A Viscous Approximation for a 2-D Steady Semiconductor or Transonic Gas Dynamic Flow: Existence Theorem for Potential Flow

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Abstract

In this paper we solve a boundary value problem in a two-dimensional domain Ω for a system of equations of Fluid-Poisson type, that is, a viscous approximation to a potential equation for the velocity coupled with an ordinary differential equation along the streamlines for the density and a Poisson equation for the electric field. A particular case of this system is a viscous approximation of transonic flow models. The general case is a model for semiconductors.

We show existence of a density ρ, velocity potential φ, and electric potential Φ in the bounded domain Ω that are $C^{1,\alpha}(\bar{\Omega})$, $C^{2,\alpha}(\bar{\Omega})$, and $W^{2,p}(\bar{\Omega})$ functions, respectively, such that ρ, φ, Φ, the speed $|\nabla\varphi|$, and the electric field $E = \nabla\Phi$ are uniformly bounded in the viscous parameter. This is a necessary step in the existing programs in order to show existence of a solution for the transonic flow problem. © 1996 John Wiley & Sons, Inc.

1. Presentation of the Model

1.1. The Semiconductor

There are many models for semiconductors. They include the quantum, kinetic, and fluid-level formulation.

The most accurate is the quantum, in which the particles are represented by wave functions solving the Schrödinger equations with a Hamiltonian that incorporates potentials due to the semiconductor lattice, Coulomb interactions, the applied bias, and particle-phonon interactions. Although such models have begun to be used for numerical simulations of small parts of the semiconductor devices where quantum effects are important, up to now it has been impossible to formulate a model for a complete device at this level.

Then there is the kinetic level of modeling and the application of Monte Carlo methods at the particle level, where the dynamics of the particles is described by the evolution of distribution functions for electrons and holes, respectively, both depending on time, position, and wave vector, where the last belongs to a periodic lattice in $\mathbb{R}^3$. This evolution is dictated by a semiclassical Boltzmann equation that incorporates the electric field and collision effects. These models are presented in Markovich and others [30]; lately there have been some analytical

Communications on Pure and Applied Mathematics, Vol. XLIX, 999–1049 (1996)
 CCC 0010-3640/96/100999-51

results regarding existence of solutions for some boundary value problems (see Poupaud [35]), but these models remain costly as numerical solutions.

Finally, there is the fluid level of modeling. First is the one based on the drift diffusion equations of parabolic type for the concentration of both carriers (electrons and holes). These models can be rigorously derived from the kinetic formulation under the assumption of low electric fields (see Golse and Poupaud [22]) and have been extensively studied from the mathematical and numerical point of view (see Jerome [24] and Markowich, Ringhofer, and Schmeiser [30] and references therein.) They give very good results for components whose typical length is of the order of a micron, but they do not seem to be valid for submicron devices or for high electric fields. Therefore, more sophisticated models have been introduced. They are based on electrohydrodynamic equations (also called "energy-balanced equations" or "extended drift diffusion models"), which are intended to take into account high field effects. These models have been obtained by closing the moment equations derived from the Boltzmann equation with a phenomenological assumption on the distribution function. The distribution is assumed to be isotropic around its mean velocity (see, for instance, Blotekjaer [4], Azoff [2], Bringer and Schön [5], and Gnudi, Odeh, and Rudan [21]). Then fluid equations are obtained with source terms modeling relaxation processes and electric field effects coupled with the Poisson equation. Some recent numerical work includes papers by Z. Chen and others [6] in two space dimensions and by Gardner ([16], [17]), Gardner, Jerome, and Shu [18], and Fatemi and others [10] in one space dimension.

Analysis addressing issues of existence, uniform bounds, and boundary layer formation for steady state, one-dimensional transonic models have been studied by Gamba ([12], [14], [15]). Ascher and others [1] presented a phase plane analysis for some special models. The isothermal, one-space-dimension, time-dependent inviscid problem has recently been analyzed by Marcati and Natalini [28], Poupaud, Rascle, and Vila [36], and Zhang [40]. In addition, Cordier and others [7] analyzed traveling waves and jump relations for an Euler-Poisson model in the quasineutral limit. Markowich [29] treated a two-dimensional, steady Euler-Poisson system in subsonic regimes under a smallness assumption of the prescribed outflow velocity (small boundary current) and under a smallness assumption of the variation of the velocity relaxation time. Degond and Markowich [8] considered a three-dimensional, potential inviscid flow model, where they proved existence and uniqueness results in a bounded domain for small Dirichlet data. Both papers deal with systems of equations that remain essentially elliptic under assumptions on the size of either the data or of the parameters under consideration.

The present paper deals with a steady state, two-dimensional fluid-level model that is an approximation to inviscid potential flow that changes type. Viscosity and friction are added, and existence of steady solutions is established for geometries corresponding to semiconductor devices.

Here, we "model a model" by tailoring the steady state electrohydrodynamic model into a solvable problem. By "solvable" we mean we can establish an existence theorem for a solution, and the implication of the theorem is that the solution can be computed. The "fixed up" model has potential gas flow into certain semiconductor regimes as a special case, just as potential transonic flow did for the full fluid equations with small viscous effects.

1.2. Transonic Flow Equations

If no electric field is present, the problem studied is a two-dimensional, steady, irrotational, compressible flow in a "channel." Here only viscosity is added. This problem becomes just a special case of the semiconductor problem.

We begin with the two-dimensional, steady state conservation laws for mass and momentum and couple them to the Poisson equation for the electric field. The principal variables are charge density ρ, velocity $\vec{u}$, pressure P, and electric field $\nabla\Phi$, where Φ is the electric potential. Thus, with $x \in \mathbb{R}^2$ the conservation laws are

$$\operatorname{div} \rho\vec{u} = 0 \tag{1.1a}$$

$$m\rho(\vec{u} \cdot \nabla)\vec{u} + \nabla P = q\rho\nabla \cdot \Phi + \vec{F} \tag{1.1b}$$

and the Poisson equation is

$$\varepsilon\,\Delta\Phi = q(\rho - C(x)). \tag{1.2}$$

The extra parameters m, q, and ε are, respectively, electron effective mass (parabolic band approximation, see [4]), the space charge constant, and the dielectric constant. The vector function $\vec{F}$ represents forces caused by viscosity or friction: We will choose $\vec{F}$ later to give us a solvable set of equations that is as consistent with the physics as has proved possible and is nowhere at wild variance with the physical problem. $C(x)$ is the doping profile function and represents the background charge. It is assumed $C(x)$ is a nonnegative step function.

In the transonic problem, $\Phi \equiv 0$ and (1.2) does not appear. The conservation of energy is replaced by taking the pressure P as a given function of $m\rho$, the mass density. This is consistent with the notion that the disturbances we are studying are weak. For example—and for simplicity—suppose we consider the fluid dynamical case with $\vec{F} = 0$ and suppose $P = P(\rho, S)$ where S is entropy. The full conservation laws then admit shocks, but a change in entropy across a shock would be third order in the strength of the shock. Thus we may take $P = P(\rho)$ provided third-order errors may be neglected.

1.3. Potential Flow and the Choice of the Viscous-Friction Force Term $\vec{F}$

If $\vec{F} \equiv 0$, we could look for an irrotational flow, $\operatorname{curl} \vec{u} = 0$, so that $\vec{u} = \nabla\varphi$ where φ is the potential and reduce the system to conservation of mass, a Bernoulli law, and Poisson's equation. An irrotational flow is again consistent with the neglect of third-order terms. What is more, it is mathematically useful because the system is reduced to fourth order (second order if $\Phi \equiv 0$ and weakly coupled if $\Phi \neq 0$). Henceforth we assume $\vec{u} = \nabla\varphi$ and choose $\vec{F}$ so that we can make a similar reduction in order. Then we check the physical consistency of this choice of $\vec{F}$. Thus a preliminary choice would be

$$\text{(1.3)} \qquad \begin{aligned} \vec{F} &= \nabla\Psi \\ \Psi &= \nu\Delta\varphi - K\varphi \end{aligned}$$

where ν is a coefficient of viscosity and K a coefficient of friction. Then with $\vec{u} = \nabla\varphi$ the momentum equations (1.1b) can be integrated to yield Bernoulli's law:

$$\frac{1}{2}m|\nabla\varphi|^2 + i(\rho) = -q\Phi + \nu\Delta\varphi + K\varphi + \text{const}$$

Here $i(\rho)$ is the enthalpy; that is, $i(\rho) = \int \frac{dP(\rho)}{\rho}$ where

$$\text{(1.4)} \qquad \begin{cases} i(\rho) \geqq 0 & \text{if } \rho \geqq 0 \\ i(\rho) = 0 & \text{otherwise,} \end{cases}$$

and $i'(\rho) > 0$ if $\rho > 0$. Furthermore, we assume either $P(\rho)$ is convex or $i''(\rho)$ and $i'''(\rho)$ are nonnegative. We also assume in this paper that the constant K is independent of ν and is adjusted by the data in order to be consistent with the inviscid problem (i.e., with $\nu = 0$).

The first term in $\vec{F}$ is $\nu\rho\Delta\vec{u}$, which is the appropriate viscous force, and the second is $\tau_p^{-1}m\rho\vec{u}$, which represents a friction term. The viscous term, with ν a constant coefficient of viscosity, is essentially the same as that of Serrin [37] or Synge [38]. The friction term is, as in Baccarani and Woderman [3] and Gnudi, Odeh, and Rudan [21], the momentum density divided by a constant-velocity relaxation time τ_p.

However, we cannot solve the boundary value problem with this choice of $\vec{F}$. To solve the system the viscous term in Ψ, that is, $\nu\Delta\varphi$, must be modified to go to infinity at zero speed like $|\nabla\varphi|^2$ and to zero at infinite speed like $|\nabla\varphi|^p$, $p \geqq 3$. For similar reasons τ_p must become infinite as $\varphi \to \pm\infty$. Thus the force term $\vec{F}$ is given by

$$\text{(1.5)} \qquad \vec{F} = \nabla\Psi, \qquad \Psi = \nu\mathscr{G}\Delta\varphi - \mathscr{R}(\varphi),$$

where $\mathscr{G}$ and $\mathscr{R}$ have the desired properties.

The case of interest is low viscosity, ν small. It turns out that in our boundary value problem, $|\nabla\varphi|$ and φ are uniformly bounded. In the range of speed and potential that occur, $\mathscr{G}(|\nabla\varphi|) \sim 1$ (except near zero speed) and $\mathscr{R}(\varphi)$ is linear. Thus the restrictions on viscous and friction coefficients have little effect on the physical interpretation of the result.

Using (1.3) for the force term, Bernoulli's law is found by integrating (1.1b) and using (1.5):

$$\text{(1.6)} \qquad \frac{m}{2}|\nabla\varphi|^2 + i(\rho) = K + q\Phi - \mathscr{R}(\varphi) + \nu\mathscr{G}\,\Delta\varphi.$$

The conservation of mass (1.1a) becomes for potential flow

$$\operatorname{div} \rho \nabla \varphi = 0 . \tag{1.7a}$$

From (1.1a) it is also possible to define a stream function ψ whose level curves describe the particle path at any point and are orthogonal to the corresponding level curves of the potential flow φ, that is, with $\rho\vec{u} = (-\psi_y, \psi_x)$,

$$\operatorname{div}(\tau \nabla \psi) = 0 \tag{1.7b}$$

where $\tau = \rho^{-1}$.

Remark: The transonic gas flow model ($\Phi \equiv 0$, $\mathcal{R}(\varphi) \equiv 0$, $\nu = 0$) still remains unsolved. Different methods have been proposed to attack that problem. Morawetz [31] proposed solving a viscous approximating system and adapted some techniques developed by DiPerna [9], Murat [33], and Tartar [39] for the use of compensated compactness in order to obtain a limit that would solve the inviscid model.

This motivated us to seek a boundary value problem for which we could prove existence and also establish the conditions that could eventually lead to the existence of a corresponding limiting solution as the parameter ν becomes very small with respect to the other constants involved in the model, that is, the existence of a corresponding limiting inviscid solution as proposed by Morawetz [31] in order to solve the transonic gas flow model. Such a program cannot be carried out without better bounds than we get for the state variables ρ and $\vec{u} = \nabla\varphi$ (see [32]).

In this paper we focus on the existence of solutions for the "viscous" potential fluid–Poisson system (1.6)–(1.7) and (1.2) in "smooth" four-sided domains and with boundary conditions relevant to the physical problem (to be described later). We prove existence of a regular solution such that the density and speed are uniformly bounded in ν, away from zero and infinity, and the electric field is also uniformly bounded in ν. Uniform bounds will be discussed in Section 4.

A physically more correct one-dimensional system (1.1)–(1.2) was solved in [12]. There it is proved that the solution ρ, v, and Φ is of uniformly bounded variation and that there exists a limiting inviscid solution ρ^0, v^0, and Φ^0 of bounded variation such that the convergence from the viscous solution to the inviscid one is pointwise and in $L^1(\Omega)$. In addition, ρ^0 and v^0 might have admissible shocks, that is, discontinuities that keep the mass and momentum flux terms as Lipschitz continuous functions that satisfy the "entropy condition"–that is, the density increases across the discontinuity in the direction of the particle path. In addition, the possible formation of boundary layers was analyzed in [12] and [14].

Remark: Setting $\nu = 0$ in (1.6) we obtain a transonic type of equation for the inviscid model. Indeed, taking $i(\rho) = K + q\Phi - \mathcal{R}(\varphi) - \frac{m}{2}|\nabla\varphi|^2$, defining $c^2 = i'(\rho)\rho = \frac{dP}{d\rho}$ as the local speed of sound, and combining with the flow equation (1.5), the potential flow function φ satisfies the PDE

$$(c^2 - m\varphi_x^2)\varphi_{xx} - 2m\varphi_x\varphi_y\varphi_{xy} + (c^2 - m\varphi_y^2)\varphi_{yy} = S(x, y, \varphi, \varphi_x, \varphi_y, \rho, \Phi, \nabla\Phi),$$

which is a mixed-type equation for φ, that is, an elliptic equation for speed values below $cm^{-1/2}$ and hyperbolic for speed values above $cm^{-1/2}$.

1.4. Further Conditions

The function R is a Lipschitz function of φ, and $\mathscr{R}(\varphi)$ is bounded above and below by constants R_U and R_L, respectively. The function $\mathscr{G}$ of the "viscous" term $\nu\mathscr{G}\Delta\varphi$ is taken as $\mathscr{G} = G(|\nabla\varphi|)f_1(\theta)$ where G and G^{-1} are locally Lipschitz functions and satisfy

$$t^2 G(t) \to C_1 > 0 \quad \text{as } t \to 0 \tag{1.8a}$$

and

$$t^{-1}G(t) \to C_2 > 0 \quad \text{as } t \to \infty \qquad \text{and} \qquad t^2\frac{G'(t)}{G^2(t)} \text{ bounded} \tag{1.8b}$$

with $\theta = \arg(\varphi_x - i\varphi_y)$. We point out here that conditions (1.8) are related to the existence problem as well as the problem of getting uniform bounds. We shall explain these facts in Sections 2, 3, and 4 in more detail.

However, since we prove that $|\nabla\varphi^\nu|$ is uniformly bounded above by a number M that depends only on the domain and the data of the boundary value problem for any $\nu \leqq \nu_0$, with ν_0 depending also on the domain and the data of the problem, a posteriori we get an existence result and upper uniform bounds even if we take a viscosity that is linear away from stagnation points, that is, by setting

$$G(|\nabla\varphi|) = 1 \quad \text{for } 0 < k_\nu < |\nabla\varphi| \leqq M^{\frac{1}{2}}$$

and extending it smoothly to $[0, \infty]$ such that growth conditions (1.8a) and (1.8b) are satisfied. This is desirable for physical reasons.

By equation (1.7a), introducing $\frac{\partial}{\partial\varphi}$ as a derivative along the streamline,

$$\Delta\varphi = -\frac{\nabla\rho}{\rho}\cdot\nabla\varphi = -(\ln\rho)_\varphi|\nabla\varphi|^2. \tag{1.9}$$

Using (1.9) in (1.6), we obtain from (1.2) and (1.5) the complete system of equations for φ, ρ, and Φ. We add the equation in ψ as a useful convenience.

$$\operatorname{div}(\rho\nabla\varphi) = 0, \qquad \operatorname{div}(\tau\nabla\psi) = 0, \tag{1.10a}$$

$$\frac{m}{2}|\nabla\varphi|^2 + i(\rho) + \mathscr{R}(\varphi) - q\Phi - K + \nu(\ln\rho)_\varphi|\nabla\varphi|^2\mathscr{G} = 0, \tag{1.10b}$$

$$\Delta\Phi = \alpha(-C(x) + \rho), \qquad \alpha = \frac{q}{\varepsilon}. \tag{1.10c}$$

Here $\tau = \rho^{-1}$.

In fact, equation (1.10b) indicates that the density is governed by the history of the flow on the streamline (it is analogous to upwind differentiation in some sense).

Remark 1.1. As in the one-dimensional model, the limiting inviscid solution may develop discontinuities that will be approximations of rapidly varying profiles governed by the viscous equations. Such a profile will dictate the nature of the entropy condition associated with the inviscid flow. In this case one sees from the one-dimensional model that density increases in the direction of flow if $\nu > 0$.

We shall consider first a more general system of equations than (1.10),

$$\operatorname{div}(\rho\nabla\varphi) = 0, \tag{1.11a}$$

$$-\nu(\ln\rho)_\varphi = f(|\nabla\varphi|^2, \theta, Q_B(\rho,\varphi,\Phi)), \tag{1.11b}$$

$$\Delta\Phi = \alpha(\rho - C(x)), \tag{1.11c}$$

where Q_B denotes the squared speed given by Bernoulli's equation (that is, equation (1.5) for $\nu = 0$) and $\theta = \arctan\frac{\varphi_y}{\varphi_x}$, the directional angle of $\nabla\varphi$ taken to be zero at some point on the inflow boundary.

In Section 2 we formulate a boundary value problem associated with (1.11), and in Section 3 we shall prove, under certain conditions on f and for sufficiently small $\nu > 0$, the existence of solutions. In particular, for $\nu \leqq m$ using (1.8), we obtain the existence of solutions ρ, φ, and Φ in $C^{1,\alpha}$, $C^{2,\alpha}$, and $W^{2,p}$, respectively.

Furthermore, $|\nabla\varphi|$ and ρ are bounded away from zero and, in addition, we establish bounds for ρ and Φ that are uniform in ν. We shall prove that $0 < i(\rho) \leqq K^*$ for all ν where K^* depends only on the Bernoulli constant K, the bounds for the function R, the domain Ω, and the data of our boundary value problem (the latter controls $\sup\Phi^\nu$.) Then Q_B takes the value $Q_B = K - i(\rho) - R(\varphi) + q\Phi$, which is the square of the speed given by Bernoulli's law as long as $Q_B > 0$, and we shall define Q_B for negative values at a later point.

Thus, any solution of system (1.11) with

$$f(|\nabla\varphi|^2, \theta, Q_B) = \left(\frac{m}{2}|\nabla\varphi|^2 - Q_B\right)\left(|\nabla\varphi|^2 G(|\nabla\varphi|)\right)^{-1} f_1(\theta) \tag{1.12}$$

yields a solution of system (1.10).

Finally, in Section 4, if f takes the form (1.12), we establish uniform bounds for φ and $|\nabla\varphi|$ for the particular case $\nu \ll m$ in an appropriate length scale.

Conclusions about the applications are sketched in Sections 5 and 6. In a separate publication they will be discussed in more detail.

2. The Boundary Value Problem, Main Result, and Proof

We consider the Fluid-Poisson system of equations given by (1.11), written as

$$\operatorname{div}(\rho\nabla\varphi) = 0 \tag{2.1a}$$

$$-\frac{\nu\nabla\rho}{\rho}\cdot\frac{\nabla\varphi}{|\nabla\varphi|^2} = -\nu(\ln\rho)_\varphi = f(|\vec{v}|^2,\theta,Q_B) \tag{2.1b}$$

$$\Delta\Phi = \alpha(\rho - C(x)) \tag{2.1c}$$

in a simple connected domain Ω to be specified below. The function φ is the fluid potential, $\vec{v}$ the corresponding velocity field, ρ a density function, and Φ an electrostatic potential.

Let

$$Q_B = \begin{cases} \frac{2}{m}(K - i(\rho) - \mathcal{R}(\varphi) + q\Phi) \\ \qquad \text{if } K - i(\rho) - \mathcal{R}(\varphi) + q\Phi \geqq 0 \\ \arctan\left(\frac{2}{m}(K - i(\rho) - \mathcal{R}(\varphi) + q\Phi)\right) \\ \qquad \text{if } K - i(\rho) - \mathcal{R}(\varphi) + q\Phi \leqq 0 \end{cases} \tag{2.2}$$

Thus, Q_B is a bounded-below function with all first derivatives locally bounded that completes the coupling of the equation (2.1c) with the rest of the system.

The function $f = f(|\nabla\varphi|^2, \theta, Q_B(\rho,\varphi,\Phi))$ satisfies

$$f \equiv 0 \text{ iff } |\nabla\varphi|^2 = Q_B \quad \text{and} \quad f(|\nabla\varphi|^2,\theta,Q_B) > 0 \text{ if } Q_B < 0 \tag{2.3a}$$

$$|f|, (|\nabla\varphi| f) \leqq C(\sup_{\bar{\Omega}} |Q_B|) \tag{2.3b}$$

$$f \text{ is locally Lipschitz in each of its variables.} \tag{2.3c}$$

Note that f, defined as in (1.12) with G as in (1.8), satisfies (2.3).

Setting $\nu = 0$ in system (2.1), the equation (2.1b) becomes

$$\frac{|\nabla\varphi|^2}{2} + i(\rho) = K - \mathcal{R}(\varphi) + q\Phi \tag{2.4}$$

with K the Bernoulli constant. Thus, equation (2.4) is a modified Bernoulli law and the system (2.1) with $\nu = 0$ is referred to as the inviscid problem, which corresponds to a problem of transonic mixed type.

2.1. About the Domain and Its Conformal Transformation

The domain Ω has a $C^{2,\alpha}$ boundary with the exception of four points called ω_i, $i = 1,\ldots,4$. We assume that the portions of the boundary curves meet at each ω_i at an angle of $\pi/2$. We call

$$\partial\Omega = \bigcup_{i=1}^{4} \partial_i\Omega,$$

with $\partial_i\Omega$ the portion of the boundary between ω_i and ω_{i+1}, $i = 1,2,3$, and $\partial_4\Omega$ the portion between ω_4 and ω_1 (see Figure 2.1).

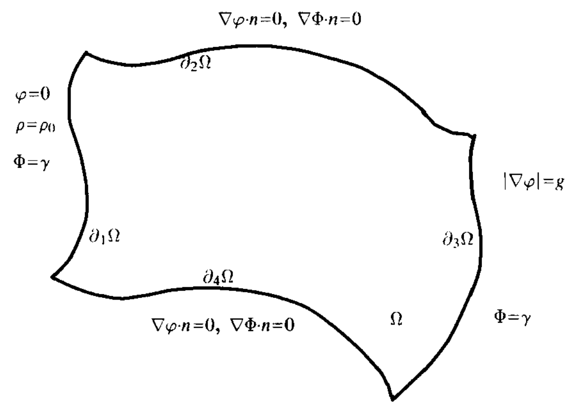

Figure 2.1. The domain Ω and boundary data.

Remark 2.1. Such a domain has the property that there is a conformal transformation $F(z)$ from Ω to R, where R is a rectangle that is unique up to translations and dilations such that $F(\omega_i)$ is a corner of R for each i. (In particular, such a conformal transformation $F(z)$ is shown in the appendix of [13].)

Let $F: \Omega \to R$ be the conformal transformation that takes Ω into a fixed rectangle R contained in $\mathbb{R}^2$. This is a one-to-one map, and F is a complex analytic function such that $|F_z|$ is smooth up to the boundary of Ω and never vanishes.

Let $\mathbf{x}' = F(\mathbf{x})$ be the real variable transformation associated with the conformal map F. Now, if $|F_{\mathbf{x}}|$ denotes the the Jacobian of the transformation, then $\sup_{\bar\Omega} |F_{\mathbf{x}}|$ and $\sup_{\bar\Omega} |F_{\mathbf{x}}^{-1}|$ are bounded depending only on the domain $\bar\Omega$.

A simple calculation shows that equations (2.1) are valid in R where

(2.5) $\nabla_{\mathbf{x}'}\varphi' = \nabla_{\mathbf{x}}\varphi|F_{\mathbf{x}}^{-1}|, \quad \nabla_{\mathbf{x}'}\rho' = \nabla_{\mathbf{x}}\rho|F_{\mathbf{x}}^{-1}|, \quad \text{and} \quad \Delta_{\mathbf{x}'}\Phi' = \Delta_{\mathbf{x}}\Phi|F_{\mathbf{x}}^{-1}|^2.$

In particular, (2.1) is transformed into

$$\begin{aligned} &\operatorname{div}(\rho'\vec{v}\,') = 0 \\ &-\nu\nabla \ln \rho' \frac{\nabla\varphi'}{|\nabla\varphi'|^2} = f((|F_{\mathbf{x}}|\,|\nabla\varphi'|)^2, \theta' - \arg F_z, Q'_B) \\ &\Delta_{\mathbf{x}'}\Phi' = \alpha(\rho - C(x)) \cdot |F_{\mathbf{x}}|^2. \end{aligned} \tag{2.6}$$

We will drop the ′ later on.

2.2. Compatibility Condition on the Data

For R_L and R_U, the lower and upper global bounds of the function $\mathcal{R}(\varphi)$, and

$$\Phi_U = \sup_{\partial_1\Omega\cup\partial_3\Omega} \gamma + \alpha \sup_{\bar\Omega} |F_{\mathbf{x}}|^2 \sup_{\bar\Omega} C(x)$$

with γ, α, and $C(x)$ given by the boundary value problem for the Poisson equation (2.1c) to be specified below and $|F_{\mathbf{x}}|$ the Jacobian of the transformation of Ω into a rectangle introduced above, we define

$$K^* = K - R_L + q\Phi_U \tag{2.7a}$$

where K is the Bernoulli constant such that it satisfies

$$K - R_U + q\Phi_L > 0, \tag{2.7b}$$

where

$$\Phi_L = \inf_{\partial_1\Omega\cup\partial_3\Omega} \gamma - \alpha \sup_{\bar\Omega} |F_{\mathbf{x}}|^2 \big(\sup_{\bar\Omega} C(x) + i^{-1}(K^*)\big),$$

and i is the enthalpy function defined in (1.4), so that i^{-1} is a well-defined, nonnegative function and K^* the number defined in (2.7a). That is, the numbers K^*, Φ_U, and Φ_L are given just in terms of the domain and the data associated with the boundary value problem for system (2.1) and are independent of ν. In particular,

$$0 < K - R_U + q\Phi_L < K - R_L + q\Phi_U = K^*.$$

We shall prove that $\sup_{\bar\Omega} Q_B \leqq K^*$ for the function Q_B defined in (2.2), as we prove first that the electric potential Φ is bounded above by Φ_U. Then, after showing that any solution ρ of system (2.1) is positive and bounded above by $i^{-1}(K^*)$ for the K^* defined in (2.7a), it follows that Φ is bounded below by Φ_L.

Remark 2.2. This compatibility condition is fundamental in order to obtain a uniform bound for the speed $|\nabla\varphi^\nu|$ in the parameter ν.

For the viscous approximation to the transonic flow model, $K = K^*$ is the value read "upstream" by presetting a density ρ_0 and speed $|v_0|$ at the boundary region $\partial_1\Omega$, so that $K = \frac{|v_0|^2}{2} + i(\rho_0) > 0$. For the semiconductor model, the Bernoulli constant K can be taken as

$$K = \frac{m}{2}|v_0|^2 + i(\rho_0) + R_U - q\Phi_L,$$

and then (2.7b) is satisfied.

2.3. About the Boundary Data

The potential flow function φ satisfies on the boundary of Ω, the domain described above,

$$\varphi = 0 \text{ on } \partial_1\Omega \text{ and } \nabla\varphi\cdot n\big|_{\partial_1\Omega}(\omega_1) < 0 \quad \text{(inflow condition)}, \tag{2.8a}$$

$$\frac{\partial\varphi}{\partial n} = 0 \text{ on } \partial_2\Omega\cup\partial_4\Omega \quad \text{(walls are streamlines)}, \tag{2.8b}$$

$$|\nabla\varphi| = g(x) \text{ on } \partial_3\Omega. \tag{2.8c}$$

The function $g(x)$ is $C^{1,\alpha}$ in $\partial_3\Omega$ and satisfies $0 < g^2(x) < K - R_U + q\Phi_L - \delta$, with δ a positive number, and n denotes the outer unit normal to $\partial\Omega$.

The density ρ is prescribed on $\partial_1\Omega$,

$$\rho\,\big|_{\partial_1\Omega} = r \quad \text{is a } C^{1,\alpha}(\partial_1\Omega)\text{-function such that} \qquad (2.9)$$
$$0 < i(r) < K - R_U + q\gamma\,\big|_{\partial_1\Omega} \leqq K^*$$

In fact, we see that equation (2.1b) is an ODE along the particle path, and the problem is well-posed if initial conditions are prescribed for ρ on all the boundary points where $\nabla\varphi\cdot n < 0$. In particular, the condition $\frac{\partial\varphi}{\partial n} = 0$ on $\partial_2\Omega\cup\partial_4\Omega$ indicates that the boundary sections $\partial_2\Omega$ and $\partial_4\Omega$ coincide with particle paths, and we shall see that $\nabla\varphi\cdot n < 0$ on $\partial_1\Omega$ and on some regions, but not all, of $\partial_3\Omega$.

We shall construct and solve a modified version of (2.1b) in an extended domain such that its solution will coincide with a solution of (2.1b) and initial data (2.9) on all the particle paths that start on $\partial_1\Omega$. At the end of this section we sketch the steps we follow in our proof.

The final boundary condition is: The electric potential Φ satisfies a Dirichlet-Neumann condition that we formulate as

$$B(\Phi) = \beta_1\Phi + \beta_2\nabla\Phi\cdot n - \gamma = 0 \quad \text{on } \partial\Omega\,. \qquad (2.10)$$

We shall assume that $B(\Phi)$ on $\partial\Omega$ allows for a solution $\Phi \in W^{2,p}(\bar{\Omega})$ for all $p \geqq 1$; that is, $B(\Phi)$ is a boundary operator that admits the maximum principle. For simplicity, we consider $\beta_2 = 0$ on $\partial_1\Omega\cup\partial_3\Omega$ and $\beta_1 = 1$ on $\partial\Omega\setminus\partial_1\Omega\cup\partial_3\Omega$ and $\beta_1 = 0$ and $\gamma = 0$ on $\partial_2\Omega\cup\partial_4\Omega$ and $\beta_2 = 1$ on $\partial\Omega\setminus\partial_2\Omega\cup\partial_4\Omega$. Finally, we take the data γ to be a $C^{2,\alpha}(\partial_1\Omega\cup\partial_3\Omega)$-function.

2.4. More About the Function f

Properties (2.3) will be sufficient to show existence of strong solutions as well as uniform bounds in ν for ρ^ν and for Φ^ν. Nevertheless, one of our goals is to show uniform bounds in ν for the speed; that is, the solution (ρ^ν,φ^ν) of (2.1) satisfying $|\nabla\varphi^\nu|$ is uniformly bounded in ν. In order to obtain these uniform bounds, we need to require more conditions on $f(|\nabla\varphi|^2,\theta,Q_B)$, where now we take f as in (1.12).

PROPERTY A. *For $f(|\nabla\varphi|^2,\theta,Q_B) = \left(|\nabla\varphi|^2 - Q_B\right)\left(|\nabla\varphi|^2 G(|\nabla\varphi|)\right)^{-1} f_1(\theta)$ with G a positive function satisfying* (1.8), *Q_B defined as in* (2.2), *and $f_1(\theta)$ positive bounded away from the zero function, the following properties hold:*

$$Q_B(0,\varphi,\Phi) - Q_B(\rho,\varphi,\Phi) \geqq 0 \qquad (2.11a)$$

$$Q_B(0,\varphi,\Phi) - Q_B(\rho,\varphi,\Phi) + \frac{\partial Q_B}{\partial\rho}\rho > 0 \qquad (2.11b)$$

$$|\nabla\varphi|\, f \to C > 0 \quad \textit{and} \quad |\nabla\varphi|^3 f_{|\nabla\varphi|^2} \ \textit{bounded} \quad \textit{as } |\nabla\varphi| \to \infty\,. \qquad (2.11c)$$

Property A will provide a sufficient condition for a ν-uniform estimate of $|\nabla\varphi^\nu|$ for ν sufficiently small provided that $Q_B(0,\varphi,\Phi)$ is uniformly bounded away from zero in $\bar\Omega$ (achieved by using a modification of a lemma due to Morawetz [31]). In fact, the compatibility condition (2.7b) yields $Q_B(0,\varphi,\Phi) = K - \mathscr{R}(\varphi) + q\Phi > K - R_U + q\Phi_L > 0$. For the given boundary value problem, we shall prove the estimate

$$|\nabla\varphi^\nu| < \left(\sup_{\bar\Omega} Q_B(0,\varphi,\Phi) + c\nu^{\frac{1}{2}}\right)\frac{\sup_{\bar\Omega}|F_{\mathbf{x}}|}{\inf_{\bar\Omega}|F_{\mathbf{x}}|} \quad \text{in } \bar\Omega$$

for $\nu \leqq \nu_0$, where c and ν_0 depend on the domain and the data of the boundary value problem.

Remark 2.3. It is easy to see that for an adiabatic gas law with $i(\rho) = \frac{\gamma}{\gamma-1}\rho^{\gamma-1} \geqq 0$, $1 < \gamma < 2$, the conditions (2.3) hold so that the viscous solution (ρ,φ,Φ) satisfies $Q_B(\rho,\varphi,\Phi) = \frac{2}{m}(K - i(\rho) - \mathscr{R}(\varphi) + q\Phi)$.

In addition, Property A is satisfied. Indeed, condition (2.11a) is equivalent to $i(\rho) \geqq 0$. Condition (2.11b) is

$$\frac{2}{m}\left(i(\rho) - i'(\rho)\rho\right) = \frac{2k}{m}\gamma\frac{2-\gamma}{1-\gamma}\rho^{\gamma-1} \geqq 0 \quad \text{iff } 1 < \gamma < 2.$$

Condition (2.11c) is satisfied if G satisfies the growth conditions stated in (1.8b).

Remark 2.4. We point out that in order to get the existence of a solution to the boundary value problem (2.1)–(2.10), we need only that f satisfies the properties listed under (2.3). In particular, if f takes the form (1.12) and G satisfies (1.8a) and (1.8b), then the growth condition (2.3b) is satisfied even if we relax the positivity condition on the constant C_2 in (1.8b). However, to obtain a uniform bound for the speed, we need for f given by (1.12) the growth condition given by (2.11c). This growth condition is a consequence of the growth condition (1.8b) for G and its derivative. In particular, the positivity of the constant C_2 becomes crucial for the ν-uniform property of the bound.

For example, if f is taken to be

$$(2.12) \qquad f(|\nabla\varphi|^2,\theta,Q_B) = \frac{f_1(\theta)(|\nabla\varphi|^2 - Q_B)}{(1+|\nabla\varphi|^2)^s}, \qquad s = \frac{3}{2},\ \inf_{\bar\Omega} f_1(\theta) > 0,$$

then f takes the form (1.12) with $G(|\nabla\varphi|) = \frac{(1+|\nabla\varphi|^2)^s}{|\nabla\varphi|^2}$, where f_1 has Lipschitz derivatives.

It is important to point out that this f satisfies (2.3) if $s \geqq \frac{3}{2}$, while this choice of G satisfies the growth conditions (1.8b) exactly for $s \leqq \frac{3}{2}$.

However, we prove that $|\nabla\varphi|$ is uniformly bounded above by a number M that depends only on the domain and the data of the boundary value problem for a $\nu \leqq \nu_0$, with ν_0 depending also on the domain and the data of the problem. Then, a posteriori we get an existence result and uniform bounds for linear viscosity away from stagnation points. In other words, we can set G to be

$$G(|\nabla\varphi|) = 1 \quad \text{for } 0 < k_\nu < |\nabla\varphi| \leqq M^{\frac{1}{2}}$$

and then extend G smoothly to $[0, \infty]$ such that growth conditions (1.8a) and (1.8b) are satisfied.

2.5. About New Variables

Due to the irrotationality of the two-dimensional steady flow, we have a potential flow function φ and, as we saw in (1.7b), a stream function ψ given by

$$\nabla\varphi = (u, v), \qquad \nabla\psi = (-\rho v, \rho u). \tag{2.13}$$

Thus we may rewrite equation (2.1a) as

$$\operatorname{div}(\rho\nabla\varphi) = 0, \qquad \operatorname{div}(\tau\nabla\psi) = 0, \qquad \tau = \rho^{-1}. \tag{2.14}$$

It is frequently helpful to use the following representation of (2.13):

$$\begin{aligned} d\varphi + i\tau\, d\psi &= w\, dz, \\ w = e^{\sigma - i\theta}, &\quad dz = dx + i\, dy. \end{aligned}$$

A computation shows that the flow equation (2.14) is transformed into the system

$$\begin{aligned} \rho\sigma_\psi - \theta_\varphi &= 0 \\ \tau\sigma_\varphi + \theta_\psi &= \tau_\varphi \end{aligned} \tag{2.15}$$

where $\sigma = \log|\nabla\varphi|$ and $\theta = \arctan\frac{\varphi_y}{\varphi_x}$, the directional angle of $\nabla\varphi$ taken to be zero at some point on the inflow boundary $\partial_1\Omega$. Note that (2.14) holds for all definitions of $\rho = \tau^{-1}$, not just one satisfying (2.1b).

We show in the following sections the existence of solutions of the boundary value problem described above for the system (2.1). We obtain the following theorem:

THEOREM 2.5. *There exists a solution $(\rho^\nu, \varphi^\nu, \Phi^\nu)$ in $C^{1,\alpha}(\bar\Omega) \times C^{2,\alpha}(\bar\Omega) \times W^{2,p}(\bar\Omega)$ of the boundary value problem of* (2.1) *with the boundary conditions* (2.8)–(2.10) *and with the conditions on the domain and the given function as described above. This solution has the following properties for sufficiently small ν:*

$$0 < k_\nu \leqq |\nabla\varphi^\nu| \leqq \bar K, \quad 0 \leqq \varphi^\nu \leqq \bar K \tag{2.16a}$$

$$0 < l_\nu \leqq \rho^\nu \leqq L^* \tag{2.16b}$$

$$|\Phi^\nu|, |\nabla\Phi^\nu| \leqq M^* \tag{2.16c}$$

for any $\nu \leqq \nu_0$, *with* $\bar{K}$, L^*, *and* M^* *constants independent of* ν.

2.6. About the Strategy of the Proof

The strategy for proving the theorem is to iterate on each equation and to have enough a priori estimates in an appropriate functional space so that we can construct a nonlinear compact mapping from a Banach space into itself which admits a fixed point that represents the desired solution. We actually use the Leray-Schauder fixed point theorem using degree theory because we have a non-convex domain in the Banach space. (We could avoid degree theory if, in (2.12), $f_1(\theta) \equiv 1$, but it is our intention to use $f_1(\theta) \neq 1$ in some future work.)

Due to the nature of equation (2.1b), we need to have robust enough bounds to keep ρ positive and bounded throughout the construction of the compact map. These bounds will be obtained from the properties (2.2) of the function f as well from coupling with equation (2.1a). Actually, it is the coupling of the first two equations that will improve the regularity in the construction of the map, so that compactness is obtained.

From now on, in order to prove existence, we work with system (2.6) where for convenience we drop the $'$ and assume Ω to be a rectangle. Note f now depends on x.

2.7. The Steps of the Proof

The steps to follow are: Give (ρ_0, φ_0) in $C^\alpha(\overline{\Omega}) \times C^{1,\alpha}(\bar{\Omega})$ such that $\rho_0|_{\partial_1\Omega}$ is a $C^{1,\alpha}$-function along the curve $\partial_1\Omega$ and $|\nabla\varphi_0| > k > 0$. Define θ_0 as the directional angle of $\nabla\varphi_0$ minus $\arg F_z$ with respect to the inflow boundary; that is, $\theta_0 = \arg(\varphi_{0_x} + i\varphi_{0_y} - F_z)$ and $\theta_0 = 0$ on $\partial_1\Omega$.

For K^* as defined in (2.7a) and for $L^* = i^{-1}(K^*)$, let

$$\tilde{\rho}_0 = \rho_0^{+,L^*} = \begin{cases} 0 & \text{if } \rho_0 \leqq 0 \\ \rho_0 & \text{if } 0 \leqq \rho_0 \leqq L^* \\ L^* & \text{if } L^* \leqq \rho_0 \end{cases}$$

Step 1. Solve

$$\begin{cases} \Delta\Phi = \alpha(\tilde{\rho}_0 - C(x))|F_{\mathbf{x}}|^2 \\ \tilde{B}(\Phi) = 0 \quad \text{on } \partial\Omega \end{cases} \tag{2.17}$$

where $\tilde{B}(\Phi)$ is the transform of $B(\Phi)$ given in (2.10), and $C(x)$ is a bounded and measurable function (usually assumed discontinuous). Thus:

LEMMA 2.6. *The solution* Φ *of* (2.7) *is* $W^{2,p}(\Omega)$ *for all* $p \geqq 1$, *and, in particular,* Φ *is* $C^{1,\alpha}(\Omega)$ *for* $0 < \alpha < 1$. *The regularity on the boundary depends on* $B(\Phi)$ *on* $\partial\Omega$.

The solution Φ is regular up to the boundary of Ω, so that $\Phi \in W^{2,p}(\bar{\Omega}) \subset C^{1,\alpha}(\bar{\Omega})$. In addition,

$$\|\Phi\|_{1,\alpha} \leqq C\left(\|\tilde{\rho}_0\|_{C^\alpha(\bar{\Omega})} + \sup_{\bar{\Omega}} |\Phi|\right) \tag{2.18a}$$

$$\begin{aligned}\Phi_L = \inf \gamma - \alpha \sup_{\bar{\Omega}} |F_{\mathbf{x}}|^2 \left(\max_{\bar{\Omega}} C(x) + \sup_{\bar{\Omega}} \tilde{\rho}_0\right) &\leqq \Phi \\ &\leqq \sup_{\bar{\Omega}} \gamma + \alpha \sup_{\bar{\Omega}} |F_{\mathbf{x}}|^2 \max_{\bar{\Omega}} C(x) = \Phi_U\end{aligned} \tag{2.18b}$$

$$|\nabla\Phi| \leqq \Phi_U + \Phi_L + \|\gamma\|_{C^{1,\alpha}(\partial\Omega)} \|F\|_{C^1(\partial\Omega)} \tag{2.18c}$$

where C depends on Ω and on the boundary data.

Step 2. For the given pair $(\tilde{\rho}_0, \varphi_0)$, a parameter t, $0 \leqq t \leqq 1$, and Φ a solution of (2.18), we solve

$$\nu\Delta\varphi = \nabla\varphi \cdot \nabla\varphi_0 t f((|F_{\mathbf{x}}|\,|\nabla\varphi_0|)^2, \theta_0, Q_B(\varphi_0, \tilde{\rho}_0, \Phi)) = -\nabla\varphi \cdot \vec{h} \tag{2.19}$$

with the prescribed data given by the transformed (2.8) data by the conformal map F.

Lemma 2.7. *There exists for each t a unique solution $\varphi_1 \in C^{2,\alpha}(\bar{\Omega})$ of the boundary value problem* (2.19) *and the transformed data associated with* (2.8) *such that*

$$\|\varphi_1\|_{C^{1,\alpha}(\bar{\Omega})} \leqq e^{\nu^{-1}KC(K^*)} F_1(K, C(K^*)) \tag{2.20a}$$

$$\|\varphi_1\|_{C^{2,\alpha}(\bar{\Omega})} \leqq e^{\nu^{-1}KC(K^*)} F_2(K,\, C(K^*), \\ \|\varphi_0\|_{C^{1,\alpha}(\bar{\Omega})} + \|\tilde{\rho}_0\|_{C^\alpha(\bar{\Omega})}), \tag{2.20b}$$

and

$$0 < k e^{-\nu^{-1}K(C(K^*))} \leqq |\nabla\varphi_1| \leqq K e^{\nu^{-1}K(C(K^*))} \tag{2.21}$$

uniformly in t, where the functions F_1 and F_2 are locally bounded and F_2 has linear growth in its third component, and the constants k and K depend on Ω, Φ_L, and Φ_U from (2.18b)*, the boundary data, the Lipschitz norms of R and G, and the conformal transformation from Ω into a fixed rectangle. The constant C depends on the bounds for f in* (2.3)*. Moreover,*

$$-\widetilde{K}_\nu \leqq \varphi_1 \leqq \widetilde{K}_\nu$$

where $\widetilde{K}_\nu$ is proportional to the diameter of Ω times the upper bound of $|\nabla\varphi_1|$.

This lemma has been already proven (see [13]), where it is shown that the transformation of the data (2.8) by a one-to-one conformal map preserves the structure of the (2.8) boundary data.

In addition, $\nabla\varphi_1 \cdot n|_{\partial_1\Omega} < 0$. However, because the boundary $\partial_3\Omega$ might not satisfy $\nabla\varphi_1 \cdot n|_{\partial_3\Omega} > 0$ at all points, it is not necessarily an "outflow" boundary. In fact, there may be points $a_i \in \partial_3\Omega$ where $\nabla\varphi_1 \cdot n|_{\partial_3\Omega}(a_i) = 0$. This possible configuration arises from the choice of the boundary data on $\partial_3\Omega$ when $|\nabla\varphi|$ is prescribed. In particular, the particle path that contains a_i, that is, the curve orthogonal to the level curve of φ_1 at a_i, becomes tangential to the boundary section $\partial_3\Omega$, where a_i is the contact point.

Therefore, if the contact between the particle path and $\partial_3\Omega$ is of first order, the $C^{2,\alpha}$-regularity of φ_1 yields the existence of a subregion $\partial\Omega'_3 \Subset \partial_3\Omega$ where

$$\nabla\varphi_1 \cdot n|_{\partial\Omega'_3} < 0 .$$

This circumstance will force us to modify the boundary value problem for the ODE defined for the density along the particle path given by the differentiable characteristic field $\nabla\varphi_1$, because we must prescribe data on $\partial\Omega'_3$. This leads us to the following:

Step 3. In order to insure that the solution of the modified ODE along the particle path defined by an extension of $\nabla\varphi_1$ to an extended rectangular domain of Ω^e is a well-posed problem, we shall define Ω^e. Consider a "one-sided" extension of Ω given by

$$\Omega^e \supset \Omega \quad \text{with } \partial_1\Omega^e = \partial_1\Omega,\ \partial_i\Omega \subset \partial_i\Omega^e, \quad i = 2,4,$$

such that $\Omega^e \backslash \Omega$ is a smaller rectangle.

LEMMA 2.8. *There exists an extension of $\nabla\varphi_1$ as a $C^{1,\alpha}$-function, denoted $\vec{b} = (b_1, b_2)$, defined in Ω^e such that $\vec{b} \cdot n|_{\partial_3\Omega^e} > 0$ and satisfying the following properties: $\vec{b} = \nabla\varphi_1$ in $\bar{\Omega}$, $\vec{b} \in C^{1,\alpha}(\bar{\Omega}^e)$ with $\|\vec{b}\|_{C^{1,\alpha}(\overline{\Omega^e})} \leqq k\|\varphi_1\|_{C^{2,\alpha}(\bar{\Omega})}$, where k is a constant that depends on k_ν and K_ν from* (2.21), *and $\vec{b} \cdot n|_{\partial_2\Omega^e \cup \partial_4\Omega^e} = 0$, $\vec{b} \cdot n|_{\partial_3\Omega^e} > 0$, and $\vec{b} \cdot n|_{\partial_1\Omega^e} < 0$. (In Section* 3 *we write $\vec{b}$ explicitly.)*

The angle θ_1^e is the directional angle of $\vec{b} - \arg F_z$ in $\overline{\Omega^e}$, and $\vec{b}$ defines in $\bar{\Omega}$ a characteristic field whose characteristic curves do not close. The last property is obtained by using the complex representation of $\nabla\varphi_1$ in the convex set $\bar{\Omega}$ but taking the continuous complex logarithm associated with $v = \varphi_{1x} - i\varphi_{1y}$ and extending v to $b = b_1 + ib_2$ to a convex domain $\bar{\Omega}$, where b has a continuous logarithm.

We shall also obtain an estimate of the smoothness of b and $\log b$, which we postpone until Section 3.

Step 4. Next, let $\vec{b}$ be the $C^{1,\alpha}(\bar{\Omega}^e)$-vector field from the lemma of Step 3, Φ the solution of problem (2.17), and Φ^e a $C^{1,\alpha}$-extension of Φ to $\overline{\Omega^e}$, $|F_{\mathbf{x}}|$ the

Jacobian of the domain transformation, and $|F_{\mathbf{x}}|^e$ a $C^{2,\alpha}$-extension to $\overline{\Omega^e}$, and θ_1^e the directional angle of the field $\vec{b} - \arg F_z$.

LEMMA 2.9. *There exists a solution ρ_1 of the ODE* (2.1b) *along the characteristic path in $\overline{\Omega^e}$ with initial data given by* (2.9) *for the given differentiable characteristic field $\vec{b} \in C^{1,\alpha}(\bar{\Omega}^e)$ constructed in Step* 3, *Lemma* 2.8, *depending on the parameter t as in* (2.19) *and for a function f that satisfies property* (2.3). *That is,*

$$(2.22)\qquad \begin{cases} \ln \rho_1 = h \\ \nabla h \cdot \vec{b} = -\dfrac{|\vec{b}|^2}{\nu} t f\left((|F_{\mathbf{x}}|^e |\vec{b}|)^2, \theta^e, Q_B(\varphi_1^e, \exp h, \Phi_0^e)\right) \\ \rho_1|_{\partial_1\Omega} = \rho_0 \in C^{1,\alpha}, \end{cases}$$

where φ_1^e and Φ_0^e are the $C^{2,\alpha}(\bar{\Omega}^e)$- and $C^{1,\alpha}(\bar{\Omega}^e)$-extensions of φ_1 and Φ_0, respectively, such that their norms are preserved in $C^{m,\alpha}(\bar{\Omega}^e)$, $m = 1, 2$, respectively.

The function $\rho_1 \subset C^{1,\alpha}(\bar{\Omega}^e)$. Also, ρ_1 is $C^{2,\alpha}(\bar{\Omega}^e)$ along the "particle" path defined by the characteristic field $\vec{b}$. The following estimates also hold:

$$(2.23a)\qquad \begin{aligned} \|\rho_1\|_{C^\alpha(\bar{\Omega})} &\leqq \|\rho_1\|_{C^\alpha(\overline{\Omega^e})} \\ &\leqq \mathcal{H}_1\big(\nu,\ K,\ C(K^*),\ \|\varphi_0\|_{C^{0,1}(\bar{\Omega})} + \|\tilde{\rho}_0\|_{C^{0,0}(\bar{\Omega})}, \\ &\qquad \|\varphi_1\|_{C^{1,\alpha}(\bar{\Omega})} + \|\Phi\|_{C^\alpha(\bar{\Omega})}\big) \end{aligned}$$

$$(2.23b)\qquad \begin{aligned} \|\rho_1\|_{C^{1,\alpha}(\bar{\Omega})} &\leqq \|\rho_1\|_{C^{1,\alpha}(\bar{\Omega}^e)} \\ &\leqq \mathcal{H}_2\big(\nu,\ K,\ C(K^*),\ \|\varphi_0\|_{C^{1,\alpha}(\bar{\Omega})} + \|\tilde{\rho}_0\|_{C^\alpha(\bar{\Omega})}, \\ &\qquad \|\varphi_1\|_{C^{2,\alpha}(\bar{\Omega})}, + \|\Phi\|_{C^{1,\alpha}(\bar{\Omega})}\big) \end{aligned}$$

and

$$(2.24)\qquad 0 < l_\nu \leqq \rho_1 \leqq L^* = i^{-1}(K^*) \quad in\ \overline{\Omega^e},$$

uniformly in t, where K is a constant depending on the domain, the data, and the Lipschitz norms of Q_B and $F_{\mathbf{x}}$ from (2.6). *In addition, $\mathcal{H}_1$ and $\mathcal{H}_2$ are smooth functions and $\mathcal{H}_2$ has linear growth in its fourth component. The lower bound l_ν depends on the bounds $\tilde{K}_\nu$ of φ_1 and the initial data, and the upper bound L^* depends on the domain and on the data of the boundary value problem but is independent of ν and $\tilde{\rho}_0$.*

Step 5. Here we use the Leray-Schauder fixed point theorem for nonconvex domain using degree theory. The proof of this theorem can be found in Leray and Schauder [26].

THEOREM 2.10. (LERAY-SCHAUDER, 1934) *Let T_t, $0 \leqq t \leqq 1$, be a continuous compact mapping from a nonconvex set $\mathcal{U}$ of the Banach space $\mathcal{B}$ into itself, and suppose there exists a constant M independent of t such that*

$$\|u\|_{\mathcal{B}} \leqq M, \quad 0 \leqq t \leqq 1,$$

for all $u \in \mathcal{U}$ satisfying $(I - T_t)u = 0$ such that

(i) *there is a $u_0 \in \mathcal{U}$ that satisfies $u_0 - T_0 u_0 = 0$ and*

(ii) *T_t is homotopic to T_0 under a compact homotopy with no fixed points on $\partial\mathcal{U}$ for all $0 < t \leqq 1$.*

Then T_1 has a fixed point in $\mathcal{U}$.

In order to apply this theorem, we take $\mathcal{B} = C^\alpha(\bar{\Omega}) \times C^{1,\alpha}(\bar{\Omega})$. Let $\mathcal{U}_{M+1,c_0}$ be the open set in $\mathcal{B}$ given by

$$\mathcal{U}_{M+1,c_0} = \{(\rho, \varphi) \in \mathcal{B}: \|(\rho,\varphi)\|_{\mathcal{B}} \leqq M + 1, \ |\nabla\varphi| > c_{0/2}\}.$$

Take a mapping T_t: $[0, 1] \times \mathcal{U} \to \mathcal{B}$ defined the following way: Given $(w, u) \in \mathcal{U}_{M+1,c_0}$, we first solve (2.17) for w^+ given, and we find the unique $\Phi \in W^{2,p}(\bar{\Omega})$ that satisfies estimates (2.18). (Set $\rho_0 = w^+$ in Step 1.)

Next, we solve (2.19) with w^{+,L^*}, u, and Φ from above, and we find a unique $\varphi \in C^{2,\alpha}(\bar{\Omega})$ that satisfies estimates (2.20) and (2.21). (Set $\tilde{\rho}_0 = w^{+,L^*}$ and $\varphi_0 = u$ in Step 2.)

Next, let $\vec{b}$ be a $C^{1,\alpha}(\bar{\Omega}^e)$-vector field from Step 3, Φ the solution of problem (2.17), and Φ^e a $C^{1,\alpha}$-extension of Φ to $\overline{\Omega^e}$, $|F_{\mathbf{x}}|$ the Jacobian of the domain transformation, and $|F_{\mathbf{x}}|^e$ a $C^{2,\alpha}$-extension to $\overline{\Omega^e}$, and θ_1^e the directional angle of the field $\vec{b}$ minus the argument of F_z.

Finally, we get a function ρ that solves (2.22) and satisfies estimates (2.23) and (2.24). We define $(\rho, \varphi) = T_t(w, u)$, $0 \leqq t \leqq 1$. We shall prove the following in Section 3:

LEMMA 2.11. *Estimates* (2.18), (2.20)–(2.21), *and* (2.23)–(2.24) *imply the a priori estimate*

$$\|\rho\|_{C^\alpha(\bar{\Omega})} + \|\varphi\|_{C^{1,\alpha}(\bar{\Omega})} \leqq \mathcal{K}_1(\nu, K, C(K^*, L^*)) \tag{2.25}$$

and

$$\|\rho\|_{C^{1,\alpha}(\bar{\Omega})} + \|\varphi\|_{C^{2,\alpha}(\bar{\Omega})} \leqq \mathcal{K}_2(\nu, K, C(K^*, L^*), \|w\|_{C^\alpha(\bar{\Omega})} + \|u\|_{C^{1,\alpha}(\Omega)}) \tag{2.26}$$

uniformly in t, $0 \leqq t \leqq 1$, and $L^* = i^{-1}(K^*)$ with K^*, as defined in (2.7a), *independent of w and u.*

From (2.25) and (2.26) it follows that T_t maps bounded sets of $\mathcal{U}_{M+1,c_0} = \mathcal{U}_{M+1,c_0} \cap (C^\alpha(\bar{\Omega}) \times C^{1,\alpha}(\bar{\Omega}))$ into bounded sets of $\mathcal{U}_{M+1,c_0} \cap (C^{2,\alpha}(\overline{\Omega}) \times C^{1,\alpha}(\bar{\Omega}))$ independently of t, where the latter one is precompact in $\mathcal{U}_{M+1,c_0}$. Thus:

LEMMA 2.12. *T_t is a compact operator.*

Moreover, the estimates of the lemmas in Steps 1 to 4 yield the following:

LEMMA 2.13.

$$\|\tilde{\rho}\|_{C^\alpha(\bar{\Omega})} \leqq M, \quad \|\tilde{\varphi}\|_{C^{1,\alpha}(\bar{\Omega})} \leqq M, \quad \text{and} \quad |\nabla\tilde{\varphi}| > k_\nu \tag{2.27}$$

if $(\tilde{\rho},\tilde{\varphi}) = T_t(\tilde{\rho},\tilde{\varphi})$, $0 \leqq t \leqq 1$, with M independent of $\tilde{\rho}$, $\tilde{\varphi}$, and t. In particular, T_t is a compact homotopy with no fixed points on $\partial\mathcal{U}_{M+1,k_\nu}$.

Finally, we show the following:

LEMMA 2.14. *T_t is a continuous operator in $\mathcal{B}$.*

Therefore by the Leray-Schauder theorem, T_1 has a fixed point $(\rho,\varphi) \in \mathcal{U}_{M+1,k_\nu} \cap C^{1,\alpha}(\bar{\Omega})\times C^{2,\alpha}(\bar{\Omega})$; that is, there is a $\Phi \in W^{2,p}(\bar{\Omega}) \subset C^{1,\alpha}(\bar{\Omega})$ such that ρ, φ, and Φ solve the boundary value problem (2.6) in a rectangular domain Ω, with boundary data given by the conformally transformed data (2.8), (2.9), and (2.10) by F.

Thus, $(\rho(F^{-1}),\varphi(F^{-1}),\Phi(F^{-1})) \in C^{1,\alpha}(\bar{\Omega})\times C^{2,\alpha}(\bar{\Omega})\times W^{2,p}(\bar{\Omega})$ solves the boundary value problem (2.1) with the data (2.8), (2.9), and (2.10) in the original domain, and Theorem 2.5 is proved except for determining that some of the bounds are independent of ν.

2.8. ν-Uniform Bounds for the Speed in the Original Domain Ω

Using the transformed equations (2.15) for state variables (σ^ν,θ^ν) in the (φ^ν,ψ^ν)-space in the domain Ω and using the conditions (2.11) on f, we shall prove a generalization of a lemma due to Morawetz [31] and obtain the estimate for the speed of the flow.

THEOREM 2.15. *The solution φ of Theorem 2.5 satisfies*

$$\begin{aligned} 0 < |\nabla\varphi|^2 &\leqq \left(Q_B(0,\varphi,\Phi) + c\nu^{\frac{1}{2}}\right)\frac{\sup_{\bar{\Omega}}|F_{\mathbf{x}}|}{\inf_{\bar{\Omega}}|F_{\mathbf{x}}|} \\ &\leqq (2K^* + c\nu^{\frac{1}{2}})\frac{\sup_{\bar{\Omega}}|F_{\mathbf{x}}|}{\inf_{\bar{\Omega}}|F_{\mathbf{x}}|} \quad \text{in } \bar{\Omega} \text{ for } \nu < \nu_0, \end{aligned} \tag{2.28}$$

where ν_0 and the constant c depend on K^* and $L^* = i^{-1}(K^*)$. The constant K^*, defined in (2.7a)*, depends on the Bernoulli constant K (i.e., the stagnation enthalpy constant) and on the domain and the data of the boundary value problem.*

We note here that in the transonic flow model $L^* = \rho^*$, the stagnation density, and $Q_B(0,\varphi,\Phi) = 2K$ is the square of the cavitation velocity given by Bernoulli's law.

2.9. Convergence as $\nu \to 0$

Let us denote the dependence on ν by a superscript, thus φ^ν. Theorem 2.15 implies that there exists a subsequence of φ^ν that converges. It is not hard to show that ρ^ν is uniformly bounded. This fact implies that there exists a convergent subsequence of φ^ν and thus we can say $\{\varphi^\nu, \psi^\nu\} \to \{\varphi^0, \psi^0\}$. However, to prove that the limit $\{\varphi^0, \psi^0\}$ represents a weak solution of the limiting equations requires better convergence. It could be arrived at via the theory of compensated compactness (see [32]) if the flow angle θ was bounded independent of ν, if the speed was uniformly bounded away from zero and the cavitation speed, and if $\big| |\nabla\varphi^\nu| - \sqrt{Q_B} \big| \to 0$.

The first two conditions are certainly satisfied physically in many situations. They probably could be satisfied here by adjusting the viscous term f artificially.

We note here the following about the assumption of the limit of the boundary conditions: The conditions $\psi = \text{const}$ and $\phi = \text{const}$ will be assumed because of the uniform bounds on the derivatives. It is not clear what happens to $|\nabla\varphi|$ on $\partial\Omega_3$ or ρ on $\partial\Omega_1$.

2.10. Possible New Directions

First, one might prove existence of steady state viscous-friction solutions in three dimensions. In principle, the methods could carry over and similar estimates for use in a fixed point theorem derived. Here there are no longer infinite entropy pairs for the inviscid case, and it is very unlikely that one can use the method of compensated compactness even by assuming some extra estimates.

Second, one could try to study steady state Navier-Stokes equations or their analogue from the same point of view. Here there would be a complication from the new boundary conditions, and it is not at all evident that the equations could be separated into such simple systems as we have here for ρ and φ.

3. Existence of a Solution

In this section we develop in detail the steps outlined in the preceding section. Here we assume that Ω is a rectangle conformally transformed from the original domain.

Let $\mathscr{B}$ be the Banach space given by $C^\alpha(\Omega) \times C^{1,\alpha}(\Omega)$. Let $\mathscr{U}_{M+1,c_0}$ be the open set in $\mathscr{B}$ given by

$$\mathscr{U}_{M+1,c_0} = \{(\rho,\varphi) \in \mathscr{B}\colon \|(\rho,\varphi)\|_{\mathscr{B}} \leqq M+1,\ |\nabla\varphi| > c_{0/2}\}.$$

Since our goal is to construct a compact, continuous map T_t from $[0,1]\times\mathscr{U}_{M+1,C_0} \subset [0,1]\times\mathscr{B}$ into $\mathscr{B}$, we let t be a parameter in $[0,1]$ and $(\rho_0,\varphi_0) \in \mathscr{U}_{M+1,C_0}$ be such that ρ_0 coincides with a $C^{1,\alpha}$-function along the curve $\partial_1\Omega$, $\varphi_0 = k_0 = \text{const}$ on $\partial_1\Omega$. We follow the steps drafted in Section 2 and prove the lemmas given without proof there. As we defined there, for K^* defined as in (2.7a) and for $L^* = i^{-1}(K^*)$, let

$$\tilde{\rho}_0 = \rho_0^{+,L^*} = \begin{cases} 0 & \text{if } \rho_0 \leqq 0 \\ \rho_0 & \text{if } 0 \leqq \rho_0 \leqq L^* \\ L^* & \text{if } L^* \leqq \rho_0 \end{cases}$$

and $\|\tilde{\rho}_0\|_{C^\alpha(\bar{\Omega})} \leqq C\|\rho_0\|_{C^\alpha(\bar{\Omega})}$, with C depending on the function i.

Step 1: Solving for the Electric Potential Φ. For the given $\tilde{\rho}_0 \in C^\alpha(\bar{\Omega})$ from above, we solve

$$\begin{cases} \alpha^{-1}\Delta\Phi = (-C(x) + \tilde{\rho}_0)|F_x|^2 \\ \tilde{B}(\Phi) = \tilde{\beta}_1\Phi + \tilde{\beta}_2\nabla\Phi\cdot n - \tilde{\gamma} = 0 \quad \text{on } \partial\Omega \end{cases} \tag{3.1}$$

where $C(x)$ is a bounded, measurable function and the boundary data are transformed from (2.10) under the conformal map F.

Problem (3.1) is a classical elliptic boundary value problem that admits a maximum principle for which Lemma 2.6 holds. We rephrase and prove Lemma 2.6 as follows:

LEMMA 3.1. *There exists a unique solution Φ of problem* (3.1) *in $W^{2,p}(\bar{\Omega})$ for all $p \geqq 1$, and the following estimate holds:*

$$\|\Phi\|_{C^{1,\alpha}(\bar{\Omega})} \leqq C\|\Phi\|_{W^{2,p}(\bar{\Omega})} \leqq \alpha C\left\{\|\tilde{\rho}_0\|_{\infty,\bar{\Omega}} + \sup_{\bar{\Omega}}|\Phi|\right\} \tag{3.2a}$$

where C depends on Ω and the boundary data. In addition, the following estimates hold:

$$\begin{aligned} \sup_{\partial\Omega}\Phi &\leqq \sup\gamma + \alpha\sup_{\bar{\Omega}}|F_x|^2\max_{\bar{\Omega}}C(x) \\ &= \Phi_U(\gamma, C(x)) \\ \inf_{\partial\Omega}\Phi &\geqq \inf\gamma - \alpha\sup_{\bar{\Omega}}|F_x|^2\left(\max_{\bar{\Omega}}C(x) + \sup_{\bar{\Omega}}\tilde{\rho}_0\right) \\ &= \Phi_L(\gamma, C(x), i^{-1}(K^*)) \\ |\nabla\Phi| &\leqq \Phi_U + \Phi_L + \|\gamma\|_{C^{1,\alpha}(\partial\Omega)}\|F\|_{C^1(\partial\Omega)}\,. \end{aligned} \tag{3.2b}$$

Note that (3.2b) can be used in (3.2a).

Proof: The existence and uniqueness of Φ and estimate (3.2a) are given by standard elliptic results (see, for instance, Gilbarg and Trudinger [19]) for the boundary value problem (3.1). We show that estimate (3.2b) is obtained by constructing appropriate super- and subsolutions.

In fact, the upper bound is obtained by constructing a supersolution independent of ρ. Let w be a solution of $\alpha^{-1}\Delta w = -C(x)|F_{\mathbf{x}}|^2$ in Ω, $\widetilde{B(w)}|_{\partial\Omega} = \widetilde{B(\Phi)}|_{\partial\Omega}$. Since $\tilde{\rho}_0 > 0$, we have

$$\alpha^{-1}\Delta w < \left(-C(\mathbf{x}) + \rho_0^+\right)|F_{\mathbf{x}}|^2 = \alpha^{-1}\Delta\Phi .$$

Using the maximum principle for elliptic problems with mixed data, it follows that $\Phi \leqq w \leqq \max_{\bar{\Omega}} w = \Phi_U$ in $\bar{\Omega}$, where $\Phi_U = \sup\gamma + \alpha \max_{\bar{\Omega}} C(x) \sup_{\bar{\Omega}} |F_{\mathbf{x}}|^2$.

Similarly, in order to get a lower bound for Φ, let w be the solution of $\Delta w = \left(-C(x) + \max_{\bar{\Omega}} \tilde{\rho}_0\right) \sup_{\bar{\Omega}} |F_{\mathbf{x}}|^2$ in Ω with $\widetilde{B(w)}|_{\partial\Omega} = \widetilde{B(\Phi)}|_{\partial\Omega}$. Then $\Delta w \geqq \Delta\Phi$ in Ω.

By the maximum principle, $\Phi \geqq w \geqq \min w = \Phi_L > -\infty$ in $\bar{\Omega}$, where

$$\Phi_L = \inf\gamma - \alpha\left(\max_{\bar{\Omega}} C(x) + \max_{\bar{\Omega}} \tilde{\rho}_0\right) \sup_{\bar{\Omega}} |F_{\mathbf{x}}|^2 .$$

The estimate for the gradient follows from the maximum-infimum norms for Φ, yielding

$$|\nabla\Phi| \leqq C\left(\Phi_U + \Phi_L + \|\gamma\|_{C^{1,\alpha}(\partial\Omega)}\right)$$

where C depends on the domain.

In particular, (3.2b) holds.

Step 2: Existence of a Potential Flow Function. Let (ρ_0, φ_0) in $\mathcal{U}_{M,C_0}$ be given such that $|\nabla\varphi_0| > k > 0$. Let θ_0 denote the directional angle of $\nabla\varphi_0$ in $\bar{\Omega}$ with respect to the inflow boundary $\partial_1\Omega$. That is, $\theta_0|_{\partial_1\Omega} = 0$ and $\theta_0 = \arg(\varphi_{0_x} + i\varphi_{0_y} - F_z)$ defined locally as a unique continuation of a $C^{1,\alpha}$-function in small enough neighborhoods. Since $|\nabla\varphi_0| > k > 0$, then $\bar{\Omega}$ can be covered by a finite number of neighborhoods depending on k, where θ_0 will be uniquely defined.

We define $\vec{h} = (h_1, h_2)$ as

$$\vec{h} = tf\left((|F_{\mathbf{x}}|\,|\nabla\varphi_0|)^2,\ \theta_0,\ Q_B(\tilde{\rho}_0, \varphi_0, \Phi)\right)\nabla\varphi_0 \tag{3.3}$$

with Φ the solution of (3.1).

The function Q_B, defined as in (2.2), satisfies

$$-\frac{\pi}{2} \leqq Q_B(\tilde{\rho}_0, \varphi_0, \Phi) \leqq K - \inf_{\bar{\Omega}} R(\phi_0) + q \sup_{\bar{\Omega}} \Phi = K - R_L + q\Phi_U = K^* \tag{3.4}$$

with K^* as defined in (2.7a), which depends only on the data of the boundary value problem.

Then, if f satisfies property (2.3b),

$$|\vec{h}| = |\mathcal{H}(\mathbf{x}, |\nabla\varphi_0|, \varphi_0, \theta_0, \tilde{\rho}_0, \Phi)| \leqq tKC(K^*) \leqq KC(K^*) \tag{3.5}$$

uniformly in t, $0 \leqq t \leqq 1$, where $\mathscr{H}$ is given by (3.3) and each component of $\mathscr{H}$ is a Lipschitz continuous function in each of its variables and is a $C^\alpha(\bar\Omega)$-vector function. In addition, $C(K^*)$ is a smooth function of K^*, and here K is a constant that depends on the domain and on the growth conditions (1.8) for the function G.

The following theorem, which yields the lemma of Step 2, solves an elliptic problem with prescribed speed in a section of the boundary.

THEOREM 3.2. *Given $\vec{h}$ as in (3.3), there exists a unique $\varphi_1 \in C^{2,\alpha}(\bar\Omega)$ that solves*

$$(3.6)\qquad \begin{cases} \Delta\varphi_1 = \frac{1}{\nu}\nabla\varphi_1\cdot\vec{h} \\ \varphi_1 = 0 \text{ on } \partial_1\Omega \text{ and } \nabla\varphi_1\cdot n|_{\partial_1\Omega}(w_1) < 0 \\ \quad\text{(inflow condition)} \\ \nabla\varphi_1\cdot n = 0 \quad \text{on } \partial_2\Omega\cup\partial\Omega_4 \\ \quad\text{(n being the exterior normal)} \\ |\nabla\varphi_1| = g(x)|F_{\mathbf{x}}|^{-2} \quad \text{on } \partial_3\Omega \end{cases}$$

where $g(x)$ on $\partial_3\Omega$ satisfies (2.8). *Moreover, the solution φ_1 satisfies the following estimates:*

$$(3.7a)\qquad \|\varphi_1\|_{C^{1,\alpha}(\bar\Omega)} \leqq e^{\nu^{-1}KC(K^*)}F_2\left(K,\ C(K^*),\ \|\varphi_0\|_{C^{0,1}(\bar\Omega)} + \|\tilde\rho_0\|_{C^0(\bar\Omega)}\right)$$

$$(3.7b)\qquad \|\varphi_1\|_{C^{2,\alpha}(\bar\Omega)} \leqq e^{\nu^{-1}KC(K^*)}F_1\left(K,\ C(K^*),\ \|\varphi_0\|_{C^{1,\alpha}(\bar\Omega)} + \|\tilde\rho_0\|_{C^\alpha(\bar\Omega)}\right)$$

$$(3.7c)\qquad \|\arg\varphi_{1_x} + i\varphi_{1_y}\|_{C^\alpha(\bar\Omega)} \leqq \frac{1}{\nu}\left(K + F_3\left(K,\ C(K^*),\ \|\varphi_0\|_{C^{0,1}(\bar\Omega)} + \|\tilde\rho_0\|_{C^0(\bar\Omega)}\right)\right)$$

$$(3.7d)\qquad \|\arg\varphi_{1_x} + i\varphi_{1_y}\|_{C^{1,\alpha}(\bar\Omega)} \leqq \frac{1}{\nu}\left(K + F_4\left(K,\ C(K^*),\ \|\varphi_0\|_{C^{1,\alpha}(\bar\Omega)} + \|\tilde\rho_0\|_{C^\alpha(\bar\Omega)}\right)\right)$$

where the F_i, $i = 1,2,3,4$, are smooth, and F_2 and F_4 have linear growth in their third component. The number K depends on $g(x)$, Ω, the data of the problem, and the Lipschitz norms of R and G.

In addition, we have the L^∞-bounds for $|\nabla\varphi_1|$ and φ_1 given by the estimates

$$(3.8a)\qquad 0 < ke^{-\nu^{-1}KC(K^*)} < |\nabla\varphi_1| < Ke^{\nu^{-1}KC(K^*)}$$

where K and k depend on Ω, $g(x)$, and C as above. Moreover,

$$-K_1 e^{\nu^{-1}C(K^*)} < \varphi_1 < K_1 e^{\nu^{-1}KC(K^*)} \tag{3.8b}$$

where K_1 depends on the diameter of Ω.

Proof: The proof of this theorem is a direct application of a theorem in a slightly more general form proved by the first author (see [13]). It is shown there that if $\vec{h} = (h_1, h_2)$ is a $C^\alpha(\bar{\Omega})$-vector function for Ω, a domain of the kind described in Section 2, then there exists a unique solution of the problem (3.6) that satisfies

$$\|\varphi_1\|_{C^{1,\alpha}(\bar{\Omega})} \leqq K e^{\nu^{-1}K\|\vec{h}\|_{\infty,\bar{\Omega}}}(1 + \|\vec{h}\|_{\infty,\bar{\Omega}}) \tag{3.9a}$$

$$\begin{aligned}\|\varphi_1\|_{C^{2,\alpha}(\bar{\Omega})} \leqq\ & K e^{\nu^{-1}K\|\vec{h}\|_{\infty,\bar{\Omega}}} \\ & \cdot \left(\|\vec{h}\|_{C^\alpha(\bar{\Omega})}\|\vec{h}\|_{\infty,\bar{\Omega}} + 1 + \|\vec{h}\|^2_{\infty,\bar{\Omega}}\right)\left(1 + \|\vec{h}\|_{\infty,\bar{\Omega}}\right),\end{aligned} \tag{3.9b}$$

where K depends on Ω and $g(x)$. Moreover, the following bounds hold for $|\nabla\varphi_1|$ and φ_1:

$$0 < k_\nu = k e^{-\nu^{-1}K\|\vec{h}\|_\infty} \leqq |\nabla\varphi_1| \leqq K e^{\nu^{-1}K\|\vec{h}\|_\infty} \tag{3.10}$$

and

$$-\widetilde{K}_\nu \leqq \varphi_1 \leqq \widetilde{K}_\nu, \tag{3.11}$$

where k and K depend on $g(x)$ and Ω. $\widetilde{K}_\nu$ depends on the diameter of Ω times the upper bound of $|\nabla\varphi_1|$.

Thus, using (3.3) and (3.5), $\vec{h}$ can be estimated by

$$\begin{aligned}\|\vec{h}\|_{C^\alpha(\bar{\Omega})} &\leqq \left\{\sup_{\bar{\Omega}}|tf|\ \|\nabla\varphi_0\|_{C^\alpha(\bar{\Omega})} + \sup_{\bar{\Omega}}|\nabla\varphi_0|\ \|tf\|_{C^\alpha(\bar{\Omega})}\right\} \\ &\leqq C(K^*)\|\nabla\varphi_0\|_{C^\alpha(\bar{\Omega})} + \sup_{\bar{\Omega}}|\nabla\varphi_0|\ \big(\|\nabla\varphi_0\|_{\infty,\bar{\Omega}}\ \|\varphi_0\|_{C^{1,\alpha}(\bar{\Omega})} \\ &\qquad + K\left(\|\tilde{\rho}_0\|_{C^\alpha(\bar{\Omega})} + \|\theta_0\|_{C^\alpha(\bar{\Omega})} + \|\Phi\|_{C^{1,\alpha}(\bar{\Omega})}\right)\big).\end{aligned} \tag{3.12}$$

uniformly in t, $0 \leqq t \leqq 1$, and K depending on the Lipschitz norm of Q_B, that is, on the Lipschitz norm of i, R, f_1, and G as well as on the constant q and the domain $\bar{\Omega}$.

Because θ_0 can be represented as $\theta_0 = \arg(\varphi_{0_x} - i\varphi_{0_y} - F_z)$, then

$$\|\theta_0\|_{C^\alpha(\bar\Omega)} \leqq \mathcal{G}\left(\sup_{\bar\Omega}|\nabla\varphi_0|\right)\|\varphi_0\|_{C^{1,\alpha}(\bar\Omega)} + \|F\|_{C^{1,\alpha}(\bar\Omega)}$$

where $\mathcal{G}$ is a smooth function of its argument.

Let $B = C(K^*)$; from (3.9b), (3.2a), and (3.2b), we can estimate the $C^{2,\alpha}$-norm of φ_1 as

$$\begin{aligned}&\|\varphi_1\|_{C^{2,\alpha}(\bar\Omega)}\\ &\leqq Ke^{\nu^{-1}KB}(1+B)\Big[K + B^2 + \Big(B + \sup_{\bar\Omega}|\nabla\varphi_0| + \mathcal{G}\sup_{\bar\Omega}|\nabla\varphi_0|)\Big)\\ &\quad\cdot B\sup_{\bar\Omega}|\nabla\varphi_0|^2(\|\varphi_0\|_{C^{1,\alpha}(\bar\Omega)} + \|\tilde\rho_0\|_{C^\alpha(\bar\Omega)})\Big]\end{aligned}$$

where K now depends on the domain Ω and the boundary data.

Since φ_0 was assumed to satisfy $\sup_{\bar\Omega}|\nabla\varphi_0| < Ke^{\nu^{-1}KB}$, we have obtained the estimate

$$\begin{aligned}&\|\varphi_1\|_{C^{2,\alpha}(\bar\Omega)}\\ &\leqq Ke^{\nu^{-1}KB}(1+B)\Big[1 + B^2 + \Big(B + \mathcal{G}(Ke^{\nu^{-1}KB}) + Ke^{\nu^{-1}KB}\Big)\\ &\quad\cdot BK^2e^{2\nu^{-1}KB}\left(\|\varphi_0\|_{C^{1,\alpha}(\bar\Omega)} + \|\tilde\rho_0\|_{C^\alpha(\bar\Omega)}\right)\Big]\\ &= e^{\nu^{-1}KC(K^*)}F_1\left(K, B, \|\varphi_0\|_{C^{1,\alpha}(\bar\Omega)} + \|\tilde\rho_0\|_{C^\alpha(\bar\Omega)}\right)\end{aligned} \tag{3.13}$$

where F_1 is smooth and has linear growth in its third component. Then (3.7b) holds. In addition, by using estimate (3.9a), estimate (3.7a) can be obtained in a similar manner.

In order to obtain estimates (3.7c)–(3.7d) for $\arg(\varphi_{1_x} - i\varphi_{1_y})$, where φ_1 is the unique solution with nonvanishing gradient of problem (3.6), we make use of the estimates obtained in [13] for representing the solution as $\varphi_{1_x} - i\varphi_{1_y} = \tilde{\mathfrak{f}}(z)e^{\mathbf{a}(z)+i\mathbf{b}(z)}$, with $\tilde{\mathfrak{f}}(z)$ a nonvanishing holomorphic function in $\bar\Omega$ that satisfies the boundary data and $\mathbf{b}(z)$ is the imaginary part of the exponential argument of the Bers-Nirenberg representation of the solution for $\nabla\varphi_1$.

It is shown in [13] that for $\vec{h} = \frac{1}{\nu}f((|F_{\mathbf{x}}|\,|\nabla\varphi_0|)^2, \theta_0, Q_B)\nabla\varphi_0$

$$\|\mathbf{b}\|_{C^{1,\alpha}(\bar\Omega)} \leqq \frac{1}{\nu}F_4(K, \|h\|_{\infty,\bar\Omega}, \|h\|_{C^\alpha(\bar\Omega)})$$

$$\|\mathbf{b}\|_{C^\alpha(\bar\Omega)} \leqq \frac{1}{\nu}F_3(K, \|h\|_{\infty,\bar\Omega})$$

and

$$\|\arg\tilde{\mathfrak{f}}(z)\|_{C^{1,\alpha}(\bar\Omega)} \leqq \frac{K}{\nu}$$

where F_3 and F_4 are smooth, F_4 has linear growth in its last component, and K_ν depends on the domain and the boundary data. Therefore, (3.7c) and (3.7d) hold when $\vec{h}$ is estimated as in (3.12).

Since $\nabla\varphi_1$ never vanishes in $\bar{\Omega}$ and $\nabla\varphi\cdot\tau = 0$ on $\partial_1\Omega$, where τ is the tangential unit vector, φ_1 increases along the orthogonal to its level curves in the direction of $\nabla\varphi_1$. Thus, if at the boundary point w_1 the condition $\nabla\varphi_1 \cdot n|_{\partial_1\Omega}(w_1) < 0$ holds, then the flow is "entering" the domain throughout that wall, that is, $\nabla\varphi_1 \cdot \eta < 0$ on $\partial_1\Omega$. Therefore $\varphi_1 > 0$ in a neighborhood of $\partial_1\Omega$, so that $\partial_1\Omega$ is an "inflow" boundary.

On the other hand, prescribing the speed, as we do on $\partial_3\Omega$, does not say anything about the direction of the flow. Thus, there might exist a set $\partial_3\Omega' \Subset \partial_3\Omega$ where the solution of problem (3.6) satisfies $\nabla\varphi_1 \cdot n|_{\partial_3\Omega'} < 0$, and hence $\partial_3\Omega$ might not be an outflow boundary.

Step 3: Extending the Field $\nabla\varphi_1$ to a Nonvanishing Field Having an Outflow Boundary. We now prove the extension lemma of Step 3, Lemma 2.8, as described in Section 2.

Let φ_1 to be the unique solution of problem (3.6) in the convex domain Ω. Then $\varphi_1 \in C^{2,\alpha}(\bar{\Omega})$, $\nabla\varphi_1$ never vanishes in $\bar{\Omega}$, $\varphi_1 = \varphi_0 = \text{const}$, and $\nabla\varphi_1 \cdot n < 0$ on $\partial_1\Omega$.

Using complex representation of the two-dimensional space, we set $v = \varphi_{1x} - i\varphi_{1y}$. Since $|v| > k_\nu > 0$, then v has a continuous complex logarithm $u = \log v = \log|\nabla\varphi| + \arg(\varphi_{1_x} - i\varphi_{1_y})$ (or $v = e^u$).

We extend u to $\overline{\Omega^e}$, the extended set of $\bar{\Omega}$, as follows: Let $\partial_3\Omega \subset \{x = 0\}$ axis. The function $\tilde{u}$ defined in $\overline{\Omega^e}$ by

$$\tilde{u}^e(x,y) = \begin{cases} u(x,y) & \text{if } (x,y) \in \bar{\Omega} \\ a_1u(0,y) + a_2u(-x,y) + a_3u(-2x,y) & \text{if } (x,y) \in \overline{\Omega^e}\setminus\bar{\Omega} \end{cases}$$

is $C^{1,\alpha}(\overline{\Omega^e})$ for $a_1 = 2$, $a_2 = -3$, and $a_3 = 1$ fixed constants.

In fact, for that choice of the a_i's, the function $\tilde{u}$ is C^2 in the x-direction across $\partial_3\Omega$ and $C^{1,\alpha}$ in the y-direction all over $\overline{\Omega^e}$. The $C^{1,\alpha}$-regularity in $\overline{\Omega^e}\setminus\bar{\Omega}$ is inherited from u in $C^{1,\alpha}(\bar{\Omega})$ because the extension involves only a linear combination of dilations of u in $\bar{\Omega}$. In addition,

$$\begin{aligned} \|\tilde{u}\|_{C^{1,\alpha}(\overline{\Omega^e})} &\leqq C\|u\|_{C^{1,\alpha}(\bar{\Omega})} \\ &\leqq C\|\log|\nabla\varphi_1|\|_{C^{1,\alpha}(\bar{\Omega})} + \|\arg(\varphi_{1_x} - i\varphi_{1_y})\|_{C^{1,\alpha}(\bar{\Omega})} \end{aligned} \tag{3.14a}$$

and

$$\begin{aligned} \|\tilde{u}\|_{C^{\alpha}(\overline{\Omega^e})} &\leqq C\|u\|_{C^{\alpha}(\bar{\Omega})} \\ &\leqq C\|\log|\nabla\varphi_1|\|_{C^{\alpha}(\bar{\Omega})} + \|\arg(\varphi_{1_x} - i\varphi_{1_y})\|_{C^{\alpha}(\bar{\Omega})} \end{aligned} \tag{3.14b}$$

where $C = \sum_{i=1}^{3} |a_i| = 6$.

Therefore, gathering this estimate along with (3.7) estimates for φ_1 and θ_1 yields

$$\begin{aligned}\|\tilde{u}\|_{C^{1,\alpha}(\overline{\Omega^e})} &\leqq C\|u\|_{C^{1,\alpha}(\bar{\Omega})} \\ &\leqq \frac{1}{\nu}\mathscr{F}_1(K,\, C(K^*),\, \|\varphi_0\|_{C^{1,\alpha}(\bar{\Omega})} + \|\tilde{\rho}_0\|_{C^{\alpha}(\Omega)})\end{aligned} \tag{3.15a}$$

and

$$\|\tilde{u}\|_{C^{\alpha}(\overline{\Omega^e})} \leqq C\|u\|_{C^{\alpha}(\bar{\Omega})} \leqq \frac{1}{\nu}\mathscr{F}_2(K, C(K^*)) \tag{3.15b}$$

where $\mathscr{F}_1$ and $\mathscr{F}_2$ are smooth, $\mathscr{F}_1$ has linear growth in the third component, and K depends on the domain and the boundary data.

Next, we take a cutoff, real-valued function ξ such that ξ is C^∞ in Ω^e, $0 \leqq \xi \leqq 1$, with $\xi = 1$ in $\bar{\Omega}$ and $\xi \equiv 0$ in a neighborhood of $\partial_3\Omega^e$. Then we set

$$\tilde{v} = \xi\tilde{u} \quad \text{in } \Omega^e. \tag{3.15c}$$

We define

$$b = b_1 + ib_2 = e^{\tilde{v}} \quad \text{or} \quad \tilde{v} = \log b. \tag{3.15d}$$

Then $\tilde{v}$ is a well-defined, continuous logarithm and $|b| \neq 0$. In addition, $\tilde{v}$ inherits the regularity of $\tilde{u}$. That is, estimates (3.15a) and (3.15b) hold for $\tilde{v}$ as well as $\tilde{u}$.

Also, if we take the supp$\,\xi$ in Ω^e very close to $\bar{\Omega}$, then

$$|b| \geqq \frac{k_\nu}{2}. \tag{3.15e}$$

Moreover, since $\tilde{v} \equiv 0$ implies $b \equiv 1$ in a neighborhood of $\partial_3\Omega^e$, then $(b_1, b_2)\cdot n|_{\partial_3\Omega^e} > 0$. This is a very important property that will yield the existence of a global solution ρ_1 of a first-order, quasi-linear boundary value problem.

Finally, we need to show that the vector field $\vec{b} = (b_1, b_2)$ satisfies

$$\begin{aligned}\|\vec{b}\|_{C^{1,\alpha}(\Omega^e)} &\leqq e^{\nu^{-1}KC(K^*)}\mathscr{F}_1(K, C(K^*), \|\varphi_0\|_{C^{1,\alpha}} + \|\tilde{\rho}_0\|_{C^{\alpha}(\Omega)}) \\ \|\vec{b}\|_{C^{\alpha}(\overline{\Omega^e})} &\leqq e^{\nu^{-1}KC(K^*)}\mathscr{F}_2(K, C(K^*), \|\varphi_0\|_{C^{0,1}(\bar{\Omega})} + \|\tilde{\rho}_0\|_{C^{0}(\bar{\Omega})}).\end{aligned}$$

Indeed,

$$\begin{aligned} b = b_1 + ib_2 &= e^{|\operatorname{Re}\tilde{v}|}(\cos(\operatorname{Im}\tilde{v}) + i\sin(\operatorname{Im}\tilde{v})) \\ &= g(\xi)e^{|\operatorname{Re}\tilde{u}|}(\cos(\operatorname{Im}\tilde{u}) + i\sin(\operatorname{Im}\tilde{u})).\end{aligned}$$

Therefore, taking the C^α-seminorms of the products that define the real and imaginary part of b and using

$$\|\vec{b}\|_{C^{1,\alpha}(\Omega^e)} \leqq \mathscr{F}_1\left(|\tilde{u}|_{\infty,\overline{\Omega^e}}, \|\tilde{u}\|_{C^{1,\alpha}(\overline{\Omega^e})}\right)$$

where $\mathscr{F}_1$ is smooth, monotonic in the first component, and with linear growth in the last component. Also,

$$\|\vec{b}\|_{C^\alpha(\overline{\Omega^e})} \leqq \mathscr{F}_2\left(|\tilde{u}|_{\infty,\overline{\Omega^e}}, \|\tilde{u}\|_{C^\alpha(\overline{\Omega^e})}\right)$$

where $\mathscr{F}_2$ is monotonic in each component.

Now putting these last estimates with the estimates from (3.14a) and (3.15b) for $\tilde{u}$ in $\overline{\Omega^e}$ yields

$$\|\vec{b}\|_{C^{1,\alpha}(\overline{\Omega^e})} \leqq e^{\nu^{-1}KC(K^*)}\widetilde{\mathscr{F}}_1(K, C(K^*), \|\varphi_0\|_{C^{1,\alpha}(\bar{\Omega})} + \|\tilde{\rho}_0\|_{C^\alpha(\Omega)}) \tag{3.16a}$$

and

$$\|\vec{b}\|_{C^\alpha(\Omega^e)} \leqq e^{\nu^{-1}KC(K^*)}\widetilde{\mathscr{F}}(K, C(K^*), \|\varphi_0\|_{C^{0,1}(\bar{\Omega})} + \|\tilde{\rho}_0\|_{C^{0,0}(\bar{\Omega})}). \tag{3.16b}$$

Finally, standard topological arguments yield the existence and regularity of $\vec{\nu} = \log\vec{b}$ such that $\vec{b}$ is a $C^{1,\alpha}(\overline{\Omega^e})$-vector field whose characteristic equations have solutions that cannot form closed curves.

Step 4: Existence of a Density ρ_1 in $C^{1,\alpha}(\overline{\Omega^e})$. We now prove Theorem 2.9, stated in Step 4. Let $\vec{b}$ be a $C^{1,\alpha}(\overline{\Omega^e})$-vector field from (3.13), Φ the solution of problem (3.1), Φ^e a $C^{1,\alpha}(\overline{\Omega^e})$-extension of Φ to $\overline{\Omega^e}$, $|F_{\mathbf{x}}|$ the Jacobian of the domain transformation, $|F_{\mathbf{x}}|^e$ a $C^{2,\alpha}(\overline{\Omega^e})$-extension of $|F_{\mathbf{x}}|$ to $\overline{\Omega^e}$, and θ_1^e the directional angle of the field $\vec{b} - \arg F_z$.

We solve for $h = h(x,y)$ the following first-order, quasi-linear equation with a directional field given by $\vec{b} = \vec{b}(x,y)$, $0 \leqq t \leqq 1$,

$$\begin{cases} \nabla h \cdot \vec{b} = -\frac{t}{\nu}|\vec{b}|^2 f\left((|F_{\mathbf{x}}|^e|\vec{b}|)^2, \theta_1^e, Q_B(\sigma, \exp h, \Phi^e)\right) = t\mathscr{F}(x,y,h) \\ h|_{\partial_1\Omega^e} = h_0 \end{cases} \tag{3.17}$$

where $h_0 = \ln\rho_0$ for $\rho_0 \in C^{1,\alpha}(\partial_1\Omega)$ given in (2.6) and σ is the value of the solution of the characteristic equations defined below by the characteristic field $\vec{b}$. Before we prove Theorem 2.9, we re-express it as follows:

THEOREM 3.3. *For each t, $0 \leqq t \leqq 1$, there exists a solution h in $C^{1,\alpha}(\overline{\Omega^e})$ of the initial value problem* (3.17) *that satisfies the following estimates for $\rho_1 = \exp h$:*

$$\|\rho_1\|_{C^\alpha(\overline{\Omega^e})} \leqq \mathscr{H}_1(\nu, K, C(K^*), \|\varphi_0\|_{C^{0,1}(\bar{\Omega})} + \|\tilde{\rho}_0\|_{C^{0,0}(\bar{\Omega})}, \|\varphi_1\|_{C^{1,\alpha}(\bar{\Omega})} + \|\Phi\|_{C^\alpha(\bar{\Omega})}) \tag{3.18a}$$

$$\|\rho_1\|_{C^{1,\alpha}(\overline{\Omega^e})} \leqq \mathscr{H}_2(\nu, K, C(K^*), \|\varphi_0\|_{C^{1,\alpha}(\bar{\Omega})} + \|\tilde{\rho}_0\|_{C^\alpha(\bar{\Omega})}, \|\varphi_1\|_{C^{2,\alpha}(\bar{\Omega})}, + \|\Phi\|_{C^{1,\alpha}(\bar{\Omega})}) \tag{3.18b}$$

and

$$0 < l_\nu \leqq \rho_1 \leqq L^* = i^{-1}(K^*) \quad in\ \overline{\Omega^e}, \tag{3.18c}$$

uniformly in t, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are smooth functions that are monotonic in the fourth variable and $\mathcal{H}_2$ is linear in its last variable. The constant K depends on the domain Ω, the boundary data of problem (2.1), *the Lipschitz norm of Q_B as a function of its variables, and the Lipschitz norm of $i'(\rho)$ and $R'(\varphi)$.*

Proof: In order to prove this theorem, we first recall the classical theory of local existence and regularity of first-order, quasi-linear equations.

Problem (3.17) is associated with the classical Cauchy problem for a first-order, quasi-linear equation. This problem consists of passing an integral curve of the direction field given by $\vec{b}$ through the initial curve $\partial_1\Omega^e$. This is a well-posed boundary value problem because $\vec{b}\cdot n|_{\partial_2\Omega^e\cup\partial_4\Omega^e} = 0$, $\vec{b}\cdot n|_{\partial_3\Omega^e} > 0$, and the solution is a function $h(x,y)$ such that the surface $z = h(x,y)$ in the xyz-space is $C^{1,\alpha}$.

We refer the reader to F. John [25], chapter 1, and Hartman [23], chapter VI, for standard methods of getting local C^1-solutions of (3.17) in Ω^e in the analysis below and for obtaining $C^{1,\alpha}$ solutions for ODEs.

We divide the proof into three parts. We first prove the local existence and uniqueness of $C^{1,\alpha}$-solutions of the Cauchy problem using the regularity of the right-hand side of (3.17) if the initial data are prescribed on a noncharacteristic curve Γ. Then the solution is shown to exist globally in Ω^e. The second part is the derivation of global supremum and infimum estimates for the solution $h(x,y)$. The last part shows $C^{1,\alpha}$-estimates independent of the parameter t.

Part 1: Let Γ be a curve in the xyz-space that lies in the integral surface $z = h(x,y)$, which we represent parametrically by $x = \gamma_1(r)$, $y = \gamma_2(r)$, and $z = \gamma_3(r)$, where r is to be chosen later and the functions $\gamma_i(r)$ are at least $C^{1,\alpha}$ in a neighborhood of a given r. By the regularity of the data for problem (3.17), the coefficients $b_1(x,y)$, $b_2(x,y)$, and $\mathcal{F}(x,y,z)$ are $C^{1,\alpha}$ in x, y, and z near $P_0 = (x_0,y_0,z_0) = (\gamma_1(r_0),\gamma_2(r_0),\gamma_3(r_0))$.

The integral surface $z = h(x,y)$ passing through Γ will consist of the characteristic curves of (3.17) starting at Γ. Accordingly, we form for each r near r_0 the unique solution

$$x = X(s,r), \qquad y = Y(s,r), \qquad z = Z(s,r), \tag{3.19}$$

of the characteristic differential equations $\frac{dx}{ds} = b_1(x,y)$ with $x(0) = x_0$ and $\frac{dy}{ds} = b_2(x,y)$ with $y(0) = y_0$, and $\frac{dz}{ds} = \mathcal{F}(x,y,z)$ with $z(0) = z_0$ for $(x_0,y_0,z_0) \in \Gamma$. The functions X, Y, and Z then satisfy

$$X_s = b_1(X,Y), \qquad Y_s = b_2(X,Y), \qquad Z_s = \mathcal{F}(X,Y,Z), \tag{3.20a}$$

identically in s and r and also satisfy the initial conditions

$$X(0,r) = \gamma_1(r), \qquad Y(0,r) = \gamma_2(r), \qquad Z(0,r) = \gamma_3(r). \tag{3.20b}$$

From the general theorems on existence and on continuous dependence on the parameters of solutions of systems of ODEs, it is easy to show that there exists a unique set of functions $X(s,r)$, $Y(s,r)$, and $Z(s,r)$ of class $C^{1,\alpha}$ for (s,r) near $(0,r_0)$ that satisfy (3.20a) and (3.20b).

Thus, equations (3.19) represent a surface $\Sigma : z = h(x,y)$ parametrized by s and r if we can change the variables x, y to s, r. Then the function h is defined by

$$z = h(x,y) = Z(S(x,y), R(x,y)) \tag{3.21}$$

where $s = S(x,y)$ and $r = R(x,y)$. This expression explicitly represents the surface Σ with, by (3.21), $x_0 = X(0,r_0)$ and $y_0 = Y(0,r_0)$.

Thus we can find solutions $s = S(x,y)$ and $r = R(x,y)$ of $x = X(S(x,y), R(x,y))$ and $y = Y(S(x,y), R(x,y))$ of class $C^{1,\alpha}$ in a neighborhood of (x_0, y_0) and satisfying $0 = S(x_0, y_0)$ and $r_0 = R(x_0, y_0)$ whenever the Jacobian

$$J = \begin{vmatrix} X_s(0,r_0) & X_s(0,r_0) \\ X_r(0,r_0) & Y_r(0,r_0) \end{vmatrix} = \begin{vmatrix} b_1(x_0,y_0) & b_2(x_0,y_0) \\ \gamma_1'(r_0) & \gamma_2'(r_0) \end{vmatrix} \geqq 0. \tag{3.22}$$

If we take the initial curve Γ to be the orthogonal curve to the characteristic field $\vec{b}$ at (x_0, y_0), then the tangent derivative to the curve Γ is given by $\gamma_1'(s_0) = -b_2(x_0, y_0)$ and $\gamma_2'(s_0) = b_1(x_0, y_0)$, so that the Jacobian condition for existence becomes

$$(b_1^2 + b_2^2)(x_0, y_0) \geqq \frac{k_\nu^2}{4} > 0 \tag{3.23}$$

by (3.15e).

Since by Theorem 3.1 this condition is satisfied for every (x_0, y_0) in Ω^e, the initial value problem has a unique solution in a neighborhood of any point in Ω^e, and the size of the neighborhood depends on the C^α-norm of $\log \vec{b}$, that is, on the C^α-norm of $\vec{b}$ and on $\inf_{\overline{\Omega^e}} |\vec{b}| \geqq \frac{k_\nu}{2}$.

Condition (3.23) ensures $C^{1,\alpha}$ local existence of problem (3.17) in neighborhoods contained in Ω along any particle paths that start at $\partial_1\Omega$, since $\vec{b}\,|_\Omega = \nabla\varphi_1$, and the right-hand side of the equation is $C^{1,\alpha}$ in x, y, and h.

The construction of a global solution h in Ω^e of problem (3.17) follows from the uniformity of the estimates given above. Take a point (x^0, y^0) in Ω^e. By condition (3.23), the characteristic equations can be solved uniquely in a neighborhood N_ν of (x^0, y^0), where $\operatorname{diam} N_\nu$ is uniform in (x,y) and depends on the $C^\alpha(\overline{\Omega^e})$-norm of $\log \vec{b} = \tilde{v}$, which can be estimated by (3.15b) for $\tilde{u}$.

Thus, let $\operatorname{diam} N_\nu = \mathcal{O}(\varepsilon)$ and $\varepsilon = \varepsilon(\nu, \|\tilde{v}\|_{C^\alpha(\overline{\Omega^e})})$. From (3.15b), $\|\tilde{v}\|_{C^\alpha(\overline{\Omega^e})} \leqq \|\tilde{u}\|_{C^\alpha(\overline{\Omega^e})}$, which is bounded depending on ν, the domain Ω and its extension, the data and boundary data of problem (2.1)–(2.10), and $C(K^*)$. Then Ω^e can be covered by $\mathcal{O}(\varepsilon^{-2})$-number of N_ν, where the diameters of the neighborhoods N_ν do not depend on the solution of the problem but just on the data associated with the boundary value problem (3.17) and on $C(K^*)$. Therefore, in at most $\mathcal{O}(\varepsilon^{-2})$-number of neighborhoods we solve the characteristic equations and obtain a global solution by overlapping rectified local solutions.

In addition, since the initial data $h_{\rm ini}$ is noncharacteristic (by condition (3.22)) and belongs to $C^{1,\alpha}(\partial_1\Omega)$, then the integral surface Σ solves (3.17) uniquely and is a $C^{1,\alpha}(\overline{\Omega}^e)$ surface.

Part 2: Next we derive the following L^∞-estimate for the solution h of problem (3.17).

LEMMA 3.4. (L^∞-ESTIMATE FOR $h = \ln\rho$ IN $\overline{\Omega^e}$) *The solution h of problem* (3.17) *satisfies*

$$\|h\|_{L^\infty(\overline{\Omega^e})} \leqq \sup_{\partial_1\Omega} \ln\rho_0 + \frac{1}{\nu}\mathcal{H}(K, C(K^*), \|\varphi_0\|_{C^{1,\alpha}(\bar\Omega)}, \|\tilde\rho_0\|_{C^\alpha(\bar\Omega)}) \tag{3.24}$$

where K depends on Ω and on the boundary data of the boundary value problem and $\mathcal{H}$ is a smooth function with linear growth in the third component. In addition, for f satisfying (2.3a)

$$0 < l_\nu \leqq \rho_1 \leqq L^* = i^{-1}(K^*) \quad in\ \overline{\Omega^e} \tag{3.25}$$

where l_ν depends on the upper bound of (3.24) *and L^* is independent of* $\sup \bar\Omega\tilde\rho_0$.

Proof: Using equation (3.17) for h, the $\|h\|_{\infty,\overline{\Omega^e}}$ can be estimated by

$$\sup_{\partial_1\Omega^e} |h_{\rm ini}| + \frac{1}{\nu} \sup_{\overline{\Omega^e_1}} |\mathcal{F}(x, |\vec b|, \theta, Q_B(\sigma, h, \Phi))| \exp(\|\log b\|_{C^\alpha(\overline{\Omega^e})}) = K_\nu .$$

From (2.3b), $\sup_{\overline{\Omega^e}} |f| \leqq \frac{kt}{\nu} C(K^*)$, where k depends on the number $\sum_{i=1}^3 |a_i|$ from estimates (3.14). Moreover, by (3.14b), (3.15b), and (3.16b), we have that $\exp\|\log b\|_{C^\alpha(\overline{\Omega^e})}$ is bounded above by $e^\nu\mathcal{H}(K, C(K^*), \|\varphi_0\|_{C^{1,\alpha}(\bar\Omega)}, \|\tilde\rho_0\|_{C^\alpha(\bar\Omega)}))$, where K depends on the domain and on the boundary data. Then (3.24) holds. In particular, $\exp(-K_\nu) \leqq \rho_1 \leqq \exp K_\nu$.

In order to show that the upper bound for ρ_1 is L^*, we need to use property (2.3a) on f combined with an argument by contradiction. We point out that the compatibility condition (2.7) assures that $K^* > 0$, so $L^* = i^{-1}(K^*) > 0$.

Let ρ_1 be a solution of (3.17). By condition (2.6) on the initial data, $\rho_1|_{\partial_1\bar\Omega} \leqq i^{-1}(K - \inf_{\bar\Omega}\mathcal{R}(\varphi) + q\sup_{\bar\Omega}\Phi) = i^{-1}(K^*)$. If the parameter $t = 0$, then the solution ρ_1 is constant along the characteristic projection curves, so that $\rho_1 \leqq \sup_{\partial_1\bar\Omega}(\rho_1|_{\partial_1\bar\Omega}) \leqq L^*$ by condition (2.9).

If the parameter t is positive, let us assume that there is a point (x_0, y_0) in $\overline{\Omega^e}$ such that $\rho_1(x_0, y_0) > L^* = i^{-1}(K^*)$. Then let $\sigma(s) = (X(s, r_0), Y(s, r_0))$ be a parametrization of a characteristic curve such that $(x_0, y_0) = (X(s_0, r_0)$,

$Y(s_0, r_0)) = \sigma(s_0)$ is the first point along the curve $\sigma(s)$ in the direction of increasing t where $\rho_1(x_0, y_0) > L^*$. Then $\frac{\partial \rho_1}{\partial s}(\sigma(s_0)) \geqq 0$, or equivalently,

$$((\rho_1)_x b_1 + (\rho_1)_y b_2)(x_0, y_0) \geqq 0. \tag{3.26}$$

On the other hand, by the definition of the function Q_B in (2.2), if $\rho_1(\sigma(s_0)) > L^*$ then

$$Q_B(\rho_1(\sigma(s_0)), \varphi_1(\sigma(s_0)), \Phi(\sigma(s_0))) < 0,$$

so that, by (2.3a),

$$tf((|F_{\mathbf{x}}|^e|\vec{b}|)^2(\sigma(s_0)),\ \theta_1^e(\sigma(s_0)),\ Q_B(\sigma(s_0)),\ \exp h(\sigma(s_0)),\ \Phi^e(\sigma(s_0))) > 0.$$

Since ρ_1 satisfies the ODE (3.17) or, equivalently, (3.20a) along the characteristic curves, then evaluating equation (3.17) at the point $\sigma(s_0)$ yields

$$\nabla\rho_1 \cdot \vec{b}(x_0, y_0) = -\frac{1}{\nu} tf((|F_{\mathbf{x}}|^e|\vec{b}|)^2, \theta_1^e, Q_B(\sigma, \exp h, \Phi^e))(x_0, y_0) < 0,$$

which contradicts (3.26). Therefore $\rho_1 \leqq L^*$ at any point in $\overline{\Omega^e}$ independently of the parameter t, so that (3.24) and thus (3.18b) hold.

Part 3: Here we develop $C^{1,\alpha}(\overline{\Omega^e})$ estimates for the solution ρ_1 of problem (3.17) and prove the lemma in Step 5 for ρ_1. The object is to obtain good enough estimates to use in the fixed point theorem (i.e., estimates (3.18)). It is enough to obtain estimates for $h = \ln \rho_1 \in C^{1,\alpha}(\overline{\Omega^e})$ for any $C^{1,\alpha}$-initial data given along orthogonal curves to the field $\vec{b}$. It is convenient to use again the coordinates s and r used in (3.19), and it is easily seen that they can be used as global coordinates.

Thus $h(x, y) = z(s, r)$ satisfies

$$z_s = \mathscr{F}(s, r, z), \qquad z(0, r) = \gamma_3(r) = h(\gamma_1(r), \gamma_2(r)). \tag{3.27}$$

Making use of estimates (3.16b), the change of coordinates $(x, y) \to (s, r)$ leads to the estimates

$$\begin{aligned} \|s, r\|_{C^0(\mathscr{B})} &\leqq K \sup_{\overline{\Omega^e}} \left(|J_{(x,y),(s,t)}|^{-1}\right) \|\vec{b}\|_{C^0(\mathscr{B})} \\ &\leqq \mathscr{G}_1(k_\nu^{-1}, K_\nu, K) \leqq \mathscr{G}_1(\nu, K, C(K^*)) \end{aligned} \tag{3.28a}$$

$$\begin{aligned} \|s, r\|_{C^\alpha(\mathscr{B})} &\leqq K\mathscr{G}_2\left(\|J^{-1}_{(x,y),(s,t)}\|_{\mathrm{Lip}},\ \|\vec{b}\|_{C^\alpha(\overline{\Omega^e})}\right) \\ &\leqq \mathscr{G}_3\left(\nu,\ K,\ C(K^*),\ \|\varphi_0\|_{C^{0,1}(\bar{\Omega})} + \|\tilde{\rho}_0\|_{C^{0,0}(\bar{\Omega})}\right) \end{aligned} \tag{3.28b}$$

$$\begin{aligned} \|s, r\|_{C^{1,\alpha}(\mathscr{B})} &\leqq K\mathscr{G}_4\left(\|J^{-1}_{(x,y),(s,t)}\|_{C^{1,1}},\ \|\vec{b}\|_{C^{1,\alpha}(\overline{\Omega^e})}\right) \\ &\leqq \mathscr{G}_5\left(\nu,\ K,\ C(K^*),\ \|\varphi_0\|_{C^{1,\alpha}(\bar{\Omega})} + \|\tilde{\rho}_0\|_{C^{\alpha}(\bar{\Omega})}\right) \end{aligned} \tag{3.28c}$$

with $\mathcal{G}_i$, $i = 1,2,3,4,5$, smooth functions of their arguments, $\mathcal{G}_5$ having linear growth in its last argument, and K depending on the data of the original boundary value problem (2.1).

In order to obtain the $C^{1,\alpha}$-estimates of h, we shall make use of the following auxiliary lemma, which is easily proven (see Appendix).

LEMMA 3.5. *Let z be a solution in a domain $\mathcal{B}$ of problem* (3.27) *with* $z(0,r) \in C^\alpha(\Gamma)$. *Then z is $C^\alpha(\bar{\mathcal{B}})$ and*

$$\|z\|_{C^\alpha(\bar{\mathcal{B}})} \leqq \|\gamma_3\|_{C^\alpha(\Gamma)} + \|s\|_{C^0(\bar{\mathcal{B}})} \exp\left\{\|s\|_{C^0(\bar{\mathcal{B}})} \|\mathcal{F}\|_{\mathrm{Lip},z}\right\} \|\mathcal{F}\|_{C^\alpha(\bar{\mathcal{B}})}. \tag{3.29}$$

Now we can show estimates (3.18a) and (3.18b).

LEMMA 3.6. *Estimates* (3.18a) *and* (3.18b) *hold for* $\rho_1 = \exp h$ *in* $\overline{\Omega^e}$, *uniformly in t.*

Proof: First, from the previous lemma we have local estimates for $\rho_1 = \exp h(x,y) = \exp z(s,r)$. Next, the boundary value problem (3.17) is well-posed in $\overline{\Omega^e}$, because $\partial_1\Omega^e = \partial_1\Omega$ is an inflow boundary, $\partial_2\Omega^e$ and $\partial_4\Omega^e$ are characteristic curves, and $\partial_3\Omega^e$ is an outflow boundary. Since the Jacobian of the change of variables $J_{(x,y),(s,r)}|_{\overline{\Omega^e}} \geqq |\vec{b}|\,|_{\overline{\Omega^e}} \geqq \frac{k_\nu}{2}$, uniformly in $\overline{\Omega^e}$, then by combining estimates (3.28a) and (3.29), we get

$$\|\rho_1\|_{C^\alpha(\overline{\Omega^e})} \leqq K\left(\|\rho_1\|_{C^\alpha(\partial_1\Omega)} + \tilde{\mathcal{G}}_1(\nu, K, C(K^*), \|\mathcal{F}\|_{\mathrm{Lip},\rho})\|\mathcal{F}\|_{C^\alpha(\overline{\Omega^e})}\right)$$

Since $\mathcal{F}(s,r) = -\frac{t}{\nu}|\vec{b}|^2 f((|F_{\mathbf{x}}|^e|\vec{b}|)^2, \theta_1^e, Q_B((s,r), \exp h, \Phi^e))$, using the estimates (3.16b) for the C^α-norm of $\vec{b}$ yields

$$\begin{aligned}\|\mathcal{F}\|_{C^\alpha(\bar{\Omega}^e)} \leqq \mathcal{H}(\nu,\ K,\ C(K^*),\ &\|\varphi_0\|_{C^{0,1}(\bar{\Omega})} + \|\tilde{\rho}_0\|_{C^{0,0}(\bar{\Omega})},\\ &\|\varphi_1\|_{C^{1,\alpha}(\bar{\Omega})} + \|\Phi\|_{C^\alpha(\bar{\Omega})})\end{aligned} \tag{3.30}$$

where $\mathcal{H}$ is smooth and K depends on $\|F_{\mathbf{x}}|_{C^\alpha(\bar{\Omega})}$ and the data of the boundary value problem. By combining these last two estimates (3.18a) holds.

Next we find a C^α-estimate for $\nabla\rho_1$. Again we use equation (3.27) for $z(s,r) = h(x(s,r), y(s,r)) = \ln \rho_1(x(s,r), y(s,r))$. We first show local estimates for $(\ln\rho_1)_s$ and $(\ln\rho_1)_r$ in $C^\alpha(\bar{\mathcal{B}})$ working in the (s,r)-plane. Then using the change of variables to the (x,y)-space and estimates (3.28a) and (3.28b), we derive C^α-estimates for $\nabla\rho_1$.

Thus, we start obtaining an estimate for $\|z_s\|_{C^\alpha(\bar{\mathcal{B}})}$. Differentiating the equation (3.27) with respect to s and replacing z_s by $\mathcal{F}$, we get

$$(z_s)_s = \frac{d}{ds}\mathcal{F}(s,r,z) = \frac{\partial\mathcal{F}}{\partial s} + \frac{\partial\mathcal{F}}{\partial z}\mathcal{F} = \widetilde{\mathcal{F}}(s,r). \tag{3.31}$$

Now, since $\frac{\partial \mathcal{F}}{\partial s}$ and $\frac{\partial \mathcal{F}}{\partial z}$ are C^α in $\bar{\mathcal{B}}$, applying Lemma 3.4 to z_s from (3.29) gives the estimate

$$\|z_s\|_{C^\alpha(\bar{\mathcal{B}})} \leqq \|\gamma_3'\|_{C^\alpha(\Gamma)} + \|s\|_{C^0(\bar{\mathcal{B}})} \left\| \frac{\partial \mathcal{F}}{\partial s} + \frac{\partial \mathcal{F}}{\partial z}\mathcal{F} \right\|_{C^\alpha(\bar{\mathcal{B}})} . \tag{3.32}$$

Thus, we need to estimate both $\frac{\partial \mathcal{F}}{\partial s}$ and $\frac{\partial \mathcal{F}}{\partial z}$ in $C^\alpha(\bar{\mathcal{B}})$. Indeed, the function $\mathcal{F}(s,r,z) = \frac{t}{\nu} f((|F_{\mathbf{x}}|^e|\vec{b}|)^2, \theta_1, Q_B)$ satisfies

$$\begin{aligned} \frac{\partial \mathcal{F}}{\partial s} &= \frac{t}{\nu}\left(\nabla f \cdot \vec{b}\right) \\ &= \frac{\partial f}{\partial(|F_{\mathbf{x}}|^e|\vec{b}|)^2}\nabla(|F_{\mathbf{x}}|^e|\vec{b}|)^2 \cdot \vec{b} + \frac{\partial f}{\partial \theta_1}\nabla\theta_1 \cdot \vec{b} + \frac{\partial f}{\partial Q_B}\nabla Q_B \cdot \vec{b}, \end{aligned} \tag{3.33}$$

and since $0 \leqq \rho_1 \leqq L^*$, we have

$$\nabla Q_B \cdot \vec{b} = -i'(\rho_1)\rho_1 \frac{t}{\nu} f((|F_{\mathbf{x}}|\,|\vec{b}|)^2, \theta_1, Q_B) - R'(s,r)\vec{b}^2 + q\nabla\Phi \cdot \vec{b}.$$

Also,

$$\frac{\partial \mathcal{F}}{\partial z} = \frac{\partial f}{\partial Q_B}(Q_B)_z = -\frac{\partial f}{\partial Q_B} i'(\rho_1)\rho_1 . \tag{3.34}$$

Recalling that f is locally Lipschitz on each of its variables, then $\frac{\partial f}{\partial(|F_{\mathbf{x}}|^e|\vec{b}|)^2}$, $\frac{\partial f}{\partial \theta_1}$, and $\frac{\partial f}{\partial Q_B}$ are bounded depending on the bounds for the arguments of f.

Also, $\rho_1 \in C^\alpha(\overline{\Omega^e})$, $|F_{\mathbf{x}}|^e|\vec{b}|$ and θ_1 are $C^{1,\alpha}(\overline{\Omega^e})$, $\Phi \in C^{1,\alpha}(\overline{\Omega^e})$, and the functions i' and R' are locally Lipschitz; therefore

$$\mathcal{F}, \frac{\partial \mathcal{F}}{\partial s}, \frac{\partial \mathcal{F}}{\partial z} \in C^\alpha(\overline{\Omega^e}). \tag{3.35}$$

In addition, collecting estimates (3.30), (3.7), (3.18a), and (3.2a) gives us

$$\begin{aligned} \left\| \frac{\partial \mathcal{F}}{\partial s} + \frac{\partial \mathcal{F}}{\partial z}\mathcal{F} \right\|_{C^\alpha(\bar{\mathcal{B}})} \leqq C \big(&\nu,\ K,\ C(K^*),\ \|\varphi_0\|_{C^{1,\alpha}(\bar{\Omega})} + \|\tilde{\rho}_0\|_{C^\alpha(\bar{\Omega})}, \\ &\|\varphi_1\|_{C^{2,\alpha}(\bar{\Omega})} + \|\Phi\|_{C^{1,\alpha}(\bar{\Omega})}\big) , \end{aligned} \tag{3.36}$$

with C a smooth function of its variables that has linear growth in its fifth variable.

Therefore, using (3.28a) and (3.36), we can estimate $(\ln \rho_1)_s$ from (3.32) to obtain

$$\begin{aligned} \|(\ln \rho_1)_s\|_{C^\alpha(\bar{\mathcal{B}})} \leqq \mathcal{C} \big(&\nu,\ K,\ C(K^*),\ \|\varphi_0\|_{C^{1,\alpha}(\bar{\Omega})} + \|\tilde{\rho}_0\|_{C^\alpha(\bar{\Omega})}, \\ &\|\varphi_1\|_{C^{2,\alpha}(\bar{\Omega})} + \|\Phi\|_{C^{1,\alpha}(\bar{\Omega})}\big) . \end{aligned} \tag{3.37}$$

Once again $\mathscr{C}$ is a smooth function of its variables that has linear growth in its fifth variable, and K depends on the $C^{1,\alpha}$-norm of $|F_{\mathbf{x}}|$ and the density r prescribed on $\partial_1\Omega$ from (2.9).

Next, we obtain an estimate for $\|z_r\|_{C^\alpha(\mathscr{B})}$. Here we differentiate equation (3.27) with respect to r, so that z_r satisfies

$$(z_r)_s = \frac{\partial\mathscr{F}}{\partial r} + \frac{\partial\mathscr{F}}{\partial z}z_r = \widetilde{\mathscr{F}}(s,r,z_r) \tag{3.38}$$

This is an ODE for z_r.

Since $\frac{\partial\mathscr{F}}{\partial r} + \frac{\partial\mathscr{F}}{\partial z}z_r$ is C^α in $\bar{\mathscr{B}}$ and Lipschitz in z_r, as we shall see below, applying Lemma 3.4 to z_r and using (3.29) yields the estimate

$$\begin{aligned}\|z_r\|_{C^\alpha(\bar{\mathscr{B}})} &\leqq \|\gamma_3'\|_{C^\alpha(\Gamma)} \\ &+ \|s\|_{C^0(\bar{\mathscr{B}})}\exp\left\{\|s\|_{C^0(\bar{\mathscr{B}})}\left\|\frac{\partial\mathscr{F}}{\partial r} + \frac{\partial\mathscr{F}}{\partial z}z_r\right\|_{\mathrm{Lip},z_r}\right\}\left\|\frac{\partial\mathscr{F}}{\partial r} + \frac{\partial\mathscr{F}}{\partial z}z_r\right\|_{C^\alpha(\bar{\mathscr{B}})}\end{aligned} \tag{3.39}$$

Thus, we need to compute in detail $\frac{\partial\mathscr{F}}{\partial r}$. As in (3.33), for $X(s,r)$ and $Y(s,r)$ the solution of the characteristic equations,

$$\begin{aligned}\frac{\partial\mathscr{F}}{\partial r} &= \frac{t}{\nu}\left(\nabla f\cdot(X_r,Y_r)\right) \\ &= \frac{\partial f}{\partial(|F_{\mathbf{x}}|^e|\vec{b}|)^2}\nabla(|F_{\mathbf{x}}|^e|\vec{b}|)^2\cdot(X_r,Y_r) \\ &+ \frac{\partial f}{\partial\theta_1}\nabla\theta_1\cdot(X_r,Y_r) + \frac{\partial f}{\partial Q_B}\nabla Q_B\cdot(X_r,Y_r)\end{aligned} \tag{3.40}$$

where here the term $\nabla Q_B\cdot(X_r,Y_r) = -i'(e^z)z_r - \nabla R(s,r)\cdot(X_r,Y_r) + q\nabla\Phi\cdot(X_r,Y_r)$.

Since X_r, Y_r satisfies the system of equations

$$\begin{aligned}(X_r)_s &= \nabla b_1\cdot(X_r,Y_r)\\ (Y_r)_s &= \nabla b_2\cdot(X_r,Y_r),\end{aligned}$$

by using a similar estimate to that in (3.29), for systems of ODEs we get

$$\|X_r,Y_r\|_{C^\alpha(\bar{\mathscr{B}})} \leqq \|\gamma_3'\|_{C^\alpha(\Gamma)} + \|s\|_{C^0(\bar{\mathscr{B}})}\exp\left\{\|s\|_{C^0(\bar{\mathscr{B}})}\|\vec{b}\|_{\infty,\mathscr{B}}\right\}\|\vec{b}\|_{C^\alpha(\bar{\mathscr{B}})}. \tag{3.41}$$

By combining estimates (3.28a), (3.28b), and (3.41) along with (3.16) for $\vec{b}$ and θ_1^e, (3.2a) for Φ, (3.34) for $\mathscr{F}$, and (3.18a) for $\rho_1 = e^z$, the right-hand side of (3.40) yields the estimate

$$\begin{aligned}\left\|\frac{\partial\mathscr{F}}{\partial r}\right\|_{C^\alpha(\bar{\mathscr{B}})} &\leqq C(\nu,K,C(K^*),\|\varphi_0\|_{C^{1,\alpha}(\bar{\Omega})} + \|\tilde{\rho}_0\|_{C^\alpha(\bar{\Omega})}, \\ &\qquad \|\varphi_1\|_{C^{2,\alpha}(\bar{\Omega})} + \|\Phi\|_{C^{1,\alpha}(\bar{\Omega})}),\end{aligned} \tag{3.42}$$

with C a smooth function of its variables that has linear growth in its fifth variable, and K depends on the data of the boundary value problem, which includes the dependence of the Lipschitz norms of the derivatives of the functions G, R, i, and f_1 and the $C^{1,\alpha}$-norm of $|F_{\mathbf{x}}|$. In addition,

$$\left\|\frac{\partial \mathscr{F}}{\partial r}\right\|_{\mathrm{Lip}, z_r} \leqq \left\|\frac{\partial f}{\partial Q_B} i'(\rho_1)\right\|_{\infty, \bar{\mathscr{B}}} \leqq C_1 \|\rho_1\|_{\infty, \bar{\mathscr{B}}} \leqq C_1 \ln L^* \tag{3.43}$$

with C_1 depending on locally Lipschitz norms of f and i'.

Then combining estimate (3.42) with estimates for (3.34) we get

$$\left\|\frac{\partial \mathscr{F}}{\partial r} + \frac{\partial \mathscr{F}}{\partial z} z_r\right\|_{C^\alpha(\bar{\mathscr{B}})} \leqq C\left(\nu, K, C(K^*), \|\varphi_0\|_{C^{1,\alpha}(\bar{\Omega})} + \|\tilde{\rho}_0\|_{C^\alpha(\bar{\Omega})}, \|\varphi_1\|_{C^{2,\alpha}(\bar{\Omega})} + \|\Phi\|_{C^{1,\alpha}(\bar{\Omega})}\right), \tag{3.44}$$

with C as in (3.42).

Next, combining (3.34) with (3.43) gives us

$$\left\|\frac{\partial \mathscr{F}}{\partial r} + \frac{\partial \mathscr{F}}{\partial z} z_r\right\|_{\mathrm{Lip}, z_r} \leqq C_2 \ln L^* \tag{3.45}$$

with C_2 depending on C_1 from (3.43).

Hence, we can estimate z_r using (3.28a), (3.44), and (3.45) in order to estimate the right-hand side of (3.39) so that

$$\|(\ln \rho_1)_r\|_{C^\alpha(\bar{\mathscr{B}})} \leqq \mathscr{C}\left(\nu, K, C(K^*), \|\varphi_0\|_{C^{1,\alpha}(\bar{\Omega})} + \|\tilde{\rho}_0\|_{C^\alpha(\bar{\Omega})}, \|\varphi_1\|_{C^{2,\alpha}(\bar{\Omega})} + \|\Phi\|_{C^{1,\alpha}(\bar{\Omega})}\right), \tag{3.46}$$

with $\mathscr{C}$ a smooth function of its variables that has linear growth in its fifth variable and K depending on the data of the boundary value problem, which includes the dependence of the Lipschitz norms of the derivatives of the functions G, R, i, f_1, the $C^{1,\alpha}$-norm of $|F_{\mathbf{x}}|$, and the density r prescribed on $\partial_1\Omega$ from (2.9).

In addition, we can get an L^∞-estimate for $\nabla_{s,r}\rho_1$ that does not depend on the $C^{1,\alpha}$-norm of $\vec{b}$ by combining estimates (2.28a) with estimates (3.16b) to obtain a bound that depends on the C^1-norm of $\vec{b}$ (that is, estimate (3.16b) uniformly in α) so that

$$\begin{aligned} \|\nabla_{s,r}\rho_1\|_{C^0(\bar{\mathscr{B}})} &\leqq \|s\|_{C^0(\overline{\Omega^\epsilon})} \|\nabla \mathscr{F}\|_{\infty, \overline{\Omega^\epsilon}} \\ &\leqq C\left(\nu, K, C(K^*), \|\varphi_0\|_{C^{1,\alpha}(\bar{\Omega})} + \|\tilde{\rho}_0\|_{C^\alpha(\bar{\Omega})}, \|\varphi_1\|_{C^{1,1}(\bar{\Omega})} + \|\Phi\|_{C^1(\bar{\Omega})}\right), \end{aligned} \tag{3.47}$$

where C has the same dependence as $\mathscr{C}$ from (3.46).

Finally, since $\nabla_{x,y}\rho_1 = J^{-1}_{(x,y),(s,r)} \cdot \nabla_{s,r}\rho_1$, by using estimates (3.28b) and (3.23c) to estimate the terms in the Jacobian of the transformation and in (3.47), we get

$$
\begin{aligned}
&\|\nabla_{x,y}\rho_1\|_{C^\alpha(\bar{\mathscr{B}})} \\
&\leqq \|s,r\|_{C^{0,1}(\bar{\mathscr{B}})}\|\nabla_{s,r}\rho_1\|_{C^\alpha(\bar{\mathscr{B}})} + \|s,r\|_{C^{1,\alpha}(\bar{\mathscr{B}})}\|\nabla_{s,r}\rho_1\|_{C^0(\bar{\mathscr{B}})} \\
&\leqq \mathscr{H}_2\big(\nu,\ K,\ C(K^*),\ \|\varphi_0\|_{C^{1,\alpha}(\bar\Omega)} + \|\tilde\rho_0\|_{C^\alpha(\bar\Omega)}, \\
&\qquad \|\varphi_1\|_{C^{2,\alpha}(\bar\Omega)} + \|\Phi\|_{C^{1,\alpha}(\bar\Omega)}\big),
\end{aligned} \tag{3.48}
$$

with, once again, $\mathscr{H}_2$ a smooth function of its variables that has linear growth in its fifth variable and K depending on the data of the boundary value problem, which includes the dependence of the Lipschitz norms of the derivatives of the functions G, R, i, and f_1, the $C^{1,\alpha}$-norm of $|F_{\mathbf{x}}|$, and the density r prescribed on $\partial_1\Omega$ from (2.9).

Therefore, taking $\mathscr{B} = \Omega^e$ as done above for the C^α-estimate of ρ_1, we get that (3.18b) holds, and then Theorem 3.3 is proven.

Step 5: The Fixed Point Map. As in Step 5 in Section 2, for any parameter t, $0 \leqq t \leqq 1$, we define a compact continuous map $T_t(w,u) = (\rho,\varphi)$ as in (2.25), that is, given by the construction from Step 1 to Step 4, for (w,u) in $\mathscr{U}_{M+1,c_0} \subset C^\alpha(\bar\Omega)\times C^{1,\alpha}(\bar\Omega) = \mathscr{B}$, where $\mathscr{U}_{M+1,c_0} = \{(\rho,\varphi)\in\mathscr{B}: \|(\rho,\varphi)\|_{\mathscr{B}} \leqq M+1, |\nabla u| > c_0/2\}$.

In order to have the map T_t precompact, we show that the a priori estimates (2.27) and (2.28) hold. In fact, from (3.7b) and (3.18b), we have that

$$
\begin{aligned}
\|\rho\|_{C^{1,\alpha}(\bar\Omega)} + \|\varphi\|_{C^{2,\alpha}(\bar\Omega)} \leqq \tilde{\mathscr{H}}\big(\nu,\ K,\ C(K^*, \sup_{\bar\Omega} w^+),& \\
\|\Phi\|_{C^{1,\alpha}(\bar\Omega)},\ \|w\|_{C^\alpha(\bar\Omega)} + \|u\|_{C^{1,\alpha}(\bar\Omega)}\big)&
\end{aligned} \tag{3.49}
$$

where K depends on the Lipschitz norms of the derivatives of f, R, G, f_1, and i as a function of ρ, the domain, and the boundary data. In addition, $\mathscr{H}$ has linear growth in its last component. Thus (2.28) holds and T_t is precompact in $\mathscr{B}$, independent of t.

Now if we are given a $(\tilde\rho,\tilde\varphi)\in\mathscr{B}$ such that $(\tilde\rho,\tilde\varphi) = T_t(\tilde\rho,\tilde\varphi)$ for $0\leqq t\leqq 1$, then $\tilde\varphi$ solves problem (3.6) and satisfies estimates (3.7)–(3.8) and $\tilde\rho$ solves problem (3.17) and satisfies estimates (3.18). Therefore, as in (3.49),

$$
\begin{aligned}
\|\tilde\rho\|_{C^{1,\alpha}(\bar\Omega)} + \|\tilde\varphi\|_{C^{2,\alpha}(\bar\Omega)} \leqq \tilde{\mathscr{H}}\big(\nu,\ K,\ C(K^*, \sup_{\bar\Omega}\tilde\rho^+),& \\
\|\tilde\rho^{+,L^*}\|_{C^\alpha(\bar\Omega)} + \|\tilde\varphi\|_{C^{1,\alpha}(\bar\Omega)} + \|\Phi\|_{C^{1,\alpha}(\bar\Omega)}\big).&
\end{aligned} \tag{3.50}
$$

uniformly in t.

By (3.18c), $0 < \tilde\rho \leqq L^* = i^{-1}(K^*)$ so that $\tilde\rho^{+,L^*} = \tilde\rho$. Then, combining this result with (3.8) yields the estimates

$$
\|\tilde\rho\|_{C^0(\bar\Omega)} \leqq i^{-1}(K^*) \quad\text{and}\quad \|\tilde\varphi\|_{C^{0,1}(\bar\Omega)} \leqq C_1 = C_1(\nu,K,C(K^*)),
$$

where K^* is as defined in (2.7a) or (3.4), and K depends only on the data of the boundary value problem for system (2.1).

Next, putting together the above estimate with (3.7a) yields

$$\|\tilde{\varphi}\|_{C^{1,\alpha}(\bar{\Omega})} \leqq \mathscr{H}(\nu, K, C_1, C(K^*)) \tag{3.51}$$

and, from (3.2a) and (3.2b),

$$\|\Phi\|_{C^{\alpha}(\bar{\Omega})} \leqq C_1 = C_1(\nu, K, C(K^*)).$$

Hence, from (3.18a) and the above estimates,

$$\|\tilde{\rho}\|_{C^{\alpha}(\bar{\Omega})} \leqq \tilde{\mathscr{H}}(\nu, K, C_1, C(K^*)). \tag{3.52}$$

Thus, from (3.50) combined with (3.51) and (3.52), the following estimate holds:

$$\|\tilde{\rho}\|_{C^{\alpha}(\bar{\Omega})} + \|\tilde{\varphi}\|_{C^{1,\alpha}(\bar{\Omega})} \leqq \tilde{\mathscr{F}}(\nu, K, C_1, C(K^*)) = \dot{M},$$

uniformly in t, where K, C_1, and $C(K^*)$ depend on Ω and the boundary data.

Therefore, the right-hand side of (3.50) is uniformly controlled by the number M, which depends on ν, the domain, the boundary data, and the growth of f but is independent of $(\tilde{\rho}, \tilde{\varphi})$ uniformly in t. In addition, $|\nabla\tilde{\varphi}| > k_\nu > 0$. Therefore, $(\tilde{\rho}, \tilde{\varphi}) \in \mathscr{U}_{M+1,k_\nu}$ and the map $I - T_t$ has no fixed points on $\partial\mathscr{U}$ for any t.

Also, the problem $I - T_0$ admits a solution in $\mathscr{U}_{M+1,k_\nu}$. Indeed, the problem

$$\begin{cases} \operatorname{div}(\tilde{\rho}_0 \nabla\tilde{\varphi}_0) = 0 \\ \nabla \ln \tilde{\rho}_0 \nabla\tilde{\varphi}_0 = 0 \end{cases}$$

has a solution $\tilde{\rho}_0 = \text{const}$ along the streamlines, so that $\tilde{\rho}_0$ is $C^{1,\alpha}$ across the streamlines and $\tilde{\varphi}_0$ is the $C^{2,\alpha}$-harmonic solution of the boundary value problem in Ω. In particular, the a priori estimate holds, so that $(\tilde{\rho}_0, \tilde{\varphi}_0)$ is in the interior of the set $\mathscr{U}_{M+1,k_\nu}$. Therefore, estimate (2.29) holds, so that T_t, for all t, is a compact homotopy with no fixed points on the boundary of $\mathscr{U}_{M+1,k_\nu}$.

Finally, in order to see that T is continuous, we take a convergent sequence (w_n, u_n) to (w_0, u_0) in the Banach space $\mathscr{B}$. Then, given an $\varepsilon > 0$, there is an $n_0 = n_0(\varepsilon)$ such that

$$\|w_n - w_0\|_{C^{\alpha}(\bar{\Omega})} + \|u_n - w_0\|_{C^{1,\alpha}(\bar{\Omega})} \leqq \varepsilon \quad \text{for all } n \geqq n_0. \tag{3.53}$$

Since the function $\vec{h}(w,u) = f(|\nabla u|^2, w, u)\nabla u$ is Lipschitz continuous in all its components,

$$\|\vec{h}(w_n, u_n) - \vec{h}(w_0, u_0)\|_{C^{\alpha}(\bar{\Omega})} \to 0 \quad \text{as } n \to \infty. \tag{3.54}$$

Now, let $(\rho_n, \varphi_n) = T(w_n, u_n)$ and $(\rho_0, \varphi_0) = T(w_0, u_0)$. We want to see that $\lim_{n\to\infty}(\rho_n, \varphi_n) = (\rho_0, \varphi_0)$. In fact, it follows from [13] that the construction of the *unique* solution of problem (3.6) defines a continuous operator $T'(w, u_n) = \varphi_n$ from $C^{\alpha}(\bar{\Omega}) \times C^{1,\alpha}(\bar{\Omega})$ to $C^{2,\alpha}(\bar{\Omega})$. Therefore

$$(3.55)\qquad \begin{aligned} \|\varphi_n - \varphi_0\|_{C^{2,\alpha}(\bar{\Omega})} &= \|T'(w_n, u_n) - T'(w_0, u_0)\|_{C^{2,\alpha}(\bar{\Omega})} \\ &\leqq \|T'(w_n, u_n) - T'(w_n, u_0)\|_{C^{2,\alpha}(\bar{\Omega})} \\ &\quad + \|T'(w_n, u_0) - T'(w_0, u_0)\|_{C^{2,\alpha}(\bar{\Omega})} \\ &\to 0 \quad \text{as } n \to 0 . \end{aligned}$$

Thus, $\|\nabla\varphi_n - \nabla\varphi_0\|_{C^{1,\alpha}(\bar{\Omega})} \to 0$, so that $X_n(s,r), Y_n(s,r)$ and $X_0(s,r), Y_0(s,r)$, the solutions of the characteristic equations defined in (3.20a) and (3.20b) that correspond to $\nabla\varphi_n$ and $\nabla\varphi_0$, respectively, satisfy

$$\|X_n - X_0\|_{C^{2,\alpha}(\bar{\Omega})} + \|Y_n - Y_0\|_{C^{2,\alpha}(\bar{\Omega})} \to 0 \quad \text{as } n \to \infty .$$

Therefore, if ρ_n and ρ_0 are $C^{1,\alpha}(\overline{\Omega^e})$-solutions constructed in (3.17), then

$$(3.56)\qquad \begin{aligned} \|\rho_n - \rho_0\|_{C^{1,\alpha}(\bar{\Omega})} &\leqq \|\rho_n(X_n, Y_n) - \rho_0(X_n, Y_n)\|_{C^{1,\alpha}(\bar{\Omega})} \\ &\quad + \|\rho_0(X_n, Y_n) - \rho_0(X_0, Y_0)\|_{C^{1,\alpha}(\bar{\Omega})} \to 0 \end{aligned}$$

as the first term vanishes by uniqueness of solutions for problem (3.17) for a given characteristic field with the same boundary data, and the second term converges to 0 because ρ_0 is $C^{1,\alpha}(\bar{\Omega})$.

From (3.55) and (3.56),

$$(3.57)\qquad \|\rho_n - \rho_0\|_{C^{1,\alpha}(\bar{\Omega})} + \|\varphi_n - \varphi_0\|_{C^{2,\alpha}(\bar{\Omega})} \to 0 \quad \text{as } n \to \infty ,$$

and this implies that the map T is continuous.

By the Leray-Schauder theorem, T has a fixed point (ρ, φ) in the set $\mathcal{U}_{M+1,k_\nu}$. Hence, for each ν fixed, (ρ^ν, φ^ν) and the corresponding Φ^ν that solves problem (3.1) and (3.2) yield a solution of the system that corresponds to the conformally transformed data from (2.1)–(2.3) with the corresponding transformed boundary data from (2.4)–(2.10) that lies in $C^{1,\alpha}(\bar{\Omega})$, $C^{2,\alpha}(\bar{\Omega})$, and $W^{2,p}(\bar{\Omega})$, $1 \leqq p < \infty$, respectively.

This is the best possible regularity if $C(x)$ is a step function. However, using a bootstrapping argument, if the doping profile function $C(x)$ is $C^\infty(\bar{\Omega})$ and the boundary sections C^∞, then ρ^ν, φ^ν, and Φ^ν are all $C^\infty(\bar{\Omega})$.

4. Uniform Bound for the Speed in $\bar{\Omega}$

Step 6. We show under the assumptions (2.8a) a ν-uniform estimate $|\nabla\varphi^\nu|$ in the original domain described in Figure 2.1.

Uniform bounds for the density ρ^ν have already been obtained in (3.18c), and combining this estimate with (3.2b) implies that the electric potential Φ^ν and the electric field are also bounded uniformly in ν.

The following restatement of Theorem 2.15 of Section 2 is a variation of a lemma due to Morawetz [31]:

THEOREM 4.1. MORAWETZ'S LEMMA (UNIFORM BOUND FOR THE SPEED) *Let* $(\rho^\nu, \varphi^\nu, \Phi^\nu)$ *be a solution of the boundary value problem* (2.1)–(2.10), *where* f *satisfies* (2.11) *with the function* G *satisfying the growth conditions* (1.8b) *and* $\inf f_1 > 0$ *in* (1.12). *Then*

$$
\begin{aligned}
|\nabla\varphi^\nu|^2 &\leqq \left\{\sup_{\bar\Omega} Q_B(0,\varphi,\Phi) + c\nu^{\frac{1}{2}}\right\} \frac{\sup_{\bar\Omega}|F_{\mathbf{x}}|}{\inf_{\bar\Omega}|F_{\mathbf{x}}|} \\
&\leqq \left(\frac{2}{m}K^* + c\nu^{\frac{1}{2}}\right)\frac{\sup_{\bar\Omega}|F_{\mathbf{x}}|}{\inf_{\bar\Omega}|F_{\mathbf{x}}|} \quad \text{in } \bar\Omega
\end{aligned}
\tag{4.1}
$$

for $\nu \leqq \nu_0$, *where* $\nu_0 = \nu_0(\|G\|_{C^1}, \|f_1\|_{C^1}, \|F\|_{C^{1,1}(\bar\Omega)})$ *and* $c = c(\|G\|_{C^1}, \|f_1^{-1}\|_{C^1}, \|F\|_{C^{1,1}(\bar\Omega)}, K^*, \|R\|_{C^{0,1}(\bar\Omega)}, \|i\|_{C^0(\bar\Omega)})$. *The map* F *is a one-to-one conformal transformation that takes* Ω *into the rectangle* Ω_R.

Remark 4.2. The term $Q_B(0,\varphi,\Phi) = \frac{2}{m}(K - R(\varphi) + q\Phi)$ is the cavitation speed given by the inviscid equation corresponding to (1.6) (Bernoulli's law).

Remark 4.3. If the enthalpy is given by the γ-law, that is, $i(\rho) = \frac{\gamma}{\gamma-1}\rho^{\gamma-1}$, $1 < \gamma < 2$, then we have shown in Section 2 that f satisfies (2.11). Consequently, we obtain the existence of a uniform bound for the speed of a steady viscous approximating model for an adiabatic gas.

Proof: The proof of Theorem 4.1 consists of two parts. The first one shows that estimate (4.1) is satisfied on any interior point of Ω; the second part shows (4.1) still holds on any boundary point.

However, in order to use standard elliptic techniques, we need to work with the conformally transformed domain Ω_R and the set of equations (2.6). Thus, let Ω_R be the rectangle given by a one-to-one conformal transformation $\mathbf{x}' = F(\mathbf{x})$ such that the interior of Ω is taken into the interior of Ω_R and the "corners" of Ω are taken into the corners of Ω_R. From (2.5), $|\nabla_{\mathbf{x}}\varphi^\nu| = |\nabla_{\mathbf{x}'}\varphi'^\nu|\,|F_{\mathbf{x}}|$.

In order to obtain the estimate on interior points, we show first that if $\mathbf{x}'_0$ is an interior point in Ω_R such that $|\nabla_{\mathbf{x}'}\varphi'^\nu|$ is maximum, then $|\nabla_{\mathbf{x}}\varphi^\nu(\mathbf{x})|^2$ is bounded by the right-hand side of (4.1). The second part consists in showing that any maximum value of $|\nabla_{\mathbf{x}'}\varphi'^\nu|$ taken on the boundary $\partial\Omega_R$ yields that $|\nabla_{\mathbf{x}}\varphi^\nu|$ cannot also be bigger than the right-hand side of (4.1) for all boundary points of $\partial\Omega$.

The proof presented here requires a ν-uniform, a priori bound on the speed $|\nabla\varphi^\nu|$. Thus, we shall prove first a lemma that yields this result. From now on we work in the domain Ω_R with the set of equations (2.6). For convenience, as in Section 3, we drop the $'$.

First, recall that the flow equation $\operatorname{div}(\rho\nabla\varphi) = 0$ is equivalent through the change of state variables $(\varphi_x, \varphi_y) \to (\sigma, \theta)$ to system (2.15) or, equivalently, if we disregard the θ-dependence, to the equation

$$(4.2)\qquad (\rho\sigma_\psi)_\psi + (\tau\sigma_\varphi)_\varphi = \tau_{\varphi\varphi} \quad \text{in } \Omega,$$

where

$$(4.3)\qquad \sigma = \log|\nabla\varphi| \quad \text{and} \quad \tau = \rho^{-1}.$$

Since $0 < \rho \leqq L^*$ in Ω_R, the operator of the left-hand side of equation (4.2) is uniformly elliptic with an ellipticity constant given by $\min\{k_\nu, (L^*)^{-1}\}$ for $k_\nu = \inf \rho$.

Now let $\mathbf{x}_0$ be an interior point in Ω_R where $|\nabla\varphi|$ has a maximum. Therefore, σ in the domain Ω_R also has an interior maximum at $\mathbf{x}_0$. By the ellipticity of equation (4.2),

$$(4.4)\qquad \tau_{\varphi\varphi}(\mathbf{x}_0) \leqq 0.$$

Now, using that $(\ln\rho)_\varphi = -(\ln\tau)_\varphi$, differentiating once more with respect to φ, we get

$$(4.5)\qquad \frac{\tau_{\varphi\varphi}}{\tau} - \frac{\tau_\varphi^2}{\tau^2} = (\ln\tau)_{\varphi\varphi} = \frac{1}{\nu}(f((|F_\mathbf{x}|\,|\nabla\varphi|)^2, \theta - \arg F_z, Q_B))_\varphi$$

or

$$\frac{\tau_{\varphi\varphi}}{\tau} = \frac{1}{\nu^2}f^2 + \frac{1}{\nu}\left\{ f_{(|F_\mathbf{x}|\,|\nabla\varphi|)^2}\left(|F_\mathbf{x}|^2\frac{\partial|\nabla\varphi|^2}{\partial\varphi} + |\nabla\varphi|^2\frac{\partial|F_\mathbf{x}|^2}{\partial\varphi}\right) + f_{\theta-\arg F_z}\left\{\frac{\partial\theta}{\partial\varphi} - \frac{\partial \arg F_z}{\partial\varphi}\right\} + f_{Q_B}\frac{\partial Q_B}{\partial\varphi}\right\}.$$

Evaluating equation (4.5) at $\mathbf{x}_0$, because $\frac{\partial|\nabla\varphi|^2(\mathbf{x}_0)}{\partial\varphi} = \frac{\partial\theta}{\partial\varphi} = 0$, we have

$$(4.6)\qquad 0 \geqq \frac{f^2}{\nu^2} + \frac{1}{\nu}\left\{ f_{(|F_\mathbf{x}|\,|\nabla\varphi|)^2}|\nabla\varphi|^2\frac{\partial|F_\mathbf{x}|^2}{\partial\varphi} + f_{Q_B}\left(\frac{\partial Q_B}{\partial\rho}\rho_\varphi - R'(\varphi) + q\Phi_\varphi\right) - f_{\theta-\arg F_z}(\arg F_z)_\varphi\right\}$$

so that replacing ρ_φ by $-\frac{1}{\nu}\rho f$ and multiplying both sides by ν^2 yields

$$(4.7)\qquad 0 \geqq f^2 - ff_{Q_B}\frac{\partial Q_B}{\partial\rho} + \nu\left\{ f_{(|F_\mathbf{x}|\,|\nabla\varphi|)^2}|\nabla\varphi|^2\frac{\partial|F_\mathbf{x}|^2}{\partial\varphi} + f_{Q_B}(-R'(\varphi) + q\Phi_\varphi) - f_{\theta-\arg F_z}(\arg F_z)_\varphi\right\}.$$

We now prove a lemma that yields a coarse uniform bound for $|\nabla\varphi^\nu|$ (well above the corresponding one on the right-hand side of (4.1)). From now on we let $\tilde{f}_1 = f_1(\theta - \arg F_z)$, where f_1 is as defined (1.12).

LEMMA 4.4. *If* $\mathbf{x}_0$ *is an interior maximum point for* $|\nabla\varphi|$ *in* Ω_R *such that* $(|F_\mathbf{x}|\,|\nabla\varphi|)^2(\mathbf{x}_0) > 2Q_B(0,\varphi,\Phi)(\mathbf{x}_0)$, *then*

$$|F_\mathbf{x}|\,|\nabla\varphi^\nu| \leqq M_1 \quad \text{for any } \nu \leqq \nu_0\,, \tag{4.8}$$

where ν_0 *and* M_1 *are numbers that depend on the data.*

Proof: At the point $\mathbf{x}_0$, adding and subtracting

$$Q_B(0,\varphi,\Phi)\frac{\tilde{f}_1(\theta)}{(|F_\mathbf{x}|\,|\nabla\varphi|)^2\,G|F_\mathbf{x}|\,|\nabla\varphi|)}$$

to and from

$$\begin{aligned} f\,&((|F_\mathbf{x}|\,|\nabla\varphi|)^2,\ \theta - \arg F_z,\ Q_B(\rho,\varphi,\Phi)) \\ &= ((|F_\mathbf{x}|\,|\nabla\varphi|)^2 - Q_B)\frac{\tilde{f}_1(\theta)}{(|F_\mathbf{x}|\,|\nabla\varphi|)^2\,G(|F_\mathbf{x}|\,|\nabla\varphi|)}\,, \end{aligned}$$

together with property (2.11a), yields

$$\begin{aligned} f\,&((|F_\mathbf{x}|\,|\nabla\varphi|)^2,\ \theta - \arg F_z,\ Q_B(\rho,\varphi,\Phi)) \\ &= \left\{\frac{(|F_\mathbf{x}|\,|\nabla\varphi|)^2}{2} + Q_B(0,\varphi,\Phi) - Q_B(\rho,\varphi,\Phi)\right\} \\ &\quad\cdot\frac{\tilde{f}_1(\theta)}{(|F_\mathbf{x}|\,|\nabla\varphi|)^2\,G(|F_\mathbf{x}|\,|\nabla\varphi|)} \\ &\geqq \frac{\tilde{f}_1(\theta)}{2G(|F_\mathbf{x}|\,|\nabla\varphi|)}\,. \end{aligned} \tag{4.9}$$

Then (2.11b) yields

$$\begin{aligned} f &- f_{Q_B}\frac{\partial Q_B}{\partial\rho}\rho \\ &\geqq \left\{Q_B(0,\varphi,\Phi) - Q_B(\rho,\varphi,\Phi) + \frac{\partial Q_B}{\partial\rho}\rho\right\}\frac{\tilde{f}_1(\theta)}{2G(|F_\mathbf{x}|\,|\nabla\varphi|)} \\ &\geqq \frac{\tilde{f}_1(\theta)}{2G(|F_\mathbf{x}|\,|\nabla\varphi|)}\,. \end{aligned} \tag{4.10}$$

Taking the product of (4.9) and (4.10) evaluated at the point $\mathbf{x}_0$ results in

$$f^2 + f f_{Q_B}\frac{\partial Q_B}{\partial\rho}\rho \geqq \left(\frac{\tilde{f}_1(\theta)}{2G(|F_\mathbf{x}|\,|\nabla\varphi|)}\right)^2. \tag{4.11}$$

Next, using the fact that the function Ω_R is globally Lipschitz together with the gradient estimate (3.2b) for Φ, we estimate

$$\begin{aligned}(4.12)\qquad &\left| f_{Q_B}\left(-R'(\varphi)+q\Phi_\varphi\right) - f_{\theta-\arg F_z}(\arg F_z)_\varphi \right| \\ &= \left((|F_{\mathbf{x}}|\,|\nabla\varphi|)^2 G(|F_{\mathbf{x}}|\,|\nabla\varphi|)\right)^{-1}\tilde{f}_1(\theta) \\ &\quad\cdot\left| R'(\varphi) + q\frac{\nabla\Phi\nabla\varphi}{|\nabla\varphi|^2} + (\ln\tilde{f}_1)'\frac{\nabla(\arg F_z)\nabla\varphi}{|\nabla\varphi|^2}\right| \\ &\leqq \left((|F_{\mathbf{x}}|\,|\nabla\varphi|)^2 G(|F_{\mathbf{x}}|\,|\nabla\varphi|)\right)^{-1}\tilde{f}_1(\theta) \\ &\quad\cdot\left(\|R\|_{C^{0,1}(\bar{\Omega}_R)} + \{qC(K^*,L^*) + \|\ln f_1\|_{C^1}\,\|F\|_{C^{1,1}(\bar{\Omega})}\}|\nabla\varphi|^{-1}\right).\end{aligned}$$

Since $|F_{\mathbf{x}}|\,|\nabla\varphi(\mathbf{x}_0)| > 2\inf Q_B(0,\varphi,\Phi)$ by the compatibility condition (2.7b), $|F_{\mathbf{x}}|\,|\nabla\varphi(\mathbf{x}_0)| \geqq 2(K - R_U + q\Phi_L) > 0$ uniformly in ν. Consequently, since $\Phi_L = \Phi_L(K^*,L^*)$,

$$\begin{aligned}(4.13)\qquad &\left| f_{Q_B}\left(-R'(\varphi)+q\Phi_\varphi\right) - \tilde{f}_1'(\arg F_z)_\varphi \right| \\ &\leqq \left((|F_{\mathbf{x}}|\,|\nabla\varphi|)^2 G(|F_{\mathbf{x}}|\,|\nabla\varphi|)\right)^{-1}\tilde{f}_1(\theta) \\ &\quad\cdot C\left(\|R\|_{C^{0,1}(\bar{\Omega}_R)}, K^*, L^*, q, \|\ln f_1\|_{C^1}, \|F\|_{C^{1,1}(\bar{\Omega})}\right).\end{aligned}$$

We also need to estimate the term $f_{(|F_{\mathbf{x}}|\,|\nabla\varphi|)^2}|\nabla\varphi|^2\frac{\partial|F_{\mathbf{x}}|^2}{\partial\varphi}$ from (4.7) at the point $\mathbf{x}_0$ as follows:

$$(4.14)\qquad \left| f_{(|F_{\mathbf{x}}|\,|\nabla\varphi|)^2}|\nabla\varphi|^2\frac{\partial|F_{\mathbf{x}}|^2}{\partial\varphi}\right| \leqq f_{(|F_{\mathbf{x}}|\,|\nabla\varphi|)^2}|F_{\mathbf{x}}|\,|\nabla\varphi|\,\left|\nabla|F_{\mathbf{x}}|\right|.$$

By the growth condition (1.8b) and by property (2.11c) for f,

$$(4.15)\qquad |F_{\mathbf{x}}|\,|\nabla\varphi|G^{-1} \to C_2^{-1} \quad\text{and}\quad \left|\left(|F_{\mathbf{x}}|\,|\nabla\varphi|\right)^3 f_{(|F_{\mathbf{x}}|\,|\nabla\varphi|)^2}\right| \leqq C_3$$

both as $|\nabla\varphi| \to \infty$. If $|F_{\mathbf{x}}|\,|\nabla\varphi^\nu|$ becomes very large near the point $\mathbf{x}_0$, by using (4.11), (4.13), and (4.14), as well as the growth at infinity from (4.15), we can estimate the right-hand side of (4.7) by

$$\begin{aligned}(4.16)\qquad 0 \geqq &\frac{\tilde{f}_1^2(\theta)C_2^{-2}}{4|\nabla\varphi|^2} + \nu\frac{\tilde{f}_1(\theta)\mathcal{O}\left(C_3, \|F_{\mathbf{x}}^{-1}\|_C^{0,1}(\bar{R})\right)}{|\nabla\varphi|^2} \\ &+ \nu\frac{\tilde{f}_1(\theta)C_2^{-1}}{2|\nabla\varphi|^3}\mathcal{O}\left(\|R\|_{C^{0,1}(\bar{R})}, K^*, L^*, q, \|\ln f_1\|_{C^1}, \|F\|_{C^{1,1}(\bar{\Omega})}\right)\end{aligned}$$

Also, if $\tilde{f}_1 > \inf f_1 = k_0 > 0$, the term $\frac{\tilde{f}_1^2(\theta)C_2^{-2}}{4|\nabla\varphi|^2}$ is nonnegative. If we then choose

$$(4.17)\qquad \nu_0 = \left|\mathcal{O}(C_3, \|F_{\mathbf{x}}^{-1}\|_{C^{0,1}(\bar{\Omega})})\right|^{-1}\frac{\tilde{f}_1(\theta)}{8C_2^2} = \nu_0(\|G\|_{C^1}, \inf f_1, \|F\|_{C^{1,1}(\bar{\Omega})}),$$

then for any $|\nabla\varphi| \geqq M_1$, where

$$(4.18)\quad M_1 = \left| \frac{1}{C_2^2 \mathcal{O}(C_3, \|F_{\mathbf{x}}^{-1}\|_{C^{0,1}(\bar\Omega)})} \cdot \mathcal{O}(\|R\|_{C^{0,1}(\bar\Omega)}, q, K^*, L^*, \|\ln f_1\|_{C^1}, \|F\|_{C^{1,1}(\bar\Omega)}) \right|,$$

which depends only on the data of the boundary value problem (2.1)–(2.10) along with condition (1.8b) and (2.11), the right-hand side of (4.16) is controlled by

$$0 \geqq \frac{\tilde f_1^2(\theta)}{16 C_2^2} \frac{1}{|\nabla\varphi|^2} \qquad \text{for all } \nu \leqq \nu_0, \quad |\nabla\varphi| > M_1,$$

which is positive. Thus we have a contradiction.

Hence we have obtained that, for any $\nu \leqq \nu_0 = \nu_0(G, f_1, \|F\|_{C^{1,1}(\bar\Omega)})$,

$$(4.19)\quad \begin{aligned} |\nabla\varphi^\nu|^2 &\leqq M_1 \\ &= M_1(\|G\|_{C^1}, K^*, \|R\|_{C^{0,1}(\bar\Omega)}, \|i\|_{C^0(\bar\Omega)}, \|\ln f_1\|_{C^1}, \|F\|_{C^{1,1}(\bar\Omega)}) \end{aligned}$$

independently of ν, so that (4.8) holds and the proof of Lemma 4.2 is now complete.

Now, we proceed to show that if $\mathbf{x}_0$ is an interior maximum for $|\nabla\varphi^\nu|$, then estimate (4.1) holds for any positive $\nu \leqq \nu_0$, with ν_0 as defined in (4.17).

Once more we use (2.11) and (4.7) in the following way: If $\mathbf{x}_0$ is an interior maximum point such that $|\nabla\varphi|^2(\mathbf{x}_0) > Q_B(0, \varphi, \Phi) + c\nu^{\frac{1}{2}}$ for $\nu \leqq \nu_0$, where ν_0 is as defined in (4.17) and c depends on the data, then inequality (4.7) does not hold.

At the point $\mathbf{x}_0$, in the same way we have obtained estimates (4.9) and (4.10), (2.11a) yields

$$(4.20)\qquad f\left(|\nabla\varphi|^2, \theta, Q_B(\rho, \varphi, \Phi)\right) \geqq c\nu^{\frac{1}{2}} \left(|\nabla\varphi|^2 G(|\nabla\varphi|)\right)^{-1} \tilde f_1(\theta),$$

and (2.11b) yields

$$(4.21)\qquad f - f_{Q_B} \frac{\partial Q_B}{\partial \rho}\rho \geqq c\nu^{\frac{1}{2}} \left(|\nabla\varphi|^2 G(|\nabla\varphi|)\right)^{-1} \tilde f_1(\theta).$$

Taking the product of (4.20) with (4.21) evaluated at the point $\mathbf{x}_0$, we get

$$(4.22)\qquad f^2 - f f_{Q_B} \frac{\partial Q_B}{\partial \rho}\rho > c^2\nu\left(|\nabla\varphi|^2 G(|\nabla\varphi|)\right)^{-2} \tilde f_1^2(\theta).$$

The term $f_{Q_B}\left(-R'(\varphi) + q\Phi_\varphi\right)$ is estimated as in (4.13), and the term

$$f_{(|F_{\mathbf{x}}|\,|\nabla\varphi|)^2} |\nabla\varphi|^2 \frac{\partial |F_{\mathbf{x}}|^2}{\partial\varphi}$$

is estimated as in (4.14). Again, as in the previous lemma, by the compatibility condition (2.7b) and the growth condition (1.8b) on G at infinity, that is, by property (2.11c) for f, we estimate the right-hand side of (4.7) by

$$
\begin{aligned}
0 &\geqq c^2 \frac{\nu}{|\nabla\varphi|^4}\left(\frac{\tilde{f}_1}{C_2}\right)^2 + \frac{\nu \tilde{f}_1}{|\nabla\varphi|^2}\left(\mathcal{O}\left(C_3, \|F^{-1}\|_{C^{1,1}(\bar{\Omega})}\right)\right. \\
&\quad \left. + \frac{\mathcal{O}\left(\|R\|_{C^{0,1}(\bar{\Omega})}, q, K^*, L^*, \|\ln f_1\|_{C^1}, \|F\|_{C^{1,1}(\bar{\Omega})}\right)}{|\nabla\varphi|\, C_2}\right) \\
&\geqq \nu\left\{c^2 \frac{1}{M_1^4}\left(\frac{\tilde{f}_1}{C_2}\right)^2 + \frac{\tilde{f}_1}{\mathcal{K}^2}\mathcal{O}\left(C_3, \|F^{-1}\|_{C^{1,1}(\bar{\Omega})}\right)\right. \\
&\quad \left. + \frac{\mathcal{O}\left(\|R\|_{C^{0,1}(\bar{\Omega})}, qK^*, L^*, \|\ln f_1\|_{C^1}, \|F\|_{C^{1,1}(\bar{\Omega})}\right)}{\mathcal{K}\, C_2}\right\},
\end{aligned}
\tag{4.23}
$$

for any $\nu \leqq \nu_0$, ν_0 from (4.17) with M_1 the bound from (4.18), and $\mathcal{K} = K - R_U + q\Phi_L > 0$ from the compatibility condition (2.7b).

Then choosing

$$
\begin{aligned}
c^2 > \frac{M_1^2 C_2}{\inf \tilde{f}_1} \,\big| &\mathcal{O}(C_3, \|F\|_{C^{1,1}(\bar{\Omega})}) \\
&+ \mathcal{O}(\|R\|_{C^{0,1}(\bar{\Omega})}, \mathcal{K}, q, K^*, L^*, \|\ln f_1\|_{C^1}, \|F\|_{C^{1,1}(\bar{\Omega})})\big|
\end{aligned}
\tag{4.24}
$$

which depends only on the data, makes the right-hand side of (4.24) nonnegative for any $\nu \leqq \nu_0$. Thus we have a contradiction. Hence, if $\mathbf{x}_0$ is an interior maximum for $|\nabla\varphi^\nu|$ in Ω_R, then

$$
(|F_{\mathbf{x}}|\, |\nabla\varphi^\nu|)^2(\mathbf{x}_0) \leqq \sup_{\bar{R}} Q_B(0, \varphi, \Phi) + c\nu^{\frac{1}{2}} \leqq \frac{2}{m}K^* + c\nu^{\frac{1}{2}}. \tag{4.25}
$$

In order to complete the proof of Theorem 4.1, we need to show that estimate (4.1) holds for all points even if the maximum is attained at the boundary of Ω_R. First, we show that $|\nabla\varphi^\nu|^2$ cannot have a maximum on $\partial_1\Omega_R$ such that $(|F_{\mathbf{x}}|\,|\nabla\varphi^\nu|)^2$ is bigger than $\sup_{\bar{R}} Q(0,\varphi,\Phi) + c\nu^{\frac{1}{2}}$. If σ takes its maximum on $\mathbf{x}_0 \in \partial_1\Omega_R$, since $(\nabla\varphi \cdot n)(\mathbf{x}_0) < 0$, then $\sigma_\varphi(\mathbf{x}_0) \leqq 0$. Using equations (2.15) in Ω_R yields $\theta = 0$, $\theta_\psi = 0$, and $\tau\sigma_\varphi = \tau_\varphi = \frac{1}{\nu}\tau f(e^{2\sigma}, \arg F_z, Q_B)$ on $\partial_1\Omega_R$.

Next, from condition (2.11a), if $(|F_{\mathbf{x}}|\,|\nabla\varphi^\nu|)^2 \geqq Q(0,\varphi,\Phi) + c\nu^{\frac{1}{2}}$ at $\mathbf{x}_0$, then $f(e^{2\sigma}, -\arg F_z, Q_B)(\mathbf{x}_0) \geqq 0$. Therefore, if $f(e^{2\sigma}, -\arg F_z, Q_B)(\mathbf{x}_0) > 0$, we get a contradiction for values of $\nu \leqq \nu_0$ with ν_0 from (4.17) and c from (4.24). If $f(e^{2\sigma}, -\arg F_z, Q_B)(\mathbf{x}_0) = 0$, by condition (2.3a) we have

$$
\begin{aligned}
(|F_{\mathbf{x}}|\,|\nabla\varphi^\nu|)^2(\mathbf{x}_0) &= Q(\rho, \varphi, \Phi)(\mathbf{x}_0) \\
&\leqq \sup_{\bar{\Omega}} Q(0, \varphi, \Phi) \quad \text{for } 0 < \rho \leqq L^*.
\end{aligned}
$$

Hence, inequality (4.25) holds on $\partial_1\Omega_R$.

Next, on $\partial_3\Omega_R$ we have that for $g(x)$, the prescribed data for $|\nabla\varphi|^2$ in the boundary region $\partial_3\Omega$ satisfies condition (2.8). Therefore

$$0 < \Big|\, |F_{\mathbf{x}}|\,|\nabla\varphi^\nu| \,\Big|^2 = g^2(x)\Big|_{\partial_3\Omega} < \mathscr{K} - \delta \quad \text{with } \mathscr{K} - \delta > 0,$$

with $\mathscr{K} = K - R_U + q\Phi_L$ for a small positive δ. Consequently,

$$\mathscr{K} - \delta < Q_B(0,\varphi,\Phi) < Q_B(0,\varphi,\Phi) + c_1\nu^{\frac{1}{2}},$$

which is a uniform value in ν. Then for any positive ν,

$$(|F_{\mathbf{x}}|\,|\nabla\varphi^\nu|)^2 < Q_B(0,\varphi,\Phi) + c_1\nu^{\frac{1}{2}} \quad \text{on } \partial_3\Omega_R,$$

so that estimate (4.25) holds on $\partial_3\Omega_R$.

Finally, we consider the boundary region $\partial_2\Omega_R \cup \partial_4\Omega_R$. Since the boundary condition $\nabla\varphi \cdot n = 0$ in $\partial\Omega$ for n the unit exterior normal, we have $\nabla\sigma \cdot n = 0$ in $\partial\Omega_R$ for a flat (zero-curvature) boundary and n now the unit exterior normal to $\partial\Omega_R$.

Therefore, reflecting the domain and equation (4.2) evenly across the boundary region and in a neighborhood of a point $\mathbf{x}_0 \in \partial_2\Omega_R \cup \partial_4\Omega_R$ where $|\nabla\varphi|$ reaches a maximum m and evaluating the resulting equation at $\mathbf{x}_0$ in the interior of the domain for this new equation, inequality (4.7) holds weakly (i.e., for integrated against nonnegative test functions). Consequently, the interior estimate (4.25) for a local maximum holds, and so (4.25) holds for all points in $\partial_2\Omega \cup \partial_4\Omega$. Hence, estimate (4.25) holds for any point in $\bar\Omega_R$, and

$$\begin{aligned} |\nabla\varphi^\nu|^2 &\leqq \left(\sup_R Q_B(0,\varphi,\Phi) + c\nu^{\frac{1}{2}}\right)\frac{1}{\inf_{\bar\Omega}|F_{\mathbf{x}}|^2} \\ &\leqq \left(\frac{2}{m}K^* + c\nu^{\frac{1}{2}}\right)\frac{1}{\inf_{\bar\Omega}|F_{\mathbf{x}}|^2} \quad \text{in } \bar\Omega_R. \end{aligned}$$

Then, for any point in $\bar\Omega$,

$$\begin{aligned} |\nabla\varphi^\nu|^2 &\leqq \sup_{\bar\Omega}|F_{\mathbf{x}}|^2\left(\sup_{\bar\Omega} Q_B(0,\varphi,\Phi) + c\nu^{\frac{1}{2}}\right)\frac{1}{\inf_{\bar\Omega}|F_{\mathbf{x}}|^2} \\ &\leqq \left(\frac{2}{m}K^* + c\nu^{\frac{1}{2}}\right)\frac{\sup_{\bar\Omega}|F_{\mathbf{x}}|^2}{\inf_{\bar\Omega}|F_{\mathbf{x}}|^2} \end{aligned} \tag{4.26}$$

for any positive $\nu \leqq \nu_0$ for ν_0 given in (4.17) and c given by (4.24).

Hence, estimate (4.1) holds for any point in $\bar\Omega$, and the proof of Theorem 4.1 is now complete.

5. Conclusions About Transonic Flow

Because $|\nabla\varphi^\nu|$ and ρ^ν are uniformly bounded from above, there exists a convergent subsequence (φ^ν, ψ^ν) with a limit (φ^0, ψ^0). However, this is not enough to establish that there exists a weak solution to the equations $\rho\varphi_x = \psi_y$, $\rho\varphi_y = -\psi_x$.

One method to prove the existence of weak solutions involves the method of compensated compactness (see Murat [33], Tartar [39], and DiPerna [9]) and the applications to mixed-type systems (see Morawetz [31], [32]), which have been used mainly for hyperbolic problems. This method would require that the speed $|\nabla\varphi^\nu|$ be uniformly bounded below from zero and above from the cavitation speed (see [32]). Furthermore, bounds on the flow angle would also be required. Although this paper provides a complete proof of existence of viscous solutions with some uniform bounds, thus enabling us to consider a convergent subsequence, there is still a major gap in showing that its limit is a solution of the inviscid problem.

Similar difficulties arise in the existence theorems of Feistauer and Nečas [11]. They show the existence of a solution for the inviscid model under the the assumption of existence of viscous solutions to a boundary value problem provided the divergence of the viscous velocity field satisfies uniform bounds in ν. Gittel [20] shows existence using a variational approach to a boundary value problem for the transonic small-disturbance equation and shows existence theorems that assume uniform bounds and entropy conditions.

We also mention the work of Klouček [27] and Nečas [34] to find entropic solutions for the transonic flow model by the method of stabilization. In these papers the authors solved a perturbed flow equation by introducing an artificial time variable. They also assumed uniform speed and entropy bounds in order to pass to the limit.

6. About the Semiconductor Device Model

We have shown the existence of a regular solution to the boundary value problem in a Fluid-Poisson system in an approximating geometry to a real device and without smallness assumptions on the size of the data. The model we have considered corresponds to a fluid-level approximation of a kinetic formulation for a particle-charged system, with a pressure law similar to the one for an isentropic gas with $1 < \gamma < 2$ and a viscous parameter ν. In this context the "viscosity" parameter is related to the mean free path and refers to the constant coefficient related to the nonconvective energy flux term, which has been modeled as a nonlinear term that involves first- and second-order derivatives of the velocity field. In addition, we have shown uniform bounds in the viscosity parameter for regimes where this parameter becomes small compared with the scaled coefficient of the acceleration term in the momentum equation.

This is a potential flow model obtained under the assumption that the velocity field aligns with the gradient field of τ, where $\tau > 0$ stands for the velocity relaxation time term. Potential flow can be justified if the initial state has zero vorticity.

In order to extend our results to the case $\gamma = 1$ (the isothermal model) modifications would be required, especially to obtain the uniform bounds. This case is often used in modeling; see [28] and [36].

In particular, for two-dimensional models in MOSFET geometry, it can be shown that the behavior of any solution at the boundary points will depend on the behavior and regularity at the boundary of the domain under the conformal transformation that takes the domain of the device into a rectangular domain where the source and drain contacts are transformed into opposite walls in the transformed domain. A singularity is expected to be found in the electric field at the boundary points corresponding to junctions of the contact regions and the oxide region.

Appendix: Proof of Lemma 3.5

From the standard theory of ODEs, (3.27) yields that z is Lipschitz in s; in particular, z is C^α in s with $0 \leqq \alpha \leqq 1$. We thus need to show that z is C^α in r. Let k be any real number. Set $\omega = z(s,r) - z(s, r+k)$ and compute the equation (3.27) for ω. That is,

$$\omega_s = \mathscr{F}(s,r,z(s,r)) - \mathscr{F}(s,r+k,z(s,r+k)). \tag{A.1}$$

If we add and subtract $\mathscr{F}(s,r,z(s,r+k))$ to (A.1), we obtain the expression

$$\begin{aligned}\omega_s &= [\mathscr{F}(s,r,z(s,r)) - \mathscr{F}(s,r,z(s,r+k))]\\ &\quad + [\mathscr{F}(s,r,z(s,r+k)) - \mathscr{F}(s,\ r+k,\ z(s,r+k))]\end{aligned} \tag{A.2}$$

Now, estimating (A.2) and using the fact that $\mathscr{F}$ is C^α in (s,r), we have

$$\omega_s \leqq \|\mathscr{F}\|_{\mathrm{Lip},z}\,|\omega| + \|\mathscr{F}\|_{C^\alpha(\bar{\mathscr{B}})}|k|^\alpha. \tag{A.3}$$

Thus, by integrating (A.3) in s from values of $s \geqq 0$ along all characteristics in $\mathscr{B}$ that initiate on Γ, we obtain the inequality

$$\omega \leqq \omega|_\Gamma + \int_{k_0}^{s} \|\mathscr{F}\|_{\mathrm{Lip},z}\,|\omega(\tau)|d\tau + s\|\mathscr{F}\|_{C^\alpha(\bar{\mathscr{B}})}|k|^\alpha. \tag{A.4}$$

Now since $\omega|_\Gamma = \gamma_3(0,r) - \gamma_3(0,r+k) \leqq \|\gamma_3\|_{C^\alpha(\Gamma)}|k|^\alpha$ and $\|s\|_{C^0(\bar{\mathscr{B}})}$ is bounded, by using Gronwall's inequality we obtain

$$|\omega| \leqq \left(\|\gamma_3\|_{C^\alpha(\Gamma)} + \|s\exp\left\{s\|\mathscr{F}\|_{\mathrm{Lip},z}\right\}\|_{C^0(\bar{\mathscr{B}})}\,\|\mathscr{F}\|_{C^\alpha(\bar{\mathscr{B}})}\right)|k|^\alpha.$$

Then we have

$$\|z\|_{C^\alpha(\bar{\mathscr{B}})} \leqq \|\gamma_3\|_{C^\alpha(\Gamma)} + \|s\|_{C^0(\bar{\mathscr{B}})}\exp\left\{\|s\|_{C^0(\bar{\mathscr{B}})}\,\|\mathscr{F}\|_{\mathrm{Lip},z}\right\}\|\mathscr{F}\|_{C^\alpha(\bar{\mathscr{B}})},$$

so that (3.29) holds.

Acknowledgments. The authors want to thank Robert V. Kohn and Louis Nirenberg for many helpful discussions. Many of the ideas developed in this paper were initially investigated by Robert V. Kohn and Cathleen S. Morawetz, which remain in the form of an unpublished manuscript. In particular, the idea of constructing a map that admits a fixed point in order to solve a system of the form (1.10a) and (1.10b) (for $\Phi \equiv 0$) was first proposed in that manuscript.

The research of the first author is supported by the National Science Foundation under Grant DMS 9206244.

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Received December 1994.