**Abstract**

For the spatially homogeneous Boltzmann equation with cutoff hard potentials it is shown that solutions remain bounded from above, uniformly in time, by a Maxwellian distribution, provided the initial data have a Maxwellian upper bound. The main technique is based on a comparison principle that uses a certain dissipative property of the linear Boltzmann equation. Implications of the technique to propagation of upper Maxwellian bounds in the spatially-inhomogeneous case are discussed.

**Key words.** Boltzmann equation – long-time behavior – Maxwellian bounds – comparison principle

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# *Upper Maxwellian bounds for the spatially homogeneous Boltzmann equation*

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## 1. Introduction and main result

The nonlinear Boltzmann equation is a classical model for a gas at low or moderate densities. The gas in a spatial domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, is modeled by the mass density function $f(x, v, t)$, $(x, v) \in \Omega \times \mathbb{R}^d$, where $v$ is the velocity variable, and $t \in \mathbb{R}$ is time. The equation for $f$ reads

$$(\partial_t + v \cdot \nabla_x) f = Q(f)\,, \tag{1}$$

where $Q(f)$ is a quadratic integral operator, expressing the change of $f$ due to instantaneous binary collisions of particles. The precise form of $Q(f)$ will be introduced below, cf. also [10, 34].

Although some of our results deal with more general situations, we will be mostly concerned with a special class of solutions that are independent of the spatial variable (spatially homogeneous solutions). In this case $f = f(v, t)$ and one can study the initial-value problem

$$\partial_t f = Q(f), \quad f|_{t=0} = f_0, \tag{2}$$

where $0 \leq f_0 \in L^1(\mathbb{R}^d)$. The spatially homogeneous theory is very well developed although not complete. In the present paper we shall solve one of the most noticeable open problems remaining in the field, by establishing the following result.

**Theorem 1.** *Assume that $0 \leq f_0(v) \leq M_0(v)$, for a. a. $v \in \mathbb{R}^d$, where $M_0(v) = e^{-a_0|v|^2+c_0}$ is the density of a Maxwellian distribution, $a_0 > 0$, $c_0 \in \mathbb{R}$. Let $f(v,t)$, $v \in \mathbb{R}^d$, $t \geq 0$ be the unique solution of equation* (2) *for hard potentials with the angular cutoff assumptions* (5), (7)*, that preserves the initial mass and energy* (12)*. Then there are constants $a > 0$ and $c \in \mathbb{R}$ such that $f(v,t) \leq M(v)$, for a. a. $v \in \mathbb{R}^d$ and for all $t \geq 0$, where $M(v) = e^{-a|v|^2+c}$.*

Before going on, let us make a few comments about the interest of these bounds. Maxwellian functions

$$M(v) = e^{-a|v|^2+b\cdot v+c}, \quad \text{with } a > 0,\, c \in \mathbb{R},\; b \in \mathbb{R}^d \text{ constants},$$

are unique, within integrable functions, equilibrium solutions of (2), and they provide global attractors for the time-evolution described by (2) (or (1), with appropriate boundary conditions). Classes of functions bounded above by Maxwellians provide a convenient analytical framework for the local existence theory of strong solutions for (1), see Grad [22] and Kaniel-Shinbrot [25]. Such bounds also play an important role in the proof of validation of the Boltzmann equation by Lanford [27], see also [10]. However, establishing the propagation of uniform bounds is generally a difficult problem, solved only in the context of small solutions in an unbounded space, see Illner-Shinbrot [24] and subsequent works [4, 21, 23, 29]. These results rely in a crucial way on the decay of solutions for large $|x|$ and on the dispersive effect of the transport term, in order to control the nonlinearity. Dispersive effects may not have such a strong influence in other physical situations, and the spatially homogeneous problem presents the simplest example of such a regime, in which case our results may be relevant.

In the spatially homogeneous case many additional properties of solutions can be established. Upper bounds with polynomial decay for $|v|$ large hold uniformly in time, see Carleman [8, 9] and Arkeryd [2]. Solutions are also known to have a lower Maxwellian bound for all positive times, even for compactly supported initial data [32]. Many results have been established that concern the behavior of the moments with respect to the velocity variable, following the work by Povzner [31], see in particular [1, 6, 12, 15, 30]. The Carleman-type estimates [2, 8, 9] were crucial in the treatment of the weakly inhomogeneous problem given in [3]. However, as also pointed out in ref. [3], Maxwellian bounds of the local existence theory [22, 25] are not known to hold on longer time-intervals, and it remains an open problem to characterize the approach to the Maxwellian equilibrium in classes of functions with exponential decay. The present work aims to at least partially remedy this situation, and to develop a technique that could be used to obtain further results in this direction.

We will next introduce the notation and the necessary concepts to make the statement of Theorem 1 more precise. We set in (2)

$$Q(f)\,(v,t) = \int_{\mathbb{R}^d} \int_{S^{d-1}} (f'_*\, f' - f_*\, f)\, B(v - v_*, \sigma)\, d\sigma\, dv_*, \tag{3}$$

where, adopting common shorthand notations, $f = f(v,t)$, $f' = f(v',t)$, $f_* = f(v_*,t)$, $f'_* = f(v'_*,t)$. Here $v$, $v_*$ denote the velocities of two particles either before or after a collision,

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\,\sigma\,, \qquad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\,\sigma\,, \tag{4}$$

are the transformed velocities, and $\sigma \in S^{d-1}$ is a parameter determining the direction of the relative velocity $v' - v'_*$. In the more general case of (1), the space variable $x$ appears (similarly to $t$ above) in each occurrence of $f$, $f_*$, $f'$, $f'_*$; we shall often omit the $t$ and $x$ variables from the notation for brevity.

Many properties of the solutions of the Boltzmann equation depend crucially on certain features of the kernel $B$ in (3). Its physical meaning is the product of the magnitude of the relative velocity by the effective scattering cross-section (see [26, §18] for terminology and explicit examples); this quantity characterizes the relative frequency of collisions between particles. Our assumptions on $B$ fall in the category of "hard potentials with angular cutoff", cf. [34]. More precisely, we assume that

$$B(v - v_*, \sigma) = |v - v_*|^{\beta}\, h(\cos\vartheta), \quad \cos\vartheta = \tfrac{(v - v_*)\cdot\sigma}{|v - v_*|}, \tag{5}$$

where $0 < \beta \leq 1$ is a constant and $h$ is a nonnegative function on $(-1, 1)$ such that

$$h(z) + h(-z) \quad \text{is nondecreasing on } (0,1) \tag{6}$$

and

$$0 \leq h(\cos\vartheta)\sin^\alpha\vartheta \leq C, \quad \vartheta \in (0,\pi), \tag{7}$$

where $\alpha < d-1$ and $C$ is a constant. Assumption (7) implies in particular that the integral $\int_{S^{d-1}} h(\cos\vartheta)\,d\sigma$ is finite; for convenience we normalize it by setting

$$\int_{S^{d-1}} h(\cos\vartheta)\,d\sigma = \omega_{d-2}\int_{-1}^{1} h(z)\,(1-z^2)^{\frac{d-3}{2}}\,dz = 1, \tag{8}$$

where $\omega_{d-2}$ is the measure of the $(d-2)$-dimensional sphere. The classical hard-sphere model in $\mathbb{R}^d$, satisfies (5) with $\beta = 1$, (6) and (7) with $\alpha = d-3$.

Notice that we can write $Q(f) = Q^+(f) - Q^-(f)$, where $Q^+(f)$ is the "gain" term, and $Q^-(f)$ is the "loss" term,

$$Q^+(f) = \int_{\mathbb{R}^d}\int_{S^{d-1}} f' f'_*\, B(v-v_*,\sigma)\,d\sigma\,dv_*, \quad Q^-(f) = (f * |v|^\beta)\,f,$$

and $*$ denotes the convolution in $v$. Because of the symmetry $\sigma \mapsto -\sigma$ in the integral defining $Q^+(f)$ we can restrict the $\sigma$-integration above to the half-sphere $\{\cos\vartheta > 0\}$ if we simultaneously replace $B(v-v_*,\sigma)$ by

$$\overline{B}(v-v_*,\sigma) := (B(v-v_*,\sigma) + B(v-v_*,-\sigma))\,1_{\{\cos\vartheta>0\}}.$$

It will be convenient to introduce the following (nonsymmetric) bilinear forms of the collision terms,

$$Q^+(f,g) = \int_{\mathbb{R}^d}\int_{S^{d-1}} f'_*\,g'\,\overline{B}(v-v_*,\sigma)\,d\sigma\,dv_*, \quad Q^-(f,g) = (f*|v|^\beta)\,g, \tag{9}$$

for which obviously $Q^{\pm}(f) = Q^{\pm}(f,f)$.

We say that a nonnegative function $f \in C([0,\infty); L^1(\mathbb{R}^d))$, such that $(1+|v|^2)f \in L^\infty((0,\infty); L^1(\mathbb{R}^d))$, is a (mild) solution of (2) if for almost all $v \in \mathbb{R}^d$

$$f(v,0) = f_0(v); \qquad f(v,t) - f(v,s) = \int_s^t Q(f)(v,\tau)\, d\tau, \tag{10}$$

for all $0 \le s < t$. Notice that the conditions on $f$ imply (in the spatially-homogeneous case!) that

$$Q^+(f),\, Q^-(f) \in L^\infty((0,\infty); L^1(\mathbb{R}^d)), \tag{11}$$

so the integral form in (10) is well-defined. This also implies that $f$ is weakly differentiable with respect to $t$ and that the differential equation (2) holds in the sense of distributions on $\mathbb{R}^d \times (0,\infty)$.

The existence of a unique solution satisfying the conservations of mass and energy,

$$\int_{\mathbb{R}^d} f(v,t)\, dv = \int_{\mathbb{R}^d} f_0(v)\, dv, \qquad \int_{\mathbb{R}^d} f(v,t)\, |v|^2\, dv = \int_{\mathbb{R}^d} f_0(v)\, |v|^2\, dv \tag{12}$$

follows from a theorem by Mischler and Wennberg [30], for all $f_0 \ge 0$ for which the above integrals are finite. The second condition in (12) is also necessary for the uniqueness [35]. For the initial data with strong decay (as in Theorem 1) one could also refer to the well-known results by Carleman, Arkeryd and DiBlasio [1, 2, 13].

The following theorem summarizes the main results about qualitative properties of solutions in the case of "hard potentials with cutoff" known before this work.

**Theorem 2.** *Let $f(v,t)$, $v \in \mathbb{R}^d$, $t \geq 0$, $(n \geq 2)$ be a solution of* (2) *that satisfies* (12), *and let the kernel $B$ in the Boltzmann operator* (3) *satisfy* (5), (7). *Then*

(i) *if $f_0 \in L^\infty(\mathbb{R}^d)$ then $f(t,\cdot) \in L^\infty(\mathbb{R}^d)$, $t \geq 0$. Moreover, if $(1+|v|)^s f_0 \in L^\infty(\mathbb{R}^d_v)$ for some $s > s_0$, then $(1+|v|)^s f(v,t) \in L^\infty(\mathbb{R}^d_v)$, $t \geq 0$. Here $s_0$ is a constant dependent on the dimension $d$.*

(ii) *if the integral of $f$ is nonzero, then for every $t_0 > 0$ there is a Maxwellian $M(v) = Ke^{-\kappa|v|^2}$, $K > 0$, $\kappa > 0$ such that*

$$f(v,t) \geq M(v), \quad t \geq t_0, \quad \text{for a. a. } v \in \mathbb{R}^d.$$

(iii) *for all $t_0 > 0$ and for all $k > 1$, the quantity $m_k(t) = \int_{\mathbb{R}^d} f(v,t)\,|v|^{2k}\,dv$ is bounded uniformly for $t \geq t_0$; moreover, this bound is uniform in $t \geq 0$ if $m_k(0) < +\infty$.*

(iv) *In the case* $d = 3$ *and* $B(v - v_*, \sigma) = c\,|v - v_*|$ *(hard spheres) or* $B(v - v_*, \sigma) = h(\frac{(v-v_*)\cdot\sigma}{|v-v_*|})$, $h \in L^1(-1, 1)$ *(pseudo-Maxwell particles) if* $f_0$ *satisfies*

$$\frac{f_0}{M_0} \in L^1(\mathbb{R}^d)$$

*for some Maxwellian* $M_0(v) = e^{-a_0|v|^2}$, $a_0 > 0$*, then there exists constants* $a > 0$*,* $C$ *such that*

$$\int_{\mathbb{R}^d} \frac{f(v,t)}{M(v)}\,dv \le C,$$

*where* $M(v) = e^{-a|v|^2}$.

Part (i) of this theorem is due to Carleman [9] in the case of the hard spheres; the general case was studied by Arkeryd in [2]. Part (ii) is due to A. Pulvirenti and Wennberg [32]. Part (iii) is due to Desvillettes [12] under the additional assumption that a moment $m_{k_0}(t)$ of order $k_0 > 1$ is finite initially; this assumption was removed by Mischler and Wennberg [30]. Earlier result by Arkeryd [1] and Elmroth [15] state that all moments remain bounded uniformly in time, once they are finite initially. Finally, part (iv) is due to Bobylev [6]; we will give an extension of this result to the class of Boltzmann kernels satisfying (5)–(7) in Section 2.

Our main contribution in the present work is to show that the estimates for the spatially homogeneous Boltzmann equation (precisely, parts (i) and (iv) of Theorem 2, together with the conservation of mass) imply Theorem 1. Since we do not use other properties of the spatially-homogeneous problem we can state our result in a more general, spatially inhomogeneous setting.

We consider solutions of (1) with the spatial domain $\Omega = \mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ (the unit hypercube with periodic boundary conditions), on an arbitrary finite time interval $[0,T]$. Spatially homogeneous solutions are then a special subclass characterized by the constant dependence on the $x$ variable. To simplify the presentation, let us assume sufficient regularity (smoothness) of the solutions $f(x,v,t)$ with respect to the $x$ and $t$ variables; this is not a restriction in the setting of Theorem 1, and the requirements of smoothness will be relaxed significantly later on to include a sufficiently wide class of weak solutions of the spatially inhomogeneous problem.

**Theorem 3.** *Let $T > 0$ and let $f \in C([0,T]; L^1(\mathbb{T}^d \times \mathbb{R}^d))$, $f \geq 0$, be a (sufficiently regular) solution of the Boltzmann equation* (1)*, with the initial condition*

$$f(x,v,0) = f_0(x,v) \leq M_0(v), \quad \textit{for a. a. } (x,v) \in \mathbb{T}^d \times \mathbb{R}^d,$$

*where $M_0(v) = e^{-a_0|v|^2+c_0}$, $a_0 > 0$, $c_0 \in \mathbb{R}$. Assume that the solution $f(x,v,t)$ satisfies the estimates*

$$\int_{\mathbb{R}^d} f(x,v,t)\,dv \;\geq\; \rho_0, \quad (x,t) \in \mathbb{T}^d \times [0,T], \tag{13}$$

*and*

$$\sup_{(x,t)\,\in\,\mathbb{T}^d\times[0,T]} \|f(x,v,t)\|_{L^\infty_v} \;\leq\; C_0, \qquad \sup_{(x,t)\in\mathbb{T}^d\times[0,T]} \int_{\mathbb{R}^d} \frac{f(x,v,t)}{M_1(v)}\,dv \leq C_1, \tag{14}$$

*where $M_1(v) = e^{-a_1|v|^2+c_1}$ and $0 < a_1 < a_0$, $c_1$, $\rho_0$, $C_0$, $C_1$ are constants. Then for any $0 < a < a_1$, for any $t \in [0,T]$*

$$f(x,v,t) \leq M(v), \quad \textit{for a. a. } (x,v) \;\in\; \mathbb{T}^d \times \mathbb{R}^d,$$

*where* $M(v) = e^{-a|v|^2+c}$, *and the constant* $c$ *depends on* $a$, $a_0$, $c_0$, $a_1$, $c_1$, $\rho_0$, $C_0$ *and* $C_1$ *only.*

**Remark 1.** The regularity assumptions in Theorem 3 are not particularly restrictive. The precise conditions in the spatially inhomogeneous case are that $f$ is a mild (renormalized) solution of (1) that is dissipative in the sense of P.-L. Lions (see Definition 1 in Section 3). A sufficient condition that is naturally satisfied in the spatially-homogeneous case is that (11) holds in addition to (10).

The plan of the paper is as follows. In Section 2 we extend property (iv) from Theorem 2 to the class of Boltzmann kernels satisfying (5)–(7). This part uses properties specific to the spatially-homogeneous problem, and develops the ideas from [5–7]. The result of Section 2 illustrates an important point that the type of behavior described by Theorem 1 is not a particular feature of the hard-sphere model, but rather a generic phenomenon that holds for a wide class of collision kernels of "hard" type. The key step occurs in Section 3: there we introduce the technique based on a comparison principle which plays a crucial role in the derivation of pointwise estimates. In Section 4 we prove a weighted bound for the collision term, based on the Carleman representation of the gain operator, which is used in the comparison argument. Finally, some classical results used throughout the text are recalled in three Appendices.

**Convention:** Throughout the text, the function $\operatorname{sign} z$ is defined as 1 for $z > 0$, $-1$ for $z < 0$ and an arbitrary fixed value in $[-1, 1]$ for $z = 0$.

## 2. Weighted $L^1$ estimates of solutions

The aim of this section is to establish the following weighted integral bound for the solution of the Boltzmann equation (2).

**Theorem 4.** *Let $f(v,t)$, $v \in \mathbb{R}^d$, $t \geq 0$ $(n \geq 2)$ be a solution of the spatially homogeneous Boltzmann equation* (2) *with the collision kernel $B$ satisfying* (5)–(7) *and with the initial datum $f_0 \geq 0$ such that*

$$\frac{f_0}{M_0} \in L^1(\mathbb{R}^d) \tag{15}$$

*for a certain Maxwellian $M_0(v) = e^{-a_0|v|^2}$, where $a_0$ is a positive constant. Then there exist constants $D$, $a > 0$, such that*

$$\int_{\mathbb{R}^d} \frac{f(v,t)}{M(v)}\, dv \leq D, \quad t \geq 0, \tag{16}$$

*where $M(v) = e^{-a|v|^2}$.*

This result was obtained by Bobylev in the case of the "hard spheres" and Maxwell molecules [5, 6]. Here we present a generalization to the case of more general kernels $B$ that satisfy (5)–(7). The basic approach that we use is based on the method of moments. We introduce the central moments of order $2k$,

$$m_k(t) = \int_{\mathbb{R}^d} f(v,t)\, |v|^{2k}\, dv, \quad k = 0, 1 \ldots, \tag{17}$$

and use the Taylor expansion $\frac{1}{M(v)} = e^{a|v|^2} = \sum_{k=0}^{\infty} \frac{|v|^{2k}}{k!}\, a^k$ to obtain

$$\int_{\mathbb{R}^d} \frac{f(v,t)}{M(v)}\, dv = \sum_{k=0}^{\infty} \frac{m_k(t)}{k!}\, a^k. \tag{18}$$

The series above converges if and only if the integral is finite. To establish the estimate (16) it is then sufficient to show that the radius of convergence of the power series in (18) remains positive, uniformly in time. To this end we will look for an estimate

$$\sup_{t\geq 0} \frac{m_k(t)}{k!} \leq C q^k, \tag{19}$$

for certain $C > 0$, $q > 0$ and for $k$ large enough; that would imply that the series (18) converges for $a < q^{-1}$, and the estimate (16) then follows.

There are two important steps to our proof. The first one is a sharp form of the Povzner lemma as presented by Bobylev [6] for hard spheres in three dimensions and extended here for the more general class of kernels satisfying (5)–(7). The next step is the study of the asymptotic behavior of the constants in the moment inequalities in which a sharper control of the constants is required for the case of the kernels with an integrable angular singularity.

Multiplying the Boltzmann equation (2) by $\Psi(|v|^2)$ where $\Psi : \mathbb{R}_+ \to \mathbb{R}$ is a convex function and integrating with respect to $v$ we obtain, after standard changes of variables,

$$\frac{d}{dt}\int_{\mathbb{R}^d} f(v,t)\,\Psi(|v|^2)\,dv = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(v,t)\,f(v_*,t)\,W_\Psi(v,v_*)\,dv\,dv_*, \tag{20}$$

where

$$W_\Psi(v,v_*) = |v-v_*|^\beta \left(G_\Psi(v,v_*) - L_\Psi(v,v_*)\right),$$

$$G_\Psi(v,v_*) = \frac{1}{2}\int_{S^{d-1}} \left(\Psi(|v'_*|^2) + \Psi(|v'|^2)\right) h\Big(\tfrac{(v-v_*)\cdot\sigma}{|v-v_*|}\Big)\,d\sigma,$$

where $h$ is as in (5), $v'_*$, $v'$ are defined in (4), and

$$L_\Psi(v,v_*) = \frac{1}{2}\left(\Psi(|v|^2) + \Psi(|v_*|^2)\right).$$

Since the expression for $G_\Psi(v,v_*)$ is clearly the most complicated part of (20) we look for a simpler upper bound. This is generally achieved by Povzner-type inequalities; the present version has the advantage of yielding an explicit constant for the moments of the "gain" term (the case $\Psi(z) = z^k$) with a "good" asymptotic behavior for $k \to \infty$.

**Lemma 1.** *(Angular averaging lemma, [6], [7].) Let* $\Psi : \mathbb{R}_+ \to \mathbb{R}$ *be convex and assume that the function* $\bar{h}(z) = \frac{1}{2}(h(z) + h(-z))$ *is nondecreasing on* $(0,1)$. *Then*

$$G_\Psi(v,v_*) \le \omega_{d-2} \int_{-1}^{1} \Psi\Big(\left(|v|^2 + |v_*|^2\right) \frac{1+z}{2}\Big)\, \bar{h}(z)\, (1-z^2)^{\frac{d-3}{2}}\, dz,$$

*where* $\omega_{d-2}$ *is the area of the unit sphere in* $\mathbb{R}^{d-1}$.

**Proof.** See [7, Lemma 1] for the case $d = 3$; the extension to general $d$ is straightforward.

The next step is to choose in (20) $\Psi(z) = z^k$, $k \ge 1$ to obtain the time-evolution of the moments $m_k(t)$. By Lemma 1 we have in that case

$$G_\Psi(v,v_*) \le a_k (|v|^2 + |v_*|^2)^k,$$

where the constant $a_k$ is given by

$$a_k = \omega_{d-2} \int_{-1}^{1} \Big(\frac{1+z}{2}\Big)^k \bar{h}(z)\, (1-z^2)^{\frac{d-3}{2}}\, dz, \qquad (21)$$

Notice that $a_1 = 1$, $a_k < 1$ for $k > 1$ and $a_k$ is strictly decreasing with increasing $k$. By (20) we then have

$$m_k'(t) \le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(v,t)f(v_*,t)\,W_k(v,v_*)\,dv\,dv_*, \tag{22}$$

where

$$\begin{aligned} W_k &= \frac{1}{2}\,|v-v_*|^\beta\big(a_k(|v|^2+|v_*|^2)^k - |v|^{2k} - |v_*|^{2k}\big) \\ &= -\,\frac{1}{2}\,(1-a_k)\,|v-v_*|^\beta\big(|v|^{2k} - |v_*|^{2k}\big) \\ &\quad + \frac{1}{2}\,a_k\,|v-v_*|^\beta\big((|v|^2+|v_*|^2)^k - |v|^{2k} - |v_*|^{2k}\big) =: -U_k + V_k. \end{aligned} \tag{23}$$

Since $a_k < 1$ for $k > 1$, the leading term $-U_k$ on the right-hand side is non-positive, and the terms in $V_k$ can be estimated using the inequalities

$|v-v_*|^\beta \le |v|^\beta + |v_*|^\beta$,

$$(|v|^2+|v_*|^2)^k - |v|^{2k} - |v_*|^{2k} \le \sum_{i=1}^{[\frac{k+1}{2}]} \binom{k}{j} (|v|^{2j}|v_*|^{2(k-j)} + |v|^{2(k-j)}|v_*|^{j}),$$

where $[\,\cdot\,]$ denotes the integer part (cf. [7]). We then have

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(v,t)f(v_*,t)\,V_k(v,v_*)\,dv\,dv_* \le a_k S_k(t),$$

where

$$S_k(t) = \sum_{j=1}^{[\frac{k+1}{2}]} \binom{k}{j} \big(m_{j+\frac{\beta}{2}}(t)\,m_{k-j}(t) + m_{k-j+\frac{\beta}{2}}(t)\,m_j(t)\big). \tag{24}$$

and

$$\begin{aligned} &\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(v,t)f(v_*,t)\,U_k(v,v_*)\,dv\,dv_* \\ &\ge (1-a_k)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(v,t)f(v_*,t)\,|v-v_*|^\beta|v|^{2k}\,dv\,dv_* \end{aligned} \tag{25}$$

To estimate the last term we will use the following lower bound for the moments of order $s \in (0,1]$.

**Lemma 2 (Cf. [6] for the case $s = 1$).** *The solution of* (2) *satisfies*

$$\int_{\mathbb{R}^d} f(v_*, t)\,|v - v_*|^s\,dv_* \geq c_s \int_{\mathbb{R}^d} f_0(v_*)\,|v - v_*|^s\,dv_*, \quad v \in \mathbb{R}^d,$$

*for any* $s \in (0, 1]$.

**Proof.** By translating the solution $f(v_*, t)$ in the velocity space, we can reduce the proof to the case $v = 0$. We will establish the estimates

$$m_s(t) \geq c_s\,m_s(0), \tag{26}$$

for $0 < s \leq 1$. Notice that $\Psi(z) = -z^s$ is a convex function. Then, by the previous computation, and using Lemma 1,

$$m_s'(t) \geq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(v,t)\,f(v_*,t)\;|v-v_*|^\beta \left(\frac{a_s}{2}\left(|v|^2+|v_*|^2\right)^s - \frac{1}{2}\left(|v|^{2s}+|v_*|^{2s}\right)\right) dv\,dv_*$$

where $a_s = \omega_{d-2}\int_{-1}^{1}(\frac{1+z}{2})^s\,\bar{b}(z)\,(1-z^2)^{\frac{d-3}{2}}\,dz > 1$. We shall estimate the integrand above in order to obtain an expression involving $m_s(t)$ and similar quantities. For this we notice that since $(x+y)^\beta \leq x^\beta + y^\beta$, for $\beta \in [0,1]$, then

$$|v - v_*|^\beta \leq (|v| + |v_*|)^\beta \leq |v|^\beta + |v_*|^\beta.$$

Also,

$$|v - v_*|^\beta \geq \big|\,|v|^\beta - |v_*|^\beta\big| \quad \text{and} \quad (|v|^2 + |v_*|^2)^s \geq \big|\,|v|^{2s} - |v_*|^{2s}\big|.$$

Therefore

$$\begin{aligned}
&|v - v_*|^\beta \left(\frac{a_s}{2}\left(|v|^2 + |v_*|^2\right)^s - \frac{1}{2}\left(|v|^{2s} + |v_*|^{2s}\right)\right) \\
&\quad \geq \frac{a_s}{2}\left(|v|^\beta - |v_*|^\beta\right)\left(|v|^{2s} - |v_*|^{2s}\right) - \frac{1}{2}\left(|v|^\beta + |v_*|^\beta\right)\left(|v|^{2s} + |v_*|^{2s}\right) \\
&\quad = \frac{a_s - 1}{2}\left(|v|^{\beta+2s} + |v_*|^{\beta+2s}\right) - \frac{a_s + 1}{2}\left(|v|^\beta|v_*|^{2s} + |v|^{2s}|v_*|^\beta\right)
\end{aligned}$$

and we obtain

$$m_s'(t) \;\geq\; (a_s-1)\, m_0\, m_{s+\frac{\beta}{2}}(t) - (a_s+1)\, m_{\frac{\beta}{2}}(t)\, m_s(t)\ .$$

In the particular case $\beta = 1$ we have

$$m_{\frac{1}{2}}'(t) \geq (a_{\frac{1}{2}}-1)\, m_0 m_1 - (a_{\frac{1}{2}}+1)\, m_{\frac{1}{2}}^2(t),$$

($m_0$ and $m_1$ are constants, by the conservation of mass and energy). Therefore,

$$m_{\frac{1}{2}}(t) \geq \min\left\{ m_{\frac{1}{2}}(0), \left(\frac{a_{\frac{1}{2}}-1}{a_{\frac{1}{2}}+1}\, m_0\, m_1\right)^{\frac{1}{2}} \right\} \geq \min\left\{ 1, \left(\frac{a_{\frac{1}{2}}-1}{a_{\frac{1}{2}}+1}\right)^{\frac{1}{2}} \right\} m_{\frac{1}{2}}(0)\ ,$$

since $m_0 m_1 \geq m_{\frac{1}{2}}(0)^2$. (This is the argument of Bobylev.) To achieve the proof for $\beta < 1$ we iterate this argument, applying it with $s = \frac{j\beta}{2}$, $j = 1\ldots$, until $\frac{(j+1)\beta}{2} \geq 1$. Consider first the case of the terminal $j$, when

$$s_0 = \frac{j\beta}{2} < 1 \leq \frac{(j+1)\beta}{2}\ .$$

In that case

$$\begin{aligned} m_{s_0}'(t) &\geq (a_{s_0}-1)\, m_0\, m_{s_0+\frac{\beta}{2}}(t) - (a_{s_0}+1)\, m_{\beta/2}(t)\, m_{s_0}(t) \\ &\geq (a_{s_0}-1)\, m_0^{2-(s_0+\frac{\beta}{2})}\, m_1^{s_0+\frac{\beta}{2}} - (a_{s_0}+1)\, m_0^{1-\frac{\beta}{2s_0}}\, m_{s_0}^{1+\frac{\beta}{2s_0}}(t) \end{aligned}$$

Therefore,

$$\begin{aligned} m_{s_0}(t) &\geq \min\left\{ m_{s_0}(0), \left(\frac{a_{s_0}-1}{a_{s_0}+1} m_0^{(\frac{1}{s_0}-1)(s_0+\frac{\beta}{2})} m_1^{s_0+\frac{\beta}{2}}\right)^{\frac{1}{1+\frac{\beta}{2s_0}}} \right\} \\ &\geq \min\left\{ 1, \left(\frac{a_{s_0}-1}{a_{s_0}+1}\right)^{\frac{1}{1+\frac{\beta}{2s_0}}} \right\} m_{s_0}(0) = \left(\frac{a_{s_0}-1}{a_{s_0}+1}\right)^{\frac{1}{1+\frac{\beta}{2s_0}}} m_{s_0}(0)\ . \end{aligned}$$

Further, take $s_1 = s_0 - \frac{\beta}{2} > 0$. Then

$$m_{s_1}'(t) \geq (a_{s_1}-1)\, m_0\, m_{s_0}(t) - (a_{s_1}+1)\, m_0^{1-\frac{\beta}{2s_1}}\, m_{s_1}^{1+\frac{\beta}{2s_1}}(t),$$

so

$$
\begin{aligned}
m_{s_1}(t) &\geq \min\left\{ m_{s_1}(0), \left( \left( \frac{a_{s_1}-1}{a_{s_1}+1} \right) m_0^{\frac{\beta}{2s_1}} m_{s_0}(t) \right)^{\frac{1}{1+\frac{\beta}{2s_1}}} \right\} \\
&\geq \min\left\{ m_{s_1}(0), \left( \left( \frac{a_{s_1}-1}{a_{s_1}+1} \right) \left( \frac{a_{s_0}-1}{a_{s_0}+1} \right)^{\frac{s_0}{s_0+\frac{\beta}{2}}} m_0^{\frac{\beta}{2s_1}} m_{s_0}(0) \right)^{\frac{1}{1+\frac{\beta}{2s_1}}} \right\} \\
&\geq \left( \frac{a_{s_1}-1}{a_{s_1}+1} \right)^{\frac{s_1}{s_1+\frac{\beta}{2}}} \left( \frac{a_{s_0}-1}{a_{s_0}+1} \right)^{\frac{s_1}{s_0+\frac{\beta}{2}}} m_{s_1}(0) \; .
\end{aligned}
$$

The rest of the proof follows by induction.

As a consequence of Lemma 2 we have

$$
\int_{\mathbb{R}^d} f(v_*, t) \, |v - v_*|^\beta \, dv_* \geq c_\beta \int_{\mathbb{R}^d} f_0(v_*) \, |v - v_*|^\beta \, dv_* \geq \nu_0 \, (1 + |v|^\beta),
$$

where $\nu_0$ is a constant depending on $\beta$ and $f_0$. Applying this estimate to (25) we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(v,t) f(v_*,t) \, U_k(v, v_*) \, dv \, dv_* \\
&\geq (1 - a_k) \, \nu_0 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(v,t)(1 + |v|^\beta) |v|^{2k} \, dv \, dv_* \geq (1 - a_k) \, \nu_0 \, m_{k+\frac{\beta}{2}}(t)
\end{aligned}
$$

Thus, we obtain for any $k \geq 1$

$$
m_k'(t) \leq -(1 - a_k) \, \nu_0 \, m_{k+\frac{\beta}{2}}(t) + a_k \, S_k(t). \tag{27}
$$

From these inequalities we see that to characterize the behavior of the moments $m_k(t)$ with $k$ integer we need to include the moments

$$
m_k(t) \quad \text{with} \quad k = j + \tfrac{\beta}{2} \, l, \quad j, l = 0, 1 \ldots \tag{28}
$$

This property and this structure of the inequalities is due to the fact that the kernel $B$ in (5) has homogeneity $|v - v_*|^\beta$. Since the total mass is conserved, $m_0(t) = m_0 = \text{const}$; we shall enumerate the rest of the moments (28) by a single index $k_n$, $n = 1, 2\ldots$, in the increasing order, and introduce the notation

$$J = \{k_n : n = 1, 2\ldots\}$$

for the index set.

The crucial next step is to obtain the control of the moments $m_k(t)$ using (27) and (24) that would establish the geometric growth for the normalized sequence (19). We introduce the normalized moments

$$z_k(t) = \frac{m_k(t)}{\Gamma(k+b)}, \quad k \in J, \tag{29}$$

where the constant $b > 0$ will be chosen below depending on $\alpha$ in (7). For $b = 1$ and $k$ nonnegative integer we have $z_k(t) = m_k(t)/k!$ which is the normalization appearing in (19).

Notice that by Stirling's formula,

$$\Gamma(k+b) \sim k^{b-1}\,\Gamma(k+1), \quad k \to \infty, \tag{30}$$

so if (19) holds for a particular $b > 0$ then it holds for any other.

By the assumptions on the initial data $f_0$, we have

$$z_k(0) \leq C_0\, q_0^k, \quad k \in J, \tag{31}$$

for any $q_0 > a_0^{-1}$ and $C_0$ large enough, where $a_0$ is the constant in (15). Further, using (27) and (29) we obtain

$$z_k'(t) \leq -(1-a_k)\,\nu_0\, m_0^{-\frac{\beta}{2k}}\,\Gamma(k+b)^{\frac{\beta}{2k}}\, z_k^{1+\frac{\beta}{2}}(t) + a_k\,\frac{S_k(t)}{\Gamma(k+b)}, \tag{32}$$

where the constant $a_k$ in (32) has been defined in (21) and we used the interpolation inequality $m_{k+\frac{\beta}{2}}(t) \geq m_0^{-\frac{\beta}{2k}}\, m_k(t)^{1+\frac{\beta}{2k}}$, which is obtained as a consequence of either Hölder or Jensen's inequality.

An estimate the sum $S_k(t)$ in (32), (24) is obtained by recalling the following result.

**Lemma 3.** *For $b > 0$ fixed set $z_k(t) = m_k(t)/\Gamma(k+b)$, $k \geq 1$. Then*

$$S_k(t) \leq C_b\, \Gamma(k + \frac{\beta}{2} + 2b)\, Z_k(t), \quad k \geq 1,$$

*where*

$$Z_k(t) = \max_{1 \leq j \leq [\frac{k+1}{2}]} \{ z_{j+\frac{\beta}{2}}(t)\, z_{k-j}(t), z_j(t)\, z_{k-j+\frac{\beta}{2}}(t) \} \tag{33}$$

*and $C_b$ is a constant depending on $b$.*

**Proof.** See [7, Lemma 4].

Using Lemma 3 the system of inequalities for the moments takes the form

$$z_k'(t) \leq -(1-a_k)\, \nu_0\, m_0^{-\frac{\beta}{2k}}\, \Gamma(k+b)^{\frac{\beta}{2k}}\, z_k^{1+\frac{\beta}{2}}(t) + a_k\, C_b\, \frac{\Gamma(k+\frac{\beta}{2}+2b)}{\Gamma(k+b)}\, Z_k(t), \tag{34}$$

for $k \in J$. We have by Stirling's formula,

$$\Gamma(k+b)^{\frac{\beta}{2k}} \sim \left(\frac{k}{e}\right)^{\beta/2} \quad \text{and} \quad \frac{\Gamma(k+\frac{\beta}{2}+2b)}{\Gamma(k+b)} \sim k^{\frac{\beta}{2}+b}, \quad k \to \infty. \tag{35}$$

In order to estimate the constant $a_k$ we recall that by (7), $\bar{h}(z) \leq C\,(1-z^2)^{-\alpha/2}$, $\alpha < d-1$. Hence, setting in (21) $s = \frac{z+1}{2}$, $\varepsilon = d - 1 - \alpha > 0$ we have

$$
\begin{aligned}
a_k &= C\,2^{-1+\varepsilon}\int_0^1 s^{k-1+\frac{\varepsilon}{2}}(1-s)^{-1+\frac{\varepsilon}{2}}\,ds \\
&= C\,2^{-1+\varepsilon}\,\frac{\Gamma(k+\frac{\varepsilon}{2})\,\Gamma(\frac{\varepsilon}{2})}{\Gamma(k+\varepsilon)} \sim C\,2^{-1+\varepsilon}\,\Gamma(\tfrac{\varepsilon}{2})\,k^{-\frac{\varepsilon}{2}}, \quad k\to\infty.
\end{aligned} \tag{36}
$$

Thus,

$$
z_k'(t) \le -A_k\, z_k^{1+\frac{\beta}{2k}}(t) + B_k\, Z_k(t), \tag{37}
$$

where $A_k \sim \bar{A}k^{\beta/2}$, $B_k \sim \bar{B}\,k^{\beta/2+b-\varepsilon/2}$, $k\to\infty$, and $\bar{A}$ and $\bar{B}$ are explicitly known constants. We fix $0 < b < \varepsilon/2$; then for a certain $c_0 > 0$, and for $k_* > 0$ large enough, we have

$$
\frac{A_k}{B_k} \ge c_0, \quad k \ge k_*. \tag{38}
$$

We next introduce some notation. Given $k = k_n \in J$ we set

$$
\bar{z}^{(k)}(t) = (z_{k_1}(t), \ldots, z_{k_{n-1}}(t)), \tag{39}
$$

which is a vector with $n-1$ components. We also notice that for $k \in J$, $k > 1 + \frac{\beta}{2}$ the term $Z_k(t)$ is of the form $F_k(\bar{z}^{(k)}(t))$, since the highest order of moment entering (33) is $k - 1 + \frac{\beta}{2}$. It is also clear the the function $F_k$ defined in this way is a continuous function of its arguments.

To complete the proof of Theorem 4 using the obtained estimates for the moments we invoke the following lemma that gives sufficient conditions for a solution of an infinite system of differential inequalities of the type (34) to propagate the geometric growth of the sequence $z_k$.

**Lemma 4.** *Given $k_* > 0$, let the sequence of nonnegative functions $z_k \in C^1([0,\infty))$, $k \in J$, satisfy*

$$
z_k'(t) \le -\,A_k\, z_k^{1+\frac{\beta}{2k}}(t) + B_k\, F_k(\bar{z}^{(k)}(t)), \quad k \in J, \quad k \ge k_* \tag{40}
$$

*and*

$$z_k(t) \le C_1\, q_1^k, \quad k \in J, \quad k < k_*, \tag{41}$$

*where* $k_* > \frac{\beta}{2}$, $C_1$ *and* $q_1$ *are positive constants,* $A_k$, $B_k$ *are positive sequences satisfying*

$$\frac{A_k}{B_k} \ge C_1^{1-\frac{\beta}{2k}}, \quad k \in J, \quad k \ge k_*, \tag{42}$$

*and* $F_k$ *are continuous functions of their arguments such that*

$$F_k(\bar{z}^{(k)}) \;\le\; C^2\, q^{k+\frac{\beta}{2}}, \quad \textit{whenever} \quad z_k \le C q^k, \quad k \in J, \quad k \ge k_*. \tag{43}$$

*We also assume that the initial sequence* $z_k(0)$ *satisfies* (31)*.*

*Then*

$$z_k(t) \le C q^k, \quad k \in J, \quad t \ge 0, \tag{44}$$

*where* $C = \max\{C_0, C_1\}$ *and* $q = \max\{q_0, q_1\}$*.*

**Proof.** We set $C = \max\{C_0, C_1\}$ and $q = \max\{q_0, q_1\}$. The proof will be achieved by induction on $k \in J$, $k \ge k_*$. For $k = k_*$ conditions (41) and (43) imply

$$z_k'(t) \le -\, A_k\, z_k^{1+\frac{\beta}{2k}}(t) + B_k\, C^2\, q^{k+\frac{\beta}{2}}.$$

By a comparison argument for Bernoulli-type ordinary differential equations (cf. [6]),

$$z_k(t) \le \max\{z_k(0), z_k^*\}, \tag{45}$$

where $z_k^*$ is determined from the equation

$$A_k\, (z_k^*)^{1+\frac{\beta}{2k}} = B_k\, C^2\, q^{k+\frac{\beta}{2}}$$

Using condition (42) it is easy to verify that $z_k^* \leq C\,q^k$, which in view of (45) and (31) implies $z_k(t) \leq Cq^k$, $k = k_*$. This provides the basis for the induction. The induction step follows by repeating the same reasoning for any $k > k_*$. So the proof of the Lemma is complete.

**Proof (of Theorem 4).** We will verify the assumptions of Lemma 4. It is straighforward to check that the moments of the solution of the Boltzmann equation (2) are continuously differentiable in time; we refer the reader to Appendix B for the details. Based on the series expansion (18) and the assumptions on the initial data (15) we can check that conditions (31) are satisfied by $z_k(0)$ with $q_0 > a_0^{-1}$. From the asymptotic equalities (35) and (36) we can find $k_* > 1 + \frac{\beta}{2}$ and $c_0 > 0$ such that (38) holds. We then obtain (42) if we take

$$\log C_1 \leq \left(1 - \frac{\beta}{2k_*}\right)^{-1} \log c_0.$$

By the results of Desvillettes [12], for each $k \in J$ the moments $m_k(t)$ are uniformly bounded in time, so the constant $q_1$ in (41) can be taken as

$$q_1 = \max_{\beta/2 \leq k < k_*} \sup_{t \geq 0} \left(\frac{z_k(t)}{C_1}\right)^{1/k}.$$

It is straightforward to check (43) using the definition of the term $Z_k(t)$ in (33). Applying Lemma 4 we we establish (44) with $q = \max\{q_0, q_1\}$ and $C = \max\{C_0, C_1\}$. By the Taylor series expansion (18) estimate (16) then holds with and $a < 1/q$ and the constant $D$ depending on $a$, $C$ and the initial mass $m_0$. This completes the proof.

## 3. Comparison principle for the Boltzmann equation

In this section we discuss the important technique of comparison that will allow us to obtain pointwise estimates of the solutions. The crucial property of the Boltzmann equation used here is a certain monotonicity of a *linear* Boltzmann semigroup. The argument is roughly as follows: if $f$ is a solution of (1), $f|_{t=0} = f_0$, and $g$ is sufficiently regular and satisfies

$$(\partial_t + v \cdot \nabla_x)\, g \geq Q(f, g), \quad g|_{t=0} = g_0, \tag{46}$$

then $u = f - g$ is a solution of

$$(\partial_t + v \cdot \nabla_x)\, u \leq Q(f, u), \quad u|_{t=0} = u_0, \tag{47}$$

where $u_0 = f_0 - g_0$. We will show that if $f$ is nonnegative (and satisfies certain minimal regularity conditions), then solutions of (47) satisfy the order-preserving property,

$$\text{if} \quad u_0 \leq 0 \quad \text{then} \quad u \leq 0 \tag{48}$$

(zero on the right-hand side can be replaced by any other solution $\widetilde{u}$ of (47)). This translates into the following estimate (comparison principle):

$$\text{if} \quad f_0 \leq g_0 \quad \text{and} \quad g \text{ satisfies (46)}, \quad \text{then} \quad f \leq g. \tag{49}$$

By reversing all inequalities we obtain a similar comparison principle that yields lower bounds of solutions.

Of course, the above scheme has to be implemented with suitable modifications. For instance, since *apriori* only limited information about $f$ is available we will require that $g$ satisfies (46) for *a class* of functions $f$ (defined by the available apriori estimates). Another important refinement is to apply the estimate (49) *locally* (in the case of Theorem 3, to a "high-velocity tail" $\{|v| \geq R\}$) since global bounds in all of the $(v,t)$-space cannot be generally obtained by this technique. We refer to Proposition 1 and the proof of Theorem 3 given below for the necessary details. In Theorem 5 we will give a rigorous statement of (49) in application to a general class of weak solutions of (1) in the sense of DiPerna and Lions [14, 28].

The basic approach leading to applications of (49) originated in the work by one of the authors [34, Sec. 6.2] in the context of lower bounds for the spatially-homogeneous equation without angular cutoff. It was also used to obtain lower bounds for solutions in a model describing inelastic collisions [18]. Compared to these earlier versions we do not require in (49) any differentiability in the $v$-variable, and we make more precise the minimal regularity conditions on $f$. It is interesting to compare the present technique with other methods based on monotonicity applied to the Boltzmann equation, in particular the one by Kaniel and Shinbrot [25] (see also [21, 24]) and the pointwise estimates by Vedenjapin [33] (the result in the latter paper follows from our approach using $g = e^{C(1+t)}$). The monotonicity property expressed by (48) has also an important relation to the concept of dissipative solutions introduced by P.-L. Lions [28].

We first explain the way to obtain (48). The bilinear form in (46), (47) is defined by

$$Q(f,u)(x,v,t) = \int_{\mathbb{R}^d}\int_{S^{d-1}} (f'_* u' - f_* u)\, B(v-v_*,\sigma)\, d\sigma\, dv_*, \qquad (50)$$

where as usual, $f'_* = f(x,v'_*,t)$, $u' = u(x,v',t)$, $f_* = f(x,v_*,t)$, $u = u(x,v,t)$. At this point we do not need to assume the kernel $B$ to satisfy (5)–(7); the argument goes through for a more general class of kernels with the usual symmetries, as described in [14], for instance.

To illustrate the general principle, consider first the case of equality in (47). Given $T>0$ we fix the function $f : \mathbb{T}^d \times \mathbb{R}^d \times [0,T] \to \mathbb{R}_+$, which we assume to be smooth in $(x,t)$, bounded and rapidly decaying for $|v|$ large. We also assume that for every $u_0 \in D \subseteq L^1(\mathbb{T}^d \times \mathbb{R}^d)$ the initial-value problem

$$(\partial_t + v\cdot\nabla_x)\, u = Q(f,u), \quad u|_{t=0} = u_0, \qquad (51)$$

has a unique solution $u \in C([0,T]; L^1(\mathbb{T}^d \times \mathbb{R}^d))$, with enough regularity so that

$$Q^+(f,|u|),\, Q^-(f,|u|) \in L^1(\mathbb{T}^d \times \mathbb{R}^d \times [0,T]). \qquad (52)$$

Thus, we have a well-defined flow map (or a semigroup)

$$\Phi_t : D \ni u_0 \mapsto u(t,\cdot,\cdot) \in L^1(\mathbb{T}^d \times \mathbb{R}^d), \quad t \in [0,T].$$

The map $\Phi_t$ can be seen to satisfy the following nonexpansive property: for any $u_0$, $\widetilde{u}_0 \in D$,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \big|\Phi_t(u_0) - \Phi_t(\widetilde{u}_0)\big|\, dv\, dx \le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |u_0 - \widetilde{u}_0|\, dv\, dx, \quad t\in[0,T]. \qquad (53)$$

Indeed, set $w = \Phi_t(u_0) - \Phi_t(\widetilde{u}_0)$; then

$$(\partial_t + v \cdot \nabla_x) w = Q(f, w) \quad \text{on} \quad \mathbb{T}^d \times \mathbb{R}^d \times (0, T)$$

in the sense of distributions, and $Q(f, w) \in L^1$ by our assumptions. By a standard argument, $\forall\, t \in [0, T]$, for a. a. $(x, v)$ the function $w^\sharp : s \mapsto w(x - (t - s)v, v, s)$, $s \in [0, T]$, is absolutely continuous, and we can apply the chain rule (see Appendix A) to obtain

$$\frac{d}{ds} |w^\sharp| \,=\, Q(f, w)^\sharp \, \mathrm{sign}\, w^\sharp, \quad s \in (0, T), \tag{54}$$

where $Q(f, w)^\sharp$ is defined similarly to $w^\sharp$. Integrating with respect to $s \in (0, t)$ and $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$ we obtain, after standard changes of variables,

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |w(x, v, t)| \, dv \, dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |w_0| \, dv \, dx + \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} Q(f, w) \, \mathrm{sign}\, w \, dv \, dx \, ds$$

where $w_0 = u_0 - \widetilde{u}_0$. We further notice that the bilinear collision term (50) satisfies

$$\int_{\mathbb{R}^d} Q(f, u) \, \mathrm{sign}\, u \, dv \leq 0, \tag{55}$$

for every $f \geq 0$ and every $u$ so that $Q^+(f, |u|)$, $Q^-(f, |u|) \in L^1$. This follows immediately from the weak form

$$\int_{\mathbb{R}^d} Q(f, u) \, \mathrm{sign}\, u \, dv = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{S^{d-1}} f_* u \, (\mathrm{sign}\, u' - \mathrm{sign}\, u) \, B \, d\sigma \, dv_* dv$$

by noticing that $u \, (\mathrm{sign}\, u' - \mathrm{sign}\, u) \leq 0$.

The same approach can be followed to obtain (48). Indeed, we have by (55) and the mass conservation

$$\int_{\mathbb{R}^d} Q(f, u) \, \tfrac{1}{2} (\mathrm{sign}\, u + 1) \, dv \, \leq 0, \tag{56}$$

where $\frac{1}{2}(\operatorname{sign} u{+}1)$ is the a. e. derivative of the Lipschitz-continuous function $u_+ = \max\{u, 0\}$. We then have

$$\frac{d}{ds}\, u_+^\sharp \;=\; Q(f,u)^\sharp\ \tfrac{1}{2}(\operatorname{sign} u + 1)^\sharp, \quad s \in (0,T),$$

and the integration yields

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u_+(x,v,t)\, dv\, dx \le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u_{0+}\, dv\, dx, \quad t \in [0,T],$$

which implies (48) for a. a. $(x, v)$.

**Remark 2.** Relation (48) can be restated as the order-preserving property of $\Phi_t$:

$$\forall\, u_0,\, \widetilde{u}_0 \in D,\;\; u_0 \le \widetilde{u}_0 \;\text{ implies }\; \Phi_t(u_0) \le \Phi_t(\widetilde{u}_0), \quad t \in [0,T]. \tag{57}$$

In fact, the equivalence of (57) and (53) follows from a general principle applied to (nonlinear) maps that preserve integral, as described by Crandall and Tartar [11]. Inequality (48) (or (57)) can then be seen as a consequence of the results in [11], the preservation of the mass $\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f\, dv\, dx$ along solutions of (47), and (53).

The following localized version of the order-preserving property will be useful for the comparison argument.

**Proposition 1.** *Let* $f, u \in C([0,T]; L^1(\mathbb{T}^d \times \mathbb{R}^d))$ *satisfy*

$$f \ge 0; \quad \partial_t u + v\cdot\nabla_x u,\; Q^+(f,u),\; Q^-(f,u) \in L^1; \quad u|_{t=0} = u_0 \le 0,$$

*and assume that for a certain (measurable) set* $U \subseteq \mathbb{T}^d \times \mathbb{R}^d \times (0,T)$,

$$\partial_t u + v\cdot\nabla_x u - Q(f,u) \;\le\; 0 \quad \text{on} \quad U,$$

*and*

$$u \le 0 \quad on \quad U^c := \big(\mathbb{T}^d \times \mathbb{R}^d \times (0,T)\big) \setminus U.$$

*Then* $u(t,\cdot,\cdot) \le 0$ *a. e. on* $\mathbb{T}^d \times \mathbb{R}^d$*, for every* $t \in [0,T]$.

**Proof.** Let $D(u) = \partial_t u + v \cdot \nabla_x u$. We obtain by arguing as above,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u_+(x,v,t)\,dv\,dx \;-\; \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u_+(x,v,0)\,dv\,dx$$
$$= \int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} D(u)\,\tfrac{1}{2}(\operatorname{sign} u + 1)\,dx\,dv\,ds.$$

We have $u_+|_{t=0} = 0$; also $\frac{1}{2}(\operatorname{sign} u + 1) = 0$ whenever $u < 0$ and $D(u) = 0$ outside of a set of zero measure in $\{u = 0\}$. Therefore, setting $U_t = \{(x,v,s) \in U : s \le t\}$ we have

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u_+(x,v,t)\,dv\,dx = \iiint_{U_t} D(u)\,dx\,dv\,ds$$
$$\le \iiint_{U_t} Q(f,u)\,dx\,dv\,ds \;=\; \int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} Q(f,u)\,\tfrac{1}{2}(\operatorname{sign} u + 1)\,dx\,dv\,ds \le 0,$$

for every $t \in [0,T]$, where we used the dissipative property (56). This shows that $u(t,\cdot,\cdot) \le 0$ almost everywhere.

Proposition 1 is sufficient to formulate the comparison principle in the generality required for Theorem 1. We will, however, give a more general statement that applies to weak solutions in the spatially inhomogeneous case. In the definition of weak solutions one has to account for the fact that the bound

$$Q(f) \in L^1_{\mathrm{loc}}(\mathbb{T}^d \times \mathbb{R}^d \times (0,+\infty))$$

is generally not available, and one has to define solutions in a sense that is weaker than distributional. The simplest way to state the definition is to require that $f \geq 0$, $f \in C([0,T]; L^1_{xv})$, $Q^\pm(f)/(1+f) \in L^1_{\mathrm{loc}}$ and the renormalized form

$$(\partial_t + v \cdot \nabla_x) \log(1+f) = Q(f)/(1+f)$$

holds in the sense of distributions, cf. [14]. Such solutions are known as renormalized. This concept can be further refined as follows, cf. [28].

**Definition 1.** We say that a renormalized solution $f$ is dissipative if $f|v|^2 \in L^\infty([0,T]; L^1_{xv})$ and for every sufficiently regular function $g : \mathbb{T}^d \times \mathbb{R}^d \times [0,T] \to \mathbb{R}$,

$$\partial_t \int_{\mathbb{R}^d} |f-g|\,dv + \mathrm{div}_x \int_{\mathbb{R}^d} |f-g|\,v\,dv \leq \int_{\mathbb{R}^d} \big(Q(f,g) - D(g)\big)\,\mathrm{sign}(f-g)\,dv, \tag{58}$$

in the sense of distributions, where $D(g) = (\partial_t + v \cdot \nabla_x) g$, and $\mathrm{sign}(0)$ is assigned an arbitrary value in $[-1,1]$.

**Remark 3.** In the above definition "sufficiently regular" precisely means that $g \in C([0,T]; L^1_{xv})$, $g|v|^2 \in L^\infty_t(L^1_{xv})$, $D(g) \in L^1_{xvt}$ and that for any $f \in C([0,T]; L^1_{xv})$ such that $f|v|^2 \in L^\infty_t(L^1_{xv})$, $Q^+(f,|g|)$, $Q^-(f,|g|) \in L^1_{xvt}$ (these conditions can be made more explicit, see [28] for details).

The formal motivation for the definition of dissipative solutions is clear: the right-hand side of the Boltzmann equation can be written as

$$Q(f) = Q(f, f-g) + Q(f,g),$$

so we have

$$(\partial_t + v \cdot \nabla_x)(f-g) = Q(f, f-g) + Q(f,g) - D(g).$$

Multiplying the above equation by $\mathrm{sign}(f-g)$ and using relation (55) (note that $f \geq 0$) we see that every sufficiently regular solution of (1) should satisfy (58).

Dissipative solutions are known to exist globally in time, for a quite general class of initial data. In fact, in [28] Lions established a large class of "dissipation inequalities" similar to (58) that hold for renormalized solutions of (1). Such solutions can also be constructed so that the local mass conservation law,

$$\partial_t \int_{\mathbb{R}^d} f\,dv + \mathrm{div}_x \int_{\mathbb{R}^d} f\,v\,dv = 0, \tag{59}$$

holds in the sense of distributions. However they need not generally satisfy the conditions $Q^+(f),\ Q^-(f) \in L^1_{\mathrm{loc}}$.

Using the order-preserving property of Proposition 1 we establish the following comparison principle for dissipative solutions of the nonlinear Boltzmann equation.

**Theorem 5.** *Let $f \in C([0,T]; L^1(\mathbb{T}^d \times \mathbb{R}^d))$ be a dissipative solution of* (1) *and let $g$ be a sufficiently regular function, such that $f|_{t=0} \leq g|_{t=0}$,*

$$\partial_t g + v \cdot \nabla_x g - Q(f,g) \geq 0 \quad \textit{on } U$$

*and $f \leq g$ on $U^c$, where $U$ is a measurable subset of $\mathbb{T}^d \times \mathbb{R}^d \times [0,T]$. Then $f \leq g$ almost everywhere on $\mathbb{T}^d \times \mathbb{R}^d$, for every $t \in [0,T]$.*

**Remark 4.** It is natural to call $g$ a (localized) upper barrier. By reversing all inequalities in the above formulation one can also obtain a similar comparison principle for the lower barrier.

**Proof.** We use the notation $D(g) = \partial_t g + v \cdot \nabla_x g$, so that

$$\partial_t \int_{\mathbb{R}^d} g\,dv + \mathrm{div}_x \int_{\mathbb{R}^d} g\,v\,dv = \int_{\mathbb{R}^d} D(g)\,dv,$$

in the sense of distributions. Using the mass conservation (59) and the identity

$$(f-g)_+ = \tfrac{1}{2}\big(|f-g| + (f-g)\big)$$

we obtain, by combining the above relations with (58),

$$\begin{aligned}\partial_t &\int_{\mathbb{R}^d} (f-g)_+\,dv + \mathrm{div}_x \int_{\mathbb{R}^d} (f-g)_+\,v\,dv \\ &\le \frac{1}{2}\int_{\mathbb{R}^d} \big(Q(f,g) - D(g)\big)\,\mathrm{sign}(f-g)\,dv - \frac{1}{2}\int_{\mathbb{R}^d} D(g)\,dv.\end{aligned}$$

Since $Q^{\pm}(f,|g|)$ are integrable, we have $\int_{\mathbb{R}^d} Q(f,g)\,dv = 0$, a. e. $(x,t)$, and therefore,

$$\begin{aligned}\partial_t &\int_{\mathbb{R}^d} (f-g)_+\,dv + \mathrm{div}_x \int_{\mathbb{R}^d} (f-g)_+\,v\,dv \\ &\le \int_{\mathbb{R}^d} \big(Q(f,g) - D(g)\big)\,\tfrac{1}{2}(\mathrm{sign}(f-g)+1)\,dv.\end{aligned} \tag{60}$$

We can choose $\mathrm{sign}(0) = -1$ in (60) to avoid estimating the integral over the set $\{f = g\}$. Since $(f-g)_+\,v \in L^1(\mathbb{T}^d \times \mathbb{R}^d \times [0,T])$ we can integrate over $x$ and $t$ to obtain

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (f-g)_+(x,v,t)\,dv\,dx \le &\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (f-g)_+(x,v,0)\,dv\,dx \\ &+ \iiint_{U_t} \big(Q(f,g) - D(g)\big)\,dx\,dv\,ds \le 0,\end{aligned} \tag{61}$$

where $U_t = \{(x,v,s) \in U : s \le t\}$ and we used that $\frac{1}{2}(\mathrm{sign}(f-g)+1)$ vanishes for $f \le g$ and that $Q(f,g) - D(g) \le 0$ on $U_t$. The inequality in (61) implies that $f \le g$, a. e. $(x,v) \in \mathbb{T}^d \times \mathbb{R}^d$, for every $t \in [0,T]$.

Theorem 5 is a crucial ingredient in the proof of Theorem 3, which we give below.

**Proof (of Theorem 3).** To apply Theorem 5 we set $U = \{(x, v, t) : |v| > R\}$, where $R$ will be chosen large enough, and $g(x, v, t) = M(v)$, where $M(v) = e^{-a|v|^2+c}$, $0 < a < a_1$ is fixed and $c > c_0$ will be chosen sufficiently large, depending on $R$. To prove that $g$ can be used as a barrier for the solution on $U$ we need to verify the inequality

$$Q^+(f, g)(x, v, t) \le Q^-(f, g)(x, v, t), \quad (x, t) \in \mathbb{T}^d \times [0, T], \quad |v| > R. \tag{62}$$

First notice that, by elementary inequalities,

$$\begin{aligned} Q^-(f, g)(x, v, t) &= M(v) \int_{\mathbb{R}^d} f(x, v_*, t)\, |v - v_*|^\beta \, dv_* \\ &\ge M(v) \Big(\rho_0 |v|^\beta - \int_{\mathbb{R}^d} f(x, v_*, t)\, |v_*|^\beta \, dv_*\Big), \end{aligned}$$

where $\rho_0$ is the constant in (13). The last term can be controlled using the estimate for the integral of $f/M_1$ from (14) as follows,

$$\int_{\mathbb{R}^d} f(x, v_*, t)\, |v_*|^\beta \, dv_* \le L \int_{\mathbb{R}^d} \frac{f(x, v_*, t)}{M_1(v_*)} \, dv_* \le L\, C_1,$$

where $L = \max_{y \ge 0} y^\beta \, e^{-a_1 y^2 + c_1}$. Thus, we have

$$Q^-(f, g)(x, v, t) \ge M(v) \big(\rho_0 |v|^\beta - L\, C_1\big).$$

The control of the "gain" term is more technical; we establish below in Lemma 5 the estimate

$$Q^+(f, g)(x, v, t) \le C\, (1 + |v|^{\beta - \varepsilon})\, M(v), \tag{63}$$

where $\varepsilon = \min\{\beta, n-1-\alpha\} > 0$. This implies that (62) holds if we set $R$ to be the largest root of the equation

$$C + LC_1 + Cy^{\beta-\varepsilon} - \rho_0 y^\beta = 0.$$

Finally, we take $c = aR^2 + \log C_0$, where $C_0$ is the constant in (14); then it is easy to verify that

$$f(x,v,t) \le C_0 \le M(v), \quad (x,t) \in \mathbb{T}^d \times [0,T], \quad |v| \le R. \tag{64}$$

The conditions $0 < a < a_1 < a_0$ and $c \ge c_0$ guarantee that we have $f(x,v,0) \le M(v)$. Together with the inequalities (62) and (64) this allows us to use Theorem 5 to conclude.

## 4. A weighted estimate for the "gain" operator

To complete the proof of Theorem 3 we prove the following weighted estimate of the linear "gain" operator. The main technique is based on Carleman's form of the "gain" term (see Appendix C).

**Lemma 5.** *Let $B : \mathbb{R}^d \times S^{d-1} \to \mathbb{R}^+$, $n \ge 2$, be a measurable function that satisfies*

$$B(u,\sigma) \;\le\; C\,(1+|u|^\beta)\,\frac{1}{|\sin\vartheta|^\alpha}\,1_{\{\cos\vartheta \ge 0\}}, \quad \cos\vartheta = \tfrac{u\cdot\sigma}{|u|},$$

*where $\beta > 0$ and $\alpha < n-1$. Define*

$$Q^+(f,g)(v) \;=\; \int_{\mathbb{R}^d}\int_{S^{d-1}} f'_*\,g'\,B(v-v_*,\sigma)\,d\sigma\,dv_*,$$

*and set* $M(v) = e^{-a|v|^2}$, $a > 0$*;* $w_\varepsilon(v) = 1 + |v|^{\beta-\varepsilon}$*, where* $\varepsilon = \min\{\beta, n - 1 - \alpha\} > 0$*. Then*

$$\Big\| \frac{Q^+(f,M)}{w_\varepsilon\, M} \Big\|_{L^\infty(\mathbb{R}^d)} \le C \Big\| \frac{f\, w_\varepsilon}{M} \Big\|_{L^1(\mathbb{R}^d)}, \tag{65}$$

*where* $C$ *is an explicitly computable constant depending on* $n$*,* $\alpha$*,* $\beta$ *and* $a$*.*

**Remark 5.** For $B$ satisfying the estimate with $\alpha = 0$ (for example, the kernel $\bar{B}$ for hard spheres in three dimensions) we have $\varepsilon = \beta$ for all $\beta \le d-1$ and the weight $w_\varepsilon(v)$ is constant. The estimate of the Lemma then takes a particularly simple form,

$$\Big\| \frac{Q^+(f,M)}{M} \Big\|_{L^\infty_v} \le C \Big\| \frac{f}{M} \Big\|_{L^1_v}.$$

For the quadratic "gain" term this implies the estimate

$$\Big\| \frac{Q^+(f)}{M} \Big\|_{L^\infty_v} \le C \Big\| \frac{f}{M} \Big\|_{L^\infty_v} \Big\| \frac{f}{M} \Big\|_{L^1_v}.$$

**Proof.** By the Carleman representation formula (Appendix C),

$$Q^+(f,M)(v) = 2^{d-1} \int_{\mathbb{R}^d} \frac{f(v'_*)}{|v - v'_*|} \int_{E_{vv'_*}} M(v')\, \frac{B(v - v_*, \sigma)}{|v - v_*|^{n-2}}\, d\pi_{v'}\,,$$

where $E_{vv'_*}$ is the hyperplane

$$\{v' \in \mathbb{R}^d \, : \, (v - v') \cdot (v - v'_*) = 0\},$$

and $d\pi_{v'}$ denotes the usual Lebesgue measure on $E_{vv'_*}$. We then have

$$\frac{Q^+(f,M)(v)}{M(v)} = \int_{\mathbb{R}^d} \frac{f(v'_*)}{M(v'_*)}\, K(v, v'_*)\, dv'_*, \tag{66}$$

where

$$K(v, v'_*) = \frac{2^{d-1}}{|v - v'_*|} \int_{E_{vv'_*}} M(v_*)\, \frac{B(v - v_*, \sigma)}{|v - v_*|^{n-2}}\, d\pi_{v'}, \tag{67}$$

and we used that, by the energy conservation,

$$\frac{M(v')\,M(v'_*)}{M(v)} = M(v_*).$$

Note that in (67) the variables $v_*$ and $\sigma$ are expressed through $v$, $v'_*$ and $v'$ as follows,

$$v_* \,=\, v'_* + v' - v, \qquad \sigma = \frac{v' - v'_*}{|v' - v'_*|}.$$

Now to establish the Lemma it suffices to verify the inequality

$$K(v, v'_*) \,\leq\, C\,(1 + |v - v'_*|^{\beta-\varepsilon}). \tag{68}$$

Indeed, since

$$1 + |v - v'_*|^{\beta-\varepsilon} \leq (1 + |v|^{\beta-\varepsilon})\,(1 + |v'_*|^{\beta-\varepsilon}),$$

then (66) and (68) imply

$$Q^+(f, M)(v) \,\leq\, C\,(1 + |v|^{\beta-\varepsilon})\,M(v) \int_{\mathbb{R}^d} \frac{f(v'_*)}{M(v'_*)}\,(1 + |v'_*|^{\beta-\varepsilon})\,dv'_*$$

which is equivalent to (65).

In the remainder of the proof we will therefore verify (68). Using the identity

$$(v - v_*)\cdot(v' - v_*) \,=\, |v - v'_*|^2 - |v - v'|^2$$

for $v' \in E_{vv'_*}$ and recalling that $B(v - v_*, \sigma)$ vanishes for $(v - v_*)\cdot\sigma < 0$ we see that the integration in (67) can be restricted to the disk

$$D_{vv'_*} = E_{vv'_*} \,\cap\, \{v' \in \mathbb{R}^d \,:\, |v - v'_*| \leq |v - v'|\}.$$

We notice that for $v' \in D_{vv'_*}$,

$$\left|\tan\frac{\vartheta}{2}\right| \,=\, \frac{|v'_* - v_*|}{|v - v'_*|}, \qquad |\vartheta| \leq \frac{\pi}{2},$$

where $\vartheta$ is the angle between the vectors $v - v_*$ and $\sigma$. This implies

$$\frac{1}{|\sin\vartheta\,|} \;\le\; \frac{1}{2}\,\frac{|v - v'_*|}{|v'_* - v_*|}$$

Thus, $K(v, v'_*) \le C\widetilde{K}(v, v'_*)$ , where

$$\widetilde{K}(v, v'_*) = \frac{2^{d-1-\alpha}}{|v - v'_*|^{1-\alpha}} \int_{D_{vv'_*}} M(v_*)\; \frac{1 + |v - v_*|^{\beta}}{|v - v_*|^{n-2}}\; \frac{1}{|v'_* - v_*|^{\alpha}}\, d\pi_{v'}\,.$$

To estimate the above expression we consider two cases.

**Case a)** $|v - v'_*| \le 1$. Since for $v' \in D_{vv'_*}$

$$|v - v'_*| \le |v - v_*| \le \sqrt{2}\,|v - v'_*|$$

we have $1 + |v - v_*|^{\beta} \le 1 + 2^{\beta/2}$ and

$$|v - v_*|^{2-n} \;\le\; |v - v'_*|^{2-n}.$$

Therefore,

$$\widetilde{K}(v, v'_*) \;\le\; \frac{2^{d-1-\alpha}(1 + 2^{\beta/2})}{|v - v'_*|^{n-1-\alpha}} \int_{D_{vv'_*}} M(v_*)\; \frac{1}{|v'_* - v_*|^{\alpha}}\, d\pi_{v'}\,.$$

Since $M(v_*) \le 1$ the last integral is estimated above by

$$\int_{D_{vv'_*}} \frac{1}{|v'_* - v_*|^{\alpha}}\, d\pi_{v'} = \int_{\{w \in \mathbb{R}^{d-1}\,:\,|w| \le |v - v'_*|\}} \frac{1}{|w|^{\alpha}}\, dw = \tfrac{\omega_{d-2}}{d-1-\alpha}\, |v - v'_*|^{d-1-\alpha},$$

if $d - 1 - \alpha > 0$, i. e. $\alpha < d - 1$. Here $\omega_{d-2}$ is the measure of the $(n-2)$-dimensional unit sphere. This implies the estimate

$$\widetilde{K}(v, v'_*) \;\le\; \frac{2^{d-1-\alpha}(1 + 2^{\beta/2})\,\omega_{d-2}}{d - 1 - \alpha}, \quad |v - v'_*| \le 1.$$

**Case b)** $|v - v'_*| > 1$. Then

$$1 + |v - v_*|^{\beta} \le 2\,|v - v_*|^{\beta} \le 2^{1+\frac{\beta}{2}}\,|v - v'_*|^{\beta},$$

and we obtain, similarly to the previous case,

$$\widetilde{K}(v, v'_*) \le \frac{2^{d-\alpha+\frac{\beta}{2}}}{|v - v'_*|^{n-1-\alpha-\beta}} \int_{D_{vv'_*}} M(v_*) \frac{1}{|v'_* - v_*|^{\alpha}} \, d\pi_{v'}.$$

Since $M(v_*)$ is a radially decreasing function of $v_* \in \mathbb{R}^d$, and so is $|v_*|^{-\alpha}$,

$$\int_{D_{vv'_*}} M(v_*) \, |v'_* - v_*|^{-\alpha} \, d\pi_{v'} \le \int_{\mathbb{R}^{d-1}} \bar{M}(w) \, |w|^{-\alpha} \, dw$$

$$\le \int_{|w|\le 1} |w|^{-\alpha} \, dw + \int_{\mathbb{R}^{d-1}} \bar{M}(w) \, dw = \frac{\omega_{d-2}}{d-1-\alpha} + \left(\frac{\pi}{a}\right)^{\frac{d-1}{2}},$$

where $\bar{M}(w) = e^{-a|w|^2}$, $w \in \mathbb{R}^{d-1}$. Since $|v - v'_*|^{\beta+\alpha-n+1} \le |v - v_*|^{\beta-\varepsilon}$ this establishes the required estimate for Case b).

## Appendix A: Some properties of weakly differentiable functions

Let $AC[a, b]$ denote the class of absolutely continuous real-valued functions defined on an interval $[a, b]$. Given $f \in AC[a, b]$ we set $[c, d] = f([a, b])$ and use the notation $\mathrm{Lip}[c, d]$ for the class of all Lipschitz continuous functions defined on $[c, d]$. Every function $\beta \in \mathrm{Lip}[c, d]$ is differentiable (in the classical sense) almost everywhere on $(c, d)$; we agree to extend this derivative to a function $\beta'$ defined everywhere on $[c, d]$ by assigning arbitrary *finite* values at the points where $\beta$ is not differentiable. The function $\beta'$ also coincides with the weak derivative of $\beta$ almost everywhere on $(c, d)$ The following chain rule was used in the arguments in Section 3.

**Proposition 2.** *Let $f \in AC[a, b]$ and $\beta \in \mathrm{Lip}[c, d]$. Then $\beta \circ f \in AC[a, b]$ and*

$$(\beta \circ f)' = (\beta' \circ f) \, f',$$

*almost everywhere on $(a, b)$.*

**Remark 6.** 1) The seeming ambiguity in the above formulation occuring since $\beta' \circ f$ can assume arbitrarily assigned values on a set of positive measure is resolved by observing that whenever this happens then $f'$ vanishes, except on a set of measure zero (see the proof below). 2) For the purposes of Section 3 we only need the chain rule for $\beta(y) = |y|$ and $\beta(y) = y_+$; these cases are covered in [17], and the proof for the case of piecewise-$C^1$ functions $\beta$ can be found in [20]. We include a short proof that applies to the general case to make the presentation in Section 3 self-contained.

**Proof.** By the definition of absolutely continuous functions,

$$\forall\, \varepsilon > 0 \;\; \exists\, \delta > 0 \;\; \text{such that} \;\; \forall\, n \in \mathbb{N}, \;\; \forall\, \{(x_j, y_j) \subseteq [a,b] : j = 1, \ldots, n\},$$

a disjoint family,

$$\sum_{j=1}^{n} |y_j - x_j| < \delta \quad \Rightarrow \quad \sum_{j=1}^{n} |f(y_j) - f(x_j)| < \varepsilon.$$

Clearly then, since

$$|\, \beta(f(y_j)) - \beta(f(x_j))\,| \le L\,|f(y_j) - f(x_j)|,$$

where $L$ is the Lipschitz constant of $\beta$, the composition $\beta \circ f$ is absolutely continuous on $[a,b]$. By Lebesgue's differentiation theorem, $f$ and $\beta \circ f$ are differentiable in the classical sense on a set with complement of measure zero in $(a,b)$. Pick $x \in (a,b)$ from this set. We will consider two cases, depending on whether $\beta$ is differentiable at $f(x)$ or not. In the first case we have

$$\begin{aligned}
&(\beta \circ f)'(x) = \lim_{h \to 0} \frac{\beta(f(x+h)) - \beta(f(x))}{h} \\
&= \lim_{h \to 0} \frac{\beta(f(x+h)) - \beta(f(x))}{f(x+h) - f(x)} \; \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \beta'(f(x)) f'(x).
\end{aligned}$$

Let us further take $A$ to be the set of $y$ such that $\beta$ is not differentiable at $f(y)$. We claim that $f'(x)$ vanishes for $x \in A$, except perhaps on a set of zero Lebesgue measure. Indeed, let $B = \{y \in A : |f'(y)| > 0\}$; then

$$B = \bigcup_{n=1}^{\infty} B_n, \quad B_n = \{y \in B : |f(z) - f(y)| \geq \tfrac{|z-y|}{n} \text{ for } |z-y| < \tfrac{1}{n}\}.$$

We prove the claim by showing that every set $B_n$ has zero measure.

Fix an $n \in \mathbb{N}$. Since $\beta$ is Lipschitz, we know that $f(A)$ is a set of measure zero. Given $\varepsilon > 0$ we can then choose the intervals $I_j$, $j = 1, \ldots$, such that

$$f(A) \subseteq \bigcup_{j=1}^{\infty} I_j \quad \text{and} \quad \sum_{j=1}^{\infty} |I_j| < \varepsilon.$$

Let $J$ be an interval of length $\frac{1}{n}$, and let $D = B_n \cap J$, $D_j = f^{-1}(I_j) \cap D$. Then, from the definition of $B_n$, $|D_j| \leq n|I_j|$; therefore, $|D| \leq n\varepsilon$ and $|B_n| \leq n^2|b-a|\varepsilon$. Since $\varepsilon$ is arbitrary this shows that $|B_n| = 0$.

We now have that for a. a. $x \in A$

$$\left|\frac{\beta(f(x+h)) - \beta(f(x))}{h}\right| \leq L \left|\frac{f(x+h) - f(x)}{h}\right|$$

for $|h|$ small enough, so $(\beta \circ f)'(x) = 0$ and $\beta'(f(x))f'(x) = 0$. This proves the claim of the Lemma for a. a. $x \in (a, b)$.

## Appendix B: Time regularity for the spatially homogeneous Boltzmann equation

We show that the solution of the Boltzmann equation (2) under the conditions of Theorem 1 is smooth with respect to time, together with its moments of any order.

For $k \geq 0$ we introduce the following weighted Lebesgue spaces

$$L^1_k(\mathbb{R}^d) = \Big\{ f \in L^1(\mathbb{R}^n) : \int_{\mathbb{R}^d} |f| \, (1+|v|^2)^k \, dv < +\infty \Big\} \qquad (.69)$$

with the norms defined by the integrals appearing in (.69). The regularity result that we used in Section 2 is the following.

**Proposition 3.** *Let $f$ be the unique solution of the Boltzmann equation* (2) *that preserves the total mass and energy. Assume that $f_0 \in L^1_k(\mathbb{R}^d)$, $k > 1 + \frac{\beta}{2}$. Then $f \in C^1\big([0,+\infty); L^1_p(\mathbb{R}^d)\big)$ for any $p < k - \frac{\beta}{2}$.*

The proof of Proposition 3 depends on the following continuity property of the nonlinear operator $Q(f)$.

**Lemma 6.** *Let the pair of positive numbers $(k,p)$ satisfy $k > p + \frac{\beta}{2}$. Then $Q(f)$ is continuous on $L^1_k(\mathbb{R}^d)$ as a mapping $L^1_k(\mathbb{R}^d) \to L^1_p(\mathbb{R}^d)$. Moreover, we have the following Hölder estimate for any $f, g \in L^1_k(\mathbb{R}^d)$*

$$\|Q(f) - Q(g)\|_{L^1_p} \leq C_p \Big( \|f-g\|_{L^1}^{1-\frac{p+\frac{\beta}{2}}{k}} + \|f-g\|_{L^1} \Big),$$

*where the constant $C_p$ depends on $p$ and on the upper bound of the $L^1_k$-norms of $f$ and $g$.*

**Proof.** Using the weak form of $Q(f)$ and $Q(g)$ we compute

$$\begin{aligned}
&\int_{\mathbb{R}^d} |Q(f) - Q(g)| \, (1+|v|^2)^p \, dv \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{S^{n-1}} (f f_* - g g_*) \, B(v - v_*, \sigma) \Big( \mathrm{sign}\big(Q(f)' - Q(g)'\big)(1+|v'|^2)^p \\
&\qquad\qquad - \mathrm{sign}\big(Q(f) - Q(g)\big)(1+|v|^2)^p \Big) \, d\sigma \, dv \, dv_* \\
&\leq \; 2^{p+1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f f_* - g g_*| \, |v - v_*|^\beta \, \big( (1+|v|^2)^p + (1+|v_*|^2)^p \big) \, dv \, dv_*
\end{aligned}$$

Since

$$|v - v_*|^\beta \,(1 + |v|^2)^p \le (1 + |v_*|^2)^{\frac{\beta}{2}} (1 + |v|^2)^p + (1 + |v|^2)^{p+\frac{\beta}{2}}$$
$$\le 2\left((1 + |v|^2)^{p+\frac{\beta}{2}} + (1 + |v_*|^2)^{p+\frac{\beta}{2}}\right)$$

and $|f f_* - g g_*| \le \frac{1}{2}\,|f - g|\,|f_* + g_*| + \frac{1}{2}\,|f + g|\,|f_* - g_*|$, we obtain

$$\|Q(f) - Q(g)\|_{L^1_p}$$
$$\le 2^{p+3} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f + g|\,|f_* - g_*| \left((1 + |v|^2)^{p+\frac{\beta}{2}} + (1 + |v_*|^2)^{p+\frac{\beta}{2}}\right) dv\, dv_*$$
$$\le 2^{p+3}\, \|f + g\|_{L^1_k} \left(\|f - g\|_{L^1_{p+\frac{\beta}{2}}} + \|f - g\|_{L^1}\right).$$

We use the interpolation inequality

$$\left(\frac{m_{k_1}(t)}{m_0}\right)^{\frac{1}{k_1}} \le \left(\frac{m_k(t)}{m_0}\right)^{\frac{1}{k}} \le \left(\frac{m_{k_2}(t)}{m_0}\right)^{\frac{1}{k_2}}, \quad k_1 \le k \le k_2, \qquad (.70)$$

with $k_1 = p + \frac{\beta}{2}$ to get

$$\|f - g\|_{L^1_{p+\frac{\beta}{2}}} \le \|f - g\|_{L^1}^{1-\frac{p+\frac{\beta}{2}}{k}} \|f - g\|_{L^1_k}^{\frac{p+\frac{\beta}{2}}{k}}$$
$$\le \left(\|f\|_{L^1_k} + \|g\|_{L^1_k}\right)^{\frac{p+\frac{\beta}{2}}{k}} \|f - g\|_{L^1}^{1-\frac{p+\frac{\beta}{2}}{k}}.$$

Substituting this bound into the previous estimate we obtain the Hölder estimate stated in the Lemma. This completes the proof.

**Proof (Proposition 3).** We fix $T > 0$. By the results of Arkeryd and Elmroth [1,15] (see part (iii) of Theorem 2), $f$ belongs to $L^\infty([0, +\infty); L^1_k(\mathbb{R}^d))$. By Lemma 6,

$$(1 + |v|^2)^p\, Q(f) \in L^1((0,T) \times \mathbb{R}^d), \quad \text{for} \quad p < k - \frac{\beta}{2} \qquad (.71)$$

The mild form of (2), together with the regularity condition (.71) imply that $f$ is weakly differentiable and $\partial_t f = Q(f)$ in the sense of distributions on $(0,T)\times\mathbb{R}^d$. Hence,

$$f \in W^{1,1}((0,T);L^1_p(\mathbb{R}^d))$$

and therefore (cf. [16, p. 286]), $f \in C\big([0,T];L^1_p(\mathbb{R}^d)\big)$. By the continuity of $Q(f)$ established in Lemma 6 it follows that $\partial_t f \in C\big([0,T];L^1_p(\mathbb{R}^d)\big)$, where $\partial_t f$ is the weak time-derivative of $f$. It is then easy to verify directly that $f$ is strongly differentiable on $(0,T)$ with values in $L^1_p(\mathbb{R}^d)$, and its derivative is continuous on $[0,T]$. Since $T$ is arbitrary, we obtain the conclusion of the Lemma.

**Remark 7.** As a consequence of Proposition 3, if the moments of all orders are finite initially then they are continuously differentiable functions of time. By iterating the argument we used in the proof above one can show that in fact then $f \in C^\infty([0,\infty);L^1_k(\mathbb{R}^d))$, for any $k \geq 0$.

## Appendix C: Carleman's representation

**Lemma 7.** *Let $Q^+(f,g)$ be defined by (9) and let $f=f(v)$ and $g=g(v)$, $v\in\mathbb{R}^d$ be smooth functions, decaying rapidly at infinity. Then*

$$Q^+(f,g)(v) = 2^{d-1}\int_{\mathbb{R}^d}\frac{f(v'_*)}{|v-v'_*|}\int_{E_{v,v'_*}}\frac{g(v')\,B(2v-v'-v'_*,\frac{v'-v'_*}{|v'-v'_*|})}{|v'-v'_*|^{d-2}}\,d\pi_{v'}\,dv'_*,$$

*where $E_{v,v'_*}$ is the hyperplane $\{v'\in\mathbb{R}^d \mid (v'-v)\cdot(v'_*-v)=0\}$ and $d\pi_{v'}$ denotes the Lebesgue measure on this hyperplane.*

**Proof.** Using the change of variables $u=v-v_*$, and recalling the definition of the delta function of a quadratic form, see [19], we have

$$Q^+(f,g)(v) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(v'_*)\,g(v')\,B(u,k)\,\delta\big(\tfrac{|k|^2-1}{2}\big)\,dk\,du, \qquad (.72)$$

where $v' = v - u + \frac{1}{2}(u + |u|k)$ and $v'_* = v - \frac{1}{2}(u + |u|k)$. We further set $z = -\frac{1}{2}(u + |u|k)$; for every $u$ fixed this defines a linear map $k \mapsto z$ with determinant $\left(\frac{|u|}{2}\right)^d$. We also have

$$k = -\frac{2z+u}{|u|} \quad \text{and} \quad \frac{|k|^2-1}{2} = \frac{|2z+u|^2-|u|^2}{2|u|^2} = \frac{2z\cdot(z+u)}{|u|^2}.$$

With this change of variables the integral in (.72) can be written as

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left(\tfrac{2}{|u|}\right)^d f(v+z)\, g(v-u-z)\, B\big(u, -\tfrac{2z+u}{|u|}\big)\, \delta\big(\tfrac{2z\cdot(z+u)}{|u|^2}\big)\, dz\, du.$$

We set $y = -z-u$; then $|u| = |y+z|$ and $\delta\big(\frac{2z\cdot(z+u)}{|u|^2}\big) = \frac{|y+z|^2}{2}\,\delta(z\cdot y)$. Further, for any test function $\varphi$,

$$\int_{\mathbb{R}^d} \delta(z\cdot y)\,\varphi(y)\,dy = |z|^{-1}\int_{z\cdot y=0} \varphi(y)\,d\pi_y,$$

where $d\pi_y$ is the Lebesgue measure on the hyperplane $\{y : z\cdot y = 0\}$. This yields

$$Q^+(f,g)(v)$$
$$= 2^{d-1}\int_{z\in\mathbb{R}^d}\int_{y\cdot z=0} f(v+z)g(v+y)\,|z|^{-1}|y+z|^{n-2}\,B\big(-y-z, \tfrac{y-z}{|y+z|}\big)\,d\pi_y\,dz$$

We now return to the original notations $v'_* = v+z$, $v' = v+y$ and perform the corresponding changes of variables to obtain the expression for $Q^+(f,g)$ stated in the Lemma.

**Remark 8.** The above result takes a particularly simple form in the case of the hard-sphere model in $\mathbb{R}^3$; in that case $B(v-v_*,\sigma) = \frac{1}{4\pi}|v-v_*|$ and

$$Q^+(f,g)(v) = \int_{\mathbb{R}^3} \frac{f(v'_*)}{\pi|v-v'|}\int_{E_{v,v'_*}} g(v')\,d\pi_{v'}\,dv'_*.$$

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