_h^{m-1})q_h^{m-1}]\partial_x e^m\\
&\leqq L_1U_T|q^m-q_h^{m-1}||\partial_x e^m|,
\end{aligned}
$$

where, by (0.3c), L_1 is the Hölder constant for the function $\mu(q)q$. Therefore,

$$|T_3| \leqq K\|e^m\|_{1,\infty}\{\|\partial_t q\|_{L^\infty(J;L^2(\Omega))}\Delta t + \|q^{m-1} - q_h^{m-1}\|_0\}\|\sigma_e^m\|_0, \tag{3.10}$$

where $K = K(U_T, L_1, \Omega)$.

Next, for the estimation of T_4 we note that

$$\varphi_{e,h}\partial e^m/\partial\tau_{e,h} - (e^m - \hat{e}^{m-1})/\hat{\Delta}t_e^m = .5\varphi_{e,h}\partial^2 e/\partial\tau_{e,h}^2[(x-\hat{x}_e^m)^2 + (\hat{\Delta}t_e^m)^2]^{1/2},$$

where $\partial^2 e/\partial\tau_{e,h}^2$ is evaluated at some point on the segment between (x, t^m) and $(\hat{x}_e^m, t^m - \hat{\Delta}t_e^m)$. Now, since

$$|x - \hat{x}_e^m| = |U_T\mu_e(q_h^{m-1})q_h^{m-1}\hat{\Delta}t_e^m| \leqq M_1 U_T \hat{\Delta}t_e^m,$$

and $\hat{\Delta}t_e^m \leqq \Delta t$, then

$$[(x-\hat{x}_e^m)^2 + (\hat{\Delta}t_e^m)^2]^{1/2} \leqq M_1 U_T \Delta t.$$

Also, (0.3b) implies that $|\varphi_{e,h}| \leqq 1 + M_1 U_T$. Thus,

$$\begin{aligned} |T_4| &= |(\varphi_{e,h}\partial e^m/\partial\tau_{e,h} - (e^m - \hat{e}^{m-1})/\hat{\Delta}t_e^m, \sigma_e^m)| \\ &\leqq K\Delta t\|\sigma_e^m\|_0, \end{aligned} \tag{3.11}$$

where $K = K(M_1, M_3, U_T, \|\partial_{xx}e\|_{L^\infty(J;L^2(\Omega))}, \|\partial_{xt}e\|_{L^\infty(J;L^2(\Omega))}, \|\partial_{tt}e\|_{L^\infty(J;L^2(\Omega))})$.

The estimation of T_5 is rather long and it will be done in four steps. Note that $\hat{\eta}_e$ vanishes on Ω_2^m. Thus,

$$\begin{aligned} |T_5| &= |((\eta_e^m - \hat{\eta}_e^{m-1})/\hat{\Delta}t_e^m, \sigma_e^m)| \\ &\leqq |((\eta_e^m - \hat{\eta}_e^{m-1})/\Delta t, \sigma_e^m)_{\Omega_1^m}| + |(\eta_e^m/\hat{\Delta}t_e^m, \sigma_e^m)_{\Omega_2^m}|. \end{aligned}$$

Next, we write the first term as the sum of the following terms:

$$\begin{aligned} ((\eta_e^m - \hat{\eta}_e^{m-1})/\Delta t, \sigma_e^m)_{\Omega_1^m} &= ((\eta_e^m - \eta_e^{m-1})/\Delta t, \sigma_e^m)_{\Omega_1^m} + ((\eta_e^{m-1} - \tilde{\eta}_e^{m-1})/\Delta t, \sigma_e^m)_{\Omega_1^m} \\ &\quad + ((\tilde{\eta}_e^m - \hat{\eta}_e^{m-1})/\Delta t, \sigma_e^m)_{\Omega_1^m} \\ &= T_{5,1} + T_{5,2} + T_{5,3}. \end{aligned}$$

A simple computation shows that

$$\Delta t|T_{5,1}| \leqq \Delta t\|\partial_t\eta_e\|_{L^\infty(J;L^2(\Omega))}\|\sigma_e^m\|_0. \tag{3.12}$$

For the estimation of $T_{5,2}$, let $\Omega_{11}^m = \{x \in \Omega_1^m : \tilde{\Delta}t_e^m = \Delta t\}$ and $\Omega_{12}^m = \Omega_1^m \backslash \Omega_{11}^m$. Next, let $F(x)$ be the function defined as follows:

$$y = F(x) = x + U_T\mu_e(q^m)q^m\Delta t, \qquad x \in \Omega_{11}^m.$$

Assumption (0.3f) implies that F is invertible for small Δt and

$$||dx/dy| - 1| \leqq U_T M_3 \Delta t \quad \text{as } \Delta t \to 0.$$

Note that $\tilde{\eta}_e = 0$ on Ω_{12}^m. Thus,

$$\Delta t T_{5,2} = \int_{\Omega_{11}^m} [\eta_e^{m-1} - \tilde{\eta}_e^{m-1}]\sigma_e^m\,dx + \int_{\Omega_{12}^m} \eta_e^{m-1}\sigma_e^m\,dx.$$

Now if $\Omega_1^* = F(\Omega_{11}^m)$, then by using an argument similar to ones in [4] and [12] and dropping momentarily m, $m-1$, and e, we can write

$$\begin{aligned}\Delta t T_{5,2} &= \int_{\Omega_{11}} \eta(x)\sigma(x)\,dx - \int_{\Omega_1^*} \eta(y)\sigma(F^{-1}(y))\left|\frac{dx}{dy}\right| dy + \int_{\Omega_{12}^m} \eta(x)\sigma(x)\,dx \\ &= \int_{\Omega_{11}\cap\Omega_1^*} \eta(y)\sigma(y) - \eta(y)\sigma(F^{-1}(y))\,dy + \int_{\Omega_{11}\cap\Omega_1^*} \eta(y)\sigma(F^{-1}(y))\left(1-\left|\frac{dx}{dy}\right|\right) dy \\ &\quad + \int_{\Omega_{11}\setminus\Omega_1^*} \eta(x)\sigma(x)\,dx + \int_{\Omega_{12}} \eta(x)\sigma(x)\,dx - \int_{\Omega_1^*\setminus\Omega_{11}} \eta(y)\sigma(F^{-1}(y))\left|\frac{dx}{dy}\right| dx.\end{aligned}$$

The first term of the expression above can be bounded by

$$\left|\int_{\Omega_{11}\cap\Omega_1^*} \eta(y)[\sigma(y)-\sigma(F^{-1}(y))]\,dy\right| \leqq K\|\eta\|_0\|\sigma\|_1 U_T M_1 \Delta t; \tag{3.13}$$

and the second term by

$$\left|\int_{\Omega_{11}\cap\Omega_1^*} \eta(y)\sigma(F^{-1}(y))\left(1-\left|\frac{dx}{dy}\right|\right) dy\right| \leqq K\|\eta\|_0\|\sigma\|_0 U_T M_3 \Delta t. \tag{3.14}$$

For the last three terms, we observe that

$$\text{meas}\,(\Omega\setminus\Omega_{11}) \leqq U_T M_1 \Delta t, \quad \text{meas}\,(\Omega\setminus\Omega_1^*) \leqq U_T M_1 \Delta t, \quad \text{meas}\,(\Omega_{12}) \leqq 2U_T M_1 \Delta t.$$

Therefore, since Ω_{11}, Ω_{12}, and Ω_1^* are contained in Ω and $\sigma|_{\partial\Omega}=0$, it follows from an argument of [4] that

$$\left|\int_{(\Omega_{11}\setminus\Omega_1^*)\cup\Omega_{12}} \eta(x)\sigma(x)\,dx\right| \leqq K\|\eta\|_0\|\sigma\|_1 U_T M_1 \Delta t \tag{3.15}$$

and

$$\left|\int_{\Omega_1^*\setminus\Omega_{11}} \eta(y)\sigma(F^{-1}(y))\left|\frac{dx}{dy}\right| dy\right| \leqq K\|\eta\|_0\|\sigma\|_1 U_T M_1 \Delta t(1+U_T M_3 \Delta t). \tag{3.16}$$

Then it follows from (3.13)-(3.16) that

$$|T_{5,2}| \leqq U_T K(M_3\|\eta_e^{m-1}\|_0\|\sigma_e^m\|_0^+(c+M_3)\|\eta_e^{m-1}\|_0\|\sigma_e^m\|_1) \tag{3.17}$$

where $K=K(M_1,\Omega_1)$.

To estimate $T_{5,3}$ write

$$\tilde{\eta}_e^{m-1} - \hat{\eta}_e^{m-1} = \int_0^1 (\partial_x \eta_e)(G(\theta))|G'(\theta)|\,d\theta$$

where $G(\theta)=(1-\theta)\tilde{x}+\theta\hat{x}$. Since, by (0.3c),

$$\begin{aligned}|G'(\theta)| = |\tilde{x}-\hat{x}| &= U_T\Delta t|\mu_e(q^{m-1})q^{m-1} - \mu_e(q_h^{m-1})q_h^{m-1}| \\ &\leqq U_T\Delta t L_1|q^{m-1}-q_h^{m-1}|,\end{aligned}$$

then

$$\begin{aligned}|T_{5,3}| &\leqq U_T L_1 \int_0^1 \left\{\int_{\Omega_1} |\partial_x\eta_e(G(\theta))||q^{m-1}-q_h^{m-1}||\sigma_e^m|\,dx\right\} d\theta \\ &\leqq K\|\eta_e\|_{1,\infty}\|q^{m-1}-q_h^{m-1}\|_0\|\sigma_e^m\|_0,\end{aligned} \tag{3.18}$$

where $K=K(U_T, L_1, \Omega)$.

To complete the estimation of T_5, we still have to analyze the term $(\eta_e^m/\hat{\Delta} t_e^m, \sigma_e^m)_{\Omega_2^m}$. First, note that

$$\Omega_2^m \subset [0, U_T M_1 \Delta t] \cup [1 - U_T M_1 \Delta t, 1].$$

Next, if $\Omega_{2,1} = \Omega_2^m \cap [0, U_T M_1 \Delta t]$, then

$$\begin{aligned} |(\eta_e^m/\hat{\Delta} t_e^m, \sigma_e^m)_{\Omega_{2,1}}| &\leqq \int_{[0, U_T M_1 \Delta t]} \left| \eta_e^m \left(\frac{x}{\hat{\Delta} t_e^m} \right) \left(\frac{\sigma_e^m}{x} \right) \right| dx \\ &\leqq (U_T M_1)^{3/2} \| \eta_e^m \|_0 \| \sigma_e^m \|_{1,\infty} \Delta t^{1/2}, \end{aligned}$$

since $\sigma|_{\partial\Omega} = 0$. Therefore

$$|(\eta_e^m/\hat{\Delta} t_e^m, \sigma_e^m)_{\Omega_{2,1}}| \leqq (U_T M_1)^{3/2} \| \eta_e^m \|_0 (\Delta t / h_d)^{1/2} \| \sigma_e^m \|_1. \tag{3.19}$$

The same estimate can be derived for the integral over the region $\Omega_{2,2} = \Omega_2^m \cap [1 - U_T M_1 \Delta t, 1]$, with the corresponding specification of $\hat{\Delta} t_e^m$.

Hence, it follows from (3.12), (3.17)-(3.19), and the equivalent estimate to (3.19) for $\Omega_{2,2}$ that

$$\begin{aligned} |T_5| \leqq K\{ [\| \partial \eta_e / \partial t \|_{L^\infty(J; L^2(\Omega))} \Delta t + M_3 \| \eta_e^{m-1} \|_0 \\ + \| \eta_e^m \|_{1,\infty} \| q^{m-1} - q_h^{m-1} \|_0] \| \sigma_e^m \|_0 \\ + [(c + M_3) \| \eta_e^{m-1} \|_0 + \| \eta_e^m \|_0 (\Delta t / h_d)^{1/2}] \| \sigma_e^m \|_1 \}, \end{aligned} \tag{3.20}$$

where $K = K(U_T, M_1, L_1, \Omega)$.

To estimate $T_6 + T_7$, let $A(q) = U_T z[\mu_e(q) + q \partial_q \mu_e(q)]$. Thus,

$$T_6 + T_7 = (A(q^m) e^m (e^m - p^m - c^m) - A(q_h^{m-1}) e_h^m (\hat{e}_h^{m-1} - \hat{p}_h^{m-1} - c^m), \sigma_e^m).$$

Next, we write

$$\begin{aligned} A(q_h^{m-1}) e_h^m (\hat{e}_h^{m-1} - \hat{p}_h^{m-1} - c^m) - A(q^m) e^m (e^m - p^m - c^m) \\ = [A(q_h^{m-1}) - A(q^m)] e_h^m (\hat{e}_h^{m-1} - \hat{p}_h^{m-1} - c^m) + A(q^m) \gamma, \end{aligned}$$

where

$$\begin{aligned} \gamma &= e_h^m (\hat{e}_h^{m-1} - \hat{p}_h^{m-1} - c^m) - e^m (e^m - p^m - c^m) \\ &= \sigma_e^m (\hat{e}_h^{m-1} - \hat{p}_h^{m-1} - c^m) - \eta_e^m (e^m - p^m - c^m) \\ &\quad + E^m [(\hat{e}_h^{m-1} - e^m) - (\hat{p}_h^{m-1} - p^m)]. \end{aligned}$$

At this point we introduce the induction hypothesis that

$$\| e_h^m \|_{0,\infty} + \| p_h^m \|_{0,\infty} \leqq M_4, \qquad m \geqq 0, \tag{3.21}$$

provided that $(\Delta t + h_d^2 + h_q^2) h_d^{-1/2}$ is bounded as Δt and $h = \max(h_d, h_q)$ tend to zero. We delay the proof of this statement to the end of the paper.

Denote by $K_{e,p}$ a bound for the L^∞-norms in space and time of e, p, E, P, e_h, p_h. Now, we want to get an estimate for the L^2-norms of $\hat{e}_h^{m-1} - e^m$ and $\hat{p}_h^{m-1} - p^m$ that is independent of the L^∞-norm of q_h. We shall derive the estimate for $\hat{e}_h^{m-1} - e^m$. The one for $\hat{p}_h^{m-1} - p^m$ can be obtained by a similar argument. To do this, write

$$\| e^m - \hat{e}_h^{m-1} \|_0 \leqq \| e^m - \tilde{e}^{m-1} \|_0 + \| \tilde{e}^{m-1} - \hat{e}_h^{m-1} \|_0.$$

By the definition of the derivative in the direction τ_e,

$$\| e^m - \tilde{e}^{m-1} \|_0 \leqq K \Delta t \left\| \frac{\partial e}{\partial \tau_e} \right\|_{L^\infty(J, L^2(\Omega))}.$$

For the other term, we write

$$\tilde{e}^{m-1}-\hat{e}_h^{m-1}=\tilde{\eta}_e^{m-1}+\tilde{E}^{m-1}-\hat{E}^{m-1}-\sigma_e^{m-1}-\hat{\sigma}_e^{m-1}+\sigma_e^{m-1}.$$

Let us first consider the term $\hat{E}^{m-1}-\tilde{E}^{m-1}$. If we proceed as in the estimate of $\hat{\eta}_e^{m-1}-\tilde{\eta}_e^{m-1}$, then

$$\|\hat{E}^{m-1}-\tilde{E}^{m-1}\|_0 \leqq U_T L_1 \|E^{m-1}\|_1 \|q^{m-1}-q_h^{m-1}\|_0.$$

Next, the same argument used to estimate T_1 leads to

$$\|\sigma_e^{m-1}-\hat{\sigma}_e^{m-1}\|_0 \leqq U_T M_1 \Delta t \|\sigma_e^{m-1}\|_1.$$

Also,

$$\|\tilde{\eta}_e^{m-1}\|_0 \leqq (1+M_3 U_t \Delta t)\|\eta_e^{m-1}\|_0,$$

where M_3 is from (0.3f). Therefore,

$$\|e^m-\hat{e}_h^{m-1}\|_0 \leqq K\Bigg[\Delta t \left\|\frac{\partial e}{\partial \tau_e}\right\|_{L^\infty(J,L^2(\Omega))}+\|E^{m-1}\|_1\|q^{m-1}-q_h^{m-1}\|_0 + \Delta t\|\sigma_e^{m-1}\|_1+(1+M_3U_t\Delta t)\|\eta_e^{m-1}\|_0+\|\sigma_e^{m-1}\|_0\Bigg],$$

where $K=K(U_T, M_1, L_1, \Omega)$.

Hence, when we combine the estimates above, it follows that

$$(3.22)\quad \begin{aligned} |(A(q^m)\gamma, \sigma_e^m)| \leqq KzM_3K_{e,p}\Bigg\{&\left(\left\|\frac{\partial e}{\partial \tau_e}\right\|+\left\|\frac{\partial p}{\partial \tau_p}\right\|_{L^\infty(J,L^2(\Omega))}\right)\Delta t \\ &+(\|E^{m-1}\|_1+\|P^{m-1}\|_1)\|q^{m-1}-q_h^{m-1}\|_0 \\ &+(\|\sigma_e^{m-1}\|_1+\|\sigma_p^{m-1}\|_1)\Delta t+\|\sigma_e^m\|_0+\|\sigma_e^{m-1}\|_0 \\ &+\|\sigma_p^{m-1}\|_0+\|\eta_e^m\|_0 \\ &+(\|\eta_e^{m-1}\|_0+\|\eta_p^{m-1}\|_0)(1+M_3U_t\Delta t)\Bigg\}\|\sigma_e^m\|_0, \end{aligned}$$

with $K=K(U_T, M_1, \Omega)$.

Next,

$$|A(q_h^{m-1})-A(q^m)| \leqq zU_T\{|\mu_e(q_h^{m-1})-\mu_e(q^m)|+|q_h^{m-1}\partial_q\mu_e(q_h^{m-1})-q^m\partial_q\mu_e(q^m)|\},$$

and it follows from (0.3c) and (0.3e) that

$$|A(q_h^{m-1})-A(q^m)| \leqq zU_T(L_2+L_3)|q^m-q_h^{m-1}|.$$

Thus, combining this last inequality with (3.21) and recalling that $L_3 \leqq M_3$, we have

$$(3.23)\quad \begin{aligned} &|([A(q_h^{m-1})-A(q^m)]e_h^m(\hat{e}_h^{m-1}-\hat{p}_h^{m-1}-c^m), \sigma_e^m)| \\ &\qquad \leqq z(K_{e,p})^2M_3K\|q^m-q_h^{m-1}\|_0\|\sigma_e^m\|_0, \end{aligned}$$

with $K=K(U_T, M_1, L_2, \Omega)$.

Therefore, inequalities (3.21) and (3.23) imply that

$$
\begin{aligned}
|T_6+T_7| \leqq zM_3K_{e,p}K \Bigg\{ & \left(\left\| \frac{\partial e}{\partial \tau_e} \right\|_{L^\infty(J,L^2(\Omega))} + \left\| \frac{\partial p}{\partial \tau_p} \right\|_{L^\infty(J,L^2(\Omega))} \right) \Delta t \\
& + (\|E^{m-1}\|_1 + \|P^{m-1}\|_1 + K_{e,p}) \|q^{m-1} - q_h^{m-1}\|_0 \\
& + K_{e,p}(\|\partial_t q\|_{L^\infty(J,L^2(\Omega))} + \|\sigma_e^{m-1}\|_1 + \|\sigma_p^{m-1}\|_1)\Delta t \\
& + \|\sigma_e^m\|_0 + \|\sigma_e^{m-1}\|_0 + \|\sigma_p^{m-1}\|_0 + \|\eta_e^m\|_0 \\
& + (\|\eta_e^{m-1}\|_0 + \|\eta_p^{m-1}\|_0)(1+M_3U_T\Delta t) \Bigg\} \|\sigma_e^m\|_0,
\end{aligned} \tag{3.24}
$$

with $K = K(U_T, M_1, L_2, \Omega)$.

Finally, to estimate T_8, we note that since

$$
|\hat{R}^{m-1} - R^m| \leqq L_R\{|\hat{e}_h^{m-1} - e^m| - |\hat{p}_h^{m-1} - p^m|\},
$$

then

$$
\begin{aligned}
|T_8| \leqq K \Bigg\{ & \left(\left\| \frac{\partial e}{\partial \tau_e} \right\|_{L^\infty(J,L^2(\Omega))} + \left\| \frac{\partial p}{\partial \tau_p} \right\|_{L^\infty(J,L^2(\Omega))} \right) \Delta t \\
& + (\|E^{m-1}\|_1 + \|P^{m-1}\|_1) \|q^{m-1} - q_h^{m-1}\|_0 \\
& + (\|\sigma_e^{m-1}\|_1 + \|\sigma_p^{m-1}\|_1)\Delta t + \|\sigma_e^m\|_0 + \|\sigma_p^m\|_0 \\
& + (\|\eta_e^{m-1}\|_0 + \|\eta_p^{m-1}\|_0)(1+M_3U_t\Delta t) \Bigg\} \|\sigma_e^m\|_0,
\end{aligned} \tag{3.25}
$$

where $K = K(L_R, U_T, M_1, L_1, \Omega)$.

Then, estimates (3.8)-(3.11), (3.20), (3.24), and (3.25) imply that

$$
\begin{aligned}
\Sigma T_i(\sigma_e^m) \leqq (zM_3K_{e,p}+1)K \Bigg\{ & \Bigg(\left\| \frac{\partial e}{\partial \tau_e} \right\|_{L^\infty(J,L^2(\Omega))} + \left\| \frac{\partial p}{\partial \tau_p} \right\|_{L^\infty(J,L^2(\Omega))} \\
& + \|\partial_t q\|_{L^\infty(J,L^2(\Omega))} + \|\sigma_e^{m-1}\|_1 + \|\sigma_p^{m-1}\|_1 + 1 \Bigg) \Delta t \\
& + K_{1,e,p}(\|q^{m-1} - q_h^{m-1}\|_0 + \Delta t) \\
& + \|\sigma_e^m\|_0 + \|\sigma_e^{m-1}\|_0 + \|\sigma_p^m\|_0 + \|\sigma_p^{m-1}\|_0 + \|\eta_e^m\|_0 \\
& + (\|\eta_e^{m-1}\|_0 + \|\eta_p^{m-1}\|_0)(1+M_3U_T\Delta t) \Bigg\} \|\sigma_e^m\|_0 \\
+ \Bigg\{ K_{e,p}(\|q^{m-1} - q_h^{m-1}\|_0 + \Delta t) + & \left(\frac{\Delta t}{h_d} \right)^{1/2} \|\eta_e^m\|_0 + \|\eta_e^{m-1}\|_0 \Bigg\} \|\sigma_e^m\|_1
\end{aligned} \tag{3.26}
$$

with $K = K(U_T, M_1, L_1, L_2, L_R, \|e\|_2, \|p\|_2, \Omega)$ and $K_{1,e,p} = K_{1,e,p}(K_{e,p}, \|e\|_{1,\infty}, \|p\|_{1,\infty})$.

Then if we combine inequalities (2.1), (3.2), (3.6), and (3.26) it follows that

$$
\begin{aligned}
& (\Delta t)^{-1}[\|\sigma_e^m\|_0^2 - \|\sigma_e^{m-1}\|_0^2] + D^*\|\sigma_e^m\|_1^2 - \left(\frac{D^*}{2} \right) \|\sigma_e^{m-1}\|_1^2 \\
& \leqq \kappa K\{M_{e,p}^2 \Delta t^2 + \|\sigma_e^m\|_0^2 + \|\sigma_e^{m-1}\|_0^2 + \|\sigma_p^{m-1}\|_0^2 + h_d^3\Delta t + (1+\|c\|_1)h_q^4 + h_d^4\},
\end{aligned} \tag{3.27}
$$

with $\kappa = z^2M_3K_{e,p}K_{1,e,p} + zK_{1,e,p}$, $K = K(U_T, M_1, L_1, L_2, L_R, \|e\|_2, \|p\|_2, \Omega)$, and

$$
M_{e,p} = \left\{ \left\| \frac{\partial e}{\partial \tau_e} \right\|_{L^\infty(J,L^2(\Omega))} + \left\| \frac{\partial p}{\partial \tau_p} \right\|_{L^\infty(J,L^2(\Omega))} + \|\partial_t q\|_{L^\infty(J,L^2(\Omega))} + 1 \right\}.
$$

An estimate for σ_p^m can be derived in a similar way. Now set

$$\|\sigma\|_j^2 = \|\sigma_e\|_j^2 + \|\sigma_p\|_j^2 \quad \text{for } j = 0 \text{ or } 1,$$

and add (3.27) and its analogue. Then

$$(\Delta t)^{-1}[\|\sigma^m\|_0^2 - \|\sigma^{m-1}\|_0^2] + D^*\|\sigma^m\|_1^2 - \left(\frac{D^*}{2}\right)\|\sigma^{m-1}\|_1^2$$
$$\leqq \kappa K\{\|\sigma^m\|_0^2 + \|\sigma^{m-1}\|_0^2 + M_{e,p}^2\Delta t^2 + h_d^3\Delta t + (1+\|c\|_1)h_q^4 + h_d^4\}.$$

Now multiply by Δt and sum on m from $m=1$ to $m=n$. If $1-\kappa K\Delta t$ is bounded below by, say, .5, then by applying the Gronwall lemma [7] it follows that

$$\max_{1\leqq m\leqq n} \|\sigma^m\|_0^2 + \sum_{m=1}^{n} \|\sigma^m\|_1^2\Delta t \leqq K \exp \kappa(\|\sigma^0\|_0^2 + \|\sigma^0\|_1^2\Delta t$$
$$+ M_{e,p}^2\Delta t^2 + h_d^3\Delta t + (1+\|c\|_1)h_q^4 + h_d^4),$$

provided that induction hypothesis (3.21) holds for $0\leqq m\leqq n-1$.

Hence, if we take e_h^0 and p_h^0 to be the linear interpolants of the initial data, i.e., E^0 and P^0, respectively, then

$$\max_{1\leqq m\leqq n} \|\sigma^m\|_0 + \left(\sum_{m=1}^{n} \|\sigma^m\|_1^2\Delta t\right)^{1/2} \leqq F\{\Delta t + h_q^2 + h_d^2 + (h_d^3\Delta t)^{1/2}\}, \tag{3.28}$$

where $F = F(K, \kappa, M_{e,p}, \|c\|_1)$, if (3.21) holds for $1\leqq m\leqq n-1$.

Let us verify (3.21). We have seen that (3.21) holds for $n=1$, so we assume it valid for $n-1$. It follows from (3.28) that

$$\max_{1\leqq m\leqq n} \|e_h^m\|_{0,\infty} \leqq \max_{1\leqq m\leqq n} \|e^m\|_{0,\infty} + h_d^{-1/2} \max_{1\leqq m\leqq n} \|\sigma_e^m\|_0,$$
$$\leqq C + Fh_d^{-1/2}\{\Delta t + h_q^2 + h_d^2 + (h_d^3\Delta t)^{1/2}\},$$

since e is bounded. A similar inequality can be derived for p_h^m. Therefore (3.21) holds for n provided that $h_d^{-1/2}\{\Delta t + h_q^2 + h_d^2\}$ to be bounded as Δt and h tend to zero.

The following theorem has then been proved.

THEOREM. *Let q, e, and p lie in $L^\infty(J, H^2(\Omega)) \cap W^{1,\infty}(\Omega)$, and let $\mu_\alpha(q)$ and $D_\alpha(q)$ be functions of q satisfying* (0.3), *for $\alpha = e$ or p. If we choose $e_h^0 = E^0$, $p_h^0 = P^0$, and let $(\Delta t + h_d^2 + h_q)h_d^{-1/2}$ be bounded as Δt and h tend to zero, then*

$$\max_n \{\|e^n - e_h^n\|_0 + \|p^n - p_h^n\|_0 + \|q^n - q_h^n\|_0\}$$
$$\leqq K \exp\left(\frac{\kappa}{2}\right)\{M_{e,p}\Delta t + (1+\|c\|_1)h_q^2 + h_d^2 + (h_d^3\Delta t)^{1/2}\},$$

with $M_{e,p} = M_{e,p}T$, where T is the final time, and K, κ, and $M_{e,p}$ are as described in (3.27).

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