On the inelastic Boltzmann equation with diffusive forcing

To Olga Ladyzhenskaya, on the occasion of her 80th Birthday

I. M. Gamba*, V. Panferov* and C. Villani†

*Department of Mathematics, The University of Texas at Austin
Austin, TX 78712-1082, USA

† UMPA, ENS Lyon, 46 allée d'Italie, F-69364 Lyon Cedex 07, France

Abstract. We discuss recent results on the study the inelastic homogeneous Boltzmann equation for hard spheres, with a diffusive term representing a random background acceleration. We show that the initial value problem has unique solutions, which become infinitely smooth and rapidly decaying after a short time, under the assumption that the data is in $L^2(\mathbb{R}^3) \cap L^1_2(\mathbb{R}^3)$ (has bounded mass and energy). In addition the time-dependent solution converges, along a subsequence of times, to a stationary solution. In addition we show that the high-velocity tails of both the stationary and time-dependent particle distribution function are overpopulated with respect to the Maxwellian distribution, and we estimate the solutions from below.

Introduction

Dynamics of granular systems has attracted a significant interest of physicists due to a variety of complex phenomena displayed by such systems. In particular, granular systems can form flows where particles move freely and exchange energy through binary collisions. An analogy between granular particles and molecules of a rarefied gas then suggests a possibility of using a kinetic theory approach [10]. Such an approach is particularly promising since it could be used to clarify a connection between the particle dynamics and the hydrodynamic description of a granular fluid.

In this paper we study a problem of formation of steady states in granular media subject to external forcing. We study a model space-homogeneous case of a system of perfect hard spheres which interact with each other by means of inelastic binary collisions. For such a system the following model based on the Enskog-Boltzmann equation was suggested by van Noije and Ernst [20]:

$$\partial_t f - \vartheta_b \Delta_v f = Q_\alpha(f, f), \quad v \in \mathbb{R}^3, \quad t > 0. \tag{1}$$

Here f is the unknown one-particle distribution function that depends on the particle velocity v and time t, and $Q_\alpha(f, f)$ is the inelastic collision operator, the details of which are presented below. The term $\vartheta_b \Delta_v f$, where ϑ_b is a physical constant, is a Fokker-Planck type operator which represents the effect of a heat bath of infinite temperature. Recent particle dynamics simulations by Bizon *et al* [2], Brillantov *et al* [7] and Moon *et al* [16] indicate the existence of states evolving to stationary ones which are not given by classical Maxwellian distributions.

The rigorous mathematical results obtained previously for the problem included constructing approximate solutions by means of formal expansions [20] in the case of hard spheres and, by different authors [8], under a simplifying assumption of pseudo-Maxwell collision operator. The approach was initiated in a work by Bobylev, Carrillo and Gamba [4]. and was used by Bobylev and Cercignani [5] to solve the pseudo-Mawell problem using the Fourier representation of the equation [5] and to study the convergence and stability to stationary states. Previously, Cercignani, Illner and Stoica [9] had showed the existence of steady weak solutions to the equation (1), for the pseudo-Maxwell collisions by means of functional analysis and fix point arguments.

Our aim here is to rigorously study the inelastic Boltzmann equation (1) and to find qualitative properties of solutions that might aid our understanding of the stationary non-equilibrium regime. We prove existence, uniqueness and regularity of time dependent solutions under the assumption that the initial data is in $L^1_2(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$. We also prove that the time-dependent solutions converge, along a sequence of times, to steady solutions, with possibly different steady states along different sequences. Finally, we find lower bounds for both stationary and transient solutions, showing that the high-velocity tails are overpopulated with respect to Maxwellians. The lower bounds have the form $K \exp(-\beta |v|^{3/2})$, with $\beta = \beta(1-\alpha^2, \vartheta_b)$, which agrees with asymptotic high-velocity solution by Van Noije and Ernst [20]. Here $K > 0$ is a constant (in the time-dependent case $K = K(t)$ is a function that may decay with time).

Our methods rely on using the parabolicity of the diffusion operator in the equation, combined with an analysis of the collision term. We extend the previously known estimates of the moments for the Boltzmann equation (Povzner inequalities) and L^p estimates due to Gustafsson [15] to the inelastic case. We establish the results on the regularity of the solutions using an approach similar to the one used by Desvillettes and Villani [12] for the homogeneous Landau equation. Uniqueness for the time-dependent problem follows by using Gronwall type of estimates, which is an approach utilized by several authors in the case of the elastic Boltzmann equation (see [1], or more recently, [18]). The lower bound estimates for the steady and time-dependent solutions are derived by classical comparison arguments for second-order PDEs.

Finally, we point out that most of the results obtained here for the hard-sphere model can be generalized in a straightforward way to the case of other "hard" interactions, which includes the Maxwell molecules model introduced by Bobylev et al. [4].

Preliminaries

The collision operator

The collision term in equation (1) can be written in the following form:

$$Q(f,f)(v) = \int_{\mathbb{R}^3} \int_{S^2} \left(\frac{1}{\alpha^2} f({}'v)\, f({}'v_*) - f(v)\, f(v_*) \right) B(u,\sigma)\, d\sigma\, dv_*, \tag{2}$$

where

$$B(u,\sigma) = \frac{1}{4\pi} |u|, \quad u = v - v_*,$$

σ is a parameter vector on the unit sphere S^2. Also, $0 < \alpha \leq 1$ is a constant coefficient of normal restitution, and the velocities ${}'v$ and ${}'v_*$ (pre-collisional velocities) are defined as follows:

$$\begin{aligned} {}'v &= \frac{w}{2} + \frac{\alpha - 1}{2\alpha} \frac{u}{2} + \frac{\alpha+1}{2\alpha} \frac{|u|}{2} \sigma, \\ {}'v_* &= \frac{w}{2} - \frac{\alpha - 1}{2\alpha} \frac{u}{2} - \frac{\alpha+1}{2\alpha} \frac{|u|}{2} \sigma. \end{aligned} \tag{3}$$

Here $w = v + v_*$ ($\frac{w}{2}$ is thereby the velocity of the center of mass).

Notice that if $\alpha = 1$, the equation becomes the classical Boltzmann equation for elastic particles. The factor $\frac{1}{\alpha^2}$ in (2) accounts for the change of the normal velocity and the Jacobian of the transformation from v, v_* to ${}'v$, ${}'v_*$.

Properties of the inelastic collision operator

Using the transformations between pre- and post-collisional velocities we obtain the following weak form of the collision operator:

$$\int_{\mathbb{R}^3} Q(f,f)\, \psi\, dv = \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} f f_* \, (\psi' + \psi'_* - \psi - \psi_*) B(u,\sigma)\, d\sigma\, dv_*\, dv, \tag{4}$$

where ψ is a suitable test function that may depend on v and t. Here we have used the common shorthand notations $\psi = \psi(v,t)$, $\psi_* = \psi(v_*,t)$, $\psi' = \psi(v',t)$, $\psi'_* = \psi(v'_*,t)$. The arguments v' and v'_*, which have the meaning of the post collisional velocities, are defined as follows:

$$
\begin{aligned}
v' &= \frac{w}{2} + \frac{1-\alpha}{2}\frac{u}{2} + \frac{1+\alpha}{2}\frac{|u|}{2}\sigma, \\
v'_* &= \frac{w}{2} - \frac{1-\alpha}{2}\frac{u}{2} - \frac{1+\alpha}{2}\frac{|u|}{2}\sigma.
\end{aligned} \tag{5}
$$

Using the weak form (4), several important properties of the equations become clear. First, it is easily checked that the mass and momentum are conserved:

$$
\int_{\mathbb{R}^d} Q(f,f)(v)\{1, v_i\}\, dv = 0, \quad i = 1, \ldots, 3. \tag{6}
$$

Next, we obtain the following relation for the dissipation of kinetic energy:

$$
\int_{\mathbb{R}^3} Q(f,f)(v)\, |v|^2\, dv = -\frac{1-\alpha^2}{16}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f\, f_* |u|^3\, dv_*\, dv, \tag{7}
$$

where we have computed the local energy dissipation as follows:

$$
|v'|^2 + |v'_*|^2 - |v|^2 - |v_*|^2 = -\frac{1-\alpha^2}{2}\frac{1-\cos\vartheta}{2}|u|^2, \tag{8}
$$

where ϑ is the angle between u and σ.

Using (6) and (7) we see that solutions of (1) formally satisfy

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3} f\, dv = 0, \\
&\frac{d}{dt}\int_{\mathbb{R}^3} f\, v\, dv = 0, \\
&\frac{d}{dt}\int_{\mathbb{R}^3} f\, |v|^2\, dv = 6 - \frac{1-\alpha^2}{16}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f f_* |u|^3\, dv_*\, dv.
\end{aligned} \tag{9}
$$

These relations will be the main source of *a priori* bounds satisfied by the solutions. Notice that the entropy decay condition, which produces an extra it a priori bound for the elastic Boltzmann equation, is generally not available in the present inelastic case. A computation of the dissipation of entropy $\int f \log f\, dv$ reveals the contribution of a nonnegative term

$$
\frac{1-\alpha^2}{2\alpha^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f f_*\, |u|\, dv\, dv_*.
$$

which appears on the right-hand side of the equation. Thus, the entropy may generally increase with time, although we can get uniform entropy bounds on every finite time interval.

Scaling

The conservation of mass and momentum imply that for every solution $f(t,v)$ the following quantities do not depend on time:

$$
\int_{\mathbb{R}^3} f(t,v)\, dv = \rho, \quad \text{and} \quad \int_{\mathbb{R}^3} f(t,v)\, v\, dv = \nu. \tag{10}
$$

Taking the function $g(t,v)$ so that

$$
f(t,v) = \frac{\rho}{\beta^3}\, g\left(\frac{t}{\alpha}, \frac{v-\nu}{\beta}\right), \tag{11}
$$

we find

$$\int_{\mathbb{R}^3} g(t,v)\,dv = 1, \quad \text{and} \quad \int_{\mathbb{R}^3} g(t,v)v\,dv = 0. \tag{12}$$

The function $g(t,v)$ will also satisfy the equation

$$\frac{1}{\alpha}\,\partial_t g - \frac{\vartheta_b}{\beta^2}\,\Delta_v g = \rho\beta\,Q(g,g). \tag{13}$$

Taking

$$\beta = \rho^{-1/3}\vartheta_b^{1/3} \quad \text{and} \quad \alpha = \rho^{-2/3}\vartheta_b^{-1/3}$$

we find that $g(t,v)$ satisfies the same equation as f but with $\vartheta_b = 1$.

Thus, the solution of the problem (1) with arbitrary ρ, ν and ϑ_b can always be reduced to solving the equation

$$\partial_t f - \Delta f = Q(f,f), \quad t > 0, \quad v \in \mathbb{R}^3, \tag{14}$$

where $f(t,v)$ has unit mass and zero momentum (12). (We renamed $g(t,v)$ back to $f(t,v)$.) This equation will be the main object of study in this paper. We will study the Cauchy problem with the initial data

$$f(v,0) = f_0(v), \tag{15}$$

such that $f_0(v)$ has unit mass and zero momentum; and also the equation satisfied by the steady (time-independent) solutions

$$-\Delta f = Q(f,f), \quad v \in \mathbb{R}^3. \tag{16}$$

The solutions with arbitrary ρ, ν and ϑ_b can then be obtained using (11).

Energy Dissipation and Moments

We state inequalities and bounds for the moments of transient and stationary solutions,

$$Y_s(t) = \int_{\mathbb{R}^3} f(t,v)\langle v\rangle^s\,dv, \quad Z_s = \int_{\mathbb{R}^3} f(v)\langle v\rangle^s\,dv,$$

respectively, where $\langle v\rangle$ denotes the weight function $(1+|v|^2)^{1/2}$ and s is a nonnegative real number. We will assume the unit mass condition $Y_0(t) = 1$ which is a consequence of (12).

The properties of the moments is an important source of information about the large-velocity behavior of solutions, and play a crucial role in the subsequent regularity analysis. For the proofs of the results stated here see [14].

The following lemma is an easy consequence of the energy dissipation relation.

Lemma 1. *Assume that a nonnegative solution f to* (14) *satisfies* (12). *Then the following differential inequality holds:*

$$\frac{d}{dt}Y_2 + k_2 Y_3 \le K_2,$$

where $K_2 = 7$ and $k_2 = \sqrt{2}\,\frac{1-\alpha^2}{16}$. Further, we have

$$Y_2(t) \le \max\left\{Y_2(0), \left(K_2/k_2\right)^{2/3}\right\},$$

uniformly for $t > 0$. For the stationary equation (16) *we obtain the a priori estimate*

$$Z_2 \le K_2/k_2.$$

The Povzner-type inequalities

Obtaining information about higher-order moments requires estimating the expressions of the type

$$|v'|^s + |v'_*|^s - |v|^s + |v_*|^s, \quad s > 2, \tag{17}$$

which appear in the weak form of the collision operator. In the case of elastic interactions the well-known Povzner estimates [19] state that for $s > 2$ and $|v| > |v_*|$, the growth for large $|v|$, $|v_*|$ in (17) is controlled above by $C|v|^{s-2}|v_*|^2$. These estimates were subsequently improved by several authors [13, 11, 18, 3, 17] and were used to get sharper versions of the moment bounds.

To generalize this type of results to the case of inelastic interactions we look for estimates of the expression

$$\mathcal{C}[\psi] = \psi(|v'|^2) + \psi(|v'_*|^2) - \psi(|v|^2) - \psi(|v_*|^2),$$

where v' and v'_* are defined by (5). and ψ is a convex function that will also be assumed to satisfy some extra conditions. Our aim is to treat the cases of

$$\psi(x) = x^p, \quad \text{and} \quad \psi(x) = (1+x)^p - 1, \quad p > 1, \tag{18}$$

and also truncated versions of such functions that will be required in the rigorous analysis of moments. Thus, we assume that in the general case the function ψ satisfies the following list of conditions:

$$\psi(x) \geq 0, \quad x > 0; \quad \psi(0) = 0; \tag{19}$$

$$\psi(x) \quad \text{is convex for} \quad x > 0; \tag{20}$$

$$\psi'(ax) \leq \eta_1(a)\, \psi'(x), \quad x > 0, \quad a > 1; \tag{21}$$

$$\psi''(ax) \leq \eta_2(a)\, \psi''(x), \quad x > 0 \quad a > 1, \tag{22}$$

where $\eta_1(a)$ and $\eta_2(a)$ are functions of a only, bounded on every finite interval of $a > 0$. The above conditions are easily verified for the functions (18). To proceed with estimates of $\mathcal{C}[\psi]$ we will need the following lemma:

Lemma 2. *Assume that $\psi(x)$ is nonnegative, smooth, convex and satisfies* (19)–(22)*. Then*

$$\psi(x+y) - \psi(x) - \psi(y) \leq A\,(\,x\,\psi'(y) + y\,\psi'(x)\,) \tag{23}$$

and

$$\psi(x+y) - \psi(x) - \psi(y) \geq b\,xy\,\psi''(x+y). \tag{24}$$

where $A = \eta_1(2)$ and $b = (2\eta_2(2))^{-1}$.

Using the above Lemma we can estimate $\mathcal{C}[\psi]$ as follows.

Lemma 3. *We can represent $\mathcal{C}[\psi]$ as a sum of two terms,*

$$\mathcal{C}[\psi] = -\mathcal{S}[\psi] + \mathcal{P}[\psi],$$

so that

$$\mathcal{P}[\psi] \leq A\,(\,|v|^2\,\psi'(|v_*|^2) + |v_*|^2\,\psi'(|v|^2)\,)$$

and

$$\mathcal{S}[\psi] \geq \kappa(\lambda,\mu)\,(\,|v|^2 + |v_*|^2)^2\,\psi''(\,|v|^2 + |v_*|^2),$$

where A is the constant in estimate (23)*, and*

$$\kappa(\lambda,\mu) = \frac{b}{4}\,\lambda^2\,(\eta_2(\lambda^{-2}))^{-1}\,(1 - \cos^2\mu),$$

where b is the constant in estimate (24)*.*

An integration with respect to the angular variable yields the following corollary:

Corollary 1. *Assume that the function $\psi(x)$ is as in Lemma 2. Then*

$$\int_{S^2} \mathcal{C}[\psi]\, d\sigma \le -k\,(\,|v|^2+|v_*|^2)^2\,\psi''(\,|v|^2+|v_*|^2) \\ +\,K\,(\,|v|^2\,\psi'(|v_*|^2)+|v_*|^2\,\psi'(|v|^2)\,).$$

where K and k are nonnegative constants depending on the weight function ψ, independent on the restitution coefficient α.

Remark. In the case $\psi(x)=x^{s/2}$, $s>2$, we recover the usual form of the Povzner inequalities for inelastic collisions:

$$\int_{S^2} \left(|v_*'|^s+|v'|^s - |v_*|^s-|v|^s\right) d\sigma \\ \le -k_s\,(|v|^s+|v_*|^s)+K_s\left(|v|^{s-2}|v_*|^2+|v|^2\,|v_*|^{s-2}\right).$$

By considering $\psi(x)=(1+x)^{s/2}-1$ we also get, for every $s>2$,

$$\int_{S^2} \left(\langle v'\rangle^s+\langle v_*'\rangle^s - \langle v\rangle^s-\langle v_*\rangle^s\right) d\sigma \\ \le -k_s\,(\langle v\rangle^s+\langle v_*\rangle^s)+K_s\,(\langle v\rangle^{s-2}\langle v_*\rangle^2+\langle v_*\rangle^{s-2}\langle v\rangle^2).$$

Here as above, the constants K_s and k_s depend on s but not on α.

Estimates for higher-order moments

The Povzner-type inequalities established above allow us to obtain the moment inequalities analogous to those obtained by Elmroth and Desvillettes [13, 11] for the classical Boltzmann equation.

Lemma 4. *Assume that $f(t,v)$ a solution to (14) that has a moment of order s bounded initially. Then, for every $s>2$,*

$$\frac{d}{dt}Y_s+k_sY_{s+1}\le K_s(Y_{s-1}+Y_{s-2})$$

where K_s and k_s are positive constants. Further,

$$\sup_{t>0} Y_s(t)\le Y_s^*=\max\left\{Y_s(0),\,\left(2K_s/k_s\right)^s\right\},$$

and for every $\tau>0$

$$\int_0^\tau Y_{s+1}(t)\,dt\le \frac{2K_s\tau+1}{k_s}Y_s^*.$$

Finally, for the stationary equation (16) we have an a priori estimate

$$Z_{s+1}\le \frac{K_s}{k_s}(Z_{s-1}+Z_{s-2}).$$

Remark. Notice that the condition on the integral of Y_{s+1} implies that if $Y_{s+1}(0)=+\infty$, then for every $\tau>0$ there is $t<\tau$ such that $Y_{s+1}(t)<+\infty$. Then, applying the lemma to Y_{s+1} we see that for every $\varepsilon>0$

$$\sup_{t>\varepsilon} Y_{s+1}(t)\le C_{\varepsilon,s},$$

so the propagation of the Y_{s+1} moment holds.

Bounds for the collision operator

We establish bounds for $Q_\alpha(f,f)$ and Q^- in the following weighted L^p spaces:

$$L^p_k(\mathbb{R}^3) = \{f \mid f(v)\langle v\rangle^k \in L^p(\mathbb{R}^3)\},$$

where $\langle v\rangle = (1+|v|^2)^{1/2}$. Our result extends the well-known L^p estimates for the classical Boltzmann equation [15] to the inelastic case.

Lemma 5. *For every $1 \le p \le \infty$ and every $k \ge 0$,*

$$\|Q_\alpha(f,g)\|_{L^p_k} \le C\left(\|f\|_{L^1_{k+1}}\|g\|_{L^p_{k+1}} + \|f\|_{L^p_{k+1}}\|g\|_{L^1_{k+1}}\right),$$

where C is a constant depending on p, k and N only.

H^1 regularity: Steady state equation

Here we use the results of Lemma 5 and the moment estimates of the preceding section to derive a priori estimates for solutions to (16) in the spaces

$$H^1_k(\mathbb{R}^3) = \{f \in L^2_k(\mathbb{R}^3) \mid \nabla f \in L^2_k(\mathbb{R}^3)\}.$$

The main tools are using the coercivity of the diffusion part, the estimates of the collision operator in L^p, and the standard interpolation theory on L^p spaces combined with the Sobolev's embedding inequality. We begin with an estimate for the gradient.

Lemma 6. *Assume that the function $f \in H^1(\mathbb{R}^3) \cap L^1_\beta(\mathbb{R}^3)$, where $\beta = \frac{5}{4}$, is a solution of* (16)*. Then*

$$\|\nabla f\|_{L^2} \le C\,A\,B^\beta, \quad \text{where} \quad A = \|f\|_{L^1_\beta}, \quad B = \|f\|_{L^1_1},$$

and C is a constant depending on the dimension.

The result of the Lemma 6 implies that the solutions to (16) gain *a priori* H^1 regularity, as soon as they are in L^1 and have the moment of order β. To see this notice that by the Sobolev embedding,

$$\|f\|_{L^6} \le C\,A\,B^\beta,$$

from which an estimate in L^2 follows by Sobolev weighted interpolation inequality using the L^1 bound, and, in fact, the solutions are in L^p for all $1 \le p \le 6$. In addition one establish *a priori* estimates for the L^2 moments of the gradient.

Lemma 7. *Assume that the solution f of equation* (16) *is in $H^1_k(\mathbb{R}^3) \cap L^1_{(k+1)\beta}(\mathbb{R}^3)$, where $k \ge 0$ and $\beta = \frac{5}{4}$. Then*

$$\|\nabla(f\langle v\rangle^k)\|_{L^2} \le C\big(A_1\,A_2^\beta + k^\gamma A_3^{1-\gamma/2}\big),$$

where

$$A_1 = \|f\|_{L^1_{(k+1)\beta}}, \quad A_2 = \|f\|_{L^1_{k+1}}, \quad A_3 = \|f\|_{L^1_{k-2/\beta}}, \quad \gamma = \frac{5}{4},$$

and C is a constant depending on the dimension 3.

Remark. Using Lemma 7 it is easy to find bounds for f in H^1_k for every $k \ge 0$. Indeed, using the inequality

$$|(\nabla f)\langle v\rangle^k|^2 \le C(\,|\nabla(f\langle v\rangle^k)|^2 + |f\nabla\langle v\rangle^k|^2)$$

and estimating the second term by interpolation between L^6 and $L^1_{k-2/\beta}$ we find an estimate for $\|(\nabla f)\langle v\rangle^k\|_{L^2}$. Further, by Sobolev weighted interpolation, it follows:

$$\|f\|_{L^2_k} \le \|f\|^\lambda_{L^6}\|f\|^{1-\lambda}_{L^1_{k/(1-\lambda)}}.$$

The estimates of f in H^1_k in terms of L^1 moments can then be obtained using Lemma 6. In particular we can notice that if the solutions are shown to have finite L^1 moments of any order, we gain apriori estimates in H^1_k with any k.

C^∞ regularity: Steady problem

Next, the following *a priori* bounds for solutions to (16) in the spaces

$$H^n_k = \{f \in L^2_k \mid \nabla^m f \in L^2_k,\ 1 \le m \le n\}$$

was established for all $1 \le n < \infty$ and all $0 \le k < \infty$. We used induction on n, with the base being given by Lemma 7. The induction step involves differentiating the equation in v.

In fact, for f and g being smooth, rapidly decaying functions, the higher-order derivatives of Q can be calculated using the following Leibniz formula (cf. [21]):

$$\partial^j Q(f,g) = \sum_{0\le l\le j} \binom{j}{l} Q(\partial^l f, \partial^{j-l} g),$$

where j and l are multi-indices $j = (j_1 \dots j_3)$, and $l = (l_1 \dots l_3)$;

$$\partial^j = \partial^{j_1}_{v_1} \dots \partial^{j_3}_{v_3},$$

and $\binom{j}{l}$ are the multinomial coefficients. Using this formula we obtain (formally) the following equations for the higher-order derivatives:

$$-\Delta\, \partial^j f = \sum_{0\le l\le j} \binom{j}{l} Q(\partial^l f, \partial^{j-l} g). \tag{25}$$

These equations are the key to the following lemma:

Lemma 8. *Any stationary solution f to* (16) *that is in $H^{n+1}_{k+\mu}(\mathbb{R}^3)$, where $n \ge 0$, $k \ge 0$ and $\mu > \frac{5}{2}$, satisfies the estimate*

$$\|\nabla^{n+1} f\|_{L^2_k} \le C\,(1 + k + \|f\|_{H^{n-1}_{k+\mu}})\,(1 + \|\nabla^n f\|_{L^2_{k+\mu}}),$$

where C is a constant depending on n and μ only.

Lemma 8 gives us a way to estimate higher-order derivatives of the solutions in terms of lower-order ones. Starting with an H^1_k bound of Lemma 7, we can obtain by induction bounds in the spaces $H^n_k(\mathbb{R}^3)$, with any n, in terms of L^1 moments. Thus, we get apriori bounds that are enough to claim that

$$f \in \bigcap_{n\ge 1,\, k\ge 0} H^n_k,$$

or in the Schwartz class $\mathcal{S}$ of rapidly decaying smooth functions.

Regularity for the time-dependent problem

In this section we apply the methods developed for the steady state solutions to the problem with time-dependence. Our first lemma is an analog of Lemma 6.

Lemma 9. *Let f be a solution to* (14) *with initial data $f_0 \in L^2(\mathbb{R}^3)$, such that f has a moment of order $\beta = \frac{3+2}{4}$ bounded uniformly in time. Then*

$$\|f(t,\cdot)\|_{L^2} \le C, \quad 0 \le t < \infty,$$

and

$$\|\nabla f(t,\cdot)\|_{L^2([0,T]\times\mathbb{R}^3)} \le C_T,$$

for every $0 \le T < \infty$.

Similar conclusions can be made about the time-dependence of L^2-moments.

Lemma 10. *Let f be a solution to* (14) *with initial data $f_0 \in L^2_k(\mathbb{R}^3)$ where $k \geq 0$, such that f has a moment of order $\beta(k+1)$, where $\beta = \frac{3+2}{4}$, bounded uniformly in time. Then*

$$\|f(t,\cdot)\|_{L^2_k} \leq C, \quad 0 \leq t < \infty,$$

and

$$\|\nabla f(t,\cdot)\|_{L^2_k([0,T]\times\mathbb{R}^3)} \leq C_T,$$

for every $0 \leq T < \infty$.

Lemma 11. *Let f be a solution to* (14) *with initial data $f_0 \in H^n_k(\mathbb{R}^3)$ where $k \geq 0$ and $n \geq 0$, such that f has a moment of order $\beta(2^n(k+\mu) - 2\mu + 1)$, where $\beta = \frac{3+2}{4}$ and $\mu > \frac{3+2}{2}$, bounded uniformly in time. Then*

$$\|f(t,\cdot)\|_{H^n_k} \leq C, \quad 0 \leq t < \infty,$$

and

$$\|f\|_{L^2([0,T],H^{n+1}_k(\mathbb{R}^3))} \leq C_T,$$

for every $0 \leq T < \infty$.

Existence and uniqueness

We establish the existence in the class of nonnegative functions from $L^1 \cap L^2$, with the bounded second L^1 moment. For the uniqueness we require an additional moment from the initial data. The results can be formulated as follows.

Theorem 12. *The Cauchy problem for the equation* (14) *with the initial data* (15) *admits a nonnegative solution*

$$f \in L^\infty([0,\infty), L^1_2 \cap L^2(\mathbb{R}^3))$$

for every nonnegative $f_0 \in L^1_2 \cap L^2(\mathbb{R}^3)$. We also have, for every $t_0 > 0$,

$$f \in C^\infty([t_0,\infty) \times \mathbb{R}^3)$$

and

$$f \in L^\infty([t_0,\infty), \mathcal{S}(\mathbb{R}^3)),$$

where $\mathcal{S}$ is the Schwartz class of rapidly decaying smooth functions.

Theorem 13. *Assume $f_0 \in L^1_3(\mathbb{R}^3)$; then the equation* (25) *with the initial condition $f(0,v) = f_0(v)$ has at most one solution.*

Lower bounds with overpopulated high energy tails

The lower bounds for both the stationary and time dependent problem can be constructed by choosing lower barrier functions for the semi-linear operator resulting from neglecting the gain term Q^+. The proof of the lower bound make use that the loss operator $Q^-(f,f)$ is pointwise bounded by a local function of the velocity times f, i.e. the inelastic Boltzmann-diffusive operator is bounded below by a linear elliptic operator, which evaluated on the positive solution f, satisfies the maximum principle. The proof works for the time-dependent equation as well as the steady one.

The next Lemma is the comparison result.

Lemma 14. *Assume $f \geq 0$ is a smooth solution to* (16) *with bounded mass ρ and energy ϑ_f respectively. Then, there are constants a, satisfying the condition above for the stationary barrier for $m = \rho$, the constant density, and ϑ, the bound for the energy, respectively, and $K = c_0 e^{-2a|v_0|^{3/2}}$, with c_0, v_0 and r_0 depending on with ρ such that*

$$f(v) \geq K e^{-2a|v|^{3/2}} \tag{26}$$

With a corresponding maximum principle for parabolic operator, we can we show, in a similar way, the lower bound control for the time dependent problem.

Lemma 15. *Assume* $f \geq 0$ *is a smooth solution to* (14) *with bounded mass and energy, uniformly in time, also denoted by* ρ *and* ϑ_f *respectively. Then, there are positive constants* $K > 0$, $a > 0$ *and* $b > 0$ *depending on* ρ *and* ϑ_f, *such that*

$$f(v,t) \geq K e^{-bt-a(1+|v|^2)^{3/4}}.$$

Further, if there exist a constant c_0 *and a ball* $B(v_0, r_0)$, *such that*

$$f(v,t) \geq c_0, \quad \text{if} \quad v \in B(v_0, r_0),$$

for all t, *then we can obtain a lower bound*

$$f(v,t) \geq K e^{-a|v|^{3/2}},$$

uniformly in time, where now $K > 0$, $a > 0$ *and* $b > 0$ *will depend on* c_0, v_0 *and* r_0.

Remark. Obtaining a pointwise upper bound using some form of maximum principle entails much more significant difficulties. Here we only notice that under the (admittedly too restrictive) assumption

$$Q^+(f,f) \leq \kappa\, Q^-(f,f), \qquad \text{for } |v| \geq R$$

with $\kappa < 1$ and $R > 0$, we can prove that

$$f(v) \leq L\, e^{-b|v|^{3/2}},$$

where L and b are positive constants which can be computed explicitly using the conditions on the barrier functions.

Another method based on using moment bounds yields an integral bound of the form

$$\int_{\mathbb{R}^3} f(v)\, e^{b\,|v|^{3/2}}\, dv$$

for the steady solution. We refer to our forthcoming work [6] for the details.

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