On the rate of relaxation for the Landau kinetic equation and related models

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Abstract

We study the rate of relaxation to equilibrium for Landau kinetic equation and some related models by considering the relatively simple case of radial solutions of the linear Landau-type equations. The well-known difficulty is that the evolution operator has no spectral gap, i.e. its spectrum is not separated from zero. Hence we do not expect purely exponential relaxation for large values of time $t > 0$. One of the main goals of our work is to numerically identify the large time asymptotics for the relaxation to equilibrium.

We recall the work of R. Strain and Y. Guo, [10, 11], who rigorously show that the expected law of relaxation is $\exp(-ct^{2/3})$ with some $c > 0$, In this manuscript, we find an heuristic way, performed by asymptotic methods, that finds this "law of two thirds", and then study this question numerically. More specifically, the linear Landau equation is approximated by a set of ODEs based on expansion in generalized Laguerre polynomials. We analyse the corresponding quadratic form and the solution of these ODEs in detail. It is shown that the solution has two different asymptotic stages for large values of time t and maximal order of polynomials N: the first one focus on intermediate asymptotics which agrees with the "law of two thirds" for moderately large values of time t and then the second one on absolute purely exponential asymptotics for very large t, as expected for linear ODEs. We believe that appearance of intermediate asymptotics in finite dimensional approximations must be typical for different classes of equations in functional spaces (some PDEs, Boltzmann equations for soft potentials, etc.) and that our methods can be applied to related problems.

Key words: Landau equation, Kinetic models, Generalized Laguerre polynomial, Spectral gap, Fractional exponential decay.

AMS subject classifications. 52.25.Dg, 52.25.Kn, 02.60.-x, 02.70.-c, 35P05, 15A18, 42C05

1 Introduction

The Landau kinetic equation is one of the most important mathematical objects in Plasma Physics [5][6]. It is also a very interesting equation from purely mathematical point of view. A brief review of the mathematical properties and main references can be found in recent papers [3][2]. The aim of the present paper is to discuss a particular question of the rate of relaxation to equilibrium for the spatially homogeneous Landau equation and its models. It was proved by Strain and Guo [10][11] that the rate cannot be faster than $\exp(-\lambda t^{2/3})$ for the "true" (Coulomb case) Landau equation. The proof of this fact is rather complicated. We shall try to make this result more "visible" below.

The generalized radial dimensionless Landau equation for the distribution function $f(x,t)$, where $x = |v|^2$, $v \in \mathbb{R}^3$, and t denote squared velocity and time respectively, reads [2]

$$\partial_t f(x,t) = x^{-\theta}\partial_x \int_0^\infty dy \min(x^{1+\theta}, y^{1+\theta})(\partial_x - \partial_y) f(x,t) f(y,t) \qquad (1)$$

where $\theta = \frac{1}{2}$ for the classical Landau equation with Coulomb interaction. The model equations with $0 \leq \theta < \frac{1}{2}$ can be also included in the consideration. Properties of these generalized Landau equations are briefly discussed in [2]. In the present paper, we confine ourselves to the simplest case of the linear Landau equation which will be introduced in Section 2.

The paper is organized as follows. In Section 2, we consider the linear Landau equation for arbitrary $\theta \in [0, \frac{1}{2}]$ and discuss the standard way to obtain the rate of relaxation in case of purely exponential decay to equilibrium. In Section 3 we approximate this equation by a set of linear ordinary differential equations (ODEs) based on expansion in generalized Laguerre polynomials with maximal order N. Then the approximate solution of the Landau equation is defined by $\exp(-tM)$, where M is a corresponding N by N matrix. Main properties of this matrix are well-known: it is symmetric and has one zero and $N-1$ positive eigenvalues. The asymptotic behaviour for large t (relaxation rate) of the solution is then defined by the smallest positive eigenvalue. This eigenvalue, however, tends for our equation to zero for increasing values of N. We show it numerically in Section 4. The only exceptional case $\theta = 0$ is exactly solvable and not very interesting for applications. Sections 5, 6 are devoted mainly to true Landau equation for Coulomb forces with $\theta = \frac{1}{2}$. First we show in Section 5 a heuristic way to guess the fractional exponential asymptotics $\exp(-at^{2/3})$ ("law of two thirds") by considering a simplified version of asymptotic solution from [2]. Finally, in Section 6, we study in detail the numerical solution of the corresponding set of N ODEs for large N (up to $N = 100$) and show how to extract the intermediate fractional exponential asymptotics. Note that for any fixed N the solution of such ODEs has purely exponential asymptotics for sufficiently large values of time. However, the intermediate stage is seen very well for large N, as discussed in Section 6.

The general question of numerical study of precise asymptotics in the case of absence of the spectral gap is difficult and important not only for the Landau equation. Therefore we hope that our approach based on numerical study of intermediate asymptotics of finite-dimensional approximations of ODEs can be useful for many applied problems.

2 The Linear Landau Equation and Quadratic Form

The linear Landau equation is an equation for a test particle, which collides with equilibriumly distributed "field" particles. It can be obtained by rewriting the homogeneous Landau equation (1) in the form of nonlinear diffusion equations for $f(x,t)$, then replacing the $f(y,t)$ in the integral terms by a constant Maxwellian, say, $M(y) = \exp(-y)$. Finally, we denote $t = (1+\theta)\tilde{t}$.

The generalized linear isotropic Landau equation reads (tildes are omitted)

$$\partial_t f(x,t) = x^{-\theta}\partial_x \left(\mathcal{D}_\theta(x)(\partial_x f(x,t) + f(x,t))\right), \quad x,t \geq 0\,, \tag{2}$$

where

$$\mathcal{D}_\theta(x) = \int_0^x y^\theta e^{-y} dy, \quad 0 \leq \theta \leq \frac{1}{2}\,. \tag{3}$$

The "true" 3-D Landau equation corresponds to $\theta = \frac{1}{2}$. The case $\theta = 0$ can be exactly solvable through the Laplace transform.

If we assume that

$$f(x,0) = f_0(x), \quad \int_0^\infty x^\theta f_0(x) dx = \Gamma(1+\theta)\,, \tag{4}$$

where the Gamma function is defined by

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx, \quad z > 0\,, \tag{5}$$

then, we can expect that, for any $x \geq 0$,

$$\lim_{t\to\infty} f(x,t) = e^{-x} . \tag{6}$$

Our goal is to study the rate of convergence.

Consider a perturbation around the equilibrium, that is,

$$f(x,t) = e^{-x}(1+\varphi(x,t)) . \tag{7}$$

Plugging this back to the generalized linear Landau equation (2) gives an equation for φ

$$x^\theta e^{-x}\partial_t\varphi(x,t) = -\mathcal{L}(\varphi)(x,t) , \tag{8}$$

where the linear operator $\mathcal{L}$ reads

$$\mathcal{L}(\varphi)(x,t) = -\partial_x[\mathcal{D}_\theta(x)e^{-x}\partial_x\varphi(x,t)] . \tag{9}$$

We define the weighted L^2 norm as

$$\|\varphi\|^2 = \int_0^\infty x^\theta e^{-x}\varphi^2 dx . \tag{10}$$

Then the equality

$$\frac{1}{2}\frac{d}{dt}\|\varphi\|^2 = -\langle\mathcal{L}(\varphi),\varphi\rangle = -\int_0^\infty \mathcal{D}_\theta(x)e^{-x}[\partial_x\varphi(x,t)]^2 dx , \tag{11}$$

where the *Dirichlet form* $\langle\mathcal{L}(\varphi),\varphi\rangle$ ($\langle\cdot,\cdot\rangle$ denotes the usual unweighted L^2 innner product) is obtained through integration by parts.

Obviously, $\mathcal{L}$ is a positive operator. Its smallest eigenvalue is 0 with multiplicity 1 with its eigenspace spanned by constant functions. In particular, we can conclude, at the formal level, that

$$\varphi(x,t) = O\left(e^{-\lambda_\theta t}\right), \quad \text{as } t\to\infty , \tag{12}$$

where λ_θ is the *spectral gap*, if exists, of operator $\mathcal{L}$, defined as the minimized Rayleigh quotient of $\mathcal{L}$

$$\lambda_\theta = \min \frac{\langle\mathcal{L}(\varphi),\varphi\rangle}{\|\varphi\|^2} \quad \text{s.t.} \int_0^\infty x^\theta e^{-x}\varphi(x)dx = 0 , \tag{13}$$

that is, φ is orthogonal to the eigenspace of eigenvalue 0.

That means, we can expect an exponential decay when the state is close to equilibrium, where the decay rate is given by $\lambda_\theta > 0$, if exists. In addition, the existence can be analytically proved for $\theta = 0$. For $\theta > 0$, we would like to study it numerically.

Remark. Note that the simplest way to solve numerically Equations (1) or (2) would be to use one of well-known finite difference schemes. There are a lot of publications on numerical solutions of the Landau-type equations based on such schemes (see e.g. [8] and references there for a review). Our goal, however, is not to construct one more numerical solution of Equation (2). The goal is to study a precise rate of relaxation of solutions to equilibrium for very large values of time (especially when $\lambda_\theta = 0$). Therefore we use below another way of discretization of Equation (2) based on Laguerre polynomials..

3 Approximation in Space of Laguerre Polynomials

We have already known that the exponential decay rate is determined by the spectral gap. Thus we need to numerically solve the constraint minimization problem (13). This is done by taking a finite-dimensional approximation space for φ, and examine the behavior for increasing dimensions of the approximate spaces.

To this goal, we introduce an orthogonal basis $\{\varphi_n(x)\}$, $n = 0, 1, \ldots$, for the weighted L^2 space with norm (10), such that $\varphi_0(x) = \text{const}$, and

$$\langle \varphi_n, \varphi_m \rangle_w = \int_0^\infty x^\theta e^{-x} \varphi_n(x) \varphi_m(x) dx = \delta_{n,m}\,, \tag{14}$$

where $\langle \cdot, \cdot \rangle_w$ denotes the weighted L^2 inner product with weight $w = x^\theta e^{-x}$.

Such requirements are perfectly satisfied by the normalized *generalized Laguerre polynomials*[1],

$$\varphi_n(x) = \frac{L_n^\theta(x)}{\|L_n^\theta\|}, \quad n = 0, 1, \ldots\,. \tag{15}$$

In particular,

$$L_0^\theta = 1, \quad L_1^\theta = 1 + \theta - x, \quad L_n^\theta = \sum_{k=0}^{n} a_k^{(n,\theta)} x^k\,, \tag{16}$$

where the coefficients

$$a_k^{(n,\theta)} = \frac{(-1)^k}{k!} \begin{pmatrix} n+\theta \\ n-k \end{pmatrix} = \frac{(-1)^k}{k!} \frac{\Gamma(n+\theta+1)}{\Gamma(n-k+1)\Gamma(\theta+k+1)}\,, \tag{17}$$

and the weighted norm of L_n^θ is given by

$$\|L_n^\theta\|^2 = \int_0^\infty x^\theta e^{-x}[L_n^\theta]^2 dx = \frac{\Gamma(n+\theta+1)}{n!}\,. \tag{18}$$

Thus, for a fixed order of approximation N, we consider the minimization problem (13) for

$$\varphi(x) = \sum_{n=1}^{N} u_n \varphi_n(x)\,, \tag{19}$$

where u_n are the coefficients. Note that, the summation starts from $n = 1$, because $\langle \varphi_0, 1\rangle_w = 0$ for any $n > 1$. This automatically fullfill the constraint in (13).

Thus, with this approximation, the Dirichlet form in (11) can be written as a quadratic form

$$\langle \mathcal{L}(\varphi), \varphi\rangle = \mathbf{u}^T G \mathbf{u}\,, \tag{20}$$

where $\mathbf{u} = (u_1, \ldots, u_N)$ is the coefficient vector. The entries G_{nm} of the symmetric matrix G, obtained by some straightforward calculations, are given by

$$\begin{aligned} G_{nm} &= \int_0^\infty \mathcal{D}_\theta(x) e^{-x} \varphi_n'(x)\varphi_m'(x) dx \\ &= \frac{1}{\|L_n^\theta\|\|L_m^\theta\|} \sum_{k=1}^{n}\sum_{l=1}^{m} k l a_k^{(n,\theta)} a_l^{(m,\theta)} (k+l-2)! S(k+l-2)\,, \end{aligned}$$

with $S(p)$ given by

$$S(p) = \sum_{j=0}^{p} \frac{\Gamma(j+\theta+1)}{2^{j+\theta+1} j!}\,. \tag{21}$$

Since the orthogonality constraint has already been taken into account (19), the smallest eigenvalue of weight matrix G will be an approximation to the spectral gap.

4 Numerical Results for Quadratic Forms

It's not hard to prove that with increasing order of approximations, the numerical spectral gap is decreasing. For example, when $N = 1$, the matrix G reduces to one single entry

$$\lambda_\theta = G_{11} = \frac{1}{1+\theta} 2^{-(1+\theta)}\,. \tag{22}$$

It can be computed analytically that for $\theta = 0$, there exists spectral gap $\lambda_0 = \frac{1}{4}$. In particular, the above first order approximation gives a rough approximation $\lambda_0 \approx \frac{1}{2}$. We compute the smallest eigenvalues of G and study its asymptotic behavior with increasing N. Here shows the results for $\theta = 0$ and $\theta = \frac{1}{2}$. For $\theta = 0$, the convergence to the analytical value $1/4$

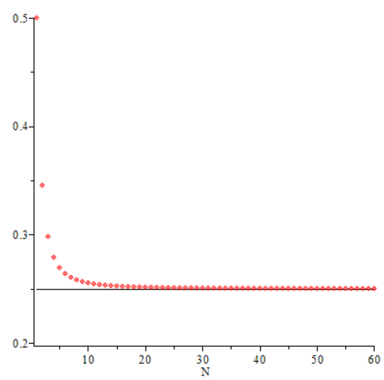

Figure 1: The numerical spectral gaps of linear Landau operator for $\theta = 0$ with increasing N

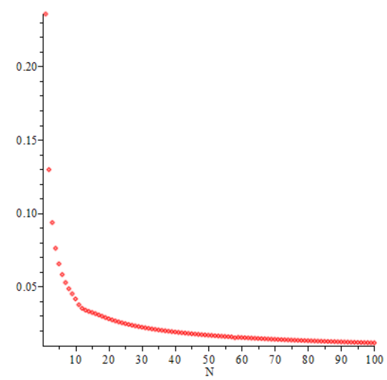

Figure 2: The numerical spectral gaps of linear Landau operator for $\theta = 1/2$ with increasing N

can be observed; for $\theta = 0.5$, the "gap" goes all the way down to zero with increased orders of basis Laguerre polynomials (Only results w.r.t N up to 100 are plotted, but "gap" will continue to decrease towards zero when order $N > 100$), which implies there is no such a spectral gap.

Remark. The definition of the entries of the matrix G involves arithmetics among numbers of enormously different magnitudes, say the factorials and Gamma functions. To avoid large error caused by the fixed-precision floating point arithmetic standard in C/C++, we include the package GNU MPFR (for GNU Multiple Precision Floating-Point Reliably [4]), which is a portable C library for arbitrary-precision binary floating-point computation with correct rounding, based on GNU Multi-Precision Library.

5 Asymptotic Form of the Equation and the "Law of Two Thirds"

Thus, the absence of spectral gap for the main case $\theta = \frac{1}{2}$ shows that it is impossible to get any estimate like $O(\exp(-\lambda t))$, with $\lambda > 0$, for the rate of relaxation. In fact, one expects to obtain the order $O(\exp(-\lambda t^{2/3}))$ based on rigorous results of Strain and Guo [10][11]. We shall see below how the "law of two thirds" can be guessed on the basis of some heuristic arguments.

Let us consider Eqn (2) for $x \to \infty$. Then

$$\mathcal{D}_\theta(x) \xrightarrow[x\to\infty]{} \int_0^\infty dx x^\theta e^{-x} = \Gamma(\theta+1), \quad 0 \le \theta \le \frac{1}{2}. \tag{23}$$

We obtain, in the domain $x \gg 1$ and with $\tilde{t} = t\Gamma(\theta+1)$,

$$\partial_{\tilde{t}} \tilde{f}(x,\tilde{t}) = x^{-\theta}(\partial_x^2 + \partial_x)\tilde{f}(x,\tilde{t}) \,, \tag{24}$$

$$\tilde{f}(x,\tilde{t}) = f(x,t). \tag{25}$$

The same asymptotic form can be obtained for the nonlinear case (see [2]). We omit the tildes for sake of simplicity and denote

$$f(x,t) = e^{-x}F(x,t)\,, \quad F(x,t) \xrightarrow[t\to\infty]{} 1. \tag{26}$$

Then, we obtain

$$F_t + x^{-\theta}F_x = x^{-\theta}F_{xx}\,, \quad x \gg 1. \tag{27}$$

We are mainly interested in the Coulomb case $\theta = \frac{1}{2}$, but the same considerations can be used for any $0 < \theta < \frac{1}{2}$. The frequency of Coulomb collisions decays as $x^{-3/2}$ for large x. Therefore, we expect that the relaxation process has two stages:

(a) fast relaxation for thermal velocities $x \simeq O(1)$ provided that the initial data has a compact support concentrated in thermal domain;

(b) slow propagation to the domain of large velocities, i.e $x \gg 1$.

Therefore, we assume that, at some time $t_0 \gg 1$, the relaxation process for small and moderately large velocities (energies) $x \in [0, x_0]$, $x_0 \gg 1$, is practically finished and

$$F(x,t_0) \approx \eta(x_0 - x), \tag{28}$$

where $\eta(x)$ stands for the unit step function. Then we need to solve the above equation for $t > t_0$. At the first approximation, we neglect the diffusion term and obtain

$$F_t + x^{-\theta} F_x = 0\,, \quad F|_{t=t_0} = \eta(x_0 - x)\,, \quad t > t_0. \tag{29}$$

Then the solution reads (traveling wave)

$$F(x,t) = \eta[x_f(t) - x]\,, \tag{30}$$

where

$$x_f(t) = [x_0 + (1+\theta)t]^{\frac{1}{1+\theta}} \tag{31}$$

corresponds to the velocity of the wave front. The corresponding distribution function reads

$$f(x,t) = e^{-x} F(x,t). \tag{32}$$

Its distance to the equilibrium distribution e^{-x} is given by

$$d(f, e^{-x}) = \|f(x,t) - e^{-x}\|_{L^\infty} = \sup_{x > x_f(t)} e^{-x} = \exp(-x_f(t)). \tag{33}$$

Hence, for large t we obtain,

$$d(f, e^{-x}) = \exp\left(-c_\theta t^{\frac{1}{1+\theta}} \left(1 + O(t^{-1})\right)\right), \tag{34}$$

where $c_\theta = (1+\theta)^{\frac{1}{1+\theta}}$. If $\theta = \frac{1}{2}$, we obtain the correct power $q = \frac{2}{3}$. This is probably the simplest heuristic way to "derive" this result from the Landau equation. The more detailed study of asymptotic Landau equation in the nonlinear case was performed in [2]. The result shows that the diffusion term influences only the shape of the wave. Therefore, the conclusion is the same: the rate of relaxation for the Landau equation (linear or nonlinear) is of order of $\exp(-ct^{2/3})$ with some $c > 0$. An interesting fact is that this conclusion can be obtained heuristically from asymptotics for large velocities.

We consider below only the value $\theta = \frac{1}{2}$ in Equation (8). The numerical evidence for the "step function"-like traveling wave $F(x,t)$ can be found in [2]. Here, under the setting of Laguerre polynomial expansion and with diffusion term being kept, we will try to numerically confirm this observation and as well verify the "two thirds" fractional relaxation rate.

6 Numerical Results for the Landau Equation

The target equation is the one for perturbation $\varphi(x,t)$, i.e Equation (8 - 9). And thus $F(x,t) = 1 + \varphi(x,t)$ in Equation (32). The $\varphi(x,t)$ will be approximated by expansion of Laguerre polynomials as in (15). We multiply both sides of Equation (8) by test function $\varphi_n(x)$, then it's not hard to obtain the following linear ODE system

$$\frac{d}{dt}U(t) = -GU \tag{35}$$

where $U = (u_1, \ldots, u_N)^T$ is the vector of coefficients; matrix G has already been analytically given in (21).

Solving the above linear system (35) is relatively simple. However, this system is extremely stiff due to the asymptotically vanishing spectral gap for matrix G, as observed in numerical experiment (see Fig. 2). Therefore, an implicit numerical method should be applied. Here, we adopt the modified backward Euler scheme

$$U(t+dt) = U(t) - \frac{dt}{2}(GU(t+dt) + GU(t)), \tag{36}$$

with dt being the time step.

Another thing worthy careful attention is, with increasing order of Laguerre polynomials and expanding tails x, the extremely contrasting magnitudes of float-pointing numbers, especially from factorials, Gamma functions and large order polynomials, will easily invalidate the computations. Based on these considerations, we choose to numerically solve (36) with Maple [7]. Maple applies hybrid symbolic-numeric approach. It supports both hardware precision and infinite precision computations. By increasing the precision, round-off error is reduced and problems of catastrophic cancellation can be avoided.

The initial function $\varphi(x, 0)$ can be arbitrarily chosen. Here, if assume the ultimate equilibrium state $\exp(-x)$, then initial $\varphi(x, 0)$ should satisfy condition (4) due to the conservation of mass. Suppose we take an initial distribution $f_0(x) = \exp(-x)(1 + \varphi(x, 0)) = a\exp(-bx)$. With $b > 1$ fixed, condition (4) gives $a = b^{3/2}$. We project $\varphi(x, 0)$ onto space of Laguerre polynomials up to order N and get

$$f_0(x) = \exp(-x)(1 + \sum_{n=1}^{N} u_n^0 \varphi_n(x)). \tag{37}$$

Note, the summation starts from $n = 1$, rather than $n = 0$, due to condition (4).

By considering the orthogonality of Laguerre polynomials (14), we can easily calculate the coefficients

$$u_n^0 = a \int_0^\infty x^{\frac{1}{2}} \exp(-bx)\varphi_n(x), \qquad n = 1 \dots N \tag{38}$$

Thus, an initial vector for system (36) would be given as $U(t = 0) = (u_1^0, \dots, u_N^0)$. To better capture the intermediate asymptotics, in the following numerical experiments, we choose $b = 6$ for a more "concentrated" initial distribution function.

Despite the above heuristic discussions in the sense of L^∞ distance in (33), the result of Strain and Guo [10][11] is expected to be valid for any reasonable norm, in particular the standard L^2 distance. We denote

$$d(x,t) = e^{-x}\varphi(x,t) = e^{-x}\sum_{n=1}^{N} u_n\varphi_n(x) \tag{39}$$

If we adopt weighted L^2 norm with weight $e^x x^{1/2}$, then

$$\|d(x,t)\|_w^2 = \sum_{n,m=1}^{N} u_n u_m \int_0^\infty e^{-x}x^{1/2}\varphi_n(x)\varphi_m(x)dx = \sum_{n=1}^{N} u_n^2 \tag{40}$$

This will be adopted in the following as the measurement of distance to equilibrium.

Fig 3 shows the traveling wave, which is the normalized solution $f(x,t)$ by equilibrium $\exp(-x)$, i.e the function $F(x,t)$ given in Equation (32). It agrees with what has been predicted in [2]. It asymptotically behaves like a step function and keeps constant width for much large time and velocities out of the thermal domain.

Fig 4 is the L^2 distance to equilibrium (40). The numerical results for the above function $d(x,t)$ show a rapid decay in the thermal domain and then slow relaxation for large velocities. This is under expectation. More detailed intermediate and large time asymptotics can be observed by exploring its double-logarithm measurements, which will be done later.

The next is to numerically verify the fractional power in the exponential decay. As discussed above, after relaxing for moderately long time, the $t^{2/3}$ term is well believed to dominate in the exponential relaxation. Then, for the moderate "far" tails, we take the double-logarithm of the distance $d(f, e^{-x})$, that is,

$$\log\left(-\log\left(d(f,e^{-x})\right)\right) \sim c + \frac{2}{3}\log t, \quad t \gg 1 \tag{41}$$

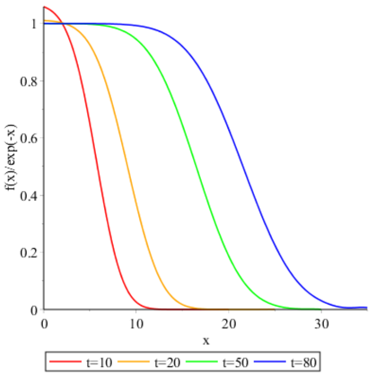

Figure 3: The travelling waves, $N = 50$

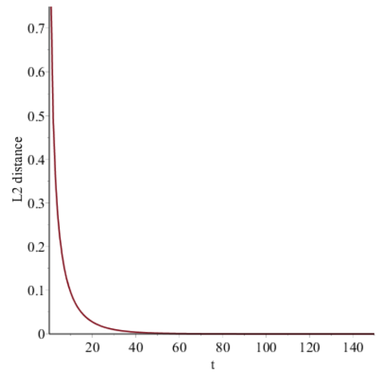

Figure 4: The distance to equilibrium, $N = 50$. Only plot up to t=150.

which is expected to be a linear relationship, with a slope being close to $\frac{2}{3}$. However, in actual numerical simulations, this fractional exponential decay is just an intermediate asymptotics, and will only last for a moderately long time. If we let the relaxation proceeds for sufficiently long time, we should expect a pure exponential decay with a decay rate determined by the spectral gap of G in (35). This decay rate, or spectral gap, will be smaller and smaller with increasing order N, as shown in Fig 2. In other words, if measured in double-log (41), there should be another linear segment after sufficiently long time, with a slope close to 1.0. And with increasing order N, it should be slower to enter the domain of pure exponential decay.

Remark Since there is no definite boundary between the fractional and pure exponential decaying realm, here is how we numerically capture in the double-log plot the two linear segments. The double-log curve will be linearly fitted starting from zero, piece by piece, with each piece spanning, e.g 100 units. During the process of sliding the fitting line toward the end of curve, we will reach several pieces with a steady (almost constant) slope. Such a segment will be treated as a line. There should be two such segments for each curve (each fixed N), representing the fractional and pure exponential decaying realm respectively. Fig 5 shows the two line segments numerically captured in the double-log plot for $N = 50$.

Fig 6 is the overlay of the double-log plots for different N. For each N, the two linear segments are visually clear, as already been discussed. The trailing lines for each N always have a slope close to 1.0, as long as the simulation time is long enough. However, as readers can see from Fig 6, as

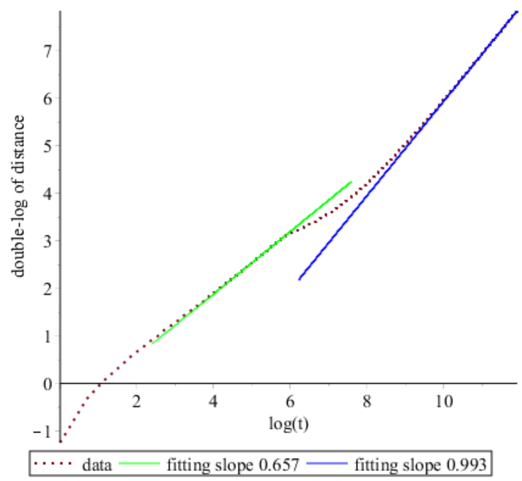

Figure 5: Two linear segments numerically captured in the double-log plot for $N = 50$, representing the fractional and pure expoential decaying realms respectively.

N increases, it takes longer time to enter the domain of pure exponential decay. This is also expected due to smaller spectral gaps.

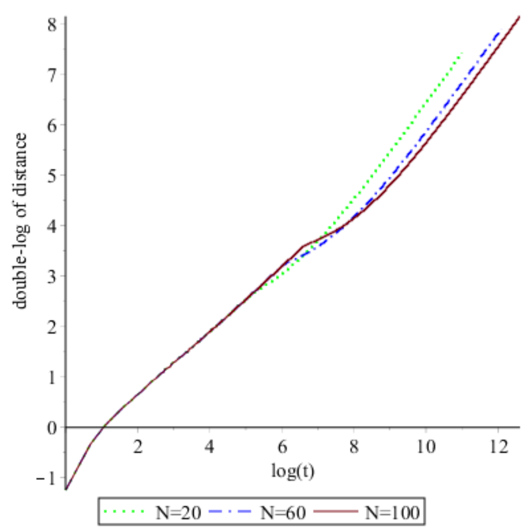

Figure 6: Overlay of double-log distance plots for $N = 20$, $N = 60$ and $N = 100$.

The slope of the intermediate line is of most interest to us. Table 1 lists the sliding fitting results introduced in the aforementioned remark. It's evident that the steady slopes of the intermediate line segments approach $\frac{2}{3}$, which numerically verifies the intermediate asymptotics of the fractional exponential decay, i.e the "law of two thirds" .

order N	20	30	40	50	60	70	80	90	100
intermediate fitting slope	0.638	0.648	0.654	0.657	0.659	0.660	0.662	0.663	0.663

Table 1: The steady fitting slopes captured for the intermeidate asymptotics.

7 Summary

This work studies the rate of relaxation to equilibrium for Landau kinetic equation and related models. Though there exists a rather complicated rigorous proof by Strain and Guo [10][11] of the "law of two thirds" for fractional exponential relaxation, we try to make this result more evident through a series of heuristic arguments as well as numerical experiments on the linear radial dimensionless generalized Landau equation. We approximate its solution by a linear combination of N generalized Laguerre polynomials, having in mind the corresponding weighted L^2 space when $N \to \infty$. The rate of (purely exponential) relaxation for corresponding system of N linear ODEs can be easily obtained through studying numerically its Rayleigh quotients for any fixed N. The difficulty, however, appears when the corresponding smallest positive eigenvalue tends to zero as $N \to \infty$. Then the limiting spectral gap does not exist. It is exactly the case for the classical Landau equation. In this case we have considered the time-dependent solution of the corresponding set of N ODEs and show numerically the existence of intermediate fractional exponential asymptotics, which is clearly separated for large N and t from the purely exponential absolute asymptotics. The intermediate asymptotics yields precisely the "law of two thirds", as clearly seen from Table 1 at the end of Section 6. This can be considered as the main result of the paper. Note that to prove it numerically we need to deal with exponentially small (for large values of time t) numbers. Such computations involve delicate handling of extremely contrasting magnitudes of float-pointing numbers. To validate such subtle computations, we make use of hybrid symbolic-numeric approach, which was impossible in previous conventional numerical treatments. Finally we note that, to the best of our knowledge, the very existence of intermediate asymptotics for a system of linear ODEs is not studied in literature. On the other hand, such asymprotics must be typical, for example, for finite dimensional approximations of kinetic equations which do not have a spectal gap. The well-known examples are linear and linearized Boltzmann equations for soft potentials with cut-off. We believe that our methods can be also applied to these and similar equations.

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