

On the Boltzmann Equation for Diffusively Excited Granular Media

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Received: 29 May 2003 / Accepted: 7 October 2003
Published online: 19 February 2004 – © Springer-Verlag 2004

Abstract: We study the Boltzmann equation for a space-homogeneous gas of inelastic hard spheres, with a diffusive term representing a random background forcing. Under the assumption that the initial datum is a nonnegative $L^2(\mathbb{R}^N)$ function, with bounded mass and kinetic energy (second moment), we prove the existence of a solution to this model, which instantaneously becomes smooth and rapidly decaying. Under a weak additional assumption of bounded third moment, the solution is shown to be unique. We also establish the existence (but not uniqueness) of a stationary solution. In addition we show that the high-velocity tails of both the stationary and time-dependent particle distribution functions are overpopulated with respect to the Maxwellian distribution, as conjectured by previous authors, and we prove pointwise lower estimates for the solutions.

0. Introduction

In recent years a significant interest has been focused on the study of kinetic models for granular flows [10, 23, 20]. Depending on the external conditions (geometry, gravity, interactions with surface of a vessel) granular systems may be in a variety of regimes, displaying typical features of solids, liquids or gases and also producing quite surprising effects [37]. Finding a systematic way to describe such systems under different conditions is a physical problem of considerable importance. At the same time, recent developments in this area gave rise to several novel mathematical models with interesting properties.

In the case of rapid, dilute flows, the binary collisions between particles may be considered the main mechanism of inter-particle interactions in the system. In such cases methods of the kinetic theory of rarefied gases, based on the Boltzmann-Enskog equations have been applied [25, 24, 21].

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A very important feature of inter-particle interactions in granular flows is their *inelastic* character: the total kinetic energy is generally not preserved in the collisions. Therefore, in order to keep the system out of the “freezing” state, when particles cease to move and the system becomes static, a certain driving mechanism, supplying the system with energy, is required. Physically realistic driven regimes include excitation from the moving boundary, through-flow of air, fluidized beds, gravity, and other special conditions. We accept a simple model for a driving mechanism, the so-called thermal bath, in which we assume that the particles are subject to uncorrelated random accelerations between the collisions. Such a model was studied in [41] in the one-dimensional case, and in [38] in general dimension.

We study the model [38] in the space-homogeneous regime, described by the following equation:

$$\partial_t f - \mu \Delta_v f = Q(f, f), \quad v \in \mathbb{R}^N, \quad t > 0. \quad (0.1)$$

Here f is the one-particle distribution function (particle density function in the phase space), which is a nonnegative function of the microscopic velocity v and the time t ; we shall assume $N \geq 2$ (dimension 1 could be treated as well but would require a few notational changes). On the right-hand side of Eq. (0.1) there is the inelastic Boltzmann-Enskog operator for hard spheres (the details of which are given below); the term $-\mu \Delta_v f$, $\mu = \text{const}$, represents the effect of the heat bath. Without loss of generality we can set $\mu = 1$ (see Sect. 1.5), which we will from now on assume. In the sequel, we shall often abbreviate Δ_v into just Δ .

One of the interesting features of the model (0.1) is the fact that it possesses nontrivial steady states described by the balance between the collisions and the thermal bath forcing. Such steady states are given by solutions of the equation

$$\mu \Delta_v f + Q(f, f) = 0, \quad v \in \mathbb{R}^N. \quad (0.2)$$

It is interesting to point out that such solutions may exist only in the case of inelastic collisions, at least in the class of integrable functions with finite second moment: see Sect. 2. Solutions of (0.2) have been studied in [38] by means of formal expansions. The same problem was also studied in [9] and in [6], for a different kind of interactions, namely the *Maxwell pseudo-particle model* [5, 26, 27], by methods of expansions and the Fourier transforms, respectively. In reference [11] the rigorous existence of radially symmetric steady solutions for the Maxwell model was established.

The aim of this study is to develop a rigorous theory for the *inelastic hard sphere* model, and to investigate the regularity and qualitative properties of the solutions. We prove that Eq. (0.1) has a unique weak solution under basic assumptions that the initial data have bounded *mass* and *kinetic energy*, and satisfy some additional conditions (bounded entropy for existence, $L^2(\mathbb{R}^N)$ for regularity, and bounded third moment in $|v|$ for uniqueness). The thermal bath (diffusion) term in (0.1) is responsible for the parabolic regularity of solutions: the weak solutions become smooth, classical solutions after an arbitrarily short time. We apply generally similar techniques, based on elliptic regularity, to treat the steady case. Finally, we establish lower bounds, for both steady and time-dependent solutions, proving that the distribution tails are “overpopulated” with respect to the Maxwellian, as was suggested in [38]. The lower bound for steady solutions is given by a “stretched exponential” $A \exp(-a|v|^{3/2})$, with $a = a(\alpha, \mu)$. In the time-dependent case the bound holds with $A = A(t)$, where $A(t)$ is a generally decaying function of time.

We emphasize that the appearance of the “3/2” exponent is a specific feature of the hard sphere model with diffusion, and could be predicted by dimensional arguments (cf. [38]). On the other hand, the Maxwell model with diffusion results in a high-velocity tail with asymptotic behavior $C \exp(-c|v|)$, see [6]. As a general rule, the exponents in the tails are expected to depend on the driving and collision mechanisms [2, 16, 17, 7]. In fact, deviations of the steady states of granular systems from Maxwellian equilibria (“thickening of tails”) is one of the characteristic features of dynamics of granular systems, and has been an object of intensive study in recent years [30, 28, 36, 33].

We remark that the “3/2” bound has rather important practical implications as well. In particular, it indicates that the approximate solutions based on the truncated expansion of the deviation from the Maxwellian into Sonine polynomials [38, 9, 33] could only be valid for moderate values of $|v|^2$. Any conclusions about the *tail behavior* drawn from such an expansion should be questioned. Indeed, since the deviation function is growing rapidly for $|v|$ large (it is in the weighted L^1 space, but not in L^2 !), the Sonine polynomial expansion should in general be expected to have poor approximation properties in this region.

The paper is organized as follows. The first section contains the preliminaries, where we introduce the inelastic collision operator and establish several basic identities which are important in the sequel. In Sect. 2 we establish the bounds for the energy and entropy of solutions. In Sect. 3 we study the moments of the distribution function by analyzing the *moment inequalities* for Eqs. (0.1) and (0.2). The key point in analyzing the moments is the so-called Povzner inequalities, well-known for the classical Boltzmann equation [35, 15, 12, 40, 4, 31], which we here extend to the case of inelastic interactions and present in a general setting of polynomially increasing convex test functions. In Sect. 4 we study the estimates of the inelastic collision operator in L^p spaces with polynomial weights, extending the results in [22] to the inelastic hard sphere case. We continue by establishing apriori regularity estimates, based on the interpolation of L^p spaces and the Sobolev-type inequalities. In Sect. 5 we present a rigorous proof of the existence and regularity of the time-dependent and steady solutions. The arguments presented there also justify the formal manipulations performed in Sects. 2, 3 and 4. In Sect. 6 we show the uniqueness for the time-dependent problem using Gronwall’s lemma. Finally, in Sect. 7 we compute lower bounds for the stationary and time-dependent solutions.

1. Preliminaries

1.1. Binary inelastic collisions. We study the dynamics of inelastic identical hard balls with the following law of interactions. Let v and v_* be the velocities of two particles before a collision, and denote by $u = v - v_*$ their relative velocity. Let the prime symbol denote the same quantities after the collision. Then we assume

$$\begin{aligned} (u' \cdot n) &= -\alpha (u \cdot n), \\ u' - (u' \cdot n)n &= u - (u \cdot n)n, \end{aligned} \tag{1.1}$$

where n is the unit vector in the direction of impact, and $0 < \alpha < 1$ is a constant called the coefficient of normal restitution. Setting $w = v + v_*$ and using the momentum conservation we can express v' and v'_* as follows:

$$v' = \frac{w}{2} + \frac{u'}{2}, \quad v'_* = \frac{w}{2} - \frac{u'}{2}. \tag{1.2}$$

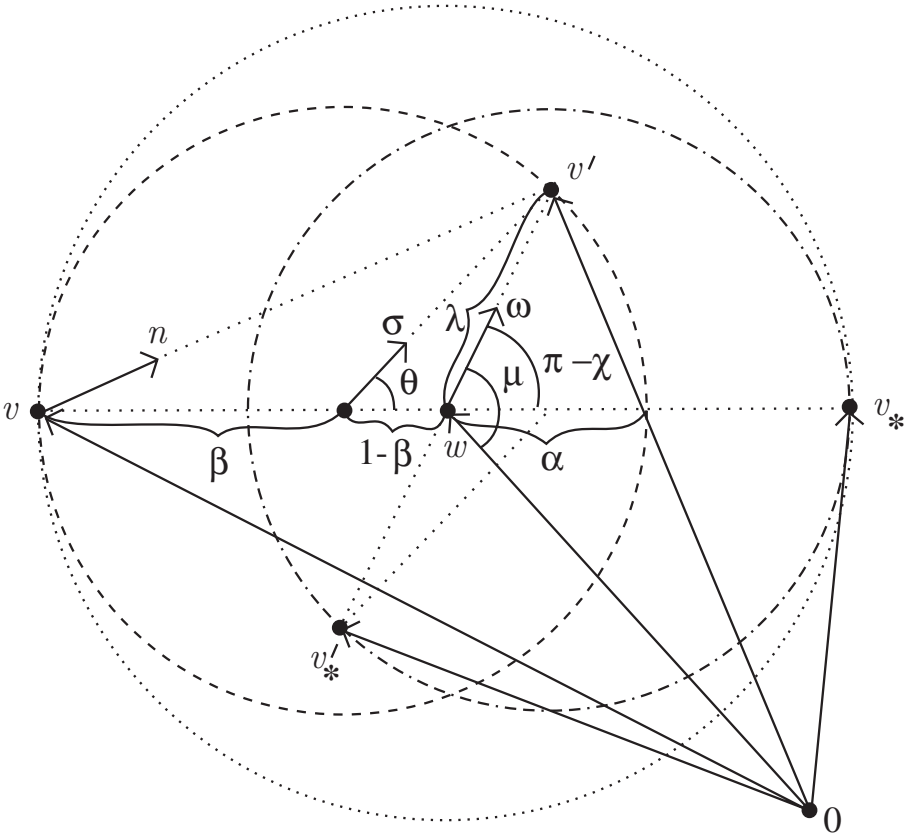


Fig. 1. A two-dimensional illustration of the collision mechanism: -----: possible locations of v' ; - - - - -: possible locations of v'_* . All lengths shown in assumption $|u|/2 = 1$. Unit vectors n, σ, ω not to scale

By substituting (1.1) into (1.2) and Eqs. (1.1), the post-collisional velocities v' and v'_* are uniquely determined by the pre-collisional ones, v and v_* , and the impact parameter n (cf. [10, 38]).

The geometry of the inelastic collisions defined by relations (1.1), (1.2) is shown in Fig. 1. For every v and v_* fixed, the sets of possible outcomes for post-collisional velocities are two (distinct) spheres of diameter $\frac{1+\alpha}{2}|u|$. Thus, it is convenient to parametrize the relative velocity after collision as follows:

$$u' = (1 - \beta)u + \beta|u|\sigma, \tag{1.3}$$

where we denoted $\beta = \frac{1+\alpha}{2}$. The relations (1.2) and (1.3) define the post-collisional velocities in terms of v, v_* and the angular parameter $\sigma \in S^{N-1}$.

1.2. Weak form of the collision operator. We define the collision operator by its action on test functions, or *observables*. Taking $\psi = \psi(v, t)$ to be a suitably regular test function, we introduce the following weak bilinear form of the collision term:

$$\int_{\mathbb{R}^N} Q(g, f) \psi \, dv = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{S^{N-1}} f g_* (\psi' - \psi) |u| b(u, \sigma) \, d\sigma \, dv \, dv_*. \quad (1.4)$$

Here and below we use the shorthand notations $f = f(v, t)$, $g_* = g(v_*, t)$, $\psi' = \psi(v', t)$, etc. The function $b(u, \sigma)$ in (1.6) is the product of the Enskog correlation factor $k(\rho, d)$ (which is a constant in the space-homogeneous case) by the differential collision cross-section, expressed in the variables u, σ . In the case of hard-sphere interactions,

$$b(u, \sigma) = k(\rho, d) \left(\frac{d}{2}\right)^{N-1} \left(\frac{1 - (v \cdot \sigma)}{2}\right)^{-\frac{N-3}{2}},$$

where $v = u/|u|$, and d is the diameter of the particles. Notice that the hard sphere cross-section depends only on the angle between u and σ , and is generally anisotropic, unless $N = 3$. Without restricting generality, by choosing the value of d accordingly, we can always assume that

$$\int_{S^{N-1}} b(u, \sigma) \, d\sigma = 1. \quad (1.5)$$

Of course, to write down the Boltzmann operator we only need $Q(f, f)$, but later on it will be sometimes convenient to work with the bilinear form $Q(g, f)$. An explicit form of Q will be given later on; however for many purposes it will be easier to work with the weak formulation which is also quite natural from the physical point of view (it is analogous to the well-known Maxwell form of the Boltzmann collision operator [39, Chap. 1, Sect. 2.3]).

In the case when $f = g$ in (1.4), we can further symmetrize and write

$$\begin{aligned} &\int_{\mathbb{R}^N} Q(f, f) \psi \, dv \\ &= \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{S^{N-1}} f f_* (\psi' + \psi'_* - \psi - \psi_*) |u| b(u, \sigma) \, d\sigma \, dv \, dv_*. \end{aligned} \quad (1.6)$$

Notice that the particular form of the inelastic collision laws enters (1.6) only through the test functions ψ' and ψ'_* .

1.3. Equations for observables and conservation relations. Using the weak form (1.6) allows us to study equations for average values of observables given by the functionals of the form $\int_{\mathbb{R}^N} f \psi \, dv$. Namely, multiplying Eq. (0.1) by a test function $\psi(v, t)$ and integrating by parts we obtain

$$\left[\int_{\mathbb{R}^N} f \psi \, dv \right]_{t=0}^{t=T} - \int_0^T \int_{\mathbb{R}^N} f (\partial_t \psi + \Delta_v \psi) \, dv \, dt = \int_0^T \int_{\mathbb{R}^N} Q(f, f) \psi \, dv \, dt. \quad (1.7)$$

With the weak form (1.6) of the collision operator, it is easy to verify formally the basic conservation relations that follow from (0.1). Namely, setting $\psi = 1$ and $\psi = v_i$ in (1.7) and assuming that $\int_{\mathbb{R}^N} f \psi \, dv$ is differentiable in t , we obtain the *conservation of mass and momentum*:

$$\frac{d}{dt} \int_{\mathbb{R}^N} f \{1, v_1, \dots, v_N\} \, dv = 0. \quad (1.8)$$

Further, taking $\psi = |v|^2$ and computing

$$|v'|^2 + |v'_*|^2 - |v|^2 - |v_*|^2 = -\frac{1 - \alpha^2}{2} \frac{1 - (v \cdot \sigma)}{2} |u|^2, \tag{1.9}$$

we obtain the following relation for the *dissipation of kinetic energy*:

$$\frac{d}{dt} \int_{\mathbb{R}^N} f |v|^2 dv = 2N - \epsilon_N \frac{1 - \alpha^2}{4} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f f_* |u|^3 dv_* dv, \tag{1.10}$$

where

$$\epsilon_N = \int_{S^{N-1}} \frac{1 - (v \cdot \sigma)}{2} b(u, \sigma) d\sigma = \text{const.}$$

Notice that, unlike the no-diffusion case, the kinetic energy is not necessarily a monotone function of time. However, it is not difficult to show using (1.10) (see Sect. 2) that the kinetic energy remains bounded for all times, provided the initial distribution function has finite energy.

Finally, Eq. (1.7) allows us to define the concept of solutions of (0.1) which we use throughout the paper. Namely, we say that a function f is a *weak solution* of (0.1) if for every $T > 0$, $f \in L^1([0, T] \times \mathbb{R}^N)$, $Q(f, f) \in L^1([0, T] \times \mathbb{R}^N)$ and (1.7) holds for every $\psi \in C^1([0, \infty), C^2(\mathbb{R}^N))$ vanishing for $t > T$. It can be shown in the usual way that if a weak solution is sufficiently smooth (say, continuously differentiable with respect to time and twice continuously differentiable with respect to velocity) and satisfies suitable decay conditions for large $|v|$, then it also is a classical solution.

1.4. Entropy identity. Taking in the weak form (1.6) $\psi = \log f$ we obtain an interesting identity for the entropy $\int_{\mathbb{R}^N} f \log f dv$. First, we compute

$$\begin{aligned} & \int_{\mathbb{R}^N} Q(f, f) \log f dv \\ &= \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{S^{N-1}} f f_* \log \frac{f' f'_*}{f f_*} |u| b(u, \sigma) d\sigma dv dv_* \\ &= \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{S^{N-1}} f f_* \left(\log \frac{f' f'_*}{f f_*} - \frac{f' f'_*}{f f_*} + 1 \right) |u| b(u, \sigma) d\sigma dv dv_* \\ & \quad + \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{S^{N-1}} (f' f'_* - f f_*) |u| b(u, \sigma) d\sigma dv dv_*. \end{aligned} \tag{1.11}$$

The last term vanishes in the elastic case $\alpha = 1$; however, as we see below, it is generally different from zero if $\alpha < 1$. To compare the integral of $f' f'_*$ to that of $f f_*$ we perform the transformation corresponding to the *inverse collision*, passing from the velocities v', v'_* to their predecessors v and v_* . Such a transformation is more easily expressed in the variables u and n . Passing to these variables, we can write the integral of $f' f'_*$ as follows:

$$d^{N-1} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{S^{N-1}_+} f' f'_* |u \cdot n| dn dv dv_*, \tag{1.12}$$

where $S_+^{N-1} = \{n \in S^{N-1} \mid u \cdot n > 0\}$. The ‘‘inverse collision’’ transformation $(v, v_*, n) \mapsto (v', v'_*, -n)$ has the Jacobian determinant equal to α [10]. Therefore, using the first of Eqs. (1.1), the integral (1.12) is computed as

$$d^{N-1} \frac{1}{\alpha^2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{S_+^{N-1}} ff_* |u \cdot n| dn dv dv_*. \tag{1.13}$$

Changing variables in the angular integral from n to σ , we rewrite (1.12) as

$$\frac{1}{\alpha^2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{S^{N-1}} ff_* |u| b(u, \sigma) d\sigma dv dv_* = \frac{1}{\alpha^2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} ff_* |u| dv dv_*. \tag{1.14}$$

In view of (1.11) and (1.14) the entropy equation becomes

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^N} f \log f dv + 4 \int_{\mathbb{R}^N} |\nabla \sqrt{f}|^2 dv \\ &= \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{S^{N-1}} ff_* \left(\log \frac{f' f'_*}{ff_*} - \frac{f' f'_*}{ff_*} + 1 \right) |u| b(u, \sigma) d\sigma dv dv_* \\ & \quad + \frac{1}{2} \left(\frac{1}{\alpha^2} - 1 \right) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} ff_* |u| dv dv_*. \end{aligned} \tag{1.15}$$

In these equations as in all the sequel, the symbol ∇ will stand for the gradient operator with respect to velocity variables. Here the first term on the right-hand side is nonpositive (notice the inequality $\log x - x + 1 \leq 0$) and similar to the entropy dissipation in the elastic case. The last term in (1.15) is a nonnegative correction term that vanishes in the elastic limit $\alpha \rightarrow 1$.

1.5. Similarity in the equations and normalization of solutions. As a consequence of (1.8), the total density (mass) and momentum (mean value) of the distribution function are equal to those of the initial distribution. We can write this as follows:

$$\int_{\mathbb{R}^N} f dv = \rho_0 = \text{const}, \quad \text{and} \quad \int_{\mathbb{R}^N} f v_i dv = \rho_0 v_{0i} = \text{const}_i, \quad i = 1, \dots, N.$$

In fact, we can always assume that $\rho_0 = 1, v_0 = 0$ and $\mu = 1$ in (0.1). Indeed, if $f(v, t)$ is such a solution to (0.1), then, for every ρ_0, v_0 and μ , the function

$$f_{\{\rho_0, v_0, \mu\}}(v, t) = \rho_0 \eta^{-N} f(t/\tau, (v - v_0)/\eta),$$

where

$$\tau = \rho_0^{-2/3} \mu^{-1/3}, \quad \text{and} \quad \eta = \rho_0^{-1/3} \mu^{1/3},$$

is a solution corresponding to the given values of ρ_0, v_0 and μ .

1.6. *Strong form of the collision operator.* Using the weak form (1.6) we can derive the usual strong form of the collision operator. We notice the obvious splitting into the “gain” and the “loss” terms,

$$Q(g, f) = Q^+(g, f) - Q^-(g, f).$$

Assuming that f is regular enough, setting $\psi(v) = \delta(v - v_0)$ in the part of (1.6) corresponding to $Q^-(g, f)$, and using (1.5) we find

$$Q^-(g, f) = \int_{\mathbb{R}^N} \int_{S^{N-1}} fg_* |u| b(u, \sigma) d\sigma dv_* = f(g * |v|).$$

To find the explicit form of $Q^+(g, f)$ we invoke the inverse collision transformation, tracing the collision history back from the pair v, v_* to their predecessors, which we denote by v' and v'_* . Setting $\psi(v) = \delta(v - v_0)$ and arguing similarly to the derivation of the entropy identity we obtain

$$Q^+(g, f) = \int_{\mathbb{R}^N} \int_{S^{N-1}} f' g'_* \frac{1}{\alpha^2} |u| b(u, \sigma) d\sigma dv_*,$$

where $f' = f(v', t)$, $g'_* = g(v'_*, t)$, and the pre-collisional velocities are defined as

$$v' = \frac{w}{2} + \frac{u}{2}, \quad v'_* = \frac{w}{2} - \frac{u}{2}, \quad \text{where } u = (1 - \gamma)u + \gamma |u|\sigma, \tag{1.16}$$

and $\gamma = \frac{\alpha+1}{2\alpha}$.

2. Basic Apriori Estimates: Energy and Entropy

In the classical theory of the elastic Boltzmann equation, the *energy conservation* and the *entropy decay* are the most fundamental facts which provide the base for every analysis. In the present setting naturally we do not have energy conservation, and the energy inequality (expressing that collisions do not increase the energy) would by no means be sufficient to compensate for that. So the key ingredient will be to replace it by the more precise *energy dissipation* estimate, as follows.

To study solutions of (0.1) and (0.2) we assume for simplicity that they satisfy the normalization conditions of unit mass and zero average; however the estimates we derive below will be by no means restricted to such solutions. We use the energy equation (1.10) and apply Jensen’s inequality for the last term to get

$$\int_{\mathbb{R}^N} f_* |u|^3 dv_* \geq \left| v - \int_{\mathbb{R}^N} f(t, v) v dv \right|^3 = |v|^3,$$

and therefore,

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f f_* |u|^3 dv_* dv \geq \int_{\mathbb{R}^N} f |v|^3 dv.$$

We then get (in the time-dependent case) the differential inequality

$$\frac{d}{dt} \int_{\mathbb{R}^N} f |v|^2 dv + k_1 \int_{\mathbb{R}^N} f |v|^3 dv \leq K_1, \tag{2.1}$$

where $K_1 = 2N$ and $k_1 = \epsilon_N \frac{1-\alpha^2}{4}$. Further, by Jensen's inequality,

$$\int_{\mathbb{R}^N} f |v|^3 dv \geq \left(\int_{\mathbb{R}^N} f |v|^2 dv \right)^{3/2},$$

and we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^N} f |v|^2 dv \leq K_1 - k_1 \left(\int_{\mathbb{R}^N} f |v|^2 dv \right)^{3/2}.$$

Thus, if $\int_{\mathbb{R}^N} f |v|^2 dv > (K_1/k_1)^{2/3}$, then $\frac{d}{dt} \int_{\mathbb{R}^N} f |v|^2 dv < 0$ and so,

$$\sup_{t \geq 0} \int_{\mathbb{R}^N} f |v|^2 dv \leq \max \left\{ \int_{\mathbb{R}^N} f_0 |v|^2 dv, (K_1/k_1)^{2/3} \right\}.$$

In the steady case the derivative term drops in (2.1), and we obtain

$$\int_{\mathbb{R}^N} f |v|^3 dv \leq \frac{K_1}{k_1}.$$

Let us introduce the following weighted L^1 spaces:

$$L^1_k(\mathbb{R}^N) = \{f \mid f \langle v \rangle^k \in L^1(\mathbb{R}^N)\}, \tag{2.2}$$

where $k \geq 0$ and $\langle v \rangle = (1 + |v|^2)^{1/2}$. We then define the norms in L^1_k as $\int_{\mathbb{R}^N} |f| \langle v \rangle^k dv$, which for f nonnegative coincide with the *moments* $\int_{\mathbb{R}^N} f \langle v \rangle^k dv$. The above argument implies apriori estimates for the steady solutions in $L^1_{\frac{1}{3}}(\mathbb{R}^N)$, and for the time-dependent ones in $L^\infty([0, \infty), L^1_2(\mathbb{R}^N))$ and $L^1_{loc}([0, \infty), L^1_3(\mathbb{R}^N))$. We emphasize that the bounds depend on α and deteriorate in the elastic limit $\alpha \rightarrow 1$. In fact, in the case $\alpha = 1$ it is easy to see formally that the kinetic energy of solutions increases linearly in time, and so, no nonnegative steady solution with finite mass and second moment is possible.

Next, using the entropy equation (1.15) we show that the entropy is bounded uniformly in time, for initial data with finite mass, kinetic energy and entropy. To obtain this, we first estimate the second term in (1.15) using the Sobolev embedding inequality: assuming for simplicity here that $N \geq 3$, we have

$$\int_{\mathbb{R}^N} |\nabla \sqrt{f}|^2 dv \geq c \|f\|_{L^{p^*}},$$

where $p^* = N/(N - 2)$. Further, we have the inequality

$$\int_{\mathbb{R}^N} f \log f dv \leq C_\epsilon \|f\|_{L^{p^*}}^\epsilon, \tag{2.3}$$

for all $\epsilon > 0$. Indeed, obviously, for every $\delta > 0$,

$$\int_{\mathbb{R}^N} f \log f dv \leq C_\delta \int_{\mathbb{R}^N} f^{1+\delta} dv. \tag{2.4}$$

Further, by Hölder's inequality, for $\delta < p^*$,

$$\|f\|_{L^{1+\delta}} \leq \|f\|_{L^1}^{1-\nu} \|f\|_{L^{p^*}}^\nu,$$

where $v = \frac{p^*\delta}{(p^*-1)(1+\delta)}$. Therefore,

$$\int_{\mathbb{R}^N} f^{1+\delta} dv \leq \|f\|_{L^{p^*}}^{\frac{p^*}{p^*-1}\delta},$$

which together with (2.4) implies (2.3). Now, coming back to estimating the terms in the entropy equation (1.15), we get

$$\frac{d}{dt} \int_{\mathbb{R}^N} f \log f dv + c_\varepsilon \left(\int_{\mathbb{R}^N} f \log f dv \right)^{1/\varepsilon} \leq C \|f\|_{L^1}^2. \tag{2.5}$$

The established bound in $L^\infty([0, \infty), L^1_2(\mathbb{R}^N))$ implies that for initial data with finite mass and energy, the right-hand side of (2.5) is bounded by a constant, and we obtain by Gronwall's lemma,

$$\sup_{t \geq 0} \int_{\mathbb{R}^N} f \log f dv \leq C \left(\int_{\mathbb{R}^N} f_0 \log f_0 dv, \|f_0\|_{L^1_2} \right).$$

Integrating (1.15) in time, we also get $\sqrt{f} \in L^2([0, T], H^1(\mathbb{R}^N))$, for every $T > 0$, which implies in particular, $f \in L^{p^*}([0, T] \times \mathbb{R}^N)$, where the constants in the estimates depend on the initial mass, energy and entropy of the solutions. For the steady solutions we obtain a particularly simple estimate $\|\nabla \sqrt{f}\|_{L^2} \leq C \|f\|_{L^1}$. As the reader will easily check, our assumption that $N \geq 3$ is just for convenience, and can easily be circumvented in dimension 2 by the Moser-Trudinger inequality, or just the local control of all L^p norms of f by $\|\nabla \sqrt{f}\|_{L^2}$, together with a moment-based localization argument.

3. Moment Inequalities

We further look for apriori estimates of the solutions in the spaces L^1_k (2.2) with $k > 2$. Such estimates will play a very important role in our study of regularity, which we perform in Sect. 4. The key technique for obtaining the necessary estimates is the so-called Povzner inequalities [35, 15, 12, 32, 4, 31] which we here extend to the inelastic case.

3.1. The Povzner-type inequalities. We take $\psi(x), x > 0$ to be a convex nondecreasing function and look for estimates of the expressions

$$q[\psi](v, v_*, \sigma) = \psi(|v'|^2) + \psi(|v'_*|^2) - \psi(|v|^2) - \psi(|v_*|^2) \tag{3.1}$$

and

$$\bar{q}[\psi](v, v_*) = \int_{S^{N-1}} (\psi(|v'|^2) + \psi(|v'_*|^2) - \psi(|v|^2) - \psi(|v_*|^2)) b(u, \sigma) d\sigma, \tag{3.2}$$

which appear in the weak form of the collision operator (1.6).

Our aim is to treat the cases of

$$\psi(x) = x^p, \quad \text{and} \quad \psi(x) = (1+x)^p - 1, \quad p > 1, \tag{3.3}$$

and also truncated versions of such functions which will be required in the rigorous analysis of moments in Sect. 5. Thus, we will require functions ψ to satisfy the following list of conditions:

$$\psi(x) \geq 0, \quad x > 0; \quad \psi(0) = 0; \tag{3.4}$$

$$\psi(x) \text{ is convex, } C^1([0, \infty)), \quad \psi''(x) \text{ is locally bounded;} \tag{3.5}$$

$$\psi'(ax) \leq \eta_1(a) \psi'(x), \quad x > 0, \quad a > 1; \tag{3.6}$$

$$\psi''(ax) \leq \eta_2(a) \psi''(x), \quad x > 0, \quad a > 1, \tag{3.7}$$

where $\eta_1(a)$ and $\eta_2(a)$ are functions of a only, bounded on every finite interval of $a > 0$. The above conditions are easily verified for the functions (3.3).

We will further establish the following elementary lemma.

Lemma 3.1. *Assume that $\psi(x)$ satisfies (3.4)–(3.7). Then*

$$\psi(x + y) - \psi(x) - \psi(y) \leq A (x \psi'(y) + y \psi'(x)) \tag{3.8}$$

and

$$\psi(x + y) - \psi(x) - \psi(y) \geq b xy \psi''(x + y), \tag{3.9}$$

where $A = \eta_1(2)$ and $b = (2\eta_2(2))^{-1}$.

Proof. To establish the first of the bounds assume that $x \geq y$. Then, since $\psi(y) \geq 0$,

$$\begin{aligned} \psi(x + y) - \psi(x) - \psi(y) &\leq \psi(x + y) - \psi(x) \\ &= \int_0^y \psi'(x + t) dt \leq \int_0^y \eta_1(2) \psi'(x) dt = A y \psi'(x). \end{aligned}$$

By symmetry we have

$$\psi(x + y) - \psi(x) - \psi(y) \leq A x \psi'(y),$$

when $x \leq y$. This proves the required inequality for all x and y . To prove the second of the bounds in the lemma, we can write, using (3.7) and the normalization $\psi(0) = 0$,

$$\begin{aligned} \psi(x + y) - \psi(x) - \psi(y) &= \int_0^y (\psi'(x + t) - \psi'(t)) dt = \int_0^y \int_0^x \psi''(t + \tau) d\tau dt \\ &\geq (\eta_2(2))^{-1} \psi''(x + y) \int_0^y \int_0^x \chi_{\{t+\tau > (x+y)/2\}} d\tau dt = (2\eta_2(2))^{-1} xy \psi''(x + y). \end{aligned}$$

This completes the proof. \square

In the sequel, we shall use some relations involving post-collisional velocities v' and v'_* . It becomes more convenient to parametrize them in the *center of mass–relative velocity* variables. We therefore set

$$v' = \frac{w + \lambda|u|\omega}{2}, \quad \text{and} \quad v'_* = \frac{w - \lambda|u|\omega}{2}, \tag{3.10}$$

where $w = v + v_*$, $u = v - v_*$, and ω is a parameter vector on the sphere S^{N-1} (see Fig. 1). We have

$$\lambda\omega = \beta\sigma + (1 - \beta)v,$$

where $\beta = \frac{1+\alpha}{2}$ and $v = u/|u|$, and therefore,

$$\lambda = \lambda(\cos \chi) = (1 - \beta) \cos \chi + \sqrt{(1 - \beta)^2(\cos^2 \chi - 1) + \beta^2}, \tag{3.11}$$

where χ is the angle between u and ω . Notice that

$$0 < \alpha \leq \lambda(\cos \chi) \leq 1,$$

for all χ . With this parametrization we have

$$\begin{aligned} |v'|^2 &= \frac{|w|^2 + \lambda^2|u|^2 + 2\lambda|u||w| \cos \mu}{4}, \\ |v_*'|^2 &= \frac{|w|^2 + \lambda^2|u|^2 - 2\lambda|u||w| \cos \mu}{4}, \end{aligned} \tag{3.12}$$

where μ is the angle between the vectors $w = v + v_*$ and ω .

Lemma 3.2. *Assume that the function ψ satisfies (3.4)–(3.7). Then we have*

$$q[\psi] = -n[\psi] + p[\psi],$$

where

$$p[\psi] \leq A(|v|^2 \psi'(|v_*|^2) + |v_*|^2 \psi'(|v|^2))$$

and

$$n[\psi] \geq \kappa(\lambda, \mu) (|v|^2 + |v_*|^2)^2 \psi''(|v|^2 + |v_*|^2).$$

Here A is the constant in estimate (3.8),

$$\kappa(\lambda, \mu) = \frac{b}{4} \lambda^2 (\eta_2(\lambda^{-2}))^{-1} \sin^2 \mu,$$

and b is the constant in estimate (3.9).

Proof. We start by setting

$$p[\psi] = \psi(|v|^2 + |v_*|^2) - \psi(|v|^2) - \psi(|v_*|^2)$$

and

$$n[\psi] = \psi(|v|^2 + |v_*|^2) - \psi(|v'|^2) - \psi(|v_*'|^2).$$

The estimate for $p[\psi]$ follows easily by (3.8). It remains to verify the lower bound for $n[\psi]$. For this we use (3.9), noticing that ψ is monotone and that $|v|^2 + |v_*|^2 \geq |v'|^2 + |v_*'|^2$. We then obtain:

$$\begin{aligned} n[\psi] &\geq \psi(|v'|^2 + |v_*'|^2) - \psi(|v'|^2) - \psi(|v_*'|^2) \\ &\geq b|v'|^2 |v_*'|^2 \psi''(|v'|^2 + |v_*'|^2) \\ &= b\zeta(v', v_*') (|v'|^2 + |v_*'|^2)^2 \psi''(|v'|^2 + |v_*'|^2), \end{aligned}$$

where

$$\zeta(v', v'_*) = \frac{|v'|^2}{|v'|^2 + |v'_*|^2} \frac{|v'_*|^2}{|v'|^2 + |v'_*|^2}.$$

Further, using (3.12), we get

$$\zeta(v', v'_*) = \frac{1}{4} \left(1 - \frac{4\lambda^2 |u|^2 |w|^2}{(\lambda^2 |u|^2 + |w|^2)^2} \cos^2 \mu \right) \geq \frac{1}{4} (1 - \cos^2 \mu) = \frac{1}{4} \sin^2 \mu.$$

Finally, noticing that

$$|v'|^2 + |v'_*|^2 = \frac{\lambda^2 |u|^2 + |w|^2}{2} \geq \lambda^2 \frac{|u|^2 + |w|^2}{2} = \lambda^2 (|v|^2 + |v_*|^2),$$

we obtain

$$n[\psi] \geq \frac{b}{4} \lambda^2 (\eta_2(\lambda^{-2}))^{-1} \sin^2 \mu (|v|^2 + |v_*|^2)^2 \psi''(|v|^2 + |v_*|^2).$$

This completes the proof of the lemma. \square

Lemma 3.2 gives us the basic formulation of the *Povzner inequality* for the considered class of test functions ψ . In the example $\psi(x) = x^p$ we have $p[\psi] \sim C(|v|^2 |v_*|^2)^{p-2} + |v_*|^2 |v|^{2p-2}$ and $n[\psi] \sim c(|v|^{2p} + |v_*|^{2p})$, outside the set where $\kappa(\lambda, \mu)$ is small (which amounts to a small set of angles). This implies that the nonpositive term $-n[\psi]$ is dominating, at least when $|v| \gg |v_*|$ or $|v_*| \gg |v|$, which are the most important regions of integration from the point of view of calculation of moments (cf. also [12, 32]). We can further simplify the inequalities and get rid of the dependence on the angular variables, by integration with respect to $\sigma \in S^{N-1}$. We then obtain the following lemma.

Lemma 3.3. *Assume that the function ψ satisfies (3.4)–(3.7). Then*

$$\bar{q}[\psi] \leq -k (|v|^2 + |v_*|^2)^2 \psi''(|v|^2 + |v_*|^2) + A (|v|^2 \psi'(|v_*|^2) + |v_*|^2 \psi'(|v|^2)),$$

where the constant A is as in Lemma 3.2, and $k > 0$ is a constant that depends on the function ψ but not on α .

Proof. For the proof we notice that $\lambda(\cos \chi)$ is pointwise decreasing as $\alpha \searrow 0$ and so,

$$\lambda(\cos \chi) \geq \cos \chi, \quad \text{for } \cos \chi > 0,$$

for all $\alpha > 0$. We then denote $\cos \theta = (v \cdot \sigma)$, $b_0(\cos \theta) = b(u, \sigma)$, and estimate the integral

$$\int_{S^{N-1}} \kappa(\lambda, \mu) b_0(\cos \theta) d\sigma \geq \int_{\{\cos \chi > \varepsilon_0, \sin \mu > \varepsilon_1, 1 - \cos \theta > \varepsilon_2\}} \kappa(\lambda, \mu) b_0(\cos \theta) d\sigma, \tag{3.13}$$

setting $\varepsilon_0, \varepsilon_1$ and ε_2 small enough. The integrand on the right-hand side of (3.13) is bounded below by a constant, and so is the area of the domain of integration. (The verification of the last statement for the condition $\sin \mu > \varepsilon_1$ is somewhat tedious and is achieved by changing the variables of integration from ω to σ : we omit the technical details.) We therefore find that the integral (3.13) is bounded below by a constant $k > 0$, independent on α . The rest of the claim is easy to verify. \square

Finally, we present estimates for the integral expression (3.2) multiplied by the relative speed, in the cases when $\psi(x)$ is given by one of the functions (3.3).

Lemma 3.4. *Take $p > 1$ and $\psi(x) = x^p$. Then*

$$|u| \bar{q} [\psi](v, v_*) \leq -k_p(|v|^{2p+1} + |v_*|^{2p+1}) + A_p(|v||v_*|^{2p} + |v|^{2p}|v_*|).$$

Also, take $\psi(x) = (1 + x)^p - 1$, then

$$|u| \bar{q} [\psi](v, v_*) \leq -k_p(\langle v \rangle^{2p+1} + \langle v_* \rangle^{2p+1}) + A_p(\langle v \rangle \langle v_* \rangle^{2p} + \langle v \rangle^{2p} \langle v_* \rangle).$$

Here the constants k_p and A_p are independent on the restitution coefficient α .

Proof. We use Lemma 3.3 and the inequalities

$$||v| - |v_*|| \leq |u| = |v - v_*| \leq |v| + |v_*|.$$

Then in the case $\psi(x) = x^p$ the bounds have the form

$$-p(p-1)k_p |u| (|v|^2 + |v_*|^2)^p + pA_p |u| (|v|^2 |v_*|^{2p-2} + |v|^{2p-2} |v_*|^2).$$

The terms appearing with the negative sign are estimated using the inequality

$$|u| (|v|^2 + |v_*|^2)^p \geq \frac{1}{2}(|v|^{2p+1} + |v_*|^{2p+1}) - \frac{1}{2}(|v||v_*|^{2p} + |v|^{2p}|v_*|).$$

For the remaining terms we have

$$|u| (|v|^2 |v_*|^{2p-2} + |v|^{2p-2} |v_*|^2) \leq C_p (|v||v_*|^{2p} + |v|^{2p}|v_*|),$$

which completes the proof of the first part of the lemma. The case $\psi(x) = (1 + x)^p - 1$ can be treated by arguing along the same lines, by using the inequalities

$$(|v|^2 + |v_*|^2)^2 \geq \frac{1}{2}(1 + |v|^2 + |v_*|^2)^2 - 1$$

and $|v| \geq (1 + |v|^2)^{1/2} - 1$. \square

3.2. Estimates for higher-order moments. The Povzner-type inequalities of Lemma 3.4 allow us to study the topics of propagation and appearance of moments. We find that results known for the classical Boltzmann equation with “hard-forces” interactions [15, 12] transfer to present case. We introduce the notation

$$Y_s(t) = \int_{\mathbb{R}^N} f \langle v \rangle^s dv,$$

and denote by \bar{Y}_s the corresponding steady moment.

Lemma 3.5. *Let f be a sufficiently regular and rapidly decaying solution of (0.1). Then, the following differential inequality holds:*

$$\frac{d}{dt} Y_s + 2k_s Y_{s+1} \leq K_s (Y_s + Y_{s-2}), \tag{3.14}$$

where K_s and k_s are positive constants. Further,

$$\sup_{t>0} Y_s(t) \leq Y_s^* = \max \{ Y_s(0), (K_s/k_s)^s \},$$

and for every $\tau > 0$

$$\int_0^\tau Y_{s+1}(t) dt \leq \frac{K_s \tau + 1/2}{k_s} Y_s^*. \tag{3.15}$$

Finally, for the steady equation (0.2) we obtain the apriori estimate

$$\bar{Y}_{s+1} \leq \frac{K_s}{2k_s} (\bar{Y}_s + \bar{Y}_{s-2}).$$

Proof of Lemma 3.5. Using the weak form of Eq. (0.1) with $\psi(v) = \langle v \rangle^s$ we find

$$\frac{d}{dt} \int_{\mathbb{R}^N} f \langle v \rangle^s dv - \int_{\mathbb{R}^N} \Delta f \langle v \rangle^s dv = \int_{\mathbb{R}^N} Q(f, f) \langle v \rangle^s dv. \tag{3.16}$$

Estimating the moments of the collision integral according to Lemma 3.4 we get

$$\int_{\mathbb{R}^N} Q(f, f) \langle v \rangle^s dv \leq -2k_s Y_{s+1} + 2A_s Y_1 Y_s.$$

The moments of the Laplacian term are computed as follows:

$$\begin{aligned} \int_{\mathbb{R}^N} f \Delta \langle v \rangle^s dv &= \int_{\mathbb{R}^N} f ((s(s-2) + sN) \langle v \rangle^{s-2} - s(s-2) \langle v \rangle^{s-4}) dv \\ &= (s(s-2) + sN) Y_{s-2} - s(s-2) Y_{s-4}. \end{aligned} \tag{3.17}$$

Combining (3.16) and (3.17) and neglecting the non-positive Y_{s-4} term, we obtain inequality (3.14) with $K_s = \max\{2A_s, s(s-2) + sN\}$.

To obtain a uniform bound for $Y_s(t)$, we use Jensen's inequality to write

$$Y_{s+1} \geq (Y_s)^{(s+1)/s}.$$

Then we find, estimating the right-hand side of (3.14) by $2K_s Y_s$,

$$\frac{d}{dt} Y_s \leq -2k_s (Y_s)^{(s+1)/s} + 2K_s Y_s.$$

Thus, $Y'_s(t) < 0$ if $Y_s > (K_s/k_s)^s$, and so, the upper bound for $\sup_{t>0} Y_s(t)$ must hold.

Further, integrating in time we obtain

$$2k_s \int_0^\tau Y_{s+1} \leq 2K_s \tau Y_s^* - Y(s) + Y(0) \leq (2K_s \tau + 1) Y_s^*,$$

which proves (3.15). Finally, the last inequality is obtained by the same arguments as (3.14) applied to the steady equation. \square

Based on the lemma just proven we can make the following conclusions about the behavior of the moments of the solutions. First, if a moment Y_s is finite initially, it *propagates*, that is, it remains bounded for the whole time-evolution. Further, the integral condition on Y_{s+1} implies the *appearance* of moments of order $s + 1$: these moments become finite after an arbitrarily short time, even if they are initially infinite (cf. [12]). Indeed, suppose that $Y_{s+1}(0) = +\infty$, then for every $\tau > 0$ there is a $t_0 < \tau$ such that $Y_{s+1}(t_0) < +\infty$. Then, applying the lemma to Y_{s+1} , starting with $t = t_0$, we obtain that for every $t_0 > 0$,

$$\sup_{t > t_0} Y_{s+1}(t) \leq C_{t_0, s},$$

which implies the above statement. The last part of the lemma implies an important statement concerning the moments of the steady solution: on the formal level, every solution that has a finite moment of order $s > 2$ has finite moments of *all positive orders*. In fact, in view of the $L^1_3(\mathbb{R}^N)$ estimate of the previous section, this implies that every solution with finite mass should have this property.

4. L^p Bounds and Apriori Regularity

In this section we study the apriori regularity of solutions to (0.1) and (0.2). The presence of the diffusion term in the equation makes it plausible that solutions to the steady equation should be smooth, and those for the time-dependent equation should gain smoothness after an arbitrarily short time. However, to realize this idea we need to make use of the particular structure of the collision term. As we will see below, the moment bounds of the previous section will also be of crucial importance. We start by establishing the bounds for the collision operator in the spaces L^p with a polynomial weight, extending the results well-known in the case of the classical Boltzmann equation, and first derived by Gustafsson [22]. Below, we shall establish these bounds by adapting the simple strategy that was suggested in [39, Chap. 2, Sect. 3.3] and later developed in [34] to establish improved L^p bounds in the elastic case.

4.1. L^p bounds for the collision operator. We will use the following weighted L^p spaces:

$$L^p_k(\mathbb{R}^N) = \{f \mid f \langle v \rangle^k \in L^p(\mathbb{R}^N)\},$$

where $\langle v \rangle = (1 + |v|^2)^{1/2}$. The necessity to introduce a weight comes from the presence of the factor $|u|$ in the hard sphere collision term (1.6). The collision operator is generally unbounded on L^p : in order to control its norm we will invoke the L^p_k norms with higher powers of $\langle v \rangle$. The precise formulation of this statement is given in next lemma.

Lemma 4.1. *For every $1 \leq p \leq \infty$ and every $k \geq 0$,*

$$\|Q(g, f)\|_{L^p_k} \leq C (\|g\|_{L^p_{k+1}} \|f\|_{L^1_{k+1}} + \|g\|_{L^1_{k+1}} \|f\|_{L^p_{k+1}}),$$

where C is a constant depending on p, k and N only.

Proof. We fix an exponent $1 \leq p \leq \infty$. It is easy to estimate the “loss” part $Q^-(g, f) = (g * |v|)f$, using the inequality

$$|g * |v|| \leq \|g\|_{L^1_1}(v),$$

from which it follows

$$\|Q^-(g, f)\|_{L^p_k} \leq \|g\|_{L^1_1} \|f\|_{L^p_{k+1}}. \tag{4.1}$$

We now turn to estimate the Q^+ term: starting from the weak form (1.6), we find

$$\begin{aligned} & \|Q^+(g, f)\langle v \rangle^k\|_{L^p} \\ &= \sup_{\|\psi\|_{L^{p'}=1}} \int_{\mathbb{R}^N} Q^+(g, f) \psi \langle v \rangle^k dv \\ &= \sup_{\|\psi\|_{L^{p'}=1}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f g_* |u| \int_{S^{N-1}} \psi' \langle v' \rangle^k b(u, \sigma) d\sigma dv dv_*. \end{aligned} \tag{4.2}$$

By using the inequalities $|u| \leq \langle v \rangle + \langle v_* \rangle$ and $\langle v' \rangle^k \leq (\langle v \rangle + \langle v_* \rangle)^k$ the integral (4.2) is bounded as

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f g_* (\langle v \rangle + \langle v_* \rangle)^{k+1} \int_{S^{N-1}} \psi' b(u, \sigma) d\sigma dv dv_*. \tag{4.3}$$

We now see that the problem comes down to estimating the integral

$$S[\psi](v, v_*) = \int_{S^{N-1}} \psi' b(u, \sigma) d\sigma$$

in either $L^\infty(\mathbb{R}^N_v, L^{p'}(\mathbb{R}^N_{v_*}))$ or $L^\infty(\mathbb{R}^N_{v_*}, L^{p'}(\mathbb{R}^N_v))$. In fact, we split $S[\psi]$ into two parts $S_+[\psi]$ and $S_-[\psi]$ and prove the bounds for each of the parts in the respective spaces. We set

$$S_\pm[\psi](v, v_*) = \int_{\{\pm u \cdot \sigma > 0\}} \psi' b(u, \sigma) d\sigma$$

and establish the bounds for S_+ and S_- in the following proposition.

Proposition 4.2. *The operators*

$$\begin{aligned} S_+ &: L^q(\mathbb{R}^N) \rightarrow L^\infty(\mathbb{R}^N_v, L^q(\mathbb{R}^N_{v_*})), \\ S_- &: L^q(\mathbb{R}^N) \rightarrow L^\infty(\mathbb{R}^N_{v_*}, L^q(\mathbb{R}^N_v)), \end{aligned}$$

are bounded for every $1 \leq q \leq \infty$.

Proof. We prove the L^q bounds by interpolation between L^∞ and L^1 . The L^∞ estimates are clear due to the boundedness of the domain of integration. To check the L^1 bounds we assume without loss of generality that $\psi \geq 0$ and calculate the L^1 norms as follows:

$$\begin{aligned} \|S_-[\psi](v, v_*)\|_{L^1(\mathbb{R}^N_{v_*})} &= \int_{\mathbb{R}^N} \int_{\{u \cdot \sigma < 0\}} \psi \left(v + \frac{\beta}{2} (-u + |u|\sigma) \right) b(u, \sigma) d\sigma du \\ &= \int_{\mathbb{R}^N} \psi(v+z) \int_{S^{N-1}} \frac{b(u(z, \sigma), \sigma) \chi_{\{(\sigma \cdot u(z, \sigma)) < 0\}}}{|J_-(u(z, \sigma), \sigma)|} d\sigma dz. \end{aligned}$$

Here, $z = v' - v = \frac{\beta}{2}(-u + |u|\sigma)$, and $J_-(u, \sigma)$ is the Jacobian of the transformation $u \mapsto z$ (for fixed σ):

$$J_-(u, \sigma) = \left(\frac{\beta}{2}\right)^N \left(-1 + \frac{(u \cdot \sigma)}{|u|}\right).$$

The condition $u \cdot \sigma < 0$ ensures that $|J_-|$ is bounded below by $\left(\frac{\beta}{2}\right)^N$, and then,

$$\|S_-[\psi](v, v_*)\|_{L^1(\mathbb{R}_{v_*}^N)} \leq \left(\frac{\beta}{2}\right)^{-N} \int_{S^{N-1}} b(u, \sigma) d\sigma \|\psi\|_{L^1} = \left(\frac{\beta}{2}\right)^{-N} \|\psi\|_{L^1},$$

for every $v \in \mathbb{R}^N$.

Similarly, for the S_+ term we have

$$\begin{aligned} & \|S_+[\psi](v, v_*)\|_{L^1(\mathbb{R}_v^N)} \\ &= \int_{\mathbb{R}^N} \int_{\{u \cdot \sigma > 0\}} \psi\left(v_* + \frac{1}{2}((2 - \beta)u + \beta|u|\sigma)\right) b(u, \sigma) d\sigma du \\ &= \int_{\mathbb{R}^N} \psi(v + z) \int_{S^{N-1}} \frac{b(u(z, \sigma), \sigma) \chi_{\{(\sigma \cdot u(z, \sigma)) > 0\}}}{|J_+(u(z, \sigma), \sigma)|} d\sigma dz, \end{aligned}$$

where now $z = v' - v_* = \frac{1}{2}((2 - \beta)u + \beta|u|\sigma)$, and

$$J_+(u, \sigma) = \left(\frac{2 - \beta}{2}\right)^N \left(1 + \frac{\beta}{2 - \beta} \frac{(u \cdot \sigma)}{|u|}\right).$$

Then, since $(u \cdot \sigma) > 0$, we can argue similarly to the previous case to obtain

$$\|S_+[\psi](v, v_*)\|_{L^1(\mathbb{R}_v^N)} \leq \left(\frac{2 - \beta}{2}\right)^{-N} \|\psi\|_{L^1},$$

uniformly in $v_* \in \mathbb{R}^N$. The statement of the proposition now follows by the Marcinkiewicz interpolation theorem. \square

End of proof of Lemma 4.1. Combining the bound (4.3) with the ones proven in Proposition 4.2 we find

$$\begin{aligned} & \int_{\mathbb{R}^N} Q(g, f) \psi dv \\ & \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f g_* (\langle v \rangle + \langle v_* \rangle)^{k+1} (S_+[\psi](v, v_*) + S_-[\psi](v, v_*)) dv dv_* \\ & \leq C_k \int_{\mathbb{R}^N} g_* \int_{\mathbb{R}^N} f (\langle v \rangle^{k+1} + \langle v_* \rangle^{k+1}) S_+[\psi](v, v_*) dv dv_* \\ & \quad + C_k \int_{\mathbb{R}^N} f \int_{\mathbb{R}^N} g_* (\langle v \rangle^{k+1} + \langle v_* \rangle^{k+1}) S_-[\psi](v, v_*) dv_* dv \\ & \leq C (\|g\|_{L^1} \|f\|_{L_{k+1}^p} + \|g\|_{L_{k+1}^1} \|f\|_{L^p} + \|f\|_{L^1} \|g\|_{L_{k+1}^p} + \|f\|_{L_{k+1}^1} \|g\|_{L^p}), \end{aligned}$$

since $\|\psi\|_{L^{p'}} = 1$. From this the conclusion of the lemma follows easily. \square

4.2. H^1 regularity: Steady state equation. We start by establishing apriori estimates for solutions to the steady equation (0.2), for which the analysis is performed in a rather more direct way than for the time-dependent problem. We first show the bounds in the Sobolev spaces with the weight $\langle v \rangle^k = (1 + |v|^2)^{k/2}$:

$$H_k^1(\mathbb{R}^N) = \{f \in L_k^2(\mathbb{R}^N) \mid \nabla f \in L_k^2(\mathbb{R}^N)\}.$$

The main tools are the coercivity of the diffusion part, the estimates of the collision operator in L^p , and the interpolation inequalities for L^p spaces. The constants in the estimates are expressed in terms of the L^1 moments. In all this section, we shall assume for simplicity that $N \geq 3$, but there is no difficulty to adapt the proofs to cover the case $N = 2$ as well. We begin with an estimate for the gradient in L^2 .

Lemma 4.3. *Assume that the function $f \in H^1(\mathbb{R}^N) \cap L_r^1(\mathbb{R}^N)$, where $r = \frac{N+2}{4}$, is a solution of (0.2). Then*

$$\|\nabla f\|_{L^2} \leq CA B^r,$$

where

$$A = \|f\|_{L_r^1}, \quad B = \|f\|_{L_1^1},$$

and C is a constant depending on the dimension.

Proof. Multiplying Eq. (0.2) by f , integrating and applying Hölder’s inequality yields

$$\int_{\mathbb{R}^N} |\nabla f|^2 dv = \int_{\mathbb{R}^N} Q(f, f) f dv \leq C \|f\|_{L^p} \|Q(f, f)\|_{L^{p'}}, \tag{4.4}$$

for all $1 \leq p \leq \infty$. We choose $p = 2^* = 2N/(N - 2)$, where 2^* is the critical Sobolev exponent, and apply the Sobolev’s embedding inequality

$$\|f\|_{L^{2^*}} \leq C \|\nabla f\|_{L^2}. \tag{4.5}$$

(Note: for $N = 3$, $2^* = 6$ and $(2^*)' = 6/5$.) Then, by Lemma 4.1,

$$\|Q(f, f)\|_{L^{(2^*)'}} \leq C \|f\|_{L_1^{(2^*)}} \|f\|_{L_1^1}. \tag{4.6}$$

To estimate $\|f\|_{L^{(2^*)}}$ we use the following interpolation inequality for weighted L^p norms (φ is any weight function), which can be easily verified using Hölder’s inequality:

$$\|f\varphi^k\|_{L^q} \leq \|f\varphi^{k_1}\|_{L^{q_1}}^v \|f\varphi^{k_2}\|_{L^{q_2}}^{1-v}, \tag{4.7}$$

where

$$\frac{v}{q_1} + \frac{1-v}{q_2} = \frac{1}{q} \quad \text{and} \quad k_1 v + k_2(1-v) = k.$$

Now, interpolating the norm in L_1^q for $q = (2^*)'$ between $q_1 = 2^*$ and $q_2 = 1$, we get

$$\|f\|_{L_1^{(2^*)'}} \leq \|f\|_{L^{2^*}}^v \|f\|_{L_1^1}^{1-v}, \tag{4.8}$$

where ν and r are determined from the following equations:

$$\frac{\nu}{2^*} + \frac{1-\nu}{1} = \frac{1}{(2^*)'} \quad \text{and} \quad r(1-\nu) = 1,$$

so that

$$\nu = \frac{N-2}{N+2} \quad \text{and} \quad r = \frac{1}{1-\nu} = \frac{N+2}{4}. \tag{4.9}$$

Combining estimates (4.4)–(4.8) we obtain the inequality

$$\|\nabla f\|_{L^2}^2 \leq C \|f\|_{L^1} \|f\|_{L_r^1}^{1-\nu} \|f\|_{L^{2^*}}^{1+\nu} \leq C B A^{1-\nu} \|\nabla f\|_{L^2}^{1+\nu}, \tag{4.10}$$

from which the conclusion of the lemma follows. \square

The result of the lemma implies a bound for the solutions in the space $H^1(\mathbb{R}^N)$. Indeed, by the Sobolev embedding,

$$\|f\|_{L^{2^*}} \leq C A B^r.$$

Interpolating between L^1 and L^{2^*} using inequality (4.7) we get a bound for the L^2 norm, which then implies a bound in H^1 . Since the constants in the estimates depend on the L_k^1 norms only, and the latter are controlled by the moments bounds, we gain an a priori control of the H^1 norm by means of the mass and the energy only. We next see that the derivatives of the solutions have an appropriate decay, so even L_k^2 norms for all $k \geq 0$ are bounded.

Lemma 4.4. *Let f be a solution of Eq. (0.2) and assume that $f \in H_k^1(\mathbb{R}^N) \cap L_{(k+1)r}^1(\mathbb{R}^N)$, where $k \geq 0$ and $r = \frac{N+2}{4}$. Then*

$$\|\nabla(f\langle v \rangle^k)\|_{L^2} \leq C(A_1 A_2^r + k^{2r} A_3),$$

where

$$A_1 = \|f\|_{L_{(k+1)r}^1}, \quad A_2 = \|f\|_{L_{k+1}^1}, \quad A_3 = \|f\|_{L_{k-2/r}^1},$$

and C is a constant depending on the dimension N .

Proof. Integrating Eq. (0.2) against $f\langle v \rangle^{2k}$ we obtain

$$\int_{\mathbb{R}^N} \nabla f \cdot \nabla(f\langle v \rangle^{2k}) dv = \int_{\mathbb{R}^N} Q(f, f) f\langle v \rangle^{2k} dv. \tag{4.11}$$

Using estimates from the previous lemma, the right-hand side can be bounded above as follows:

$$\|Q(f, f)\langle v \rangle^k\|_{L^{(2^*)'}} \|f\langle v \rangle^k\|_{L^{2^*}} \leq C \|f\langle v \rangle^{k+1}\|_{L^{(2^*)'}} \|f\langle v \rangle^{k+1}\|_{L^1} \|f\langle v \rangle^k\|_{L^{2^*}}. \tag{4.12}$$

Interpolating as in (4.7) we find

$$\|f\langle v \rangle^{k+1}\|_{L^{(2^*)'}} \leq \|f\|_{L^{2^*}}^\nu \|f\|_{L_{(k+1)r}^1}^{1-\nu}, \tag{4.13}$$

where ν and r are as defined in (4.9). Therefore, combining (4.12) with (4.13) we bound the right hand side of (4.11) by

$$C\left(\|f\|_{L_1^{(k+1)r}}^{1-\nu} \|f\|_{L_1^{k+1}}\right)\|f\langle v\rangle^k\|_{L^{2*}}^{1+\nu} \leq CA_1^{1-\nu} A_2\|\nabla(f\langle v\rangle^k)\|_{L^2}^{1+\nu}. \tag{4.14}$$

The integral on the left-hand side of (4.11) is estimated as follows:

$$\begin{aligned} \int_{\mathbb{R}^N} \nabla f \cdot \nabla(f\langle v\rangle^{2k}) dv &= \int_{\mathbb{R}^N} |\nabla(f\langle v\rangle^k)|^2 dv - \int_{\mathbb{R}^N} f^2 |\nabla\langle v\rangle^k|^2 dv \\ &\geq \|\nabla(f\langle v\rangle^k)\|_{L^2}^2 - k^2 \|f\|_{L_{k-1}^2}^2. \end{aligned} \tag{4.15}$$

Further, interpolating the L_{k-1}^2 norm between L^{2*} and L^1 we get

$$\|f\langle v\rangle^{k-1}\|_{L^2} \leq C\|f\langle v\rangle^k\|_{L^{2*}}^\lambda \|f\langle v\rangle^{k^2}\|_{L^1}^{1-\lambda} \leq C\|\nabla(f\langle v\rangle^k)\|_{L^2}^\lambda \|f\|_{L_{k_2}^1}^{1-\lambda},$$

where

$$\frac{\lambda}{2^*} + \frac{1-\lambda}{1} = \frac{1}{2}, \quad \text{so that} \quad \lambda = \frac{N}{N+2} = \frac{1+\nu}{2}, \tag{4.16}$$

and

$$k-1 = \lambda k + (1-\lambda)k_2, \quad \text{so that} \quad k_2 = k - \frac{1}{1-\lambda} = k - 2/r.$$

Gathering the above inequalities and noticing that $2\lambda = 1 + \nu$ we obtain:

$$\|\nabla(f\langle v\rangle^k)\|_{L^2}^2 \leq C(A_1^{1-\nu} A_2 + k^2 A_3^{1-\nu})\|\nabla(f\langle v\rangle^k)\|_{L^2}^{1+\nu}.$$

Dividing by the norm of the gradient to the power $1 + \nu$ we get

$$\|\nabla(f\langle v\rangle^k)\|_{L^2} \leq (CA_1^{1-\nu} A_2 + k^2 A_3^{1-\nu})^{\frac{1}{1-\nu}}.$$

Noticing that $\frac{1}{1-\nu} = r$ and using the inequality $(x + y)^r \leq C_r(x^r + y^r)$ we arrive at the conclusion of the lemma. \square

Using the lemma just proven we find bounds for solutions f in H_k^1 for every $k \geq 0$. Indeed, using the inequality

$$|(\nabla f)\langle v\rangle^k|^2 \leq C(|\nabla(f\langle v\rangle^k)|^2 + |f\nabla\langle v\rangle^k|^2)$$

and interpolating in the second term between L^{2*} and $L_{k-2/r}^1$ we get

$$\|\nabla f\|_{L_k^2}^2 \leq C(\|\nabla(f\langle v\rangle^k)\|_{L^2}^2 + \|\nabla(f\langle v\rangle^k)\|_{L^2}^{1+\nu} \|f\|_{L_{k-2/r}^1}^{1-\nu}),$$

from which an estimate in terms of the L^1 moments follows. Further, by the interpolation inequality (4.7),

$$\|f\|_{L_k^2} \leq \|f\|_{L^{2*}}^\lambda \|f\|_{L_{k/(1-\lambda)}^1}^{1-\lambda},$$

and so, in view of our earlier remarks, the norm in L_k^2 is also estimated in terms of L^1 moments only. Summarizing the results obtained so far, the solutions are controlled apriori in $H_k^1(\mathbb{R}^N)$ for any $k \geq 0$ in terms of mass and kinetic energy only.

4.3. *Schwartz class regularity: Steady problem.* Our next aim is now to establish a priori bounds for solutions to (0.2) in the spaces

$$H_k^n(\mathbb{R}^N) = \{f \in L_k^2(\mathbb{R}^N) \mid \nabla^m f \in L_k^2(\mathbb{R}^N), 1 \leq m \leq n\},$$

for all $1 \leq n < \infty$ and all $0 \leq k < \infty$. We use induction on n , differentiating the equation in v in each step. The base of the induction is given by Lemma 4.4. We recall the following rule for differentiating the collision integral.

Proposition 4.5. *Let f and g be smooth, rapidly decaying functions of v . Then*

$$\nabla Q(g, f) = Q(\nabla g, f) + Q(g, \nabla f).$$

Proof. We use the splitting into the “gain” and “loss” terms, $Q(g, f) = Q^+(g, f) - Q^-(g, f)$. Since $Q^-(g, f) = f(g * |v|)$, the differentiation rule for the “loss” term is obvious. To prove the proposition for the “gain” term $Q^+(g, f)$ we represent it as follows, using (1.16):

$$Q^+(g, f) = \int_{\mathbb{R}^N} \int_{S^{N-1}} f\left(v + \frac{u-u'}{2}\right) g\left(v - \frac{u+u'}{2}\right) \frac{1}{\alpha^2} |u| b(u, \sigma) d\sigma du.$$

Since u' is a function of u and σ only, the statement follows by differentiation under the integral sign. \square

Remark. The above statement is in fact a corollary of the following abstract statement which can be proven very easily: Let Q be a bilinear operator commuting with translations, continuously differentiable; then $\nabla Q(g, f) = Q(\nabla f, g) + Q(f, \nabla g)$. Thus, the differentiation formula of Proposition 4.5 can be seen as a consequence of the translation invariance of Q .

As a direct corollary of Proposition 4.5, higher-order derivatives of Q can be calculated using the following Leibniz formula:

$$\partial^j Q(g, f) = \sum_{0 \leq l \leq j} \binom{j}{l} Q(\partial^{j-l} g, \partial^l f),$$

where j and l are multi-indices $j = (j_1 \dots j_N)$, and $l = (l_1 \dots l_N)$;

$$\partial^j = \partial_{v_1}^{j_1} \dots \partial_{v_N}^{j_N},$$

and $\binom{j}{l}$ are the multinomial coefficients. Thus, for every multi-index j , by formal differentiation of (0.2) we obtain the following equations for higher-order derivatives:

$$-\Delta \partial^j f = \sum_{0 \leq l \leq j} \binom{j}{l} Q(\partial^{j-l} g, \partial^l f). \tag{4.17}$$

By applying the methods developed in Lemmas 4.3 and 4.4 to Eq. (4.17) we arrive at the following result.

Lemma 4.6. *Let f be a solution to (0.2), such that $f \in H_{k+\mu}^{n+1}(\mathbb{R}^N)$, with $n \geq 0$, $k \geq 0$ and $\mu > 1 + \frac{N}{2}$. Then*

$$\|\nabla^{n+1} f\|_{L_k^2} \leq C (1 + k + \|f\|_{H_{k+\mu}^{n-1}}) (1 + \|\nabla^n f\|_{L_{k+\mu}^2}),$$

where C is a constant depending on n and N only.

Proof. Taking a multi-index j with $|j| = n$, multiplying Eq. (4.17) by $\partial^j f \langle v \rangle^{2k}$ and integrating by parts we obtain:

$$\int_{\mathbb{R}^N} \nabla \partial^j f \cdot \nabla (\partial^j f \langle v \rangle^{2k}) dv = \sum_{0 \leq l \leq j} \binom{j}{l} \int_{\mathbb{R}^N} Q(\partial^{j-l} f, \partial^l f) \partial^j f \langle v \rangle^{2k} dv. \tag{4.18}$$

Similarly to (4.15), the left-hand side can be written as

$$\|\nabla(\partial^j f \langle v \rangle^k)\|_{L^2}^2 - \|\partial^j f \nabla \langle v \rangle^k\|_{L^2}^2.$$

Each integral on the right-hand side of (4.18) can be bounded above by using Cauchy-Schwartz's inequality and Lemma 4.1 as follows:

$$\begin{aligned} \int_{\mathbb{R}^N} Q(\partial^{j-l} f, \partial^l f) \partial^j f \langle v \rangle^{2k} dv &\leq \|Q(\partial^{j-l} f, \partial^l f)\|_{L_k^2} \|\partial^j f\|_{L_k^2} \\ &\leq C \|\partial^l f\|_{L_{k+1}^1} \|\partial^{l-j} f\|_{L_{k+1}^2} \|\nabla^n f\|_{L_{k+1}^2}^2 \|\partial^j f\|_{L_k^2}. \end{aligned}$$

Now, the L^1 norms can be estimated as follows:

$$\|\partial^l f\|_{L_{k+1}^1} \leq \|\langle v \rangle^{1-\mu}\|_{L^2} \|\partial^l f\|_{L_{k+\mu}^2} \leq C \|\partial^l f\|_{L_{k+\mu}^2}$$

as soon as $\mu > 1 + \frac{N}{2}$. Gathering the above estimates we obtain:

$$\begin{aligned} &\|(\partial^j f \langle v \rangle^k)\|_{L^2}^2 \\ &\leq \|\partial^j f \nabla \langle v \rangle^k\|_{L^2}^2 + C \|\partial^j f\|_{L_k^2} \sum_{0 \leq l \leq j} \binom{j}{l} \|\partial^l f\|_{L_{k+\mu}^2} \|\partial^{l-j} f\|_{L_{k+1}^2} \\ &\leq k^2 \|\partial^j f\|_{L_{k-1}^2}^2 + C \|f\|_{L_{k+\mu}^2} \|\partial^j f\|_{L_{k+\mu}^2}^2 + C \|\partial^j f\|_{L_{k+\mu}^2} \|f\|_{H_{k+\mu}^{n-1}}^2 \\ &\leq k^2 \|\partial^j f\|_{L_{k+\mu}^2}^2 + C(1 + \|f\|_{H_{k+\mu}^{n-1}}^2) \|\partial^j f\|_{L_{k+\mu}^2}^2 + C(1 + \|\partial^j f\|_{L_{k+\mu}^2}^2) \|f\|_{H_{k+\mu}^{n-1}}^2. \end{aligned}$$

Since

$$\nabla(\partial^j f \langle v \rangle^k) = (\nabla \partial^j f) \langle v \rangle^k + k \partial^j f |v| \langle v \rangle^{k-2},$$

we obtain

$$\|\nabla \partial^j f\|_{L_k^2}^2 \leq C(1 + k^2 + \|f\|_{H_{k+\mu}^{n-1}}^2)(1 + \|\partial^j f\|_{L_{k+\mu}^2}^2).$$

Taking the sum over all j with $|j| = n$ implies the estimate of the lemma. \square

Lemma 4.6 gives us a way to estimate higher-order derivatives of solutions in terms of lower-order ones. Thus, provided we have a solution to (0.2) that has all H_k^1 norms bounded in terms of mass and energy (as we assumed in the previous section), we can derive bounds in H_k^2 for every k , and then proceed by induction, obtaining bounds in $H_k^n(\mathbb{R}^N)$, for all n and all $k \geq 0$. We then obtain

$$f \in \bigcap_{n \geq 1, k \geq 0} H_k^n = \mathcal{S},$$

where \mathcal{S} is the Schwartz class of rapidly decaying smooth functions. Notice that the bounds in each of the spaces $H_k^n(\mathbb{R}^N)$ can be expressed in terms of mass and energy of the solutions.

4.4. *Regularity for the time-dependent problem.* An analysis of the regularity of the time-dependent solutions can be performed in the same vein as for the steady problem. Using the estimates obtained in the previous section in combination with the Gronwall lemma will give us results for the time-dependent equation (0.1). Our first lemma is an analog of Lemma 4.3.

Lemma 4.7. *Let f be a sufficiently regular solution to (0.1) with the initial condition $f(\cdot, 0) = f_0 \in L^2(\mathbb{R}^N)$, such that f has a moment of order $r = \frac{N+2}{4}$ bounded uniformly in time. Then*

$$\|f(\cdot, t)\|_{L^2} \leq C, \quad 0 \leq t < \infty,$$

and

$$\|\nabla f\|_{L^2((0, T] \times \mathbb{R}^N)} \leq C_T,$$

for every $0 \leq T < \infty$, where the constants C and C_T depend only on N , $\|f_0\|_{L^2}$ and $\sup_{t \geq 0} \|f(\cdot, t)\|_{L^1_r}$.

Proof. Integrating Eq. (0.1) against f we get, arguing similarly to the case of the steady problem:

$$\frac{1}{2} \frac{d}{dt} \|f\|_{L^2}^2 + \|\nabla f\|_{L^2}^2 \leq K(t) \|\nabla f\|_{L^2}^{1+\nu}, \tag{4.19}$$

where $K(t) = C \|f\|_{L^1} \|f\|_{L^1_r}^{1-\nu}$ and $\nu = \frac{N-2}{N+2}$ as in (4.9). By interpolation and Sobolev embedding,

$$\|f\|_{L^2} \leq \|f\|_{L^{2^*}}^\lambda \|f\|_{L^1}^{1-\lambda} \leq C \|\nabla f\|_{L^2}^\lambda \|f\|_{L^1}^{1-\lambda},$$

where $\lambda = \frac{N}{N+2}$ as given by (4.16). Therefore,

$$\|\nabla f\|_{L^2}^2 \geq k \|f\|_{L^2}^{2/\lambda}, \tag{4.20}$$

where $k = C^{-1} \|f\|_{L^1}^{(\lambda-1)/\lambda}$ is a constant. Distributing the term $\|\nabla f\|_{L^2}^2$ in (4.19) equally between the left and the right-hand sides and using inequality (4.20) we obtain

$$\frac{d}{dt} \|f\|_{L^2}^2 + k \|f\|_{L^2}^{2/\lambda} \leq -\|\nabla f\|_{L^2}^2 + 2K(t) \|\nabla f\|_{L^2}^{1+\nu}. \tag{4.21}$$

The function $X \mapsto -X^2 + 2K(t)X^{1+\nu}$, appearing on the right-hand side of (4.21) has a global maximum $(1 + \nu)^{2r-1} (1 - \nu) K(t)^{2r} = C K(t)^{2r}$, so we obtain

$$\frac{d}{dt} \|f\|_{L^2}^2 + k \|f\|_{L^2}^{2/\lambda} \leq C K(t)^{2r} \leq C \bar{K}^{2r}, \tag{4.22}$$

where $\bar{K} = \sup_{t \geq 0} K(t) \leq \sup_{t \geq 0} \|f\|_{L^1_r}^2$. Applying Gronwall's lemma argument to (4.22) we then obtain a bound of the L^2 norm of f in terms of $\|f_0\|_{L^2}$ and $\sup_{t \geq 0} \|f\|_{L^1_r}$. Further, integrating (4.19) over time, we get

$$\|\nabla f\|_{L^2((0, T] \times \mathbb{R}^N)}^2 \leq C + \bar{K} \int_0^T \|\nabla f\|_{L^2}^{1+\nu} dt \leq C + \bar{K} T^{(1-\nu)/2} \|\nabla f\|_{L^2((0, T] \times \mathbb{R}^N)}^{1+\nu},$$

which proves the second claim of the lemma. \square

Similar results can be established about the time-dependence of the L_k^2 norms of the solutions.

Lemma 4.8. *Let f be a sufficiently regular solution to (0.1), with initial data $f_0 \in L_k^2(\mathbb{R}^N)$, where $k \geq 0$, and such that f has a moment of order $r(k + 1)$, where $r = \frac{N+2}{4}$, bounded uniformly in time. Then*

$$\|f(\cdot, t)\|_{L_k^2} \leq C, \quad 0 \leq t < \infty,$$

and

$$\|\nabla f\|_{L_k^2([0, T] \times \mathbb{R}^N)} \leq C_T,$$

for every $0 \leq T < \infty$, where the constants C and C_T depend on N , $\|f_0\|_{L_k^2}$ and $\sup_{t \geq 0} \|f(\cdot, t)\|_{L_{r(k+1)}^1}$ only.

Proof. Multiplying the equation by $f\langle v \rangle^{2k}$ and integrating we obtain:

$$\frac{1}{2} \frac{d}{dt} \|f\|_{L_k^2}^2 + \int_{\mathbb{R}^N} \nabla f \cdot \nabla (f\langle v \rangle^{2k}) dv = \int_{\mathbb{R}^N} Q(f, f) f\langle v \rangle^{2k} dv.$$

Following the steps of the proof of Lemma 4.4 and distributing the term $\|\nabla(f\langle v \rangle^k)\|^2$ evenly between the left and right-hand sides we obtain the following differential inequality:

$$\begin{aligned} & \frac{d}{dt} \|f\langle v \rangle^k\|_{L^2}^2 + \|\nabla(f\langle v \rangle^k)\|_{L^2}^2 \\ & \leq -\|\nabla(f\langle v \rangle^k)\|_{L^2}^2 + \left(CA_1(t)^{1-\nu} A_2(t) + k^2 A_3(t)^{1-\nu} \right) \|\nabla(f\langle v \rangle^k)\|_{L^2}^{1+\nu}, \end{aligned} \quad (4.23)$$

where $A_1(t)$, $A_2(t)$, and $A_3(t)$ are the moments defined in Lemma 4.4. The uniform bounds of the moments imply that the right-hand side of (4.23) is bounded above by a constant. The left-hand side is estimated below as

$$\frac{d}{dt} \|f\langle v \rangle^k\|_{L^2}^2 + c\|f\langle v \rangle^k\|_{L^2}^{2/\lambda}$$

analogously to (4.19). Thus, by a Gronwall-type argument we obtain that the L_k^2 -norm of f is bounded uniformly in time. Integrating (4.23) over time we also get the second claim of the lemma. \square

Finally, we establish the following analog of Lemma 4.6 which will allow us to study the regularity of higher-order derivatives.

Lemma 4.9. *Let f be a solution to (0.1) with initial data $f_0 \in H_k^n(\mathbb{R}^N)$ where $k \geq 0$ and $n \geq 0$, such that f has a moment of order $r^* = r(2^n(k + \mu) - 2\mu + 1)$, where $r = \frac{N+2}{4}$ and $\mu > \frac{N+2}{2}$, bounded uniformly in time. Then*

$$\|f(\cdot, t)\|_{H_k^n} \leq C, \quad 0 \leq t < \infty,$$

and

$$\|f\|_{L^2([0, T], H_k^{n+1}(\mathbb{R}^N))} \leq C_T,$$

for every $0 \leq T < \infty$, where the constants C and C_T depend on N , $\|f_0\|_{H_k^n}$ and $\sup_{t \geq 0} \|f(\cdot, t)\|_{L_{r^*}^1}$ only.

Proof. We will use induction on n . The case $n = 0$ is already proven in Lemma 4.8. Assuming that the statement of the lemma holds for $n - 1$, we differentiate the equation in v and argue as in the proof of Lemma 4.6, obtaining the following inequality:

$$\frac{1}{2} \frac{d}{dt} \|\nabla^n f\|_{L^2_k}^2 + \|\nabla^{n+1} f\|_{L^2_k}^2 \leq C(1 + k^2 + \|f\|_{H^{n-1}_{k+\mu}}^2)(1 + \|\nabla^n f\|_{L^2_{k+\mu}}^2).$$

We estimate $\|\nabla^n f\|_{L^2_{k+\mu}}^2$ integrating by parts and using Young’s inequality (cf. [13]):

$$\|\nabla^n f\|_{L^2_{k+\mu}}^2 \leq \delta \|\nabla^{n+1} f\|_{L^2}^2 + C_\delta \|\nabla^{n-1} f\|_{L^2_{(k+\mu)}}^2.$$

Then, since we assumed f to be bounded in $H^{n-1}_{2(k+\mu)}$ we get

$$\frac{1}{2} \frac{d}{dt} \|\nabla^n f\|_{L^2_k}^2 \leq -\|\nabla^{n+1} f\|_{L^2_k}^2 + C_1(n, k)\delta \|\nabla^{n+1} f\|_{L^2}^2 + C_2(n, k, \delta).$$

Choosing δ suitably small, we obtain the conclusion by Gronwall’s lemma. \square

Lemmas 4.7 – 4.9 allow us to make the following conclusions about the regularity of solutions to (0.1). Provided a sufficient number of moments is initially available, the H^n regularity of the initial data is preserved with time. Moreover, the established bounds for the derivatives in $L^2([0, T] \times \mathbb{R}^N)$ imply that after an *arbitrarily short time* the derivatives $\partial^j f(\cdot, t)$ of *any order* are in $L^2(\mathbb{R}^N)$, and then they propagate in time. Thus, on the level of apriori estimates we find that the solutions become immediately infinitely smooth in v and decay faster than any negative power for $|v|$ large.

We can also see that the solutions are infinitely differentiable in t . Indeed, in view of the established H^n_k regularity we have $f(\cdot, t) \in \mathcal{S}(\mathbb{R}^N)$ for $t > 0$, and then Eq. (0.1) implies $\partial_t f(\cdot, t) \in \mathcal{S}(\mathbb{R}^N)$, for every $t > 0$. Differentiating the equation in time and proceeding by induction we find also that $\partial_t^m f(\cdot, t) \in \mathcal{S}(\mathbb{R}^N)$, for every $m = 1, 2, \dots$, and for every $t > 0$. The time derivatives also remain bounded uniformly in time.

5. Existence

We next proceed with a rigorous proof of existence that will also justify the formal manipulations performed in the derivation of apriori inequalities.

Theorem 5.1. *For every $f_0 \geq 0$, $f_0 \in L^{\frac{1}{2}} \cap L \log L(\mathbb{R}^N)$ there exists a nonnegative weak solution*

$$f \in L^\infty([0, \infty), L^1_2(\mathbb{R}^N)), \quad f \log f \in L^\infty([0, \infty), L^1(\mathbb{R}^N))$$

to Eq. (0.1), with the initial condition $f(\cdot, 0) = f_0$. Furthermore, if in addition $f_0 \in L^1_r \cap L^2(\mathbb{R}^N)$, where $r = \max\{2, \frac{N+2}{4}\}$, then for every $t_0 > 0$,

$$f \in C^\infty_b([t_0, \infty), \mathcal{S}(\mathbb{R}^N)),$$

where C^∞_b denotes the class of functions with bounded derivatives of any order, and \mathcal{S} is the Schwartz class of rapidly decaying smooth functions. In particular, for $t > 0$, f is a classical solution of (0.1).

Theorem 5.2. *For every $\rho > 0$ there exists a nonnegative solution f to (0.2),*

$$f \in \mathcal{S}(\mathbb{R}^N), \quad \text{satisfying} \quad \int_{\mathbb{R}^N} f \, dv = \rho.$$

Furthermore, every nonnegative solution in $L^1_r \cap L^2(\mathbb{R}^N)$, where $r = \max\{2, \frac{N+2}{4}\}$ is in fact in $\mathcal{S}(\mathbb{R}^N)$.

Proof of Theorem 5.1. We assume that the initial datum f_0 is in $C^\infty(\mathbb{R}^N)$ and has compact support (we will remove this assumption in the end of the proof). We also introduce a truncation in the collision term by replacing the factor $|u|$ in (1.6) by

$$|u|_{m,M} = m + \min\{|u|, M\}, \tag{5.1}$$

and $m > 0, M > 0$ are truncation parameters. We then denote by $Q_{m,M}(f, f)$ the corresponding collision operator.

The first step of the proof will be to find approximating solutions which we define using the following truncated problem:

$$\begin{aligned} \partial_t f - \Delta_v f &= Q_{m,M}(f, f), \quad v \in \mathbb{R}^3, \quad t \in [0, T], \\ f(0, v) &= f_0(v), \end{aligned} \tag{5.2}$$

where m, M and T are fixed positive parameters. We will denote by f solutions to (5.2), keeping in mind that they generally depend on m and M .

The solutions will be constructed by applying a fixed point argument to the following approximation scheme:

$$\begin{aligned} \partial_t f - \Delta_v f + Mf &= Q_{m,M}(g, g) + Mg, \quad v \in \mathbb{R}^3, \quad t \in [0, T], \\ f(v, 0) &= f_0(v). \end{aligned} \tag{5.3}$$

Here g is a nonnegative function from $L^\infty([0, T], L^1_2 \cap L^2(\mathbb{R}^N))$, which for every $t > 0$ has unit mass and zero average.

Denoting by h the right hand side of Eq. (5.3) we notice that $h \geq 0$, for every $g \geq 0$, due to the truncation of the kernel. Indeed,

$$h = Q_{m,M}(g, g) + Mg \geq -g(g * |v|_{m,M}) + Mg \geq 0. \tag{5.4}$$

Further, by analogy with Lemma 4.1 we can estimate $Q_{m,M}(g, g)$ as follows:

$$\|Q_{m,M}(g, g)\|_{L^p_k} \leq C_M \|g\|_{L^1_k} \|g\|_{L^p_k}, \quad 1 \leq p \leq \infty \tag{5.5}$$

(there will be no loss of moments since the kernel $B_{m,M}$ is bounded). Therefore,

$$h \in L^\infty([0, T], L^1_2 \cap L^2(\mathbb{R}^N)),$$

as soon as g is in the same space. The unique weak solution $f \in L^\infty([0, T], L^1_2 \cap L^2(\mathbb{R}^N))$ of (5.3) is then obtained from the following integral representation:

$$f(v, t) = f_0(v) * E(v, t) + \int_0^t h(v, \tau) * E(v, t - \tau) \, d\tau, \tag{5.6}$$

where $*$ denotes the convolution in v , and $E(v, t)$ is the fundamental solution of (5.3):

$$E(v, t) = \frac{1}{(4\pi t)^{N/2}} e^{-\frac{|v|^2}{4t} - Mt}.$$

The H^2 regularity of f is then guaranteed by the classical parabolic regularity result [29, Sect. 3.3], and we have the bound

$$\|f\|_{H^2([0, T] \times \mathbb{R}^N)} \leq C_M(\|h\|_{L^2([0, T] \times \mathbb{R}^N)} + \|f_0\|_{H^1(\mathbb{R}^N)}). \tag{5.7}$$

We denote by \mathcal{T} the operator that maps g into f . We next establish that for a certain choice of constants A_1 and A_2 this operator maps the set

$$B = \left\{ f \in L^1([0, T] \times \mathbb{R}^N) \mid f \geq 0, \int_{\mathbb{R}^N} f \, dv = 1, \int_{\mathbb{R}^N} f v \, dv = 0, \right. \\ \left. \int_{\mathbb{R}^N} f |v|^2 \, dv \leq A_1, \int_{\mathbb{R}^N} f^2 \, dv \leq A_2^2, \text{ for a. a. } t \in [0, T] \right\} \tag{5.8}$$

into itself. Indeed, the nonnegativity of f is evident from the integral representation (5.6), since $h \geq 0$. The mass and momentum normalization conditions follow easily, since for $g \in B$ the collision term $Q_{m, M}(g, g)$ integrates to zero when multiplied by 1 or v . It remains to verify the last two conditions in (5.8).

For the first of these conditions, multiplying the Eq. (5.3) by $|v|^2$ and integrating by parts we obtain:

$$\frac{d}{dt} \int_{\mathbb{R}^N} f |v|^2 \, dv + M \int_{\mathbb{R}^N} f |v|^2 \, dv \\ \leq 2N + M \int_{\mathbb{R}^N} g |v|^2 \, dv - k \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} g g_* |v|^2 |v - v_*|_{m, M} \, dv \, dv_* \\ \leq 2N + (M - mk) \int_{\mathbb{R}^N} g |v|^2 \, dv, \tag{5.9}$$

where $k = \epsilon_N(1 - \alpha^2)/4$. Therefore, taking g so that

$$\int_{\mathbb{R}^N} g |v|^2 \, dv \leq A'_1 = \frac{2N}{mk},$$

yields the differential inequality

$$\frac{d}{dt} \int_{\mathbb{R}^N} f |v|^2 \, dv + M \int_{\mathbb{R}^N} f |v|^2 \, dv \leq M A'_1.$$

Then, by Gronwall's lemma,

$$\int_{\mathbb{R}^N} f |v|^2 \, dv \leq \max \left(A'_1, \int_{\mathbb{R}^N} f_0 |v|^2 \, dv \right).$$

Therefore, setting

$$A_1 = \max \left(A'_1, \int_{\mathbb{R}^N} f_0 |v|^2 \, dv \right),$$

we obtain the required estimate.

To obtain a bound of f in L^2 we integrate the equation against f and use the inequality (5.5) to estimate $Q_{m,M}(g, g)$:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} f^2 dv + \int_{\mathbb{R}^N} |\nabla f|^2 dv + M \int_{\mathbb{R}^N} f^2 dv \\ & \leq C_M \|g\|_{L^1} \|g\|_{L^2} \|f\|_{L^2} + M \|g\|_{L^2} \|f\|_{L^2}. \end{aligned} \tag{5.10}$$

By Sobolev’s embedding and interpolation,

$$\|\nabla f\|_{L^2} \geq K \|f\|_{L^{2^*}} \geq K \|f\|_{L^2}^{1/\lambda} \|f\|_{L^1}^{-(1-\lambda)/\lambda},$$

where $0 < \lambda < 1$ is as in (4.16). Therefore, dividing (5.10) by $\|f\|_{L^2}$ and taking into account that $\|g\|_{L^1} = \|f\|_{L^1} = 1$, we get

$$\frac{d}{dt} \|f\|_{L^2} + K^2 \|f\|_{L^2}^{2/\lambda-1} + M \|f\|_{L^2} \leq (C_M + M) \|g\|_{L^2}.$$

Using the inequality

$$K^2 x^{2/\lambda-1} \geq \frac{1}{\varepsilon} x - K_\varepsilon,$$

true for all $\varepsilon > 0$, we find

$$\frac{d}{dt} \|f\|_{L^2} + (1/\varepsilon + M) \|f\|_{L^2} \leq K_\varepsilon + (C_M + M) \|g\|_{L^2}.$$

We then get by Gronwall’s lemma:

$$\|f\|_{L^2} \leq \max(\|f_0\|_{L^2}, \beta + \gamma \|g\|_{L^2}). \tag{5.11}$$

where

$$\beta = \frac{K_\varepsilon}{1/\varepsilon + M} \quad \text{and} \quad \gamma = \frac{C_M + M}{1/\varepsilon + M}. \tag{5.12}$$

Choosing $\varepsilon < 1/C_M$ we get $\gamma < 1$. Therefore, we obtain the inequality $\|f\|_{L^2} \leq A_2$ if we set

$$A_2 = \max(\|f_0\|_{L^2}, \beta/(1 - \gamma)).$$

It is straightforward to verify that the set B is convex and closed in the strong topology of $L^1([0, T] \times \mathbb{R}^N)$, using Fatou’s lemma and the fact that the second moment in $|v|$ is uniformly bounded for $g \in B$. Further, the uniform in time bounds assumed in the definition of B imply the continuity of $Q_{m,M}(g, g)$ in L^1 . We can then deduce easily that the solution operator \mathcal{T} itself is continuous, based on the representation (5.6). Finally, the bound for the second moment and the regularity estimate (5.7) imply that the operator \mathcal{T} maps B into its compact subset. By the Schauder theorem, this proves the existence of a fixed point for \mathcal{T} in B , which is thereby a weak solution $f_{m,M} \in L^\infty([0, T], L^1_2 \cap L^2(\mathbb{R}^N))$ of (5.2).

Our next goal is to pass to the limit as $M \rightarrow \infty$ and then as $m \rightarrow 0$, to recover the solutions with the “hard sphere” collision kernel. To this end, we will show that the bounds set forth in the apriori estimates hold for the fixed point solutions, and are uniform in M (and m). First of all, using the computation (5.9) it is easy to conclude that

the second moment is bounded uniformly in M , as soon as $m > 0$. Indeed, we obtain the following inequality for $f = f_{m,M}$,

$$\frac{d}{dt} \int_{\mathbb{R}^N} f |v|^2 dv \leq 2N - km \int_{\mathbb{R}^N} f |v|^2 dv, \tag{5.13}$$

so the required bound follows by Gronwall’s lemma.

Further, we see that for every $m > 0$, $M > 0$ and for every $T > 0$, the solutions are in $L^\infty([0, T], L^1_{2p}(\mathbb{R}^N))$, for every $p > 1$. To see this we take $K > 0$ and introduce the truncated function

$$\Psi_{p,K}(x) = \begin{cases} x^p, & 0 \leq x < K \\ K^p + p K^{p-1}(x - K), & x \geq K. \end{cases}$$

Then $\Psi_{p,K}(x)$ is convex in x , continuously differentiable, and has a bounded second derivative. It also verifies conditions (3.4)–(3.7), so Lemma 3.2 applies. Taking $\Psi_{p,K}(|v|^2)$ as a test function in the weak form of (5.2) and arguing as in Lemma 3.4, we get

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^N} f \Psi_{p,K}(|v|^2) + p(p-1)k_p m \int \int_{\{|v|^2 + |v_*|^2 \leq K\}} f f_* |v|^{2p} dv dv_* \\ & \leq 2pAM \int_{\mathbb{R}^N} f |v|^2 dv \int_{\mathbb{R}^N} f |v|^{2p-2} dv + 2p(2p-2+N) \int_{\mathbb{R}^N} f |v|^{2p-2} dv. \end{aligned} \tag{5.14}$$

Therefore, if we take $1 < p \leq 2$, we can pass to the limit as $K \rightarrow \infty$ in (5.14) using the monotonicity with respect to K and the bound of f in L^1_{2p} . This implies

$$f \in L^\infty([0, T], L^1_{2p}(\mathbb{R}^N)),$$

for every $1 < p \leq 2$, with bounds generally dependent on m and M . By induction, the same property is extended to every $p \geq 0$.

We see further that the bounds in L^1_{2p} are in fact independent on M . Indeed, estimating the middle term in (5.14) using the inequality

$$|v - v_*|_{m,M} \leq m + |v - v_*|$$

and following the arguments of Lemma 3.4 we get

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^N} f |v|^{2p} dv + p(p-1)k_p m \int_{\mathbb{R}^N} f |v|^{2p} dv \\ & \leq A_p \int_{\mathbb{R}^N} f |v| dv \int_{\mathbb{R}^N} f |v|^{2p} dv \\ & \quad + \left(pm \int_{\mathbb{R}^N} f |v|^2 dv + 2p(2p-2+N) \right) \int_{\mathbb{R}^N} f |v|^{2p-2} dv. \end{aligned} \tag{5.15}$$

This implies that for every $T > 0$ fixed and every $p \geq 0$, the bounds of $f = f_{m,M}$ in $L^\infty([0, T], L^1_{2p}(\mathbb{R}^N))$ are independent of M .

Using the established L^1_{2p} bounds and the fact that $f \in H^2([0, T] \times \mathbb{R}^N)$ we can make rigorous the arguments of Lemma 4.7 and then proceed as in Lemmas 4.8-4.9 obtaining that

$$f \in L^\infty([0, T], H^{2p}_r(\mathbb{R}^N)),$$

for every $n = 1, 2, \dots$, and every $p \geq 0$, with bounds independent on M . This will allow us to pass to the limit as $M \rightarrow \infty$ in the weak form and to show that the limit solutions satisfy the equation with the kernel

$$(m + |u|) b(u, \sigma).$$

We can then substitute the computation (5.15) by the argument of Lemma 3.4 and find the bounds in $L^\infty([0, T], L^1_{2p}(\mathbb{R}^N))$ that are independent on m and T . Arguing as above we can then pass to the limit as $m \rightarrow 0$. The limit solution obtained in this step will then satisfy the equation with the “hard sphere” kernel.

Finally, in order to treat the problem with the initial data $f_0 \in L^1_2 \cap L \log L(\mathbb{R}^N)$ we can take a sequence $f^n \in C^\infty_0(\mathbb{R}^N)$ that converges to f_0 in $L^1(\mathbb{R}^N)$. Then, since the constants in the bounds for the energy and entropy from Sect. 2 are independent of n , we can pass to the weak L^1 -limit in the equations. The fact that the bounds of the solutions are independent of T allows us to continue the obtained solutions to $[0, 2T]$, and by induction, to $[0, \infty)$.

To study the regularity of solutions with L^2 initial data we use the parabolic regularity of the equation [29] to find that $f \in H^1([0, T] \times \mathbb{R}^N)$, for any $T > 0$. Using this fact in combination with the bound $f \in L^\infty([0, \infty), L^1_r(\mathbb{R}^N))$ we can make rigorous the argument of Lemma 4.7 and then proceed as in Lemmas 4.8 and 4.9 to find the infinite differentiability of the solutions. \square

We now turn our attention to the steady equation (0.2) and give a proof of Theorem 5.2. One of the possible approaches consists of adapting the arguments developed above for the time-dependent case. In fact, as a careful reader will easily check, practically all arguments in the above proof apply to the steady equation: the Gronwall lemma arguments will be replaced by the inequalities obtained by dropping the time-derivative terms. The only point that would need more careful attention is the moment estimate (5.15), which is *not* uniform in T . It can be replaced by a more elaborate argument for the moment bounds in the case of the truncated collision kernel. We will, however, take another approach, which will allow us to obtain the existence of the steady problem as a consequence of the regularization properties of the time-dependent equation.

Proof of Theorem 5.2. The proof of Theorem 5.1 enables us to construct a semigroup on the convex set C made of those functions in $L^1_2 \cap L^2(\mathbb{R}^N)$ with unit mass and zero mean. Denote it by $(S_t)_{t \geq 0}$. Our bounds imply that for all $t > 0$, the range of S_t is compact in C . Therefore, for all n the equation

$$f_n = S_{2^{-n}} f_n$$

is solvable by Schauder’s theorem. Since $f_n = S_1 f_n$, the sequence f_n is contained in a fixed compact of C , namely $S_1(C)$. We can therefore extract a subsequence which converges towards some f . Now for all $k \leq n$ we have

$$f_n = S_{2^{-k}} f_n$$

(because 2^{-k} is a multiple of 2^{-n}), and we can pass to the limit as $n \rightarrow \infty$ using the continuity of the semigroup, thereby obtaining

$$f = S_{2^{-k}} f, \quad \text{for all } k \geq 0.$$

Therefore $f = S_t f$ for every t which is a sum of inverse powers of 2. Since the set of such times forms a dense subset of \mathbb{R}_+ and since the semigroup is continuous with respect to t , we conclude that

$$f = S_t f, \quad \text{for all } t \geq 0.$$

This ends the proof. \square

6. Uniqueness by Gronwall’s Lemma

We next show that under the assumption that the initial data has the moment of order 3 finite, the solution to the time-dependent problem is unique. The proof uses an argument based on a certain cancellation property of the collision operator multiplied by $\text{sgn}(f)$ [1, 14] (see also [32] for discussion). We show that this property yields the desired result for the operator with inelastic collisions as well.

Theorem 6.1. *Assume that $f_0 \in L^1_3(\mathbb{R}^N)$; then Eq. (0.1) with the initial condition $f(\cdot, 0) = f_0$ has at most one solution.*

Proof. Assume that f and g are solutions of (0.1), with the same initial data f_0 . Set $h = f - g$ and $H = f + g$. Then h satisfies the equation

$$\partial_t h - \Delta_v h = \frac{1}{2} (Q(h, H) + Q(H, h)), \tag{6.1}$$

with the homogeneous initial data. Now take a function $\psi_\varepsilon(x)$, a continuous approximation of $\text{sgn}(x)$. We can take

$$\psi_\varepsilon(x) = \begin{cases} -1, & x \leq -\varepsilon \\ x/\varepsilon, & -\varepsilon < x \leq \varepsilon \\ 1, & x > \varepsilon. \end{cases}$$

Multiplying Eq. (6.1) by $\psi_\varepsilon(h) (1 + |v|^2)$ and integrating by parts we get

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^N} h \psi_\varepsilon(h) (1 + |v|^2) dv + \frac{1}{2\varepsilon} \int_{\{|h| \leq \varepsilon\}} |\nabla h|^2 (1 + |v|^2) dv \\ - 2N \int_{\mathbb{R}^N} \phi_\varepsilon(h) dv = \frac{1}{2} \int_{\mathbb{R}^N} (Q(h, H) + Q(H, h)) \psi_\varepsilon(h) (1 + |v|^2) dv, \end{aligned}$$

where

$$\phi_\varepsilon(x) = \int_0^x \psi(t) dt = \begin{cases} -x + \varepsilon/2, & x \leq -\varepsilon \\ x^2/2\varepsilon, & -\varepsilon < x \leq \varepsilon \\ x - \varepsilon/2, & x > \varepsilon. \end{cases}$$

To estimate the right-hand side we can adapt the argument that is known to work in the case of the elastic Boltzmann equation (cf. [1, 14]). Passing to the weak form we get:

$$\begin{aligned} & \int_{\mathbb{R}^N} (Q(h, H) + Q(H, h)) \psi_\varepsilon(h) (1 + |v|^2) dv \\ &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{S^{N-1}} H_* \{h \psi_\varepsilon(h') (1 + |v'|^2) + h \psi_\varepsilon(h'_*) (1 + |v'_*|^2) \\ & \quad - h \psi_\varepsilon(h) (1 + |v|^2) - h \psi_\varepsilon(h_*) (1 + |v_*|^2)\} |u| b(u, \sigma) d\sigma dv dv_*. \end{aligned} \tag{6.2}$$

Since $|v'|^2 + |v'_*|^2 \leq |v|^2 + |v_*|^2$, we can estimate the integrals of the first two terms in the braces as follows:

$$\begin{aligned} & (2 + |v|^2 + |v_*|^2 - c_N \frac{1 - \alpha^2}{4} |v - v_*|^2) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |h| H_* \\ & \quad \times (2 + |v|^2 + |v_*|^2) |v - v_*| dv dv_*. \end{aligned}$$

Subtracting the third term in the integral (6.2) and noticing that

$$|h \psi_\varepsilon(h) - |h|| \leq |h| \chi_\varepsilon(h),$$

where $\chi_\varepsilon(x)$ is the characteristic function of the interval $[-\varepsilon, \varepsilon]$, we obtain the estimate for the first three terms:

$$\begin{aligned} & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |h| H_* (1 + |v_*|^2) |v - v_*| dv dv_* \\ & \quad + \int_{\mathbb{R}^N} H_* \int_{\{|h| \leq \varepsilon\}} h (1 + |v|^2) |v - v_*| dv dv_*. \end{aligned}$$

The fourth term in (6.2) contributes with another integral like the first one above, so we finally get

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^N} h \psi_\varepsilon(h) (1 + |v|^2) dv \\ & \leq 2N \int_{\mathbb{R}^N} \phi_\varepsilon(h) dv + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |h| H_* (1 + |v_*|^2) |v - v_*| dv dv_* \\ & \quad + \frac{1}{2} \int_{\mathbb{R}^N} H_* \int_{\{|h| \leq \varepsilon\}} h (1 + |v|^2) |v - v_*| dv dv_*. \end{aligned}$$

Passing to the limit as $\varepsilon \rightarrow 0$, we find:

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^N} |h| (1 + |v|^2) dv \\ & \leq 2N \int_{\mathbb{R}^N} |h| dv + \int_{\mathbb{R}^N} |h| (1 + |v|^2)^{1/2} dv \int_{\mathbb{R}^N} H (1 + |v|^2)^{3/2} dv \\ & \leq C \int_{\mathbb{R}^N} |h| (1 + |v|^2) dv, \end{aligned}$$

since H is assumed to be bounded in $L^1_3(\mathbb{R}^N)$. Now since $h(0, v) = 0$ it follows by Gronwall's lemma that $h(t, v) = 0$ for all times. \square

Remark. The uniqueness result of Theorem 6.1 is most certainly suboptimal. We believe that the uniqueness could be obtained in the class of initial conditions with finite mass and energy, with no additional assumptions, similarly to the classical Boltzmann equation [32]. The main technical obstacle for such a result is extending the Povzner inequalities in the case of inelastic collisions to the class of slowly growing piecewise linear functions ψ studied in [32]. We believe that this can be overcome with a more careful analysis of the inelastic collision mechanism.

7. Lower Bounds with Overpopulated High Energy Tails

In this section we obtain pointwise lower estimates of solutions to (0.1) and (0.2) showing that the behavior of the high-energy tails of solutions is controlled from below by “stretched exponentials” $A \exp(-a|v|^{3/2})$. The bounds are established by using the comparison principle based on the parabolic (elliptic) structure of the equations. The following proposition establishes the particular role played by the “stretched exponentials”: they can be used as barrier functions in the comparison principle.

Proposition 7.1. *Let $g(v)$ be a nonnegative function with finite mass $\rho_0 = \int_{\mathbb{R}^N} g \, dv$ and moment of order one, $\rho_1 = \int_{\mathbb{R}^N} g |v| \, dv$. Then for every $r > 0$, and every $K > 0$, there is a constant $a > 0$ such that the function*

$$h(v) = K e^{-a|v|^{3/2}},$$

satisfies

$$\Delta h - Q^-(g, h) \geq 0, \quad \text{for all } |v| > r. \tag{7.1}$$

Further, choosing $b > 0$ large enough, the function

$$h(v, t) = K e^{-bt - a(1+|v|^2)^{3/4}}$$

satisfies

$$-\partial_t h + \Delta h - Q^-(g, h) \geq 0, \tag{7.2}$$

for all $t > 0$ and all $v \in \mathbb{R}^N$.

Proof. To prove inequality (7.1) we fix an $r > 0$, compute

$$\Delta h = \left(\frac{9}{4} a^2 |v| - \frac{3(2N-1)}{4} a |v|^{-1/2} \right) h,$$

and use the estimate

$$Q^-(g, h) = h (g * |v|) \leq (\rho_1 + \rho_0 |v|) h,$$

to obtain

$$\Delta h - Q(g, h) \geq \left(\left(\frac{9}{4} a^2 - \rho_0 \right) |v| - \rho_1 - \frac{3(2N-1)}{4} a |v|^{-1/2} \right) h. \tag{7.3}$$

If $\frac{9}{4}a^2 \geq \rho_0$, the factor on right-hand side of (7.3) attains its minimum for $|v| = r$. Therefore, inequality (7.1) holds for every $a \geq a^*$, where a^* is the positive root of the quadratic equation

$$\frac{9r}{4}a^2 - \frac{3(2N-1)r^{-1/2}}{4}a - (\rho_0r + \rho_1) = 0.$$

For the time-dependent operator, setting $\langle v \rangle = (1 + |v|^2)^{\frac{1}{2}}$, we obtain

$$\Delta h - \partial_t h = \left(\frac{3a}{4} |v|^2 \langle v \rangle^{-1} (3a + \langle v \rangle^{-1/2}) - \frac{3Na}{2} \langle v \rangle^{-1/2} + b \right) h. \quad (7.4)$$

Choosing a so that $\frac{9}{4}a^2 \geq \rho_0$ and then $b \geq \frac{3Na}{2} + \rho_0 + \rho_1$ we obtain inequality (7.2) and complete the proof. \square

The established property of the function $h(v)$ is used in next lemma to obtain a comparison result for the steady equation.

Lemma 7.2. *Let $f \in L^1_1(\mathbb{R}^N)$ be a nonnegative smooth solution to (0.2), with the mass $\rho_0 > 0$. Then, there is a constant $K > 0$, such that*

$$f(v) \geq Ke^{-2a|v|^{3/2}}, \quad (7.5)$$

for all $v \in \mathbb{R}^N$, where a is a constant as in Proposition 7.1.

Proof. Assuming the smoothness of the solution to (0.2), there is a constant $c_0 > 0$ and a ball $B(v_0, r_0)$ with $v_0 \in \mathbb{R}^N$ and $r_0 > 0$, such that

$$f(v) \geq c_0 > 0, \quad \text{if } v \in B(v_0, r_0). \quad (7.6)$$

The value of c_0 (as well as r_0 and v_0) depend on the solution f and use the fact that $\rho_0 > 0$.

Since Eq. (0.2) is translation invariant, we can take $g(v) = f(v + v_0)$; then

$$\Delta g - Q^-(g, g) = \Delta g - (g * |v|)g \leq 0. \quad (7.7)$$

Applying Proposition 7.1 to the function $g(v)$ with $r = r_0$ we find the barrier function $h(v) = c_0 \exp(-a|v|^{3/2})$, for which we have

$$\Delta h - Q^-(g, h) = \Delta h - (g * |v|)h \geq 0, \quad \text{for } |v| > r_0 \quad (7.8)$$

and

$$g(v) \geq h(v), \quad \text{for } |v| \leq r_0.$$

Therefore, letting $U(v) = g(v) - h(v)$, subtracting (7.8) from (7.7) we obtain the inequality

$$\Delta U - (g * |v|)U \leq 0, \quad |v| > r_0.$$

To prove that $U(v) \geq 0$ everywhere we apply a form of a strong maximum principle (see, for example, [19]) to the operator

$$\mathcal{L}U = \Delta U - v(U + h)U.$$

We can reduce the problem to proving that $U \geq 0$ in a bounded domain. Indeed, the decay conditions on f imply that for every $\varepsilon > 0$ we can find $R > 0$ such that $|U(v)| < \varepsilon$ if $|v| \geq R$. Then we have

$$\mathcal{L}(U + \varepsilon) = \mathcal{L}U - \varepsilon v(g) \leq 0, \quad r_0 < |v| < R$$

and $U + \varepsilon > 0$ for $|v| = r_0$ and $|v| = R$. The strong maximum principle then implies that $U + \varepsilon \geq 0$ for all $r_0 \leq |v| \leq R$. Letting ε go to zero we get

$$U \geq 0, \quad \text{for all } |v| \geq r_0.$$

In view of the inequality (7.6) this implies

$$g(v) \geq c_0 e^{-a|v|^{3/2}},$$

or, applied to the function $f(v)$,

$$f(v) \geq c_0 e^{-a|v-v_0|^{3/2}} \geq K e^{-2a|v|^{3/2}},$$

with $K = c_0 e^{-a|v_0|^{3/2}}$. This completes the proof of the lemma. \square

By using a version of the maximum principle for the parabolic operator, we obtain, in a similar fashion, the pointwise lower bound for the time dependent problem.

Lemma 7.3. *Let $f \in L^\infty([0, \infty), L^1_2(\mathbb{R}^N))$ be a nonnegative smooth solution to (0.1) with the initial data $f_0 \geq c_0 \exp(-a_0|v|^{3/2})$. Then, there are positive constants K , b and a , generally depending on the solution, such that*

$$f(v, t) \geq K e^{-bt - a|v|^{3/2}},$$

for all $t > 0$ and all $v \in \mathbb{R}^N$. Further, if there is a constant c_1 and a ball $B(v_0, r_0)$, such that

$$f(v, t) \geq c_1, \quad \text{if } v \in B(v_0, r_0),$$

for all t , then the lower bound

$$f(v, t) \geq K e^{-a|v|^{3/2}},$$

holds uniformly in time, where now $K > 0$, $a > 0$ and $b > 0$ will depend on c_1 , v_0 and r_0 .

Proof. To prove the first statement of the lemma we use the second part of Proposition 7.1 and repeat the comparison arguments of Lemma 7.2 taking

$$h(v, t) = K e^{-bt - a(v)^{3/2}}$$

and using the strong maximum principle for the parabolic operator on $U = f(v + v_0) - h(v)$,

$$\mathcal{L}U = \Delta U - v(f)U - \partial_t U.$$

For the second part, the additional assumption made on f allows us to repeat the proof of Lemma 7.2 using the function h from (7.4). \square

Remark. It is natural to conjecture that solutions to (0.2) should satisfy a pointwise upper bound of the type $K' \exp(-a'|v|^{3/2})$, for certain values of a' and K' . A recent result by Bobylev *et al.* [7] shows that such a bound holds “in the L^1 sense”, namely, for certain $b > 0$,

$$\int_{\mathbb{R}^N} f(v) \exp(b|v|^{3/2}) dv < +\infty.$$

Further, an application of the maximum principle formulated in [39] allows one to reduce the problem to obtaining an estimate of the type

$$Q^+(f, h) = o(Q^-(f, h)), \quad |v| \rightarrow \infty,$$

where f is a solution of (0.2) and $h = K \exp(-a'|v|^{3/2})$. We will return to this problem in a forthcoming paper [18].

8. Concluding Remarks

We studied the existence, uniqueness and regularity for the time-dependent equation (0.1) and the existence, regularity for the steady equation (0.2). An important problem that remained beyond the scope of our study is the convergence of the time-dependent solutions to the steady ones as time approaches infinity. In fact, this remains a serious open problem, since no Lyapunov functional for the time-evolution is known to exist. A number of other interesting questions can be raised in connection to the obtained results. Are the steady states unique up to a normalization? Do the steady solutions necessarily have radial symmetry? (This can be expected from the rotation invariance of the equations; the existence and regularity of radial solutions can be obtained by applying our analysis to the reduced one-dimensional problem, as in [11], or just by working in spaces of radially symmetric functions.)

We hope that the methods developed in the present work for the case of diffusion forcing could be useful for studying other problems involving the Boltzmann (Enskog) collision terms with other collision and driving mechanisms. In particular, a generalization to the case of a heat bath including a friction term seems to be rather straightforward. (The lower bounds in that case are expected to be Maxwellians.) It is also likely that applying the techniques of this paper should yield results for problems with the normal restitution coefficient dependent on the relative velocity [3, 8], which would allow us to study a broader range of physical phenomena.

Another problem worth studying is the (quasi-)elastic limit $\alpha \rightarrow 1$. The steady states for the Boltzmann equation with elastic interactions ($\alpha = 1$) and vanishing diffusion ($\mu = 0$) are Maxwellians, while for every $\mu > 0$ and every $\alpha < 1$ we have a “3/2” lower bound. Obtaining quantitative information on the transition to the Maxwellian steady states would be valuable. We hope to address some of these questions in our future work.

Acknowledgements. A number of people have contributed in a very important way to the development of the present paper. We would like to thank C. Bizon, S. J. Moon, J. Swift and H. Swinney for discussions concerning the physical aspects of the problem. We are also thankful to A. V. Bobylev, J. A. Carrillo, C. Cercignani and S. Rjasanow for fruitful discussions and a number of suggestions that helped us to improve the presentation. The first author has been supported by NSF under grant DMS 9971779 and by TARP under grant 003658-0459-1999; the third author has been supported by the HYKE European network, contract HPRN-CT-2002-00282. Support from the Texas Institute for Computational and Applied Mathematics/Austin is also gratefully acknowledged.

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Communicated by H.-T. Yau