The Cauchy problem for the quantum Boltzmann equation for bosons at very low temperature

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Abstract

The system that describes the dynamics of a Bose-Einstein Condensate (BEC) consists of a quantum Boltzmann equation of the excitation distribution function and the Gross-Pitaevskii equation of the condensate wave function. We solve the Cauchy problem for the quantum Boltzmann equation that approximates the evolution of the distribution function of the excitations - thermal cloud, at the temperature regime that is very low compared to the Bose-Einstein Condensation critical temperature. Such an equation has a cubic kinetic transition probability kernel. We develop the existence and uniqueness result by means of abstract ODE's theory in Banach spaces by characterizing an invariant bounded, convex, closed subset $\mathcal{S}$ of the positive cone associated with the Banach space $C^1\big([0,\infty);L^1(|p|dp)\big)$. The subset $\mathcal{S}$ depends on the kinetic transition probability kernel structure as well as the interaction law for bosons. It also depends on the shown propagation and creation of polynomial moments accounting for high-energy tails in the sense of L^1. In addition, we show the scaled summability of polynomial moments by studying the propagation and generation of Mittag-Leffler moments. These estimates imply the unique solution has an exponential decaying high energy-tail in the sense of L^1.

Keywords Quantum kinetic theory, low-temperature Bose particles, spin-Peierls model, Mittag-Leffler moments, abstract ODE theory.

MSC: 82C10, 82C22, 82C40.

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1 Introduction

After the first Bose-Einstein Condensate (BEC) was produced by Cornell, Wieman, and Ketterle, which led them to the 2001 Nobel Prize in Physics [2, 3], there has been an explosion of research on BECs and cold bosonic gases. Above the condensation temperature, the dynamic of a bose gas is determined by the Uehling-Uhlenbeck kinetic equation introduced in [46]; see for instance [20, 21] for interesting results and list of references. The first proof of BECs was done in [33]. Below the condensation temperature, the bosonic gas dynamics is governed by a system that couples a quantum Boltzmann and a Gross-Pitaevskii equations. In such a system, the wave function of the BEC follows the Gross-Pitaevskii equation and the quantum Boltzmann equation describes the evolution of the density function of the excitations. The system was first derived by Kirkpatrick and Dorfmann in [31, 32], using a Green function approach and was revisited by Zaremba-Nikuni-Griffin and Gardiner-Zoller et. al. in [26, 28, 30, 48]. It has then been developed and studied extensively in the last two decades by several authors (see [10, 27, 39, 44], and references therein). In [42], Spohn gives a heuristic derivation for the one-dimensional version of the system, using an perturbation theory for the Uehling-Uhlenbeck equation. A formal derivation, for the full three-dimensional case, is done in [41]. The first step to a rigorous derivation may take the ideas generated from the works [15, 19], in combination with techniques from quantum field theory [41].

The dynamics of the excitations is the object of study in the present paper, more specifically, we are interested in the dynamics of dilute Bose gases at very low temperature under the assumption of reference [18, 22, 31, 32], that is, the BEC is very stable and contains a sizeable number of atoms, the interaction between excited atoms is small, being the dominant interaction the one between excited atoms and the BEC. The evolution of the space homogeneous probability density distribution function $f := f(t,p)$, with $(t,p) \in [0,\infty)\times\mathbb{R}^3$, for p the momenta state variable, of such Bose gases can be described by the following bosonic quantum Boltzmann equation:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = n_c Q[f]\,, \qquad f(0,\cdot) = f_0\,, \tag{1.1}$$

where the interaction operator is defined as

$$\begin{aligned} Q[f] &:= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathrm{d}p_1 \mathrm{d}p_2 \big[R(p,p_1,p_2) - R(p_1,p,p_2) - R(p_2,p_1,p)\big]\,,\\ R(p,p_1,p_2) &:= \\ &|\mathcal{M}(p,p_1,p_2)|^2 \big[\delta\left(\frac{\omega(p)}{k_BT} - \frac{\omega(p_1)}{k_BT} - \frac{\omega(p_2)}{k_BT}\right)\delta(p-p_1-p_2)\big]\\ &\qquad \times \big[f(p_1)f(p_2)(1+f(p)) - (1+f(p_1)(1+f(p_2))f(p)\big]\,. \end{aligned} \tag{1.2}$$

where $\beta := \frac{1}{k_BT} > 0$ is the physical constant depending on the Boltzmann constant k_B, and the temperature of the quasiparticles T at equilibrium. The term $\mathcal{M}(p,p_1,p_2)$ is the transition probability and the particle energy $\omega(p)$ is given by the Bogoliubov dispersion law:

$$\omega(p) = \left[\frac{gn_c}{m}|p|^2 + \left(\frac{|p|^2}{2m}\right)^2\right]^{1/2}, \tag{1.3}$$

where $p \in \mathbb{R}^3$ is the momenta, m is the mass of the particles, g is the interaction coupling constant and n_c is the density of particles in the BEC. Notice that $n_c = n_c(t) = |\Psi|^2(t)$, in which Ψ is the wave function of the

BEC. The wave function Ψ, as mentioned above, satisfies the cubic nonlinear Schrodinger equation and the evolution of the condensate density distribution n_c, under some further assumptions, follows the following differential equation (cf. [6, 42, 47])

$$\begin{cases} \frac{\mathrm{d}n_c}{\mathrm{d}t} & = -n_c \int_{\mathbb{R}^3} Q[f]dp, \\ n_c(0) & = n_0\,, \end{cases} \tag{1.4}$$

or, equivalently

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\log n_c & = -\int_{\mathbb{R}^3} Q[f]dp, \\ \log(n_c(0)) & = \log(n_0). \end{cases} \tag{1.5}$$

However, in the scope of our paper, we only focus on the study of the quantum Boltzmann equation and leave the coupling quantum Boltzmann equation - nonlinear Schrodinger equation topic for future research. We, therefore, impose the following condition on the density distribution of the condensate

$$\begin{aligned} n \in C^1[0,\infty), \quad & \text{and there exists constants } \underline{n}_c,\, \overline{n}_c > 0 \text{ such that} \\ & \underline{n}_c < \ n(t) < \overline{n}_c, \quad \forall t \in [0,\infty). \end{aligned} \tag{1.6}$$

This assumption is physically meaningful. It says that the condensate does not vanish, and its density distribution is uniformly bounded from above and below in time.

The collision operator Q describes the interaction between the condensed and the excited atoms. The corresponding equilibrium distribution f_∞ of the collisional equation (1.1)-(1.2) has the form

$$f_\infty(p) = \frac{1}{e^{\beta\omega(p)} - 1}\,, \tag{1.7}$$

for $\beta = (k_BT)^{-1}$, as is usually referred as a Bose-Einstein distribution. In this work, we restrict the range of the temperature T, the condensate density n_c, and the interaction coupling constant g to values for which k_BT is much smaller than $(gn_c/m)^{1/2}$, i.e. a *cold gas regime.* Under this condition, the dispersion law $\omega(p)$ in (1.3) is approximated by

$$\frac{1}{k_BT}\left[\frac{gn_c}{m}|p|^2 + \left(\frac{|p|^2}{2m}\right)^2\right]^{1/2} \approx \frac{c}{k_BT}|p|, \quad \text{where } c := \sqrt{\frac{gn_c}{m}}\,,$$

when $(gn_c/m)^{1/2}(k_BT)^{-1} = O(1)$ and $k_BT \ll 1$. In particular, the energy will be now defined by the classical phonon dispersion law (still using the same notation), see [14, 18, 29, 40]

$$\omega(p) = c|p|, \quad \text{for } c := c(t) = \sqrt{\frac{g n_c(t)}{m}}\,. \tag{1.8}$$

Under this *very cold gas regime*, the transition probability $\mathcal{M}$ is approximated by

$$|\mathcal{M}|^2 = \kappa |p||p_1||p_2| \tag{1.9}$$

where

$$\kappa = \frac{9c}{64\pi^2 m n_c}. \tag{1.10}$$

We observe that $O(\sqrt{\underline{c}}) \le c(t) \le O(\sqrt{\overline{c}})$ and $O(\sqrt{\overline{c}}^{-1}) \le \kappa \le O(\sqrt{\underline{c}}^{-1})$ uniformly in time.

Different from previous mathematical works [6, 4, 7, 8], we do not truncate the transition probability $|\mathcal{M}|^2$ from above, or assume that it is cut-off near the origin. Thus, we perform the analysis in the whole momentum space, not in a piece of it or the torus [43], requiring a detailed control of the solution's tails.

Notice that in the pioneering experiments [2, 3, 9], one can observe the growth of the condensate after fast evaporative cooling. Equation (1.1)-(1.2) is the main term that leads to the growth of the BEC. Moreover, the kinetic equation (1.1)-(1.2) is also used to describe phonon interactions in anharmonic crystal lattices, first derived in this context by Peierls [37, 38], then by several other authors [14, 43].

In particular the linearization of the Quantum Boltzmann equation (1.1)-(1.2) about Bose-Einstein states is perform by setting

$$f(t,p) = f_\infty(p) + f_\infty(p)\big(1 + f_\infty(p)\big)\Omega(t,p), \tag{1.11}$$

evaluated into collision operator in (1.2) and restricting the evaluation to the linear terms. The resulting linearized equation was obtained in [23]

$$f_\infty(p)\big(1 + f_\infty(p)\big)\frac{\partial \Omega}{\partial t}(t,p) = -M(p)\Omega(t,p) + \int_{\mathbb{R}^3} dp'\, \mathcal{U}(p,p')\Omega(t,p'), \tag{1.12}$$

for some explicit function $M(p)$ and measure $\mathcal{U}(p,p')$. The Cauchy problem and the convergence toward equilibrium of such linearized model (1.12) were addressed in the aforementioned reference. The discrete theory of the equation, based on a dynamical system approach, was done in [16]. In reference [36], it has been proved that positive classical solutions of the model have a Gaussian in momenta barrier from below.

From now on, and without loss of generalization for the existence and uniqueness results as well as high energy tails behavior, we assume the temperature $T \ll 1$, such that $0 < \underline{c}(k_B T)^{-1} < \overline{c}(k_B T)^{-1} < 1$ in the reduced phonon dispersion law (1.8), and so the quantum collisional integral (1.2) becomes

$$
\begin{aligned}
Q[f] &:= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathrm{d}p_1 \mathrm{d}p_2 \big[R(p, p_1, p_2) - R(p_1, p, p_2) - R(p_2, p_1, p) \big] \,, \\
R(p, p_1, p_2) &:= |p||p_1||p_2| \big[\delta\left(|p| - |p_1| - |p_2| \right) \delta(p - p_1 - p_2) \big] \\
&\quad \times \big[f(p_1) f(p_2)(1 + f(p)) - (1 + f(p_1)(1 + f(p_2)) f(p) \big] \,.
\end{aligned}
\tag{1.13}
$$

Clearly, from the interaction law $p = p_1 + p_2$ and $|p| = |p_1| + |p_2|$ modeled in the collision operator by the singular Dirac delta masses, this cubic collisional form (1.13) is reduced into a quadratic one, that can be split in the difference of two positive quadratic operators, as will be shown in the existence result.

In addition the low temperature quantum collisional form (1.13) can be split into *gain* and *loss* operator forms

$$
\begin{aligned}
Q[f](t, p) &= Q^+[f](t, p) - Q^-[f](t, p) \\
&= Q^+[f](t, p) - f(t, p)\, \nu[f](t, p),
\end{aligned}
\tag{1.14}
$$

as is done with the classical Boltzmann operator acting on an $f(t, v)$, for binary elastic interactions, when the transition probability (or collision kernel) is an integrable function with respect to the scattering angle as much as is integrable respect with a velocity v_*, the interacting with the velocity v in the binary process.

Here, the gain operator is also defined by the positive contributions in the total rate of change in time of the collisional form $Q(f)(t, p)$ in (1.13),

$$
\begin{aligned}
Q^+[f](t, p) &:= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathrm{d}p_1 \mathrm{d}p_2 |p||p_1||p_2| \delta(p - p_1 - p_2) \\
&\times \delta(|p| - |p_1| - |p_2|) f(t, p_1) f(t, p_2) + 2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathrm{d}p_1 \mathrm{d}p_2 |p||p_1||p_2| \\
&\quad \times \delta(p_1 - p - p_2) \delta(|p_1| - |p| - |p_2|) \big[2 f(t, p) f(t, p_1) + f(t, p_1) \big] \,.
\end{aligned}
\tag{1.15}
$$

In analog, the loss operator models the negative contributions in the total rate of change in time of same collisional form $Q(f)(t, p)$. It is local in $f(t, p)$ and so written $Q^-[f] := f\, \nu[f]$, where $\nu[f](t, p)$, referred as the *collision* *frequency or attenuation coefficient*, defined by

$$\begin{aligned}
\nu[f](t,p) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathrm{d}p_1\mathrm{d}p_2|p||p_1||p_2|\delta(p-p_1-p_2) \\
\times\, \delta(|p|-|p_1|-|p_2|)\big[2f(t,p_1)+1\big] + 2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathrm{d}p_1\mathrm{d}p_2|p||p_1||p_2| \\
\times\, \delta(p_1-p-p_2)\delta(|p_1|-|p|-|p_2|)f(t,p_2)\,,
\end{aligned} \tag{1.16}$$

is nonlocal in $f(t,p)$.

Remark 1.1 *In order to grant the split of the collision operator in gain and loss parts, it is necessary that $\nu[f](t,p)$ is well defined. This is secured whenever solutions have at least the second moment finite throughout the evolution. This property will be automatically granted by the proofs of creation and propagation of statistical moments in Section 4 and the corresponding existence theorem in Section 5.*

Thus, our goal is to study the Cauchy problem of radial solutions for the quantum Boltzmann gas model at low temperature (1.1)-(1.13), or equivalently by (1.14, 1.15, 1.16). In addition we will show that the unique solutions of this Cauchy problem have exponential decaying tails in the sense of $L^1(\mathbb{R}^3)$, which we referred as Mittag-Leffler moments. This is the first step to solve an equation of the kind without cut-off assumptions in the transition probability kernels.

The existence and uniqueness arguments we use in this manuscript are based on techniques developed in the last few years for the classical Boltzmann equation in [11, 24, 25, 45]. We point out that the propagation or generation of polynomial moments is what enable us to find a natural space to show existence and uniqueness of solutions for equation using an abstract ODE theory, without need of bounded initial entropy. The propagation and generation of Mittag-Leffler moments follows from adapting the techniques recently developed in [45].

A technical difficulty in the analysis is the fact that the natural conservation law for the model is energy conservation, that is, the solution's first moment, whereas the homogeneity of the kinetic potential kernel in the model is 3. Due to this fact, it is essential to perform high moment analysis which, in contrast, it is not central for the Cauchy problem in the classical Boltzmann equation, refer to [5, 25, 35].

The organization of the paper is as follows.

- Section 2 presents the weak formulation and recall the main conservation laws as well entropy estimate and corresponding analog to an H-Theorem for (1.1) with the low temperature regime collisional form (1.13).
- The next two sections regard the Cauchy problem and high energy tail behavior, which will be fully developed in context of radially symmetric solutions.
    - Section 3 is devoted to a key *a priori* estimate on the moments of equation (1.1 1.13) which will be used several times along the paper, Proposition 3.1.
    - Section 4 proves the creation and propagation of polynomial moments, Theorem 4.1 by means of Proposition 3.1.
    - Section 5 poses in Theorem 5.1 for a general framework of solving evolution problems in Banach space by listing sufficient conditions for existence and uniqueness of solutions to the initial value problem associate to (1.1, 1.13). It also proves that such conditions are satisfied for radially symmetric solutions under natural conditions fulfilled by the creation and propagation of polynomial moments estimates obtained in Section 4. Existence is based on a Hölder estimate and a condition of the sub-tangent type for Q, see Theorem 5.2. Uniqueness is based on a one-side Lipschitz estimate.
- Section 6 addresses the propagation and creation of Mittag-Leffler moments for the solution to (1.1, 1.13), proven in Theorems 6.1 and 6.2.
- Section 7 is the Appendix showing the proof of Theorem 5.1 for a general framework for abstract ODE theory to solve first order initial value problems in Banach spaces. This proof is inspired by the unpublished work by A. Bressan in [13].

2 Conservation of energy and momentum

The following properties hold for the low temperature quantum collisional form (1.13) remarking that, for notational convenience, we will usually omit the time variable t unless some stress is necessary in the context.

Proposition 2.1 (Weak Formulation) *For any suitable test function φ, the following weak formulation holds for the collision operator* (1.13)

$$\begin{aligned}
\int_{\mathbb{R}^3} dp\, n_c\, Q[f](p)\varphi(p) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} dp\, dp_1\, dp_2\, n_c\, |p||p_1||p_2|\delta(p-p_1-p_2) \\
&\quad \times \delta(|p|-|p_1|-|p_2|)\Big[f(p_1)f(p_2)-f(p_1)f(p)-f(p_2)f(p)-f(p)\Big] \\
&\qquad \times \Big[\varphi(p)-\varphi(p_1)-\varphi(p_2)\Big] \\
&= 2\pi\int_{\mathbb{R}^3} dp_1 \int_{\mathbb{R}^+} d|p_2|\, n_c\, \big|p_1+|p_2|\widehat{p_1}\big||p_1||p_2|^3\Big[f(p_1)f(|p_2|\widehat{p_1}) \\
&\quad - f(p_1)f(p_1+|p_2|\widehat{p_1}) - f(|p_2|\widehat{p_1})f(p_1+|p_2|\widehat{p_1}) - f(p_1+|p_2|\widehat{p_1})\Big] \\
&\quad \times \Big[\varphi(p_1+|p_2|\widehat{p_1})-\varphi(p_1)-\varphi(|p_2|\widehat{p_1})\Big]
\end{aligned} \tag{2.1}$$

Furthermore, for radially symmetric functions $f(p) := f(|p|)$ and $\varphi(p) := \varphi(|p|)$, the following holds true

$$\begin{aligned}
\int_{\mathbb{R}^3} dp\, n_c\, Q[f](p)\varphi(p) &= 8\pi^2\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} d|p_1|\, d|p_2|\, n_c(|p_1|+|p_2|)\, |p_1|^3|p_2|^3 \times \\
&\Big[f(|p_1|)f(|p_2|)-f(|p_1|)f(|p_1|+|p_2|)-f(|p_2|)f(|p_1|+|p_2|) \\
&\quad - f(|p_1|+|p_2|)\Big] \times \Big[\varphi(|p_1|+|p_2|)-\varphi(|p_1|)-\varphi(|p_2|)\Big]\,.
\end{aligned} \tag{2.2}$$

Proof. In this proof we use the short-hand $\int := \int_{\mathbb{R}^9} \mathrm{d}p\, \mathrm{d}p_1\, \mathrm{d}p_2$. First, observe that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \mathrm{d}p\, n_c\, Q[f](p)\varphi(p) = \\
&\int n_c|p||p_1||p_2|\delta(p-p_1-p_2)\delta(|p|-|p_1|-|p_2|)R(p,p_1,p_2)\varphi(p) \\
&\quad - \int n_c|p||p_1||p_2|\delta(p-p_1-p_2)\delta(|p|-|p_1|-|p_2|)R(p_1,p,p_2)\varphi(p) \\
&\qquad - \int n_c|p||p_1||p_2|\delta(p-p_1-p_2)\delta(|p|-|p_1|-|p_2|)R(p_2,p_1,p)\varphi(p)\,.
\end{aligned} \tag{2.3}$$

Second, interchanging variables $p \leftrightarrow p_1$ and $p \leftrightarrow p_2$,

$$\int n_c|p||p_1||p_2|R(p_1,p,p_2)\varphi(p) = \int n_c|p||p_1||p_2|R(p,p_1,p_2)\varphi(p_1)\,, \tag{2.4}$$

and

$$\int n_c|p||p_1||p_2|R(p_2,p_1,p)\varphi(p) = \int n_c|p||p_1||p_2|R(p,p_1,p_2)\varphi(p_2)\,. \tag{2.5}$$

Combining (2.3), (2.4), (2.5), we get the first equality in (2.1). Now, evaluate the Dirac in $p = p_1 + p_2$ (conservation of momentum) to obtain

$$\begin{aligned}\int_{\mathbb{R}^3} \mathrm{d}p\, n_c\, Q[f](p)\varphi(p) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} n_c\,|p_1+p_2||p_1||p_2|\delta(|p_1+p_2|-|p_1|-|p_2|)\\ \Big[f(p_1)f(p_2) - f(p_1)f(p_1+p_2) - f(p_2)f(p_1+p_2) - f(p_1+p_2)\Big]\\ \times\Big[\varphi(p_1+p_2) - \varphi(p_1) - \varphi(p_2)\Big]\mathrm{d}p_1\,\mathrm{d}p_2\,.\end{aligned} \tag{2.6}$$

Now, observe that

$$\big\{(p_1,p_2)\in\mathbb{R}^6\big||p_1+p_2|-|p_1|-|p_2| = 0\big\} = \big\{(p_1,p_2)\in\mathbb{R}^6\big|1-\widehat{p_1}\cdot\widehat{p_2} = 0\big\}\,.$$

Then, the following identity holds for any continuous function $F(p_2)$

$$\begin{aligned}&\int_{\mathbb{R}^3}\mathrm{d}p_2\,F(p_2)\,\delta\big(|p_1+p_2|-|p_1|-|p_2|\big) = \int_{\mathbb{R}^3}\mathrm{d}p_2\,F(p_2)\,\delta\big(1-\widehat{p_1}\cdot\widehat{p_2}\big)\\ &\quad= \int_{\mathbb{R}^+}|p_2|^2\mathrm{d}|p_2|\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\sin(\theta)\mathrm{d}\theta\,F\big(p_2(\cos(\theta),\sin(\phi))\big)\,\delta\big(1-\cos(\theta)\big)\\ &\quad= \int_{\mathbb{R}^+}|p_2|^2\mathrm{d}|p_2|\int_0^{2\pi}\mathrm{d}\phi\int_{-1}^{1}\mathrm{d}s\,F\big(p_2(-s,\sin(\phi))\big)\delta(1+s)\\ &\quad= 2\pi\int_{\mathbb{R}^+}|p_2|^2\mathrm{d}|p_2|\,F(|p_2|\widehat{p_1})\,.\end{aligned}$$

In the last step we used that $p_2(1,\sin(\phi)) = |p_2|\widehat{p_1}$. Using this identity in (2.6) proves the second equality in (2.1). Finally, for radially symmetric functions $f(p) := f(|p|)$ and $\varphi(p) := \varphi(|p|)$, one simply uses that $\big|p_1 + |p_2|\widehat{p_1}\big| = |p_1| + |p_2|$ and polar coordinates in the p_1-integral to obtain (2.2) ■

Corollary 2.1 (Conservation laws) *If f is a solution of* (1.1)-(1.13), *it formally conserves momentum and energy*

$$\int_{\mathbb{R}^3} dp\,f(t,p)\,p = \int_{\mathbb{R}^3} dp\,f_0(p)\,p\,, \tag{2.7}$$

$$\int_{\mathbb{R}^3} dp\,f(t,p)\,|p| = \int_{\mathbb{R}^3} dp\,f_0(p)\,|p|\,. \tag{2.8}$$

Remark 2.1 *Since f is the density function of the thermal cloud, the mass is not conserved due to the fact that atoms could move in and out of the condensate. In order words, the total mass of the system thermal cloud - condensate is unchanged as time evolves, but the mass of each component of the system the thermal cloud and the condensate is not conserved. Now, let us look at the system that couples the two equations* (1.1) *and* (1.4). *Integrating Equation* (1.1) *in p and taking the sum with the second equation* (1.5)*, we obtain*

$$\frac{d}{dt}\left(\int_{\mathbb{R}^3} f(t,p)dp \ + \ n_c(t)\right) \ = \ 0, \tag{2.9}$$

which confirms that the total mass of the whole system is conserved.

Corollary 2.2 (H-Theorem) *If $f(t,p)$ is a solution of* (1.1)-(1.13)*, then*

$$\frac{d}{dt}\int_{\mathbb{R}^3} dp\Big[f(p)\log f(p) - \big(1+f(p)\big)\log\big(1+f(p)\big)\Big] \leq 0.$$

A radially symmetric equilibrium of the equation has the following form

$$f(p) = \frac{1}{e^{\alpha\,\omega(p)}-1}, \quad \textit{for some } \ \alpha > 0. \tag{2.10}$$

Proof. We observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \mathrm{d}p\Big[f(p)\log f(p) - \big(1+f(p)\big)\log\big(1+f(p)\big)\Big] = \int_{\mathbb{R}^3} \mathrm{d}p\,\partial_t f(p)\log\left(\frac{f(p)}{f(p)+1}\right).$$

In addition, we can rewrite

$$\begin{aligned}\int_{\mathbb{R}^3} \mathrm{d}p\, n_c Q[f](p)\varphi(p) = \int_{\mathbb{R}^9} & n_c|p||p_1||p_2|\delta(p-p_1-p_2)\delta(|p|-|p_1|-|p_2|) \\ & \times \big(1+f(p)\big)\big(1+f(p_1)\big)\big(1+f(p_2)\big) \\ \times \left(\frac{f(p_1)}{f(p_1)+1}\frac{f(p_2)}{f(p_2)+1} - \frac{f(p)}{f(p)+1}\right) & \Big[\varphi(p)-\varphi(p_1)-\varphi(p_2)\Big]\mathrm{d}p\mathrm{d}p_1\mathrm{d}p_2.\end{aligned}$$

Choosing $\varphi(p) = \log\left(\frac{f(p)}{f(p)+1}\right)$ we obtain, in the case of equality, that

$$\frac{f(p_1)}{f(p_1)+1}\frac{f(p_2)}{f(p_2)+1} - \frac{f(p)}{f(p)+1} = 0,$$

or equivalently, putting $h(p) = \log\left(\frac{f(p)}{f(p)+1}\right)$, we get

$$h(p_1) + h(p_2) = h(p). \tag{2.11}$$

The fact that $h(\cdot)$ is radially symmetric yields $h(p) = -\alpha\,\omega(p)$, for all $p \in \mathbb{R}^3$ and some positive constant α. This proves the claim. ■

3 *A priori* estimates on a solution's moments

The aim of the following sections is to consider radially symmetric solutions of (1.1)-(1.13) that lie in $\mathcal{C}\big([0,\infty); L^1(\mathbb{R}^3, |p|^k \mathrm{d}p)\big)$ where

$$L^1(\mathbb{R}^3, |p|^k \mathrm{d}p) := \Big\{ f \text{ measurable} \mid \int_{\mathbb{R}^3} \mathrm{d}p\, |f(p)||p|^k < \infty,\ k \geq 1 \Big\}.$$

That is, in sections 3 and 4 the *a priori* estimates *assume* the existence of a radially symmetric solution enjoying time continuity in such Lebesgue spaces for k sufficiently large. Define the solution's moment of order k as

$$\mathcal{M}_k\langle f\rangle(t) := \int_{\mathbb{R}^3} \mathrm{d}p f(t,p)|p|^k\,. \tag{3.1}$$

When f is as radially symmetric function $f(t,p) = f(t,|p|)$, one can use spherical coordinates to reduce the integral with respect to $\mathrm{d}p$ on $\mathbb{R}^3$ to an integral on $\mathbb{R}_+$ with respect to $\mathrm{d}|p|$. As a consequence,

$$\mathcal{M}_k\langle f\rangle(t) = \left|\mathbb{S}^2\right| \int_{\mathbb{R}^+} \mathrm{d}|p| f(t,|p|)|p|^{k+2}\,.$$

Thus, it will be convenient for notation purposes to introduce and work with what we call "line-moments"

$$m_k\langle f\rangle(t) := \int_0^\infty \mathrm{d}|p|\, f(t,|p|)|p|^k\,. \tag{3.2}$$

Observe that $\mathcal{M}_k\langle f\rangle = |\mathbb{S}^2| m_{k+2}\langle f\rangle$.

We are going to use the definition of moments in two contexts: In one hand, in sections 3, 4 and 6 we always consider the moment applied to a given *radial solution of the equation.* Thus, there is no harm to omit the function dependence and just write $\mathcal{M}_k(t)$, $\mathcal{M}_k$, $m_k(t)$ or m_k to denote moments and line-moments for simplicity. In the other hand, in section 5 we will use moments as norms of the spaces $L^1(\mathbb{R}^3, |p|^k \mathrm{d}p)$, as a consequence, the functional dependence will be important. In addition, time dependence will not be key in this section, thus, we will write line-moments as $m_k\langle\cdot\rangle$. Note that according to the conservation law (2.8) and assuming initial energy finite, the following equivalent estimates hold

$$\mathcal{M}_1(t) = \mathcal{M}_1(0) < \infty\,, \qquad m_3(t) = m_3(0) < \infty\,.$$

Before entering into details, let us explain the necessity of considering radially symmetric solutions of the equation (1.1) in the following arguments. Choosing $\varphi(p) = |p|^k$ in the weak formulation Proposition 2.1, one is lead to estimate terms of the form

$$\int_{\mathbb{R}^3} \mathrm{d}p_1 f(t,p_1)|p_1|^i \int_{\mathbb{R}_+} \mathrm{d}|p_2| f(t,|p_2|\widehat{p_1})|p_2|^j\,, \qquad i,\, j \in \mathbb{N}.$$

These terms are not estimated by products of moments of f unless the function is radially symmetric. In such a case this particular term simply writes as a product of line-moments of f, namely $|\mathbb{S}^2| m_{i+2}\langle f\rangle\, m_j\langle f\rangle$. This technical issue will be central in finding closed *a priori* estimates in terms of line-moments of solutions.

Proposition 3.1 (Line-Moment Ordinary Differential Inequalities) *For $1/k \le \gamma \le 1$, $k > 1$, we have the following a priori estimate on the moments valid with some universal constants C_1 and C_2*

$$\begin{aligned}\frac{d}{dt} m_{k\gamma+2}(t) \le C_1 \sum_{i=1}^{\left[\frac{k+1}{2}\right]} \binom{k}{i} \Big(& m_{i\gamma+4} m_{3+(k-i)\gamma} \\ & + m_{i\gamma+3} m_{4+(k-i)\gamma}\Big)(t) - C_2\, m_{k\gamma+8}(t)\,.\end{aligned} \tag{3.3}$$

In order to prove Proposition 3.1, we first need the following lemmata.

Lemma 3.1 *For $k > 3$, we have the following equation for m_k*

$$\begin{aligned}\frac{d}{dt} m_k(t) = 8\pi^2 \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} dr_1\, dr_2\, n_c(r_1+r_2) r_1^3 r_2^3 \big[f(t,r_1)f(t,r_2) \\ - 2f(t,r_1)f(t,r_1+r_2) - f(t,r_1+r_2)\big] \times \left[|r_1+r_2|^{k-2} - r_1^{k-2} - r_2^{k-2}\right].\end{aligned} \tag{3.4}$$

Proof. Take $\varphi(p) = |p|^{k-2}$ as a test function and use Proposition 2.1, recall that the line-moment m_k is equivalent to $\mathcal{M}_{k-2}$, to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} m_k = 2\pi \int_{\mathbb{R}^3}\int_{\mathbb{R}_+} \mathrm{d}p_1\, \mathrm{d}|p_2|\, n_c \left|p_1 + |p_2|\widehat{p_1}\right| |p_1||p_2|^3 \Big[f(t,p_1)f(t,|p_2|\widehat{p_1}) \\ - f(t,p_1)f(t,p_1+|p_2|\widehat{p_1}) - f(t,|p_2|\widehat{p_1})f(t,p_1+|p_2|\widehat{p_1}) - f(t,p_1+|p_2|\widehat{p_1})\Big] \\ \times \Big[\big||p_1| + |p_2|\big|^{k-2} - |p_1|^{k-2} - |p_2|^{k-2}\Big]\,.\end{aligned}$$

In addition $f(p) := f(|p|)$, which leads to

$$
\frac{\mathrm{d}}{\mathrm{d}t} m_k = 8\pi^2 \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \mathrm{d}|p_1|\, \mathrm{d}|p_2|\, n_c(|p_1| + |p_2|)\, |p_1|^3 |p_2|^3 \times
\Big[f(t,|p_1|)f(t,|p_2|) - f(t,|p_1|)f(t,|p_1|+|p_2|) - f(t,|p_2|)f(t,|p_1|+|p_2|)
- f(t,|p_1|+|p_2|)\Big] \times \Big[|p_1+p_2|^{k-2} - |p_1|^{k-2} - |p_2|^{k-2}\Big] .
$$

This estimate completes the proof of this Lemma 3.1. ■

Lemma 3.2 (From Ref. [12]) *Assume that* $k > 1$*, let* $\left[\frac{k+1}{2}\right]$ *denote the integer part of* $\frac{k+1}{2}$*. Then for all* $a, b > 0$*, the following inequality holds*

$$
\sum_{i=1}^{\left[\frac{k+1}{2}\right]-1} \binom{k}{i} (a^i b^{k-i} + a^{k-i} b^i)
\leq (a+b)^k - a^k - b^k \leq \sum_{i=1}^{\left[\frac{k+1}{2}\right]} \binom{k}{i} (a^i b^{k-i} + a^{k-i} b^i) . \tag{3.5}
$$

Proof. (of Proposition 3.1) For simplicity we omit t, the time variable, in the argument of this proof. From (3.4), we eliminate the negative term $-2f(t,r_1)f(t,r_1+r_2)$ and take into account the fact that

$$
|r_1 + r_2|^{k\gamma} - r_1^{k\gamma} - r_2^{k\gamma} > 0 ,
$$

to get

$$
\frac{\mathrm{d}}{\mathrm{d}t} m_{k\gamma+2}(t) \leq C(\pi) \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \mathrm{d}r_1 \mathrm{d}r_2 n_c(r_1+r_2) r_1^3 r_2^3 \big[f(t,r_1)f(t,r_2) -
- f(t,r_1+r_2)\big] \times \big[|r_1+r_2|^{k\gamma} - r_1^{k\gamma} - r_2^{k\gamma}\big] . \tag{3.6}
$$

By applying the inequality

$$
|r_1 + r_2|^{k\gamma} \leq (|r_1|^\gamma + |r_2|^\gamma)^k , \tag{3.7}
$$

with $1/k \le \gamma \le 1$ into (3.6), it yields

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} m_{k\gamma+2}(t) \ &\le \ C(\pi, \overline{n_c}) \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \mathrm{d}r_1 \mathrm{d}r_2 (r_1 + r_2)\, r_1^3\, r_2^3\, f(t, r_1) f(t, r_2) \times \\
&\big[(|r_1|^\gamma + |r_2|^\gamma)^k - r_1^{k\gamma} - r_2^{k\gamma}\big] - C(\pi, \underline{n_c}) \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \mathrm{d}r_1 \mathrm{d}r_2 (r_1 + r_2) r_1^3 r_2^3 \\
&\quad \times f(t, r_1 + r_2) \big[|r_1 + r_2|^{k\gamma} - r_1^{k\gamma} - r_2^{k\gamma}\big] .
\end{aligned} \tag{3.8}$$

In order to obtain (3.3), we estimate the two terms on the right hand side of (3.8). Using Lemma 3.2 with $a = r_1^\gamma$ and $b = r_2^\gamma$, the first term can be estimated as follows

$$\begin{aligned}
&\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \mathrm{d}r_1 \mathrm{d}r_2 (r_1 + r_2)\, r_1^3\, r_2^3 \Big[\big(|r_1|^\gamma + |r_2|^\gamma\big)^k - r_1^{k\gamma} - r_2^{k\gamma}\Big] f(t, r_1) f(t, r_2) \\
&\le \int_{\mathbb{R}_+^2} \mathrm{d}r_1 \mathrm{d}r_2 (r_1 + r_2) r_1^3 r_2^3 \sum_{i=1}^{\left[\frac{k+1}{2}\right]} \binom{k}{i} \Big(r_1^{i\gamma} r_2^{(k-i)\gamma} + r_1^{(k-i)\gamma} r_2^{i\gamma}\Big) f(t, r_1) f(t, r_2) ,
\end{aligned}$$

which, by a simple expansion process, can be bounded by

$$\begin{aligned}
&\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \mathrm{d}r_1 \mathrm{d}r_2 \sum_{i=1}^{\left[\frac{k+1}{2}\right]} \binom{k}{i} \Big(r_1^{i\gamma+4} r_2^{3+(k-i)\gamma} + r_1^{i\gamma+3} r_2^{4+(k-i)\gamma} \\
&\qquad + r_1^{(k-i)\gamma+4} r_2^{i\gamma+3} + r_1^{(k-i)\gamma+3} r_2^{i\gamma+4}\Big) f(t, r_1) f(t, r_2) \\
&\le 2 \sum_{i=1}^{\left[\frac{k+1}{2}\right]} \binom{k}{i} \Big(m_{i\gamma+4}\, m_{3+(k-i)\gamma} + m_{i\gamma+3}\, m_{4+(k-i)\gamma}\Big)(t) .
\end{aligned} \tag{3.9}$$

Note that in the above inequality, we only use the definition of $m_{i\gamma+3}$, $m_{i\gamma+4}$, $m_{(k-i)\gamma+3}$, and $m_{(k-i)\gamma+4}$. Regarding the second term on the right side of (3.8), we rewrite it using the change of variables $r_1 + r_2 \to r$ and $r_1 \to r - r_2$

$$\begin{aligned}
&- \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \mathrm{d}r_1 \mathrm{d}r_2 (r_1 + r_2) r_1^3 r_2^3 \big[|r_1 + r_2|^{k\gamma} - r_1^{k\gamma} - r_2^{k\gamma}\big] f(t, r_1 + r_2) \\
&= \int_0^\infty \int_0^r \mathrm{d}r_2 \mathrm{d}r\, r\, (r - r_2)^3\, r_2^3\, \big[|r - r_2|^{k\gamma} + r_2^{k\gamma} - |r|^{k\gamma}\big] f(t, r) .
\end{aligned} \tag{3.10}$$

Set

$$I := \int_0^r \mathrm{d}r_2\, r\, (r - r_2)^3\, r_2^3\, \Big[|r - r_2|^{k\gamma} + r_2^{k\gamma} - |r|^{k\gamma}\Big] .$$

Then, by (3.7), $I \leq 0$. By the change of variables $r_2 \to r - r_2$, one gets the following identity

$$\int_0^r \mathrm{d}r_2 (r - r_2)^{3+k\gamma} r_2^3 = \int_0^r \mathrm{d}r_2 (r - r_2)^3 r_2^{3+k\gamma} ,$$

which implies the equality

$$I \;\; = \;\; \int_0^r \mathrm{d}r_2 (r - r_2)^3 r_2^3 \big[2 r_2^{k\gamma} - r^{k\gamma} \big] \, . \tag{3.11}$$

Develop $(r - r_3)^3$ in the above integral, the following equality holds

$$\begin{aligned} I &= \int_0^r \mathrm{d}r_2 (r - r_2)^3 r_2^3 \big[2 r_2^{k\gamma} - r^{k\gamma} \big] \\ &= \int_0^r \mathrm{d}r_2 \big[r^3 - 3 r_2 r^2 + 3 r_2^2 r - r_2^3 \big] \big[2 r_2^{k\gamma+3} - r^{k\gamma} r_2^3 \big] \\ &= \int_0^r \mathrm{d}r_2 \big[2 r_2^{k\gamma+3} r^3 - 6 r_2^{k\gamma+4} r^2 + 6 r_2^{k\gamma+5} r - 2 r_2^{k\gamma+6} \\ &\quad - r^{k\gamma+3} r_2^3 + 3 r^{k\gamma+2} r_2^4 - 3 r^{k\gamma+1} r_2^5 + r_2^6 r^{k\gamma} \big] = -C r^{k\gamma+7} , \end{aligned} \tag{3.12}$$

where the last equality follows by evaluating the integral of $\mathrm{d}r_2$ in $(0, r)$. Since $I \leq 0$, the constant C is explicit and positive. Combining (3.10), (3.11), (3.12), we get the following equation for the second term on the right hand side of (3.8)

$$\begin{aligned} - \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} & \mathrm{d}r_1 \mathrm{d}r_2 (r_1 + r_2) \, r_1^3 \, r_2^3 \left[|r_1 + r_2|^{k\gamma} - r_1^{k\gamma} - r_2^{k\gamma} \right] f(t, r_1 + r_2) \\ & = -C \int_0^\infty r^{k\gamma+8} f(t, r) dr = -C \, m_{k\gamma+8} \, . \end{aligned} \tag{3.13}$$

Putting together (3.6),(3.9) and (3.13), we obtain the ordinary differential line-moments inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} m_{k\gamma+2} \leq C \sum_{i=1}^{\left[\frac{k+1}{2}\right]} \binom{k}{i} \Big(m_{i\gamma+4} m_{3+(k-i)\gamma} + m_{i\gamma+3} m_{4+(k-i)\gamma} \Big) - C' m_{k\gamma+8} .$$

that shows inequality (3.3). Thus, the proof of Proposition 3.1 is now complete.

∎

4 Creation and propagation of polynomial moments

Let us write the main result of this section.

Theorem 4.1 *Suppose that $f_0(p) = f_0(|p|)$, $m_3(0) < \infty$ and $m_k(t)$ defined in (3.2). Then, there exists a constant $C_k(\mathfrak{h}_3)$ that depends only on $\mathfrak{h}_3 := \mathfrak{h}_3(m_3(0))$, and on k such that we have the following creation of the k^{th} line moment*

$$m_k(t) \le C_k(\mathfrak{h}_3)\left(1 - e^{-C_k t}\right)^{-\frac{k-3}{6}}\ , \quad \forall\, k > 3\,. \tag{4.1}$$

Moreover, if $m_k(0) < \infty$, we have the following propagation of the k^{th} line moment

$$m_k(t) \le \max\left\{m_k(0), C_k(\mathfrak{h}_3)\right\}. \tag{4.2}$$

Lemma 4.1 (Moment interpolation) *The line-moment $m_k = m_k(t)$ satisfies*

$$m_\rho \le m_{\rho_1}^{\gamma} m_{\rho_2}^{1-\gamma}, \tag{4.3}$$

where the positive constants $\rho, \rho_1, \rho_2, \gamma$ satisfy $0 < \rho_1 \le \rho \le \rho_2$, $0 < \gamma < 1$, and $\rho = \gamma\rho_1 + (1-\gamma)\rho_2$.

Proof. The proof of this statement is straightforward. Indeed, Hölder's inequality imply

$$\begin{aligned} m_{\rho_1}^{\gamma} m_{\rho_2}^{1-\gamma} &= \left(\int_{\mathbb{R}_+} \mathrm{d}r\, |r|^{\rho_1} f(r)\right)^{\gamma} \left(\int_{\mathbb{R}_+} \mathrm{d}r\, |r|^{\rho_2} f(r)\right)^{1-\gamma} \\ &\ge \int_{\mathbb{R}_+} \mathrm{d}r\, |r|^{\rho_1\gamma + \rho_2(1-\gamma)} f(r) \ge \int_{\mathbb{R}_+} \mathrm{d}r\, |r|^{\rho} f(r) \ge m_\rho\,. \end{aligned}$$

■

Proof. (of Theorem 4.1) In this proof, we will use Lemma 3.1 with $\gamma = 1$ which reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t} m_{k+2} \le C_1 \sum_{i=1}^{\left[\frac{k+1}{2}\right]} \binom{k}{i} \Big(m_{i+4} m_{3+(k-i)} + m_{i+3} m_{4+(k-i)}\Big) - C_2 m_{k+8}\,,$$

where C_1 and C_2 are some universal positive constants. For the sake of simplicity, we shift $k+2 \to k$ in the above inequality to get

$$\frac{\mathrm{d}}{\mathrm{d}t} m_k \le C_1 \sum_{i=1}^{\left[\frac{k-1}{2}\right]} \binom{k-2}{i} \big(m_{i+4} m_{1+(k-i)} + m_{i+3} m_{2+(k-i)}\big) - C_2 m_{k+6}\,. \tag{4.4}$$

From (4.4), our goal is to construct a differential inequality for $m_k = m_k(t)$ from which the boundedness of m_k could be deduced. In order to do that, we will estimate the right hand side of(4.4) by some function of m_k, which leads to a uniform in time upper bound of m_k. First, let us start bounding the right hand side of (4.4) by estimating the term $m_{i+4}m_{1+k-i}$ with Hölder's inequality,

$$m_{i+4} \leq m_3^{\frac{k+2-i}{k+3}} m_{k+6}^{\frac{i+1}{k+3}} = Cm_{k+6}^{\frac{i+1}{k+3}},$$

where we notice that, by the conservation of energy (2.8), m_3 and $m_3^{\frac{k+1-i}{k+2}}$ are constants. Multiplying m_{i+4} by m_{1+k-i} and using Young's inequality

$$m_{i+4}m_{1+k-i} \leq Cm_{k+6}^{\frac{i+1}{k+3}} m_{1+k-i} \leq \frac{m_{k+6}^{\frac{(i+1)p}{k+3}} \epsilon^p}{p} + \frac{m_{1+k-i}^q}{q\epsilon^q}. \tag{4.5}$$

We set $q = \frac{k+3}{k+2-i}$ and $p = \frac{k+3}{i+1}$ and choose $\epsilon > 0$ in the sequel. The quantity m_{1+k-i} could be bounded by Hölder's inequality again

$$m_{1+k-i} \leq m_k^{\frac{k-i-2}{k-3}} m_3^{\frac{i-1}{k-3}}.$$

Therefore, from (4.5) and the aforementioned bound on m_{1+k-i}, we obtain the estimate for the term $m_{i+4}m_{1+k-i}$ on the right side of (4.4)

$$m_{i+4}m_{1+k-i} \leq \frac{m_{k+6}\epsilon^p}{p} + \frac{m_k^{\frac{(k+3)(k-i-2)}{(k+2-i)(k-3)}}}{q\epsilon^q}. \tag{4.6}$$

Since

$$\frac{1}{2} < \frac{(k+3)(k-i-2)}{(k+2-i)(k-3)} < \frac{k-1}{k-3},$$

an interpolation argument applied to inequality (4.6) leads to

$$m_{i+4}m_{1+k-i} \leq \frac{m_{k+6}\epsilon^p}{p} + C\frac{m_k^{1/2}}{q\epsilon^q} + C\frac{m_k^{\frac{k-1}{k-3}}}{q\epsilon^q}, \tag{4.7}$$

where C is some positive constant that can vary from line to line. Second, we continue estimating the right side of (4.4) by controling the term $m_{i+3}m_{2+k-i}$. We consider two cases: (1) $i \geq 2$ (then $2+k-i \leq k$), and (2) $i = 1$ (then $i+3 = 4 \leq k$). Let us start with the latter.

Case (2). Using Hölder inequality (4.3) and the conservation of momentum on m_3

$$m_{2+k-i} \leq m_3^{\frac{4+i}{k+3}} m_{k+6}^{\frac{k-i-1}{k+3}} = Cm_{k+6}^{\frac{k-i-1}{k+3}}.$$

Multiplying the this inequality by m_{i+3} and employing Hölder's inequality again, we have

$$m_{i+3}m_{2+k-i} \leq Cm_{i+3}m_{k+6}^{\frac{k-i-1}{k+3}} \leq \frac{m_{i+3}^r}{r\epsilon^r} + \frac{m_{k+6}^{\frac{s(k-i-1)}{k+3}}\epsilon^s}{s}, \tag{4.8}$$

where we set $s = \frac{k+3}{k-1-i}$ and $r = \frac{k+3}{i+4}$. Since $i+3 \leq k$, we can use Hölder's inequality

$$m_{i+3} \leq m_k^{\frac{i}{k-3}} m_3^{\frac{k-3-i}{k-3}} .$$

One concludes that

$$m_{i+3}m_{2+k-i} \leq \frac{m_{k+6}\epsilon^s}{s} + \frac{m_k^{\frac{i}{k-3}\frac{k+3}{i+4}}}{r\epsilon^r} = \frac{m_{k+6}\epsilon^s}{s} + \frac{m_k^{\frac{k+3}{5(k-3)}}}{r\epsilon^r} . \tag{4.9}$$

For Case (1) a similar argument is made to conclude that

$$m_{i+3} \leq m_3^{\frac{k+3-i}{k+3}} m_{k+6}^{\frac{i}{k+3}} = Cm_{k+6}^{\frac{i}{k+3}} .$$

Multiplying m_{i+3} by m_{2+k-i} and using Young's inequality

$$m_{i+3}m_{2+k-i} \leq Cm_{k+6}^{\frac{i}{k+3}}m_{2+k-i} \leq \frac{m_{k+6}^{\frac{is'}{k+3}}\epsilon^{s'}}{s'} + \frac{m_{2+k-i}^{r'}}{r'\epsilon^{r'}} ,$$

where we set $r' = \frac{k+3}{k+3-i}$ and $s' = \frac{k+3}{i}$. The quantity m_{2+k-i} can be bounded as

$$m_{2+k-i} \leq m_k^{\frac{k-i-1}{k-3}} m_3^{\frac{i-2}{k-3}} .$$

Therefore, we obtain the estimate for the term $m_{i+3}m_{2+k-i}$ for the right side of (4.4)

$$m_{i+3}m_{2+k-i} \leq \frac{m_{k+6}\epsilon^p}{p} + \frac{m_k^{\frac{(k+3)(k-i-1)}{(k+3-i)(k-3)}}}{q\epsilon^q}.$$

Since

$$\frac{1}{2} < \frac{(k+3)(k-i-1)}{(k+3-i)(k-3)} < \frac{k-1}{k-3},$$

we can interpolate to conclude that

$$m_{i+3}m_{2+k-i} \leq \frac{m_{k+6}\epsilon^{s'}}{s'} + C\frac{m_k^{\frac{1}{2}}}{r'\epsilon^{r'}} + C\frac{m_k^{\frac{k-1}{k-3}}}{r'\epsilon^{r'}} . \tag{4.10}$$

Combining (4.4), (4.5), (4.9) and (4.10), we get

$$\frac{\mathrm{d}}{\mathrm{d}t} m_k \le C(\epsilon) m_{k+6} + C'(\epsilon)\Big[m_k^{\frac{k-1}{k-3}} + m_k^{\frac{k+3}{5(k-3)}} + m_k^{\frac{1}{2}}\Big] - C'' m_{k+6}, \qquad (4.11)$$

where $C(\epsilon)$ and $C'(\epsilon)$ are positive constants satisfying $C(\epsilon) \to 0$ and $C'(\epsilon) \to \infty$ as $\epsilon \to 0$, and C'' is a positive constant depending only on $\mathfrak{h}_3 := m_3(0)$. Notice also that $C(\epsilon)$ and $C'(\epsilon)$ also depend on k. For $\epsilon > 0$ sufficiently small, the constant $C(\epsilon)$ is absorbed by C'' and we infer from (4.11) that

$$\frac{\mathrm{d}}{\mathrm{d}t} m_k \le C_k\Big[m_k^{\frac{k-1}{k-3}} + m_k^{\frac{k+3}{5(k-3)}} + m_k^{\frac{1}{2}}\Big] - \frac{C''}{2} m_{k+6}\,, \qquad (4.12)$$

for some $C_k > 0$ depending only on $k > 3$. In order to obtain a differential inequality for m_k, it remains to estimate m_{k+6}. Indeed, using Hölder's inequality (4.3)

$$m_{k+6}^{\frac{k-3}{k+3}} m_3^{\frac{6}{k+3}} \ge m_k\,,$$

which implies $m_{k+6} \ge m_k^{\frac{k+3}{k-3}}$. As a consequence, from (4.12) we finally arrive to

$$\frac{\mathrm{d}}{\mathrm{d}t} m_k(t) \ \le\ C_k\Big[m_k^{\frac{k-1}{k-3}} + m_k^{\frac{k+3}{5(k-3)}} \ +\ m_k^{\frac{1}{2}}\Big](t) \ -\ \frac{C''}{2} m_k^{\frac{k+3}{k-3}}(t)\,. \qquad (4.13)$$

By Young inequality, there are positive constants $C(\epsilon)$ and ϵ such that

$$m_k^{\frac{k-1}{k-3}} \le \epsilon m_k^{\frac{k+3}{k-3}}(t) + C(\epsilon), \qquad m_k^{\frac{k+3}{5(k-3)}} \le \epsilon m_k^{\frac{k+3}{k-3}}(t) + C(\epsilon),$$

and by Cauchy inequality

$$m_k^{\frac{1}{2}} \le \frac{1}{2} m_k + \frac{1}{2}.$$

Combining the above inequalities, for ϵ small, with (4.13) we conclude that there are positive constants, still denoted by C_k and $C''/2$, such that

$$\frac{\mathrm{d}}{\mathrm{d}t} m_k(t) \ \le\ C_k\big(1 +\ m_k(t)\big) \ -\ C'' m_k^{\frac{k+3}{k-3}}(t)\,. \qquad (4.14)$$

By comparing (4.14) with the solution of the Bernoulli equation

$$\frac{\mathrm{d}}{\mathrm{d}t} Y(t) \ \le\ C_k Y(t) \ -\ C'' Y^{\frac{k+3}{k-3}}(t)\,,$$

which is

$$Y(t) = \Big[\big(Y(0)e^{-C_k t}\big)^{-\frac{6}{k-3}} + \frac{C''}{C_k}\big(1 - e^{-\frac{C_k 6t}{k-3}}\big)\Big]^{-\frac{k-3}{6}}$$
$$\leq C_k(\mathfrak{h}_3)\big(1 - e^{-\frac{C_k 6t}{k-3}}\big)^{-\frac{k-3}{6}},$$

where $C_k(\mathfrak{h}_3) := \big(C_k/C''\big)^{\frac{k-3}{6}}$ is a constant depending linearly on $\sqrt{\overline{n}_c}$ and $\sqrt{\underline{n}_c}$, since C'' depends only on $\mathfrak{h}_3 = m_3(0)$ and C_k only on k. Hence inequality (4.1) holds. In addition, if the initial k^{th} line-moment $m_k(0)$ is finite, then clearly the bound may be improved at $t = 0$, and $m_k(t)$ clearly satisfies inequality (4.2). ■

5 The Cauchy Problem

This section is devoted to show existence and uniqueness of positive solutions of the initial value problem associated to equation (1.14), (1.15) and (1.16), which corresponds the to solutions of the initial value problem for equation (1.1)-(1.13) where the collision operator has a transition probability given by $|\mathcal{M}|^2 = \kappa|p||p_1||p_2|$ from (1.9) for $p = p_1 + p_2$ and $|p| = |p_1| + |p_2|$.

The approach we use is based on an abstract framework for solving ODE's in Banach spaces applied in this context to find uniqueness of non-negative homogeneous radially symmetric solutions of the quantum Boltzmann equation for bosons at very low temperature in $L^1\big(\mathbb{R}^3, |p|\mathrm{d}p\big)$, the set of measurable functions, integrable w.r.t. the measure $|p|\mathrm{d}p$.

More specifically, we have the following theorem, whose proof can be found in the Appendix 7.

Theorem 5.1 *Let $E := (E, \|\cdot\|)$ be a Banach space, $\mathcal{S}$ be a bounded, convex and closed subset of E, and $Q : \mathcal{S} \to E$ be an operator satisfying the following properties:*

Hölder continuity condition

$$\big\|Q[f] - Q[g]\big\| \leq C\|f - g\|^{\beta}, \quad \beta \in (0,1), \quad \forall\, f, g \in \mathcal{S}\,, \tag{5.1}$$

Sub-tangent condition

$$\liminf_{h\to 0^+} h^{-1}\, dist\big(f + hQ[f], \mathcal{S}\big) = 0, \qquad \forall\, f \in \mathcal{S}\,, \tag{5.2}$$

and, one-sided Lipschitz condition

$$\big[Q[f]-Q[g], f-g\big] \le C\|f-g\|, \qquad \forall\, f,g \in \mathcal{S}\,, \tag{5.3}$$

where $\big[\varphi,\phi\big] := \lim_{h\to 0^-} h^{-1}\big(\|\phi+h\varphi\|-\|\phi\|\big)$.

Suppose that $n=n(t)$ *is a continuous function in* $C^1([0,\infty)$ *and* n *is bounded uniformly from below and above by positive constants* $\underline{n}$ *and* $\overline{n}$.
Then the equation

$$\partial_t f = nQ[f] \ \text{on}\ [0,\infty)\times E, \qquad f(0)=f_0\in\mathcal{S} \tag{5.4}$$

has a unique solution in $C^1((0,\infty),E)\cap C([0,\infty),\mathcal{S})$.

This theorem is an extension of Theorem A.1 proposed in [13] by Bressan in the context of solving the classical elastic Boltzmann equation for hard spheres in 3 dimensions. We point out that [13] does not properly show that (5.2) is satisfied in that case. For completeness of this manuscript we rewrite Bressan's unpublished proof in the Appendix. Bressan's needed techniques can be found in [34].

Indeed, referring to the argument given in [1], using conditions (5.1) and (5.2) combined with [34, Theorem VI.2.2] one has that conditions (C1), (C2) and (C3) in [34, pg. 229] are satisfied and hence, together with (5.3), all needed conditions for the existence and uniqueness theorem [34, Theorem VI.4.3] for ODEs in Banach spaces are fulfilled.

For our particular case, we need to identify a suitable Banach space and a corresponding bounded, convex and closed subset $\mathcal{S}$.

Indeed, choosing $E = L^1\big(\mathbb{R}^3, |p|\mathrm{d}p\big)$, the choice of the subspace $\mathcal{S}$, defined below in (5.5), specifically depend on the estimates to solutions of the quantum Boltzmann equation (1.14), (1.15) and (1.16), whose collisional operator satisfy conditions (5.1), (5.2) and (5.3) when the transition probability (1.9) is given by $|\mathcal{M}|^2 = \kappa|p||p_1||p_2|$ for $p=p_1+p_2$ and $|p|=|p_1|+|p_2|$.

More specificallly, such subset $\mathcal{S}\subset L^1\big(\mathbb{R}^3, |p|\mathrm{d}p\big)$ is characterized by the Hölder continuity and sub-tangent conditions (5.1) and (5.2), respectively, (to be shown next in subsection 5.2), and it is defined as follows:

$$\mathcal{S} \ := \ \Big\{ f\in L^1\big(\mathbb{R}^3,|p|\mathrm{d}p\big)\ \big|\ \textbf{i.}\ f \text{ nonnegative \& radially symmetric}\,,$$
$$\textbf{ii.}\ m_3\langle f\rangle \ = \ \int_{\mathbb{R}_+} d|p|\, f(|p|)|p|^3 \ = \ \mathfrak{h}_3\,, \tag{5.5}$$
$$\textbf{iii.}\ m_{10}\langle f\rangle \ = \ \int_{\mathbb{R}_+} d|p|\, f(|p|)|p|^{10} \ \le \ \mathfrak{h}_{10}\Big\},$$

where $\mathfrak{h}_3$ is an arbitrary initial energy, and the specific $\mathfrak{h}_{10}$ is defined below in (5.29). We are now in conditions to state and prove the existence and uniqueness theorem.

Theorem 5.2 (Existence and Uniqueness) *Let* $f_0(p) = f_0(|p|) \in \mathcal{S}$. *Then, equation* (1.1)-(1.13) *with* (1.9) *has a unique momentum and energy conservative solution*

$$0 \le f(t,p) = f(t,|p|) \in \mathcal{C}\big([0,\infty);\mathcal{S}\big) \cap \mathcal{C}^1\big((0,\infty);L^1\big(\mathbb{R}^3,|p|dp\big)\big). \qquad (5.6)$$

Proof. The proof of this theorem consists of verifying the three conditions (5.1), (5.2), and (5.3) in Subsections 5.1, 5.2, and 5.3, respectively. We start first with the Hölder continuity condition.

5.1 Hölder Estimate for Q

Recall the definition of $m_k\langle f\rangle$, the k^{th}-line-moment of a radially symmetric $f(p) := f(|p|)$

$$m_k\langle f\rangle := \int_{\mathbb{R}_+} \mathrm{d}p\, f(|p|)|p|^k\,, \qquad k \ge 0\,, \qquad (5.7)$$

and observe that $m_3\langle |f|\rangle$ is equivalent to the usual norm for a radially symmetric function in $L^1\big(\mathbb{R}^3,|p|\mathrm{d}p\big)$.

Lemma 5.1 (Hölder continuity) *The collision operator*

$$Q : \mathcal{S} \to L^1\big(\mathbb{R}^3,|p|\,dp\big)$$

is Hölder continuous, with the following Hölder estimate

$$m_3\big\langle |Q[f]-Q[g]|\big\rangle \le A_1\, m_3\big\langle |f-g|\big\rangle^{\frac{1}{7}} + A_2\, m_3\big\langle |f-g|\big\rangle\,, \qquad (5.8)$$

valid for all f, $g \in \mathcal{S}$. *The constants* A_i, *for* $i = \{1,2\}$, *depend only on* $\mathfrak{h}_3$ *and* $\mathfrak{h}_{10}$.

Proof. We first observe that for any $f \in \mathcal{S}$, properties **i.** and **ii.** in (5.5) yield the interpolation estimates shown in (4.3) for moments $m_5\langle f\rangle \le \mathcal{C}_5$ and $m_6\langle f\rangle \le \mathcal{C}_6$, with $\gamma = \frac{2}{7}$ and $\gamma = \frac{3}{7}$ and positive constants depending only on $\mathfrak{h}_3$ and $\mathfrak{h}_{10}$, respectively.

Next, in order to estimate the $L^1\big(\mathbb{R}^3,|p|\mathrm{d}p\big)$-norm of the difference of the collision operator on any pair of functions f and g in $\mathcal{S}$, we use the weak formulation shown in Proposition 2.1 applied to the test function $\varphi(p) = \text{sign}\big(Q[f]-Q[g]\big)(p)$, yielding the identity

$$\begin{aligned}
\int_{\mathbb{R}^3} \mathrm{d}p\, \big|Q[f]-Q[g]\big|(p)|p| &= \int_{\mathbb{R}^3} \mathrm{d}p\, \big(Q[f]-Q[g]\big)(p)\text{sign}\big(Q[f]-Q[g]\big)(p)|p| \\
&= \int_{\mathbb{R}^9} \mathrm{d}p\,\mathrm{d}p_1\mathrm{d}p_2\, |p\,p_1p_2|\delta(p-p_1-p_2)\delta(|p|-|p_1|-|p_2|) \\
&\quad \times \Big[f(p_1)f(p_2) - 2f(p_2)f(p) - f(p) - g(p_1)g(p_2) + 2g(p_2)g(p) + g(p)\Big] \\
&\quad \times \Big[|p|\text{sign}\big(Q[f]-Q[g]\big)(p) - |p_1|\text{sign}\big(Q[f]-Q[g]\big)(p_1) \\
&\qquad\qquad - |p_2|\text{sign}\big(Q[f]-Q[g]\big)(p_2)\Big]\,.
\end{aligned}$$

So, using the triangle inequality, it follows

$$\begin{aligned}
&\int_{\mathbb{R}^3} \mathrm{d}p\, \big|Q[f]-Q[g]\big|(p)|p| \\
&\leq \int_{\mathbb{R}^9} \mathrm{d}p\,\mathrm{d}p_1\mathrm{d}p_2\, |p\,p_1p_2|\delta(p-p_1-p_2)\delta(|p|-|p_1|-|p_2|) \\
&\quad \times \Big|f(p_1)f(p_2) - 2f(p_2)f(p) - f(p) - g(p_1)g(p_2) + 2g(p_2)g(p) + g(p)\Big| \\
&\qquad\qquad \times \Big[|p| + |p_1| + |p_2|\Big].
\end{aligned} \tag{5.9}$$

Hence, using the same change of coordinates (3.10) used to obtained the a priori moment's estimates, now applied to the above inequality (5.9), yields

$$\begin{aligned}
&\int_{\mathbb{R}_+} \mathrm{d}r\, \big|Q[f]-Q[g]\big|(r)r^3 \leq \\
&C\int_0^\infty\int_0^r \mathrm{d}r_2\mathrm{d}r\, |r-r_2|^3|r_2|^3 r\big|f(r-r_2)f(r_2) - 2f(r_2)f(r) - f(r) \\
&\qquad - g(r-r_2)g(r_2) + 2g(r_2)g(r) + g(r)\big|\big(|r| + |r-r_2| + |r_2|\big),
\end{aligned} \tag{5.10}$$

where C is a explicit positive constant that varies from line to line. Now, since $|r| + |r-r_2| + |r_2| = 2r$ in the $0 \leq r_2 \leq r$ domain of integration, the simplified expression follows

$$\begin{aligned}
&\int_{\mathbb{R}_+} \mathrm{d}r\big|Q[f]-Q[g]\big|(r)r^3 \leq \\
&C\int_0^\infty\int_0^r \mathrm{d}r\mathrm{d}r_2 r^2|r-r_2|^3|r_2|^3\big|f(r-r_2)f(r_2) - 2f(r_2)f(r) - f(r) \\
&\qquad - g(r-r_2)g(r_2) + 2g(r_2)g(r) + g(r)\big| \\
&= Q_1 + Q_2 + Q_3\,,
\end{aligned} \tag{5.11}$$

where the Q_i, with $i \in \{1, 2, 3\}$, are defined by

$$
\begin{aligned}
&Q_1[f, g] := \\
&C \int_0^\infty \int_0^r \mathrm{d}r_2 \mathrm{d}r\, r^2 |r - r_2|^3 |r_2|^3 \big| f(r - r_2) f(r_2) - g(r - r_2) g(r_2) \big| \,,
\end{aligned}
\tag{5.12}
$$

$$
Q_2[f, g] := C \int_0^\infty \int_0^r \mathrm{d}r_2 \mathrm{d}r\, r^2 |r - r_2|^3 |r_2|^3 \big| f(r_2) f(r) - g(r_2) g(r) \big| \,, \tag{5.13}
$$

and

$$
Q_3[f, g] := C \int_0^\infty \int_0^r \mathrm{d}r_2 \mathrm{d}r\, r^2 |r - r_2|^3 |r_2|^3 \big| f(r) - g(r) \big| \,. \tag{5.14}
$$

Therefore, the proof of the Hölder estimate for the collision operator follows from estimating these three terms.

Estimating Q_1. First, splitting $f(r - r_2) f(r_2) - g(r - r_2) g(r_2)$ as the sum of $f(r - r_2)(f(r_2) - g(r_2))$ and $g(r_2)(f(r - r_2) - g(r - r_2))$ and applying the triangle inequality from (5.12) yields

$$
\begin{aligned}
Q_1[f, g] \le\, & C \int_0^\infty \int_0^r \mathrm{d}r_2 \mathrm{d}r\, r^2 |r - r_2|^3 |r_2|^3 |f(r - r_2)| |f(r_2) - g(r_2)| \\
&+ C \int_0^\infty \int_0^r \mathrm{d}r_2 \mathrm{d}r\, r^2 |r - r_2|^3 |r_2|^3 |g(r_2)| |f(r - r_2) - g(r - r_2)| \,.
\end{aligned}
\tag{5.15}
$$

Exchanging variables $r - r_2 \to r_1$, the right side of (5.15) is bounded by

$$
\begin{aligned}
\int_{\mathbb{R}_+} \mathrm{d}r \big| Q_1[f] - Q_1[g] \big| (r) r^3 \le\, & C \int_{\mathbb{R}_+^2} \mathrm{d}r_1 \mathrm{d}r_2 (r_1 + r_2)^2 r_1^3 r_2^3 |f(r_1)| |f - g|(r_2) \\
&+ C \int_{\mathbb{R}_+^2} \mathrm{d}r_1 \mathrm{d}r_2 (r_1 + r_2)^2 r_1^3 r_2^3 |g(r_2)| |f - g|(r_1) \,.
\end{aligned}
$$

Next, using the inequality $(r_1 + r_2)^2 \le 2(r_1^2 + r_2^2)$, the right hand side integral is simplifies to

$$
\begin{aligned}
Q_1[f, g] \le\, & C \int_{\mathbb{R}_+^2} \mathrm{d}r_1 \mathrm{d}r_2 \big(r_1^5 r_2^3 + r_1^3 r_2^5 \big) |f(r_1)| |f(r_2) - g(r_2)| \\
&+ C \int_{\mathbb{R}_+^2} \mathrm{d}r_1 \mathrm{d}r_2 \big(r_1^5 r_2^3 + r_1^3 r_2^5 \big) |g(r_2)| |f(r_1) - g(r_1)| \qquad (5.16) \\
\le\, & C \left(\mathfrak{h}_3 + \mathcal{C}_5 \right) \int_{\mathbb{R}_+} \mathrm{d}r |f(r) - g(r)| \big(|r|^3 + |r|^5 \big) \,,
\end{aligned}
$$

where last inequality holds by the propagation of moments estimate

$$\int_{\mathbb{R}_+} \mathrm{d}r\, r^3 \max\{f,g\}(r) \leq \mathfrak{h}_3\,, \quad \int_{\mathbb{R}_+} \mathrm{d}r\, r^5 \max\{f,g\}(r) \leq \mathcal{C}_5\,. \tag{5.17}$$

Finally, using Hölder inequality

$$\int_{\mathbb{R}_+} \mathrm{d}r\, |f(r)-g(r)||r|^5 \leq \Big(\int_{\mathbb{R}_+} \mathrm{d}r\, |f(r)-g(r)||r|^3 \Big)^{1/3} \\ \times \Big(\int_{\mathbb{R}_+} \mathrm{d}r\, |f(r)-g(r)||r|^6 \Big)^{2/3} \leq \mathcal{C}_6^{2/3} \Big(\int_{\mathbb{R}_+} \mathrm{d}r\, |f(r)-g(r)||r|^3 \Big)^{1/3},$$

leads to estimate for the term Q_1 as follows,

$$\begin{aligned} Q_1[f,g] \leq C\, \mathfrak{h}_3\, \mathcal{C}_6^{2/3} \Big(\int_{\mathbb{R}_+} \mathrm{d}r\, |f(r)-g(r)||r|^3 \Big)^{1/3} \\ + C\, \mathcal{C}_5 \int_{\mathbb{R}_+} \mathrm{d}r |f(r)-g(r)||r|^3\,, \end{aligned} \tag{5.18}$$

where, we recall, the constants $\mathcal{C}_5$ and $\mathcal{C}_6$ are controlled by $\mathfrak{h}_3$ and $\mathfrak{h}_{10}$.

Estimating Q_2. Expressing $f(r_2)f(r) - g(r_2)g(r)$ as the sum of $(f(r_2) - g(r_2))f(r)$ and $g(r_2)(f(r)-g(r))$ we estimate (5.13) as

$$\begin{aligned} Q_2[f,g] \leq C \int_0^\infty \int_0^r \mathrm{d}r_2 \mathrm{d}r\, r^2 |r-r_2|^3 |r_2|^3 |f(r_2)-g(r_2)||f(r)| \\ + C \int_0^\infty \int_0^r \mathrm{d}r_2 \mathrm{d}r\, r^2 |r-r_2|^3 |r_2|^3 |f(r)-g(r)||g(r_2)|\,. \end{aligned} \tag{5.19}$$

Since $|r-r_2| \leq |r|$, we obtain from (5.19) that

$$\begin{aligned} Q_2[f,g] &\leq C \int_0^\infty \int_0^r \mathrm{d}r_2 \mathrm{d}r\, |r|^5 |r_2|^3 |f(r_2)-g(r_2)||f(r)| \\ &\quad + C \int_0^\infty \int_0^r \mathrm{d}r_2 \mathrm{d}r\, |r|^5 |r_2|^3 |f(r)-g(r)||g(r_2)| \\ &\leq C\mathfrak{h}_3 \int_{\mathbb{R}_+} \mathrm{d}r\, |f(r)-g(r)||r|^5 + C\, \mathcal{C}_5 \int_{\mathbb{R}_+} \mathrm{d}r\, |f(r)-g(r)||r|^3\,, \end{aligned} \tag{5.20}$$

where we have used in the last inequality (5.17). By the same argument as (5.18), we get

$$Q_2[f,g] \leq C\, \mathfrak{h}_3\, \mathcal{C}_6^{2/3} \Big(\int_{\mathbb{R}_+} \mathrm{d}r |f(r)-g(r)||r|^3 \Big)^{1/3}$$

$$+ C\,\mathcal{C}_5 \int_{\mathbb{R}_+} \mathrm{d}r|f(r) - g(r)||r|^3 \,. \tag{5.21}$$

Estimating Q_3. Integrating in r_2, we can rewrite (5.14) as an integral in r only

$$Q_3[f,g] = C \int_{\mathbb{R}_+} \mathrm{d}r|f(r) - g(r)||r|^9 \,, \tag{5.22}$$

where C is some other universal constant. Thus, using Hölder inequality as in (4.3) on $|f-g|(r)$ with $\gamma = \frac{6}{7}$, one obtains

$$\begin{aligned} C^{-1}\, Q_3[f,g] &= \int_{\mathbb{R}_+} \mathrm{d}r\,|f-g|(r)|r|^9 \\ &\leq \Big(\int_{\mathbb{R}_+} \mathrm{d}r|f-g|(r)|r|^{10} \Big)^{6/7} \times \Big(\int_{\mathbb{R}_+} \mathrm{d}r\,|f-g|(r)|r|^3 \Big)^{1/7} \\ &\leq (2\mathfrak{h}_{10})^{6/7} \Big(\int_{\mathbb{R}_+} \mathrm{d}r\,|f-g|(r)|r|^3 \Big)^{1/7} . \end{aligned} \tag{5.23}$$

Therefore, estimate (5.8) follows by gathering (5.18), (5.21) and (5.23). ■

5.2 Sub-tangent condition

This condition, jointly with the Hölder continuity, characterize the subset $\mathcal{S} \subset L^1\big(\mathbb{R}^3, |p|\mathrm{d}p\big)$ defined in (5.5).

First, we show that the collision operator Q can be split as the sum of a gain and a loss operators, as mentioned earlier in (1.14)

$$Q[f] = Q^+[f] - f\,\nu[f] \,,$$

provided $\nu[f]$ is finite whenever $f \in \mathcal{S}$. Indeed, this property follows by the nature of the interaction law (i.e. the form of the singular mass term in the integrand) and transition probability $\mathcal{M}$, since

$$\begin{aligned} \nu[f](p) &= \int_{\mathbb{R}^3} \mathrm{d}p_1 |p||p_1||p-p_1|\delta(|p| - |p_1| - |p-p_1|)\big[2f(p_1) + 1\big] \\ &\quad + 2\int_{\mathbb{R}^3} \mathrm{d}p_2 |p||p+p_2||p_2|\delta(|p+p_2| - |p| - |p_2|) f(p_2) \\ &= \int_0^{|p|} \mathrm{d}r\,|p|r^3(|p| - r)\big[2f(r) + 1\big] + 2\int_{\mathbb{R}_+} \mathrm{d}r|p|(|p| + r) r^3 f(r) \\ &\quad \leq C\,|p|\,\big(m_3\langle f\rangle^{\frac{5}{4}} + m_4\langle f\rangle + |p|^5\big)\,, \end{aligned} \tag{5.24}$$

and, therefore,

$$\big|\nu[f](p)\big| \le C(\mathfrak{h}_3, \mathfrak{h}_{10})|p|\big(1+|p|^5\big)\,, \quad \forall f \in \mathcal{S}\,. \tag{5.25}$$

The sub-tangent condition (5.2) follows as a corollary of next Proposition 5.1.

Proposition 5.1 *Fix $f \in \mathcal{S}$. Then, for any $\epsilon > 0$, there exists $h_1 := h_1(f,\epsilon) > 0$, such that the ball centered at $f + hQ[f]$ with radius $h\,\epsilon > 0$ intersects $\mathcal{S}$, that is,*

$$B(f + hQ[f], h\epsilon) \cap \mathcal{S},\ \textit{is non-empty for any}\ 0 < h < h_1.$$

Proof. First, set $\chi_R(p)$ the characteristic function of the ball of radius $R > 0$ and introduce the truncated function $f_R(p) := \chi_R(p) f(p)$, then set $w_R := f + hQ[f_R]$.

We can control w_R from below to show it is possible to find an h_1 such that w_R remains non-negative for as long $0 < h < h_1$. Indeed, for any $f \in \mathcal{S}$ its truncation $f_R(p) \in \mathcal{S}$ as well, and since Q^+ is a positive operator,

$$\begin{aligned} w_R &= f + Q^+[f_R] - hf_R\,\nu[f_R] \ge f - hf_R\,\nu[f_R] \\ &\ge f\Big(1 - h\,C(\mathfrak{h}_3, \mathfrak{h}_{10})R\big(1+|R|^5\big)\Big) \ge 0 \end{aligned} \tag{5.26}$$

for any $0 < h < h_1 := 1/C(\mathfrak{h}_3, \mathfrak{h}_{10})R\big(1+|R|^5\big)$. In addition, since $f_R \in \mathcal{S}$, $Q[f_R] \in L^1\big(\mathbb{R}^3, |p|\mathrm{d}p\big)$ by Lemma 5.1, and, as a consequence, $w_R \in L^1\big(\mathbb{R}^3, |p|\mathrm{d}p\big)$ as well. Moreover, by conservation of energy $\int_{\mathbb{R}^3}\mathrm{d}p\,Q[f_R])\,|p|^3 = 0$, yielding

$$\begin{aligned} m_3\langle w_R\rangle &= \int_{\mathbb{R}^3}\mathrm{d}p\,w_R(|p|)|p|^3 = \int_{\mathbb{R}^3}\mathrm{d}p\,\big(f + hQ[f_R]\big)|p|^3 \\ &= \int_{\mathbb{R}^3}\mathrm{d}p\,f(|p|)|p|^3 = \mathfrak{h}_3\,, \end{aligned} \tag{5.27}$$

with $\mathfrak{h}_3$ independent of the parameter R. In particular, w_R satisfies,uniformly in R, property **i.** in the characterization of the $\mathcal{S}$ defined in (5.5).

Finally we need to show that w_R also satisfies property **ii.** in the set $\mathcal{S}$. First, recall the *a priori* estimate for developed in (4.13) for the line-moment inequalities, namely

$$\begin{aligned} &\int_{\mathbb{R}^3}\mathrm{d}p\,Q[f]|p|^k \le \mathcal{L}_k\big(m_k\langle f\rangle\big) \ := \\ &\quad C_k\Big[m_k\langle f\rangle^{\frac{k-1}{k-3}} + m_k\langle f\rangle^{\frac{k+3}{5(k-3)}} + m_k\langle f\rangle^{\frac{1}{2}}\Big] - \frac{C''}{2}m_k\langle f\rangle^{\frac{k+3}{k-3}}\,, \end{aligned} \tag{5.28}$$

holds for any $k > 3$ and C_k only depending on k, and C'' only depending on $m_3\langle f\rangle = \mathfrak{h}^3$. Note that the map $\mathcal{L}_k : [0,\infty) \to \mathbb{R}$ has only one root, denoted as $\mathfrak{h}_*^k$, at which $\mathcal{L}_k$ changes from positive to negative for any $k > 3$. Note that this root only depends on $\mathfrak{h}^3$ and k. Thus, it is always the case that

$$\int_{\mathbb{R}^3} \mathrm{d}p\, Q[f]|p|^k \leq \mathcal{L}_k\big(m_k\langle f\rangle\big) \leq \max_{0\leq x\leq \mathfrak{h}_*^k}\{\mathcal{L}_k(x)\}, \qquad f \in \mathcal{S}.$$

Fix $k = 10$ and define

$$\mathfrak{h}_{10} := \mathfrak{h}_*^{10} + \max_{0\leq x\leq \mathfrak{h}_*^{10}}\{\mathcal{L}_{10}(x)\}. \tag{5.29}$$

For any $f \in \mathcal{S}$, we have two possibilities: $m_{10}\langle f\rangle \leq \mathfrak{h}_*^{10}$, or $m_{10}\langle f\rangle > \mathfrak{h}_*^{10}$. For the former, it readily follows that

$$\begin{aligned} m_{10}\langle w_R\rangle = \int_{\mathbb{R}^3} \mathrm{d}p\, w_R(|p|)|p|^{10} &= \int_{\mathbb{R}^3} \mathrm{d}p\, \big(f + hQ[f_R]\big)|p|^{10} \\ &\leq \mathfrak{h}_*^{10} + h \max_{0\leq x\leq \mathfrak{h}_*^{10}}\{\mathcal{L}_{10}(x)\} \leq \mathfrak{h}_{10}, \end{aligned}$$

where in the last inequality we have assumed $h \leq 1$ without loss of generality.

For the latter, we can choose $R := R(f)$ sufficiently large such that $m_{10}\langle f_R\rangle \geq \mathfrak{h}_*^{10}$, and therefore,

$$\int_{\mathbb{R}^3} \mathrm{d}p\, Q[f_R]|p|^{10} \leq \mathcal{L}_{10}\big(m_{10}\langle f_R\rangle\big) \leq 0\,.$$

As a consequence,

$$m_{10}\langle w_R\rangle = \int_{\mathbb{R}^3} \mathrm{d}p\, \big(f + hQ[f_R]\big)|p|^{10} \leq \int_{\mathbb{R}^3} \mathrm{d}p\, f|p|^{10} \leq \mathfrak{h}_{10}\,.$$

The conclusion is that for any $f \in \mathcal{S}$, it is always the case that

$$m_{10}\langle w_R\rangle \leq \mathfrak{h}_{10}\,, \tag{5.30}$$

which ensures that w_R satisfies property **ii.** of the set $\mathcal{S}$ in (5.5). We infer, thanks to (5.26), (5.27) and (5.30), that $w_R \in \mathcal{S}$ for any $0 < h < h_*$ where

$$h_* = \min\Big\{1, 1/\big(C(\mathfrak{h}_3)R(f)\big(1 + |R(f)|^5\big)\big)\Big\}\,. \tag{5.31}$$

The argument ends using the Hölder estimate from Lemma 5.1 to obtain

$$h^{-1}m_3\big\langle|f{+}hQ[f] - w_R|\big\rangle = m_3\big\langle|Q[f] - Q[f_R]|\big\rangle$$

$$\le A_1\, m_3\big\langle |f-f_R|\big\rangle^{\frac{1}{7}} + A_2\, m_3\big\langle |f-f_R|\big\rangle \le \epsilon\,,$$

for $R := R(\epsilon)$ sufficiently large. Then, $w_R \in B(f+hQ[f],h\epsilon)$ for this choice. Thus, choosing $R=\max\{R(f),R(\epsilon)\}$ and $h_1 := h_1(f,\epsilon)$ as in (5.31) one concludes that $w_R \in B(f+hQ[f],h\epsilon)\cap\mathcal{S}$. Consequently,

$$h^{-1}\mathrm{dist}\big(f+hQ[f],\mathcal{S}\big) \le \epsilon\,,\qquad \forall\, 0<h<h_1\,.$$

The proof of Proposition 5.1 is now complete. ■

5.3 One-side Lipschitz condition

Using dominate convergence theorem one can show that

$$\big[\varphi,\phi\big] \le \int_{\mathbb{R}^3} \mathrm{d}p\, \varphi(p)\mathrm{sign}(\phi)|p|\,.$$

Thus, the one-side Lipschitz condition is met after proving the following lemma showing a Lipschitz condition for quantum-Boltzmann operator. The following proof, which yields a uniqueness results, is in the same spirit of the original Di Blassio [17] uniqueness proof for initial value problem to the homogeneous Boltzmann equation for hard spheres, using data with enough initial moments.

Lemma 5.2 (Lipschitz condition) *Assume $f,g\in\mathcal{S}$. Then, there exists constant $C := C(\mathfrak{h}_3,\mathfrak{h}_{10})>0$ such that*

$$\int_{\mathbb{R}^3} dp\,\big(Q[f](p)-Q[g](p)\big)\,sign(f-g)\big(|p|^1+|p|^2\big) \le C\, m_3\big\langle |f-g|\big\rangle\,.$$

Proof. We start with the identity valid for radial functions $f := f(|p|)$ and $\varphi := \varphi(|p|)$

$$\begin{aligned}\int_{\mathbb{R}^3} \mathrm{d}p\, Q[f](p)\varphi(p) = 2(2\pi)^2 \int_0^\infty\!\!\int_0^\infty \mathrm{d}r_1\mathrm{d}r_2(r_1+r_2)r_1^3r_2^3\\ \times\big[\varphi(r_1+r_2)-\varphi(r_1)-\varphi(r_2)\big]R(f)(r_1,r_2)\,,\end{aligned}$$

where

$$R(f)(r_1,r_2) := f(r_1)f(r_2) - 2f(r_1)f(r_1+r_2) - f(r_1+r_2)\,.$$

Thus,

$$\int_{\mathbb{R}^3} \mathrm{d}p\, \big(Q[f](p) - Q[g](p)\big)\varphi(p) = 2(2\pi)^2 \int_0^\infty \int_0^\infty \mathrm{d}r_1 \mathrm{d}r_2 (r_1 + r_2) r_1^3 r_2^3 \\ \times \big[\varphi(r_1 + r_2) - \varphi(r_1) - \varphi(r_2)\big]\big(R(f)(r_1, r_2) - R(g)(r_1, r_2)\big)\,, \tag{5.32}$$

where, by definition

$$R(f)(r_1, r_2) - R(g)(r_1, r_2) = \big(f(r_1)f(r_2) - g(r_1)g(r_2)\big) \\ - 2\big(f(r_1)f(r_1 + r_2) - g(r_1)g(r_1 + r_2)\big) - \big(f(r_1 + r_2) - g(r_1 + r_2)\big)\,.$$

Now, let us particularize for $\varphi := \varphi_k = |\cdot|^k \mathrm{sign}(f - g)$, with $k \in \{1, 2\}$, and control each of the natural 3 terms appearing in the right side of (5.32). For the first, use simply $|\varphi_k| \leq |\cdot|^k$ to obtain

$$\big(f(r_1)f(r_2) - g(r_1)g(r_2)\big)\big[\varphi_k(r_1 + r_2) - \varphi_k(r_1) - \varphi_k(r_2)\big] \\ \leq \Big(\big|f(r_1) - g(r_1)\big|f(r_2) + g(r_1)\big|f(r_2) - g(r_2)\big|\Big)\big[|r_1 + r_2|^k + |r_1|^k + |r_2|^k\big].$$

Since $|r_1 + r_2|^k + |r_1|^k + |r_2|^k \leq 2\big(r_1 + r_2\big)^k$, it readily follows that

$$\int_0^\infty \int_0^\infty \mathrm{d}r_1 \mathrm{d}r_2 (r_1 + r_2) r_1^3 r_2^3 \\ \times \big[\varphi(r_1 + r_2) - \varphi(r_1) - \varphi(r_2)\big]\big(f(r_1)f(r_2) - g(r_1)g(r_2)\big) \\ \leq 2^{k+1} m_3\big\langle f + g\big\rangle\, m_{k+4}\big\langle |f - g|\big\rangle + 2^{k+1} m_{k+4}\big\langle f + g\big\rangle\, m_3\big\langle |f - g|\big\rangle\,. \tag{5.33}$$

Similar argument for the second term, together with the change of variable $r_1 + r_2 \to r_2$, leads to

$$-2\int_0^\infty \int_0^\infty \mathrm{d}r_1 \mathrm{d}r_2 (r_1 + r_2) r_1^3 r_2^3 \\ \times \big[\varphi(r_1 + r_2) - \varphi(r_1) - \varphi(r_2)\big]\big(f(r_1)f(r_1 + r_2) - g(r_1)g(r_1 + r_2)\big) \\ \leq 2\, m_3\big\langle g\big\rangle\, m_{k+4}\big\langle |f - g|\big\rangle + 2\, m_{k+4}\big\langle f\big\rangle\, m_3\big\langle |f - g|\big\rangle\,. \tag{5.34}$$

Now, the absorption (third) term is nonpositive for $k = 1$ since

$$-\big(f(r_1 + r_2) - g(r_1 + r_2)\big)\big[\varphi_1(r_1 + r_2) - \varphi_1(r_1) - \varphi_1(r_2)\big] \\ \leq \big|f(r_1 + r_2) - g(r_1 + r_2)\big|\big[|r_1| + |r_2| - |r_1 + r_2|\big] = 0\,.$$

In addition, for $k = 2$ it follows that

$$-\big(f(r_1 + r_2) - g(r_1 + r_2)\big)\big[\varphi_2(r_1 + r_2) - \varphi_2(r_1) - \varphi_2(r_2)\big]$$

$$\leq \big|f-g\big|(r_1+r_2)\big[|r_1|^2+|r_2|^2-|r_1+r_2|^2\big] = -2r_1r_2\big|f-g\big|(r_1+r_2)\,.$$

In turn, this leads to

$$\begin{aligned}
&-\int_0^\infty\int_0^\infty \mathrm{d}r_1\mathrm{d}r_2(r_1+r_2)r_1^3r_2^3\big[\varphi(r_1+r_2)-\varphi(r_1)-\varphi(r_2)\big]\big(f-g\big)(r_1+r_2)\\
&\quad\leq -2\int_0^\infty\int_0^\infty \mathrm{d}r_1\mathrm{d}r_2(r_1+r_2)r_1^4r_2^4\big|f(r_1+r_2)-g(r_1+r_2)\big| \qquad (5.35)\\
&\quad= -2\int_0^\infty \mathrm{d}r\, r|f-g|(r)\int_0^r \mathrm{d}r_1 r_1^4(r-r_1)^4 = -C\,m_{10}\big\langle|f-g|\big\rangle\,,
\end{aligned}$$

for some universal $C>0$. Gathering (5.33), (5.34) and (5.35) we conclude that for $f,\, g\in\mathcal{S}$

$$\begin{aligned}
&\int_{\mathbb{R}^3}\mathrm{d}p\,\big(Q[f](p)-Q[g](p)\big)\big(|p|^1+|p|^2\big)\mathrm{sign}(f-g)\leq c_1\,m_3\big\langle|f-g|\big\rangle\\
&\qquad + c_2\,m_5\big\langle|f-g|\big\rangle + c_3\,m_6\big\langle|f-g|\big\rangle - C\,m_{10}\big\langle|f-g|\big\rangle \leq c_4\,m_3\big\langle|f-g|\big\rangle\,,
\end{aligned}$$

where the constants c_i, with $i\in\{1,2,3,4\}$, depend on $\mathfrak{h}_3$ and $\mathfrak{h}_{10}$. The last inequality follows noticing that $c_1r^3+c_2r^5+c_3r^6-Cr^{10}\leq c_4r^3$ for any $r\geq 0$. ■

The proof of Theorem 5.2 is now completed, as an application of Theorem 5.4, where the three conditions (5.1), (5.2), and (5.3) have been verified in Subsections 5.1, 5.2, and 5.3, respectively.

■

6 Mittag-Leffler moments

6.1 Propagation of Mittag-Leffler tails

In this section we are interested in studying the propagation and creation of Mittag-Leffler moments of order $a\in[1,\infty)$ and rate $\alpha>0$ for radially symmetric solutions built in section 5. This concept of Mittag-Leffler tails was introduced recently in [45] and it is a generalization of the classical exponential tails for hard potentials in Boltzmann equations. We perform the analysis using standard moments $\mathcal{M}_k$ stressing that same estimates are valid for line moments since $\mathcal{M}_k = |\mathbb{S}^2|\,m_{k+2}$ in the context of radially symmetric solutions. In terms of infinite sums, see [45], this is equivalent to control the integral

$$\int_{\mathbb{R}^3} \mathrm{d}p\, f(t,p)\mathcal{E}_a(\alpha^a|p|) = \sum_{k=1}^{\infty} \frac{\mathcal{M}_k(t)\alpha^{ak}}{\Gamma(ak+1)}\,, \tag{6.1}$$

where

$$\mathcal{E}_a(x) := \sum_{k=1}^{\infty} \frac{x^k}{\Gamma(ak+1)} \approx e^{x^{1/a}} - 1\,, \qquad x \gg 1\,. \tag{6.2}$$

We have excluded the term $k = 0$ to account for the fact that equation (1.1)-(1.13) does not conserves mass. For convenience define for any $\alpha > 0$ and $a \in [1,\infty)$ the partial sums

$$\mathcal{E}_a^n(\alpha,t) := \sum_{k=1}^{n} \frac{\mathcal{M}_k(t)\alpha^{ak}}{\Gamma(ak+1)} \quad \text{and} \quad \mathcal{I}_{a,\rho}^n(\alpha,t) := \sum_{k=1}^{n} \frac{\mathcal{M}_{k+\rho}(t)\alpha^{ak}}{\Gamma(ak+1)}\,, \quad \rho > 0\,.$$

This notation will be of good use throughout this section.

Theorem 6.1 (Propagation of Mittag-Leffler tails) *Let f be a solution of (1.1)-(1.13) in $\mathcal{S}$ associated to the initial condition $f_0 \geq 0$, $a \in [1,\infty)$, and suppose that there exists positive α_0 such that*

$$\int_{\mathbb{R}^3} dp\, f_0(p)\, \mathcal{E}_a(\alpha_0^a|p|) \leq 1\,.$$

Then, there exists positive constant $\alpha := \alpha(\mathcal{M}_1(0),\alpha_0,a)$ such that

$$\int_{\mathbb{R}^3} dp\, f(t,p)\, \mathcal{E}_a(\alpha^a|p|) \leq 2\,, \qquad \forall\, t \geq 0\,. \tag{6.3}$$

Lemma 6.1 (From Ref. [45]) *Let $k \geq 3$, then for any $a \in [1,\infty)$, we have*

$$\sum_{i=1}^{\left[\frac{k+1}{2}\right]} \binom{k}{i} B\big(ai+1, a(k-i)+1\big) \leq C_a (ak)^{-1-a},$$

for some constant C_a depending on a.

Lemma 6.2 (Moment interpolation)

$$\mathcal{M}_\rho \leq \mathcal{M}_{\rho_1}^{\gamma}\mathcal{M}_{\rho_2}^{1-\gamma}, \tag{6.4}$$

where the positive constants $\rho, \rho_1, \rho_2, \gamma$ satisfy $0 < \rho_1 \leq \rho \leq \rho_2$, $0 < \gamma < 1$, and $\rho = \gamma\rho_1 + (1-\gamma)\rho_2$.

Remark 6.1 *Contrary to section 4, we will work in this section with the moments $\mathcal{M}_k$ rather than work with the line-moments m_k. It turns out to be clearer in terms of notation.*

Lemma 6.3 *Let $\alpha > 0$, $a \in [1, \infty)$. Then, the following estimate holds*

$$\sum_{k=k_0}^{n} \sum_{1}^{\left[\frac{k+1}{2}\right]} \binom{k}{i} \Big(\mathcal{M}_{i+2}\mathcal{M}_{1+(k-i)} + \mathcal{M}_{i+1}\mathcal{M}_{2+(k-i)} \Big) \frac{\alpha^{ak}}{\Gamma(ak+1)} \le C_a \frac{ak_0 + 1}{(ak_0)^{1+a}} \, \mathcal{E}_a^n \, \mathcal{I}_{a,3}^n \,, \qquad n \ge k_0 \ge 1 \,, \tag{6.5}$$

with universal constant C_a depending only on a.

Proof. First, we estimate the sum of the left side of (6.5) by controlling the sum $\mathcal{M}_{i+2}\mathcal{M}_{1+(k-i)} + \mathcal{M}_{i+1}\mathcal{M}_{2+(k-i)}$ with $2\mathcal{M}_i\mathcal{M}_{k-i+3}$ for any $i \ge 3$. This can be done using Hölder's inequality (6.4)

$$\mathcal{M}_{i+2} \le \mathcal{M}_i^{\frac{k+1-2i}{k+3-2i}} \mathcal{M}_{k-i+3}^{\frac{2}{k+3-2i}} \quad \text{and} \quad \mathcal{M}_{1+(k-i)} \le \mathcal{M}_i^{\frac{2}{k+3-2i}} \mathcal{M}_{k-i+3}^{\frac{k+1-2i}{k+3-2i}} \,.$$

Thus, the product of these terms is controlled by

$$\mathcal{M}_{i+2}\mathcal{M}_{1+(k-i)} \le \mathcal{M}_i\mathcal{M}_{k-i+3} \,.$$

Similarly, from (6.4), the following inequalities also hold

$$\mathcal{M}_{i+1} \le \mathcal{M}_i^{\frac{k-2i+2}{k+3-2i}} \mathcal{M}_{k-i+3}^{\frac{1}{k+3-2i}} \quad \text{and} \quad \mathcal{M}_{2+(k-i)} \le \mathcal{M}_i^{\frac{1}{k+3-2i}} \mathcal{M}_{k-i+3}^{\frac{k-2i+2}{k+3-2i}} \,,$$

which lead to the estimate

$$\mathcal{M}_{i+1}\mathcal{M}_{2+(k-i)} \le \mathcal{M}_i\mathcal{M}_{k-i+3} \,.$$

As a consequence,

$$\mathcal{M}_{i+2}\mathcal{M}_{1+(k-i)} + \mathcal{M}_{i+1}\mathcal{M}_{2+(k-i)} \le 2\mathcal{M}_i\mathcal{M}_{3+(k-i)} \,.$$

Therefore, it readily follows that

$$\mathcal{J} := \sum_{k=k_0}^{n} \sum_{i=1}^{\left[\frac{k+1}{2}\right]} \binom{k}{i} \Big(\mathcal{M}_{i+2}\mathcal{M}_{1+(k-i)} + \mathcal{M}_{i+1}\mathcal{M}_{2+(k-i)} \Big) \frac{\alpha^{ak}}{\Gamma(ak+1)} \le 2 \sum_{k=k_0}^{n} \sum_{i=1}^{\left[\frac{k+1}{2}\right]} \binom{k}{i} \mathcal{M}_i\mathcal{M}_{3+(k-i)} \frac{\alpha^{ak}}{\Gamma(ak+1)} \,. \tag{6.6}$$

Using the following identities for the Beta and Gamma functions

$$B(ai+1,a(k-i)+1) = \frac{\Gamma(ai+1)\,\Gamma(a(k-i)+1)}{\Gamma(a(i+1)+a(k-i)+1)} = \frac{\Gamma(ai+1)\,\Gamma(a(k-i)+1)}{\Gamma(ak+2)},$$

and the identity $\alpha^{ak} = \alpha^{\alpha i}\alpha^{a(k-i)}$, we deduce from (6.6) that

$$\mathcal{J} \le 2\sum_{k=k_0}^{n}\sum_{i=1}^{[\frac{k+1}{2}]}\binom{k}{i}\frac{\mathcal{M}_i\alpha^{ai}}{\Gamma(ai+1)}\frac{\mathcal{M}_{k-i+3}\alpha^{a(k-i)}}{\Gamma(a(k-i)+1)} \times B(ai+1,a(k-i)+1)\frac{\Gamma(ak+2)}{\Gamma(ak+1)}. \tag{6.7}$$

Since $\Gamma(ak+2) = (ak+1)\Gamma(ak+1)$, the term $\frac{\Gamma(ak+2)}{\Gamma(ak+1)}$ in (6.7) can be reduced to $ak+1$. That is,

$$\mathcal{J} \le 2\sum_{k=k_0}^{n}(ak+1) \times \sum_{i=1}^{[\frac{k+1}{2}]}\binom{k}{i}\frac{\mathcal{M}_i\alpha^{ai}}{\Gamma(ai+1)}\frac{\mathcal{M}_{k-i+3}\alpha^{a(k-i)}}{\Gamma(a(k-i)+1)}B(ai+1,a(k-i)+1). \tag{6.8}$$

Also, each component in the sum on the right side of (6.8) can be bounded as

$$\sum_{i=1}^{[\frac{k+1}{2}]}\binom{k}{i}\frac{\mathcal{M}_i\alpha^{ai}}{\Gamma(ai+1)}\frac{\mathcal{M}_{k-i+3}\alpha^{a(k-i)}}{\Gamma(a(k-i)+1)}B(ai+1,a(k-i)+1) \le \sum_{i=1}^{[\frac{k+1}{2}]}\frac{\mathcal{M}_i\alpha^{ai}}{\Gamma(ai+1)}\frac{\mathcal{M}_{k-i+3}\alpha^{a(k-i)}}{\Gamma(a(k-i)+1)}\sum_{j=1}^{[\frac{k+1}{2}]}\binom{k}{j}B(aj+1,a(k-j)+1),$$

which implies, by Lemma 6.1, that

$$\sum_{i=1}^{[\frac{k+1}{2}]}\binom{k}{i}\frac{\mathcal{M}_i\alpha^{ai}}{\Gamma(ai+1)}\frac{\mathcal{M}_{k-i+3}\alpha^{a(k-i)}}{\Gamma(a(k-i)+1)}B(ai+1,a(k-i)+1) \le \frac{C_a}{(ak)^{1+a}}\sum_{i=1}^{[\frac{k+1}{2}]}\frac{\mathcal{M}_i\alpha^{ai}}{\Gamma(ai+1)}\frac{\mathcal{M}_{k-i+3}\alpha^{a(k-i)}}{\Gamma(a(k-i)+1)}. \tag{6.9}$$

Combining (6.8) and (6.9) yields the estimate on $\mathcal{J}$

$$\mathcal{J} \le 2C_a \sum_{k=k_0}^{n} \frac{ak+1}{(ak)^{1+a}} \sum_{i=1}^{\left[\frac{k+1}{2}\right]} \frac{\mathcal{M}_i \alpha^{ai}}{\Gamma(ai+1)} \frac{\mathcal{M}_{k-i+3}\alpha^{a(k-i)}}{\Gamma(a(k-i)+1)} \,. \tag{6.10}$$

Notice that $\frac{ak+1}{(ak)^{1+a}}$ decreases towards 0 as k increases to infinity. Therefore, from (6.10) one concludes that

$$\begin{aligned}
&\sum_{k=k_0}^{n} \sum_{i=1}^{\left[\frac{k+1}{2}\right]} \binom{k}{i} \Big(\mathcal{M}_{i+2}\mathcal{M}_{1+(k-i)} + \mathcal{M}_{i+1}\mathcal{M}_{2+(k-i)} \Big) \frac{\alpha^{ak}}{\Gamma(ak+1)} \\
&\qquad \le 2C_a \frac{ak_0+1}{(ak_0)^{1+a}} \sum_{k=k_0}^{n} \sum_{i=1}^{\left[\frac{k+1}{2}\right]} \frac{\mathcal{M}_i \alpha^{ai}}{\Gamma(ai+1)} \frac{\mathcal{M}_{k-i+3}\alpha^{a(k-i)}}{\Gamma(a(k-i)+1)} \qquad (6.11) \\
&\le 2C_a \frac{ak_0+1}{(ak_0)^{1+a}} \sum_{i=1}^{n} \frac{\mathcal{M}_i \alpha^{ai}}{\Gamma(ai+1)} \sum_{i=1}^{n} \frac{\mathcal{M}_{i+3}\alpha^{ai}}{\Gamma(ai+1)} \le C_a \frac{ak_0+1}{(ak_0)^{1+a}} \mathcal{E}_a^n \, \mathcal{I}_{a,3}^n .
\end{aligned}$$

■

Lemma 6.4 *The following control is valid for any $\alpha > 0$ and $a \in [1, \infty)$*

$$\mathcal{I}_{a,6}^n(\alpha, t) \ge \frac{1}{\alpha^3} \mathcal{E}_a^n(\alpha, t) - \frac{1}{\alpha^{5/2}} \mathcal{M}_1 \mathcal{E}_a(a - 1/2) \,. \tag{6.12}$$

Proof. Observe that

$$\mathcal{I}_{a,6}^n(\alpha, t) = \sum_{k=1}^{n} \frac{\mathcal{M}_{k+6}(t)\alpha^{ak}}{\Gamma(ak+1)} \ge \sum_{k=1}^{n} \int_{\{|p| \ge \frac{1}{\sqrt{\alpha}}\}} \mathrm{d}p \, \frac{|p|^{k+6}\alpha^{ak}}{\Gamma(ak+1)} f(t,p) \,.$$

Note that in the set $\{|p| \ge \frac{1}{\sqrt{\alpha}}\}$ one has $|p|^{k+6} \ge \frac{|p|^k}{\alpha^3}$, therefore

$$\begin{aligned}
\mathcal{I}_{a,6}^n(\alpha, t) &\ge \frac{1}{\alpha^3} \sum_{k=1}^{n} \int_{\{|p| \ge \frac{1}{\sqrt{\alpha}}\}} \mathrm{d}p \, \frac{|p|^k \alpha^{ak}}{\Gamma(ak+1)} f(t,p) \\
&= \frac{1}{\alpha^3} \Bigg(\sum_{k=1}^{n} \int_{\mathbb{R}^3} \mathrm{d}p \, \frac{|p|^k \alpha^{ak}}{\Gamma(ak+1)} f(t,p) - \sum_{k=1}^{n} \int_{\{|p| < \frac{1}{\sqrt{\alpha}}\}} \mathrm{d}p \, \frac{|p|^k \alpha^{ak}}{\Gamma(ak+1)} f(t,p) \Bigg) .
\end{aligned}$$

In the set $\{|p| < \frac{1}{\sqrt{\alpha}}\}$ one has $|p|^k < |p| \alpha^{-(k-1)/2}$, consequently

$$\mathcal{I}_{a,6}^n(\alpha, t) \ge \frac{1}{\alpha^3} \Bigg(\mathcal{E}_a^n(t) - \sum_{k=1}^{n} \int_{\mathbb{R}^3} \mathrm{d}p \, \frac{\alpha^{-(k-1)/2}\alpha^{ak}}{\Gamma(ak+1)} f(t,p)|p| \Bigg)$$

$$= \frac{1}{\alpha^3}\mathcal{E}_a^n(t) - \frac{\mathcal{M}_1}{\alpha^{5/2}}\sum_{k=1}^{n}\frac{\alpha^{(a-1/2)k}}{\Gamma(ak+1)}\,.$$

Since

$$\sum_{k=1}^{n}\frac{\alpha^{(a-1/2)k}}{\Gamma(ak+1)} \le \sum_{k=1}^{\infty}\frac{\alpha^{(a-1/2)k}}{\Gamma(ak+1)} = \mathcal{E}_a(a-1/2)\,,$$

estimate (6.12) follows. ■

Proof. (of Theorem 6.1) The proof consists in showing that for any $a \in [1,\infty)$, there exists positive constant α such that

$$\mathcal{E}_a^n(\alpha,t) \le 2, \quad \forall\, t \ge 0,\, \forall\, n \in \mathbb{N}\backslash\{0\}. \tag{6.13}$$

For this purpose we define for sufficiently small $\alpha > 0$, chosen in the sequel, the sequence of times

$$T_n := \sup\left\{t \ge 0 \,\middle|\, \mathcal{E}_a^n(\alpha,\tau) \le 2, \forall\, \tau \in [0,t]\right\}$$

and prove that $T_n = +\infty$. This sequence of times is well-defined and positive. Indeed, for any $\alpha \le \alpha_0$

$$\mathcal{E}_a^n(\alpha,0) = \sum_{k=1}^{n}\frac{\mathcal{M}_k(0)\alpha^{ak}}{\Gamma(ak+1)} \le \sum_{k=1}^{n}\frac{\mathcal{M}_k(0)\alpha_0^{ak}}{\Gamma(ak+1)} = \int_{\mathbb{R}^3}\mathrm{d}p\, f_0(p)\mathcal{E}_a(\alpha_0^a|p|) \le 1\,.$$

Since each term $\mathcal{M}_k(t)$ is continuous in t, the partial sum $\mathcal{E}_a^n(\alpha,t)$ is also continuous in t. Therefore, $\mathcal{E}_a^n(\alpha,t) \le 2$ in some nonempty interval $(0,t_n)$ and, thus, T_n is well-defined and positive for every $n \in \mathbb{N}$.

Now, let us establish a differential inequality for the partial sums that implies $T_n = +\infty$. Note that (3.3), with $\gamma = 1$, implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_k \le C_1\sum_{i=1}^{\left[\frac{k+1}{2}\right]}\binom{k}{i}\Big(\mathcal{M}_{i+2}\mathcal{M}_{1+(k-i)} + \mathcal{M}_{i+1}\mathcal{M}_{2+(k-i)}\Big) - C_2\mathcal{M}_{k+6}\,.$$

Multiplying the above inequality by $\frac{\alpha^k}{\Gamma(ak+1)}$ and summing with respect to k in the interval $k_0 \le k \le n$, with $k_0 \ge 1$ to be chosen later on sufficiently large,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=k_0}^{n}\frac{\mathcal{M}_k\,\alpha^k}{\Gamma(ak+1)} \le C_1\sum_{k=k_0}^{n}\sum_{i=1}^{\left[\frac{k+1}{2}\right]}\binom{k}{i}\Big(\mathcal{M}_{i+2}\mathcal{M}_{1+(k-i)}\\ + \mathcal{M}_{i+1}\mathcal{M}_{2+(k-i)}\Big)\frac{\alpha^k}{\Gamma(ak+1)} - C_2\sum_{k=k_0}^{n}\frac{\mathcal{M}_{k+6}\,\alpha^k}{\Gamma(ak+1)}\,.\end{aligned} \tag{6.14}$$

We observe that the sum on the left side of (6.14) will become $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_a^n(\alpha,t)$ after adding

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=1}^{k_0-1}\frac{\mathcal{M}_k\,\alpha^k}{\Gamma(ak+1)} \leq C(k_0,\alpha_0,a) < \infty \tag{6.15}$$

to this expression. The latter inequality holds due to the choice $\alpha \leq \alpha_0$ and the control of moments (3.3). Therefore, from (6.14) and (6.15), we obtain the differential inequality

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_a^n(\alpha,t) \leq C_1 \sum_{k=k_0}^{n}\sum_{i=1}^{\left[\frac{k+1}{2}\right]}\binom{k}{i}\big(\mathcal{M}_{i+2}\mathcal{M}_{1+(k-i)} + \\ \mathcal{M}_{i+1}\mathcal{M}_{2+(k-i)}\Big)\frac{\alpha^k}{\Gamma(ak+1)} - C_2\sum_{k=k_0}^{n}\frac{\mathcal{M}_{k+6}\,\alpha^k}{\Gamma(ak+1)} + C(k_0,\alpha_0,a).\end{aligned} \tag{6.16}$$

Let us now estimate the sum on the right side of (6.16). We deduce from Theorem 4.1 that

$$\sum_{k=1}^{k_0}\frac{\mathcal{M}_{k+6}\,\alpha^k}{\Gamma(ak+1)} \leq \sum_{k=1}^{k_0}\frac{\mathcal{M}_{k+6}\,\alpha_0^k}{\Gamma(ak+1)} \leq C(k_0,\alpha_0,a)\,,$$

which leads to the following estimate for (6.16)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_a^n(\alpha,t) \leq C_1 \sum_{k=k_0}^{n}\sum_{i=1}^{\left[\frac{k+1}{2}\right]}\binom{k}{i}\Big(\mathcal{M}_{i+2}\mathcal{M}_{1+(k-i)} + \\ \mathcal{M}_{i+1}\mathcal{M}_{2+(k-i)}\Big)\frac{\alpha^k}{\Gamma(ak+1)} - C_2\sum_{k=1}^{n}\frac{M_{k+6}\,\alpha^k}{\Gamma(ak+1)} + C(k_0,\alpha_0,a)\,.\end{aligned} \tag{6.17}$$

By the definition of $\mathcal{I}_{a,6}^n$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_a^n(\alpha,t) \leq C_1 \sum_{k=k_0}^{n}\sum_{i=1}^{\left[\frac{k+1}{2}\right]}\binom{k}{i}\Big(\mathcal{M}_{i+2}\mathcal{M}_{1+(k-i)} + \\ \mathcal{M}_{i+1}\mathcal{M}_{2+(k-i)}\Big)\frac{\alpha^k}{\Gamma(ak+1)} - C_2\,\mathcal{I}_{a,6}^n + C(k_0,\alpha_0,a).\end{aligned} \tag{6.18}$$

Thus, thanks to Lemma 6.3, we have the control on (6.18)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_a^n \leq C_a\frac{ak_0+1}{(ak_0)^{1+a}}\mathcal{E}_a^n\,\mathcal{I}_{a,3}^n - C_2\,\mathcal{I}_{a,6}^n + C(k_0,\alpha_0,a)\,. \tag{6.19}$$

We now estimate the right hand side of (6.19) starting with the term $\mathcal{I}^n_{a,3}$. Using Cauchy inequality $|p|^3 \le \frac{1}{2} + \frac{1}{2}|p|^6$, then

$$\mathcal{M}_{k+3} \le \frac{1}{2}\mathcal{M}_k + \frac{1}{2}\mathcal{M}_{k+6}\,, \qquad k \ge 0\,.$$

Multiplying this inequality with $\frac{\alpha^{ak}}{\Gamma(ak+1)}$ and summing with respect to k in the interval $0 \le k \le n$ yields

$$\mathcal{I}^n_{a,3} \le \frac{1}{2}\mathcal{E}^n_a + \frac{1}{2}\mathcal{I}^n_{a,6}\,.$$

Since we are considering $t \in [0, T_n]$ one has $\mathcal{E}^n_a \le 2$ and, as a result, the following inequality is valid

$$\mathcal{I}^n_{a,3} \le 1 + \frac{1}{2}\mathcal{I}^n_{a,6}\,.$$

This implies from (6.19) the estimate on

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^n_a \le 2C_a\frac{ak_0+1}{(ak_0)^{1+a}}\big(1 + \tfrac{1}{2}\mathcal{I}^n_{a,6}\big) - C_2\mathcal{I}^n_{a,6} + C(k_0,\alpha_0,a)\,. \tag{6.20}$$

Choosing k_0 sufficiently large, the term $2C_a\frac{ak_0+1}{2(ak_0)^{1+a}}\mathcal{I}^n_{a,6}$ is absorbed by $\frac{C_2}{2}\mathcal{I}^n_{a,6}$. Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^n_a \le -\frac{C_2}{2}\mathcal{I}^n_{a,6} + C(\mathcal{M}_1,\alpha_0,a)\,. \tag{6.21}$$

Recall that C_2 only depends on the energy $\mathcal{M}_1 = \mathcal{M}_1(0)$, thus, k_0 only depends on the initial energy and a. Let us estimate the right side of (6.21) in terms of $\mathcal{E}^n_a$. Lemma 6.4 provides a lower bound on $\mathcal{I}^n_{a,6}$ in terms of $\mathcal{E}^n_a$ which can be used in (6.21) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^n_a \le -\frac{C_2}{2\alpha^3}\mathcal{E}^n_a + \frac{C_2}{2\alpha^{5/2}}\mathcal{M}_1\mathcal{E}_a(a-1/2) + C(\mathcal{M}_1,\alpha_0,a)\,.$$

Integrating the differential inequality

$$\mathcal{E}^n_a \le 1 + \frac{2\alpha^3}{C_2}\Big(\frac{C_2}{2\alpha^{5/2}}\mathcal{M}_1\mathcal{E}_a(a-1/2) + C(\mathcal{M}_1,\alpha_0,a)\Big) < 2\,, \quad t\in[0,T_n]\,, \tag{6.22}$$

provided that $\alpha := \alpha(\mathcal{M}_1,\alpha_0,a) > 0$ is such that

$$\frac{2\alpha^3}{C_2}\left(\frac{C_2}{2\alpha^2}\mathcal{M}_1\mathcal{E}_a(a-1/2) + C(\mathcal{M}_1,\alpha_0,a)\right) < 1\,.$$

Given the continuity of $\mathcal{E}_a^n(\alpha,t)$ with respect to t, estimate (6.22) contradicts the maximality of T_n, unless $T_n = +\infty$. Therefore, $\mathcal{E}_a^n(\alpha,t) \le 2$ for $t \in [0,\infty)$ and $n \in \mathbb{N}\backslash\{0\}$. Now taking the limit as $n \to \infty$ and using the definition of Mittag-Leffler moments of order $a \in [1,\infty)$ and rate $\alpha > 0$, as defined in (6.1), yields

$$\int_{\mathbb{R}^3} \mathrm{d}p\, f(t,p)\mathcal{E}_a(\alpha^a|p|) = \lim_{n\to\infty} \mathcal{E}_a^n(\alpha,t) \le 2\,.$$

This concludes the argument. ■

6.2 Creation of exponential tails

Theorem 6.2 *Let f be a positive solution of* (1.1)-(1.13) *in $\mathcal{S}$. Then, there exists constant $\alpha > 0$ depending only on $\mathcal{M}_1(0)$ such that*

$$\int_{\mathbb{R}^3} dp\, f(t,p)|p|e^{\alpha \min\{1,t^{\frac{1}{6}}\}|p|} \le \frac{1}{2\alpha}, \quad \forall\, t \ge 0. \tag{6.23}$$

Proof. Thanks to equation (4.1) we have the control

$$m_k(t) \le C_k(\mathfrak{h}_3)\big(1 - e^{-C_k t}\big)^{-\frac{k-3}{6}}, \ \ \forall\, k > 3\,.$$

This implies that

$$\mathcal{E}_1^n(t^{\frac{1}{6}}\alpha,t) = \int_{\mathbb{R}^3} \mathrm{d}p\, f(t,p)\mathcal{E}_1^n\big(t^{\frac{1}{6}}\alpha|p|\big) \le C_n(\alpha)\, t^{\frac{1}{6}}\,, \quad \alpha > 0\,. \tag{6.24}$$

Fix parameters $\alpha, \vartheta \in (0,1]$ and define

$$T_n := \sup\Big\{t \in [0,1] \big| \mathcal{E}_1^n(t^{\frac{1}{6}}\alpha,t) \le t^{\frac{1-\vartheta}{6}}\Big\}\,.$$

We proof that for sufficiently small $\alpha > 0$ depending only on $m_3(0)$, $T_n = 1$ for all $n \in \mathbb{N}$ and $\vartheta \in (0,1]$. One notices first that $T_n > 0$ for each n thanks to (6.24). Also, for $n \ge k_0 \ge 1$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=k_o}^{n} \mathcal{M}_k(t)\frac{(t^{\frac{1}{6}}\alpha)^k}{k!} = \sum_{k=k_o}^{n} \mathcal{M}_k'(t)\frac{(t^{\frac{1}{6}}\alpha)^k}{k!} + \frac{\alpha}{6t^{\frac{5}{6}}}\sum_{k=k_o}^{n} \mathcal{M}_k(t)\frac{(t^{\frac{1}{6}}\alpha)^{k-1}}{(k-1)!}\,. \tag{6.25}$$

Observe that for the last term in the right side of (6.25)

$$\frac{\alpha}{6t^{\frac{5}{6}}}\sum_{k=k_o}^{n} \mathcal{M}_k(t)\frac{(t^{\frac{1}{6}}\alpha)^{k-1}}{(k-1)!}$$

$$= \frac{\alpha}{6t^{\frac{5}{6}}} \sum_{k=k_o+6}^{n} \mathcal{M}_k(t) \frac{(t^{\frac{1}{6}}\alpha)^{k-1}}{(k-1)!} + \frac{\alpha}{6t^{\frac{5}{6}}} \sum_{k=k_o}^{k_o+5} \mathcal{M}_k(t) \frac{(t^{\frac{1}{6}}\alpha)^{k-1}}{(k-1)!}$$

$$= \frac{\alpha^6}{6} \sum_{k=k_o}^{n-6} \mathcal{M}_{k+6}(t) \frac{(t^{\frac{1}{6}}\alpha)^{k}}{(k+5)!} + \frac{\alpha}{6t^{\frac{5}{6}}} \sum_{k=k_o}^{k_o+5} \mathcal{M}_k(t) \frac{(t^{\frac{1}{6}}\alpha)^{k-1}}{(k-1)!}$$

$$\leq \frac{\alpha^6}{6} \sum_{k=k_o}^{n} \mathcal{M}_{k+6}(t) \frac{(t^{\frac{1}{6}}\alpha)^{k}}{k!} + \frac{\alpha^{k_o}}{t^{\frac{5}{6}}} C(k_o, m_3(0)) .$$

Thus, arguing as in (6.14)-(6.19) we conclude that for the quantities

$$\mathcal{E}_1^n := \mathcal{E}_1^n(t^{\frac{1}{6}}\alpha, t) , \quad \mathcal{I}_{1,6}^n := \mathcal{I}_{1,6}^n(t^{\frac{1}{6}}\alpha, t) ,$$

it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_1^n \leq \frac{C}{k_o}\mathcal{E}_1^n \mathcal{I}_{1,3}^n - \big(C_2 - \frac{\alpha^6}{6}\big)\mathcal{I}_{1,6}^n + \frac{\alpha}{t^{\frac{5}{6}}} C(k_o, m_3(0)) , \tag{6.26}$$

for a universal constant $C > 0$ and constant $C_2 > 0$ depending only $m_3(0)$. Using that

$$\mathcal{I}_{1,3}^n \leq \frac{\mathcal{E}_1^n}{2} + \frac{\mathcal{I}_{1,6}^n}{2}$$

and the definition of T_n, it follows from (6.26)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_1^n \leq \frac{C}{2k_o} - \big(C_2 - \frac{\alpha^6}{6} - \frac{C}{2k_o}\big)\mathcal{I}_{1,6}^n + \frac{\alpha}{t^{\frac{5}{6}}} C(k_o, m_3(0)) , \quad 0 < t \leq T_n . \tag{6.27}$$

Now fix $k_o \in \mathbb{N}$ and $\alpha \in (0, 1]$ such that

$$\frac{C}{2k_o} \leq \frac{C_2}{4} , \qquad \frac{\alpha^6}{6} \leq \frac{C_2}{4} ,$$

to conclude from (6.27) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_1^n \leq \frac{C}{2k_o} - \frac{C_2}{2}\mathcal{I}_{1,6}^n + \frac{\alpha}{t^{\frac{5}{6}}} C(k_o, m_3(0)) , \quad 0 < t \leq T_n . \tag{6.28}$$

Also observe that

$$\mathcal{I}_{1,6}^n = \sum_{k=1}^{n} \mathcal{M}_{k+6}(t) \frac{(t^{\frac{1}{6}}\alpha)^{k}}{k!}$$

$$
\begin{aligned}
&= \frac{1}{t\alpha^6}\sum_{k=7}^{n+6}\mathcal{M}_k(t)\frac{(t^{\frac{1}{6}}\alpha)^k}{(k-6)!} \geq \frac{1}{t\alpha^6}\sum_{k=7}^{n}\mathcal{M}_k(t)\frac{(t^{\frac{1}{6}}\alpha)^k}{k!} \\
&= \frac{1}{t\alpha^6}\mathcal{E}_1^n - \frac{1}{t\alpha^6}\sum_{k=1}^{6}\mathcal{M}_k(t)\frac{(t^{\frac{1}{6}}\alpha)^k}{k!} \geq \frac{1}{t\alpha^6}\mathcal{E}_1^n - \frac{C(m_3(0))}{t^{\frac{5}{6}}\alpha^5}\,.
\end{aligned}
$$

Together with (6.28), this leads finally to

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_1^n \leq \frac{C}{2k_o} + \frac{C(k_o, m_3(0))}{t^{\frac{5}{6}}\alpha^5} - \frac{C_2}{2t\alpha^6}\mathcal{E}_1^n\,, \quad 0 < t \leq T_n\,.
$$

Thus, using a comparison principle for ode's, we can choose $\alpha > 0$ sufficiently small, say

$$
\alpha := C_2\Big[\frac{C}{k_o} + 2C(k_o, m_3(0))\Big]^{-1}
$$

to deduce that $\mathcal{E}_1^n < t^{\frac{1}{6}}$. That is,

$$
\int_{\mathbb{R}^3} \mathrm{d}p\, f(t,p)\mathcal{E}_1^n(t^{\frac{1}{6}}\alpha|p|) < t^{\frac{1}{6}}\,, \quad 0 \leq t \leq T_n\,.
$$

Time continuity of $\mathcal{E}_1^n$ and the maximality of T_n imply that $T_n = 1$ for all $n \geq 1$ and $\vartheta \in (0,1]$. In particular, sending $\vartheta \to 0$ and, then, $n \to \infty$ one arrives to

$$
\int_{\mathbb{R}^3} \mathrm{d}p\, f(t,p)\mathcal{E}_1(t^{\frac{1}{6}}\alpha|p|) \leq t^{\frac{1}{6}}\,, \qquad 0 \leq t \leq 1\,.
$$

Furthermore, this estimate shows that

$$
\int_{\mathbb{R}^3} \mathrm{d}p\, f(1,p)\mathcal{E}_1(\alpha|p|) \leq 1.
$$

Then, using Theorem 6.1, the exponential moment propagates for $t > 1$, and choosing $\alpha > 0$ sufficiently small

$$
\int_{\mathbb{R}^3} \mathrm{d}p\, f(t,p)\mathcal{E}_1(\alpha|p|) \leq 1\,, \qquad t \geq 1\,.
$$

The result follows after noticing that

$$
\mathcal{E}_1(t^{\frac{1}{6}}\alpha|p|) \geq t^{\frac{1}{6}}\alpha|p|e^{t^{\frac{1}{6}}\frac{\alpha}{2}|p|}, \qquad 0 \leq t \leq 1.
$$

Since $m_3(0) = |\mathbb{S}^2|^{-1}\mathcal{M}_1(0)$ for radially symmetric functions, the dependence in the constants can be expressed in terms of the initial energy $\mathcal{M}_1(0)$

only. ■

Acknowledgements. This work has been partially supported by NSF grants DMS 143064 and RNMS (Ki-Net) DMS-1107444. In addition, M.-B Tran has been supported by ERC Advanced Grant DYCON. The authors would like to thank D.Heinzen, L. Reichl, M. Raizen and R. Dorfman for fruitful discussions on the topic. Support from the Institute of Computational Engineering and Sciences (ICES) at the University of Texas Austin is gratefully acknowledged. The research was carried on while M.-B. Tran and R. Alonso were visiting ICES.

7 Appendix: Proof of Theorem 5.1

Our proof follows the same lines of the argument to solve ODEs in Banach spaces proposed by A. Bressan in [13]. The proof is divided into three steps:

Step 1. Since $\mathcal{S}$ is bounded, there exists a uniform bound C_Q of $Q(u)$, for all u in $\mathcal{S}$. Let τ be in $[0,\infty)$ and u be in $\mathcal{S}$, there exists $h_{u,\tau} > 0$ such that for $0 < h < h_{u,\tau}$ and for all $\epsilon > 0$ sufficiently small,

- the intersection $B(u+n(\tau)hQ(u),\epsilon)\cap\mathcal{S}\backslash\{u+n(\tau)hQ(u)\}$ is non-empty;
- and, from properties on $n(t)$ as stated in Theorem 5.1,

$$|n(\tau+s)-n(\tau)| \le \frac{\epsilon}{2C_Q}, \forall s\in[0,h]. \tag{7.1}$$

In addition, since $n = n(t) \le \overline{n}$, then we can estimate $n(\tau)\|Q(u)-Q(v)\| \le \frac{\epsilon}{4}$, if $\|u-v\| \le \overline{n}(C_Q+1)h$. Hence, take w to be a point inside $B(u+n(\tau)hQ(u),\epsilon)\cap\mathcal{S}\backslash\{u+n(\tau)hQ(u)\}$ satisfying

$$\|w-u-n(\tau)hQ(u)\| \le \frac{\epsilon h}{4}.$$

We consider the linear map

$$s\mapsto\rho(s) = u+\frac{s(w-u)}{h}, \qquad s\in[0,h].$$

By the convexity of $\mathcal{S}$, $\rho(s)\in\mathcal{S}$ for all s in $[0,h]$. Moreover, since $\dot\rho(s) = \frac{w-u}{h}$,

$$\|\dot\rho(s)-n(\tau)Q(u)\| \le \frac{\epsilon}{4}.$$

Now, we can see that

$$\|\rho(s)-u\| = \left\|\frac{s(w-u)}{h}\right\| \le \|w-u\| \le n(\tau)h\|Q(u)\| + \frac{\epsilon h}{4} < \overline{n}(C_Q+1)h,$$

which implies

$$n(\tau)\|Q(\rho(s))-Q(u)\| \le \frac{\epsilon}{4}, \quad \forall\, s\in[0,h].$$

Therefore,

$$\|\dot{\rho}(s)-n(\tau)Q(\rho(s))\| \le \frac{\epsilon}{2}, \quad \forall\, s\in[0,h]. \tag{7.2}$$

Using (7.1), we deduce that

$$\|\dot{\rho}(s)-n(s)Q(\rho(s))\| \le \epsilon, \quad \forall\, s\in[0,h]. \tag{7.3}$$

A consequence of this fact is that

$$\|\dot{\rho}(s)\| \le 1+\overline{n}\, C_Q \tag{7.4}$$

for all s in $[0,h]$ and $\epsilon<1$.

Step 2. From Step 1, we have proved the existence of solution ρ to the equation (7.3) on an interval $[0,h]$. From this solution, we carry on the following process.

(1) We start with the solution ρ, defined on $[0,h]$ of (7.3).

(2) Suppose that the solution ρ of (7.3) is constructed on $[0,\tau]$. Since $\rho(\tau)\in\mathcal{S}$, by the same process as in Step 1, the solution ρ could be extended to $[\tau,\tau+h_\tau]$.

(3) Suppose that the solution ρ of (7.3) is constructed on a series of intervals $[0,\tau_1]$, $[\tau_1,\tau_2]$, $\cdots$, $[\tau_n,\tau_{n+1}]$, $\cdots$. Moreover, suppose the increasing sequence $\{\tau_n\}$ is bounded. Set

$$\tau = \lim_{n\to\infty}\tau_n.$$

Since $G(\rho)$ is bounded by C_G on $[\tau_n,\tau_{n+1}]$ for all $n\in\mathbb{N}$, $\dot{\rho}$ is bounded by $\epsilon+C_G$ on $[0,\tau)$. Therefore, we can define $\rho(\tau)$ satisfying

$$\rho(\tau)=\lim_{n\to\infty}\rho(\tau_n), \quad \dot{\rho}(\tau)=\lim_{n\to\infty}\dot{\rho}(\tau_n),$$

which implies that ρ is a solution of (7.3) on $[0,\tau]$.

By (3) of this process, we can see that if the solution ρ, constructed as above, is defined on $[0,T)$, it could be extended to $[0,T]$. Suppose that $[0,T]$ is the maximal closed interval that ρ could be constructed, by Step 2 of the process, ρ could be extended to a larger interval $[T, T+T_h]$, which means that ρ can be constructed on the whole interval $[0,\infty)$.

Step 3. Let us now consider two sequences of approximate solutions u^ϵ, w^ϵ, where ϵ tends to 0. From Step 1 and Step 2, one can see that the time interval $[0,T]$ can be decomposed into

$$\left(\bigcup_\gamma I_\gamma\right)\bigcup\mathfrak{N},$$

where I_γ are countably many open intervals and $\mathfrak{N}$ is of measure 0.
Taking the derivative of the difference $\|u^\epsilon(t)-w^\epsilon(t)\|$ gives

$$\begin{aligned}\frac{d}{dt}\|u^\epsilon(t)-w^\epsilon(t)\| &= \Big[u^\epsilon - w^\epsilon, \dot{u}^\epsilon(t)-\dot{w}^\epsilon(t)\Big]_- \\ &\leq \Big[u^\epsilon - w^\epsilon, \dot{u}^\epsilon(t)-\dot{w}^\epsilon(t)\Big]_- + 2\epsilon \\ &\leq L\|u^\epsilon(t)-w^\epsilon(t)\| + 2\epsilon,\end{aligned}$$

which yields

$$\|u^\epsilon(t)-w^\epsilon(t)\| \to 0 \quad \text{as} \quad \epsilon \to 0\,,$$

and we have the convergence $u^\epsilon \to u$ uniformly on $[0,T]$. The function u is, then, a solution of our equation.

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