On some properties of kinetic and hydrodynamic equations for inelastic interactions

A. V. Bobylev[*], J. A. Carrillo[†], I. M. Gamba[‡]

February 11, 2000

Abstract

We investigate a Boltzmann equation for inelastic scattering in which the relative velocity in the collision frequency is approximated by the thermal speed. The inelasticity is given by a velocity variable restitution coefficient. This equation is the analogous to the Boltzmann classical equation for Maxwellian molecules. We study the homogeneous regime using Fourier analysis methods. We analyze the existence and uniqueness questions, linearized operator around Dirac delta function, self-similar solutions and moment equations. We clarify the conditions under which self-similar solutions describe the asymptotic behavior of the homogeneous equation. We obtain formally a hydrodynamic description for near elastic particles under the assumption of constant and variable restitution coefficient. We describe the linear long-wave stability/instability for homogeneous cooling states.

1 Introduction

The aim of this paper is to clarify some questions concerning principal properties of kinetic equations for granular media. We use the well-known and wide-accepted model given by the generalized Boltzmann-Enskog equation for a dense gas of inelastic spheres as the basis of our study. For the sake of reader's convenience we describe a brief scheme of its derivation in Section 2.

We use a velocity dependent restitution coefficient which characterizes the inelasticity of collisions. The constant restitution coefficient case leads to well-known unrealistic physical states in some applications. In fact, the restitution coefficient may depend on relative velocity in such a way that collisions with small relative velocity are close to

[*]M. V. Keldysh Institute of Applied Mathematics, Russian Academy of Sciences, Moscow, RUSSIA.

[†]Department of Mathematics, University of Texas at Austin, 78712 Austin-Texas, USA; on leave from Departamento de Matemática Aplicada, Facultad de Ciencias, Universidad de Granada, 18071 Granada, SPAIN.

[‡]Department of Mathematics, University of Texas at Austin, 78712 Austin-Texas, USA.

be elastic. This type of restitution coefficient is more realistic and it has been used in molecular dynamics simulation of oscillated granular media [5]. Besides, it has been proved that a dynamical model with the constant coefficient leads to inelastic collapse [2, 11]. On the other hand, the collapse does not occur for variable restitution coefficient with appropriate behavior for small values of relative speed [14].

Dealing with dense gases of granular media we always assume (directly or indirectly) that the mean free path is relatively small since we are outside of the Boltzmann-Grad limit. This implies a transition to a certain hydrodynamic regime. At the formal (physical) level of description this transition can be made by different ways (Grad method [16], Chapman-Enskog expansion [15], etc).

However, in all cases, we need to answer the following questions: (a) how to describe large-time asymptotics of solutions to the spatially homogeneous equation; and (b) what can be said about main properties of corresponding limit equations (hydrodynamics). Our work can be considered as an attempt to partly clarify these questions, especially in the case of variable restitution coefficient. In particular, Sections 3-6 are devoted to the spatially homogeneous problem and Section 7-8 to some stability/instability properties of the limit dissipative Euler equations.

An extensive survey of physical literature can be found in [13]. The existence theory for the inelastic hard-sphere Boltzmann-Enskog equation was analyzed in [12]. A detail theory of one-dimensional granular flows was recently developed in [2, 3, 4]. These and other cited publications are only a small part of a huge literature related to flows of inelastic granular materials. We refer to [9] and the references therein for a survey of granular flow kinetic theory.

A key idea of our approach is to use a simplified (pseudo-maxwellian) version of the Boltzmann-Enskog equation. Physical reasons for such a simplification are described in detail at the end of Section 2. This simplification is especially effective in the spatially homogeneous case. In order to analyze this spatially homogeneous equation we use the Fourier transform method applied in the Boltzmann equation for Maxwellian molecules by one of the authors [6, 7].

Here, the main difference is the absence of conservation of energy at the level of the collision mechanism, and thus, at the level of the Boltzmann equation. In fact, a Maxwellian distribution can not be a solution of the inelastic Boltzmann equation. However, the Dirac delta distribution is obviously in the kernel of the collision operator. Therefore, with the Fourier method, we can describe the spectrum of the linearized homogeneous equation around the Dirac delta distribution.

In addition, we clarify the existence of self-similar solutions for this model. This type of solutions have been considered before for the inelastic hard-sphere Boltzmann equation [8, 15]. They have been called homogeneous cooling states. The nearly elastic case was studied in [15]. In the present model we state and prove the precise conditions under which these self-similar solutions exist, and then describe the large-time behavior of the system. These results are accomplished using eigenfunction expansion of the solutions in the Fourier transformed equation. Finally, we study the moment equations for this system. For constant fixed restitution coefficient the large-time asymptotics of the system are given by the Dirac delta distribution. We analyze two cases under which self-similar solutions are asymptotically relevant. These cases are: a) nearly elastic particles with constant or variable restitution coefficient and b) small temperature. In both cases, the self-similar solution is near a Maxwellian distribution for a specific asymptotics involving restitution coefficient and time (see section 6.1). Therefore, we can make a formal transition to hydrodynamics in the spatially inhomogeneous simplified equation. The resulting hydrodynamic description, valid for a) and b), is the usual Euler equations for gas-dynamics with a dissipative term in the temperature equation due to the inelasticity. Finally, we study the stability/instability of the homogeneous cooling state in the hydrodynamic description. In fact, in the constant restitution coefficient case the homogeneous cooling state is unstable. In the variable case, we find that the system is linearly stable.

2 Pseudo-Maxwellian model for dissipative hard spheres

In this section we introduce the basic model we will treat in this paper. Our starting point is the Boltzmann-Enskog equation for inelastic hard spheres neglecting rotational degrees of freedom. For the sake of completeness and reader's convenience we include a short review of this model. A wider discussion can be found in [9] and the references therein.

Assume that we are studying the dynamics of N perfect spheres of diameter $\sigma > 0$ such that they perform inelastic collisions. If (x, v) and $(x - \sigma n, w)$ are the states of two particles before a collision, where $n \in S^2$ is the unit vector along the center of both spheres, the postcollisional velocities are found assuming that the total momentum is preserved but we loose some part of the normal relative velocity, that is,

$$n \cdot (v' - w') = -e((v - w) \cdot n)$$

where $0 < e \leq 1$ is called the restitution coefficient. Using both information we construct the postcollisional velocities as

$$v' = \frac{1}{2}(v + w) + \frac{u'}{2} \tag{2.1}$$

$$w' = \frac{1}{2}(v + w) - \frac{u'}{2} \tag{2.2}$$

where $u' = u - (1 + e)(u \cdot n)n$, $u = v - w$ and $u' = v' - w'$. Let us denote by v^* and w^* the precollisional velocities corresponding to v and w. Let us remark that the coefficient of restitution can be a function of the normal relative velocity $|(v - w) \cdot n|$.

Following the standard procedures of kinetic theory [9, 10], we deduce the Boltzmann-Enskog equation for inelastic hard spheres

$$\frac{\partial f}{\partial t} + (v \cdot \nabla_x) f = \sigma^2 Q_B(f, f) \tag{2.3}$$

where the collision operator is given by

$$
\begin{aligned}
Q_B(f,f) &= \int_{\mathbb{R}^3}\int_{S^2}((v-w)\cdot n)_+\Big[\frac{1}{e}JG(x,x+\sigma n|\rho)f(t,x,v^*)f(t,x+\sigma n,w^*) \\
&\qquad -G(x,x-\sigma n|\rho)f(t,x,v)f(t,x-\sigma n,w)\Big]\,dn\,dw
\end{aligned}
$$

and

$$\rho(t,x)=\int_{\mathbb{R}^3} f(t,x,v)\,dv\ .$$

Here, J is the Jacobian of the transformation (v,w) into (v^*,w^*) and G is the statistical correlation function between the particles. We refer to [9] for a deeper discussion on the meaning of the function $G\geq 1$. Let us only remark the function G is related to corrections to the molecular chaos hypothesis for dense gases and its value is 1 for a rarefied gas, i. e., the joint probability density of two particles $f^{(2)}$ is given by

$$f^{(2)}(t,x,v,y,w)=G\Big(x,y\mid\rho(t,x)\Big)f(t,x,v)f(t,y,w). \tag{2.4}$$

We can write $Q_B(f,f)$ in a more convenient way. Since v',w' can be written as

$$
\begin{aligned}
v' &= \frac{1}{2}(w+v)+\frac{1+e}{4}(u-2n(u\cdot n))+\frac{1-e}{4}u \\
w' &= \frac{1}{2}(w+v)-\frac{1+e}{4}(u-2n(u\cdot n))-\frac{1-e}{4}u
\end{aligned}
$$

then we can use the identity

$$\int_{S^2}(u\cdot n)_+\ \varphi(n(u\cdot n))\,dn=\frac{|u|}{4}\int_{S^2}\varphi\left(\frac{u-|u|n}{2}\right)\,dn$$

for any function φ to write $Q_B(f,f)$ as

$$
\begin{aligned}
Q_B(f,f) &= \frac{1}{4}\int_{\mathbb{R}^3}\int_{S^2}\Big[|v^*-w^*|JG(x,x+\sigma\omega|\rho)f(t,x,v^*)f(t,x+\sigma\omega,w^*) \\
&\qquad -|v-w|G(x,x-\sigma\omega|\rho)f(t,x,v)f(t,x-\sigma\omega,w)\Big]\,dn\,dw
\end{aligned}
$$

where

$$\omega=\frac{u-|u|n}{|u-|u|n|}=\frac{m-n}{|m-n|}\quad\text{with } m=\frac{u}{|u|}.$$

Here, v^*,w^* are the precollisional velocities associated to the collision mechanism

$$v' = \frac{1}{2}(v+w)+\frac{1-e}{4}(v-w)+\frac{1+e}{4}|v-w|n \tag{2.5}$$

$$w' = \frac{1}{2}(v+w)-\frac{1-e}{4}(v-w)-\frac{1+e}{4}|v-w|n \tag{2.6}$$

In case of non-constant restitution coefficient $e = e(u \cdot n)$ in the initial equation, the coefficient reads now as

$$e = e\left(\frac{|u - |u|n|}{2}\right) = e\left(|u|\sqrt{\frac{1 - m \cdot n}{2}}\right).$$

The equation (2.3) is often considered as the basic kinetic equation for granular media. On the other hand, this equation is a fairly rough mathematical model of a real physical process. Even without mentioning rotational effects and deviations from spherical shape of particles, one can realize that the formula (2.4) is very rough outside of the Boltzmann-Grad limit. This formula certainly does not account for non-Markovian effects of repeating collisions. Moreover, the model of inelastic scattering based on empirical restitution coefficient is also a very rough approximation. Thus, we have the equation (2.3) which is (a) a very rough approximation from physical point of view and (b) quite complicated equation from mathematical point of view. These obvious considerations lead to the idea of using a simplified version of Eq. (2.3). In particular, the following simplification seems reasonable.

Consider $u(t,x)$ and $\theta(t,x)$ the bulk velocity and temperature defined by f, that is,

$$\rho(t,x)u(t,x) = \int_{\mathbb{R}^3} vf\, dv$$

and

$$3\rho(t,x)\theta(t,x) = \int_{\mathbb{R}^3} |v - u(t,x)|^2 f\, dv\ .$$

Therefore, let us take the approximation

$$|v - w| \simeq S\sqrt{\theta(t,x)} \tag{2.7}$$

with certain constant S for the precollisional velocities. The constant can be chosen in such a way that the Euler equations (Section 7) for the model are identical to the Euler equations for the initial kinetic equation (2.3). Similarly, we assume now that $e = e(\theta) = \tilde{e}(\sqrt{\theta})$, with an appropriate choice of the function $\tilde{e}(\sqrt{\theta})$, and as a consequence $J = \frac{1}{e}$.

This assumption on the collision frequency as a function of the kinetic temperature appear in the engineering literature on fluidization models of granular media [13].

Therefore our pseudo-Maxwellian model is given by

$$\frac{\partial f}{\partial t} + (v \cdot \nabla_x)f = Q(f,f) \tag{2.8}$$

where

$$\begin{aligned} Q(f,f) \quad = \quad & A(t,x)\int_{\mathbb{R}^3}\int_{S^2}\Big[\frac{1}{e}G(x, x+\sigma\omega|\rho)f(t,x,v^*)f(t,x+\sigma\omega,w^*) \\ & -G(x,x-\sigma\omega|\rho)f(t,x,v)f(t,x-\sigma\omega,w)\Big]\,dn\,dw \end{aligned}$$

and

$$A(t,x) = \frac{\sigma^2}{4} S\sqrt{\theta(t,x)}$$

with the collision mechanism given by (2.5)-(2.6). The collision operator can be written as

$$\begin{aligned} Q(f,f) &= 4A(t,x)\int_{\mathbb{R}^3}\int_{S^2}((v-w)\cdot n)_+\Big[\frac{1}{e^2}\frac{G(x,x+\sigma\omega|\rho)}{|v^*-w^*|}f(t,x,v^*)f(t,x+\sigma\omega,w^*) \\ &\quad -\frac{G(x,x-\sigma\omega|\rho)}{|v-w|}f(t,x,v)f(t,x-\sigma\omega,w)\Big]\,dn\,dw \end{aligned}$$

with the collision mechanism given by (2.1)-(2.2).

In this work we will study in detail the homogeneous case of equation (2.8). We call (2.8) a pseudo-maxwellian model because of the analogy to the Maxwellian models in the classical elastic case in which the cross-section does not depend on the relative velocity. From the point of view of the applications this approximation is as rough as the equation (2.3), so we prefer (2.8) because it is a more simplified model.

The homogeneous equation corresponding to (2.8) is

$$\frac{\partial f}{\partial t} = B(t)Q(f,f) \tag{2.9}$$

where

$$Q(f,f) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{S^2}\left[f(t,v^*)f(t,w^*)\frac{1}{e} - f(t,v)f(t,w)\right]\,dn\,dw \tag{2.10}$$

and

$$B(t) = \pi S\sigma^2\sqrt{\theta(t)}\;G(\rho) = B\sqrt{\theta(t)}\;G(\rho) = \tilde{B}\sqrt{\theta(t)} \tag{2.11}$$

The corresponding weak form of $Q(f,f)$ is given by the integral

$$(Q,\psi) = \frac{1}{4\pi}\int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} f(t,w)f(t,v)[\psi(v')-\psi(v)]\,dv\,dw\,dn$$

where $\psi \in C_0^\infty(\mathbb{R}^3)$ and v' is computed by

$$v' = \frac{1}{2}(v+w) + \frac{1-e}{4}(v-w) + \frac{1+e}{4}|v-w|n \tag{2.12}$$

and $e = e(\theta(t))$. In the next section, we will study the initial value problem for (2.9).

3 Initial value problem

Let us consider equation (2.9) with the collision term given by (2.10)-(2.11) and v' given by (2.12). First, we rescale in time by defining

$$d\tau = \tilde{B}\sqrt{\theta(t)}\,dt, \tag{3.1}$$

then

$$\tau = \tilde{B}\int_0^{\tau} \sqrt{\theta(\tilde{\tau})}\, d\tilde{\tau}.$$

Thus, we obtain the equation

$$\frac{\partial f}{\partial \tau} = Q(f,f), \tag{3.2}$$

where $Q(f,f)$ acts in weak form as

$$(Q,\psi) = \frac{1}{4\pi}\int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} f(v)f(w)[\psi(v')-\psi(v)]\, dv\, dw\, dn \tag{3.3}$$

with v' given by (2.12) and $e=e(\theta)$. Let us remark that equation (3.3) contains the gain and the loss term together. From now on, we will denote again by t the time variable, and since

$$\int Q(f,f)\, dv = \int vQ(f,f)\, dv = 0,$$

then it is assumed, without loss of generality, that

$$\int_{\mathbb{R}^3} f(t,v)\, dv = 1\ , \qquad \int_{\mathbb{R}^3} vf(t,v)\, dv = 0$$

for any $t \geq 0$. The temperature $\theta(t)$ is defined as usual by

$$\theta(t) = \frac{1}{3}\int_{\mathbb{R}^3} |v|^2 f(t,v)\, dv.$$

We will denote by f_0 the initial data for equation (3.2).

A detailed analytic theory of the classical elastic ($e \equiv 1$) case was developed in [6, 7]. We use here the same approach based in Fourier transform method. For the sake of simplicity, we will skip the time dependence of the functions in some of the equations below. Let us introduce the characteristic function

$$\varphi(t,k) = \int_{\mathbb{R}^3} f(t,v)\exp(-i(k\cdot v))\, dv\ , \qquad k\in\mathbb{R}^3$$

then, using (3.2)-(3.3) we obtain

$$\frac{\partial\varphi}{\partial t} = (Q, \exp(-i(k\cdot v))) = \mathcal{L}(\varphi,\varphi)$$

where

$$\mathcal{L}(\varphi,\varphi) = \frac{1}{4\pi}\int_{\mathbb{R}^3\times\mathbb{R}^3} f(v)f(w)\exp\Big\{-i(k\cdot U) - i\frac{1-e}{4}(k\cdot u)\Big\} F(k,u)\, dv\, dw\ ,$$

$U = \frac{1}{2}(v+w)$, $u=(v-w)$ and

$$F(k,u) = \int_{S^2}\Big[\exp\Big\{-\frac{1+e}{4} i(k\cdot n)|u|\Big\} - \exp\Big\{-\frac{1+e}{4} i(k\cdot u)\Big\}\Big]\, dn\ . \tag{3.4}$$

It is easy to see that the product $(k \cdot |u|n)$ can be replaced by $(u \cdot |k|n)$ in the first term of (3.4) because the function $F(k,u)$ is isotropic and only depends on the values of $|k|, |u|$ and $k \cdot u$ (see [6, 7]). Proceeding with the replacement and interchanging the order of integration we obtain

$$\mathcal{L}(\varphi,\varphi) = \mathcal{L}_+(\varphi,\varphi) - \mathcal{L}_-(\varphi,\varphi)$$

where

$$\mathcal{L}_-(\varphi,\varphi) = \varphi(t,0)\varphi(t,k)$$

and

$$\mathcal{L}_+(\varphi,\varphi) = \frac{1}{4\pi}\int_{S^2\times\mathbb{R}^3\times\mathbb{R}^3} f(v)f(w)\exp\{H(k,v,w)\}\,dv\,dw\,dn$$

with

$$H(k,v,w) = -iv\left(\frac{k}{2} + \frac{1-e}{4}k + \frac{1+e}{4}|k|n\right) + iw\left(-\frac{k}{2} + \frac{1-e}{4}k + \frac{1+e}{4}|k|n\right)$$

Therefore, the Fourier transformed gain operator is given by

$$\mathcal{L}_+(\varphi,\varphi) = \frac{1}{4\pi}\int_{S^2}\varphi\left(\frac{3-e}{4}k + \frac{1+e}{4}|k|n\right)\varphi\left(\frac{1+e}{4}(k-|k|n)\right)\,dn.$$

In particular, we have proved the following lemma.

Lemma 3.1 *The Fourier transformed equation corresponding to* (3.2) *is given by*

$$\frac{\partial\varphi}{\partial t} = \frac{1}{4\pi}\int_{S^2}\{\varphi(t,k_+)\varphi(t,k_-) - \varphi(t,0)\varphi(t,k)\}\,dn \tag{3.5}$$

with

$$k_- = \frac{1+e}{4}(k-|k|n)\ , \qquad k_+ = k - k_-\ , \qquad \varphi(t,0) = 1,$$

$$[\nabla_k\varphi](t,0) = 0\ , \qquad e = e(\theta(t))\ , \qquad \theta(t) = -\frac{1}{3}[\Delta_k\varphi](t,0)$$

and $$\varphi(0,k) = \varphi_0(k) = \int_{\mathbb{R}^3}\exp(-i(k\cdot v))f_0(v)\,dv.$$

Remark 3.2 Assuming that $f(v) \le A\exp(-\alpha|v|^2)$, the complex Fourier variable $k = i\tilde{k}$ (two-side Laplace transform) can be used in order to obtain the same equation for the positive function $\tilde{\varphi}(\tilde{k}) = \varphi(i\tilde{k})$.

We see from Lemma 3.1 that our simplification of the model leads to a very simplified homogeneous equation that can be completely described in Fourier transform formulation. The rest of this section is devoted to review, very quickly, the existence and uniqueness theory for the Fourier transformed equation (3.5).

The existence theorems for the classical elastic case ($e = 1$) were proved by Morgenstern, Wild and others (see [6] and references therein). These results can be easily generalized to the inelastic case. Consider a change in time variable of the type

$$\tau = 1 - \exp(-t) \ , \qquad \varphi(t,k) = \exp(-t)\Phi(\tau,k),$$

then (3.5) leads to

$$\frac{\partial \Phi}{\partial \tau} = \mathcal{L}_+(\Phi,\Phi)$$

with $\Phi(k,0) = \varphi_0(k)$. Solutions in power series expansion of the type

$$\Phi(\tau,k) = \sum_{n=0}^{\infty} \Phi_n(k)\tau^n$$

are given by a simple recurrent sequence of equalities

$$\begin{aligned} \Phi_0 &= \varphi_0 \\ \Phi_{n+1} &= \frac{1}{n+1}\sum_{k=0}^{n} \mathcal{L}_+(\Phi_k,\Phi_{n-k}) \ , \qquad n \geq 0 \end{aligned} \tag{3.6}$$

Noting that $|\varphi_0| \leq 1$, we obtain $|\Phi_n| \leq 1$ for any $n \geq 0$. Then the series (3.6) converges uniformly on $\tau \in [0,1)$. This series is usually called the Wild's sum in the Fourier representation.

Using the same ideas as in [6] section 13, we can obtain the following result. Let us define $f(t,v)$ to be a solution of (3.2) if its Fourier transform is a characteristic function for any $t \geq 0$ and solves (3.5).

Proposition 3.3 *Problem* (3.5) *with* $f_0 \in L^1(\mathbb{R}^3)$

$$f_0 \geq 0 \ , \qquad \int_{\mathbb{R}^3} f_0(v)\,dv = 1 \qquad \textit{and} \qquad \int_{\mathbb{R}^3} |v|^2 f_0(v)\,dv = 3\theta_0 < \infty,$$

has a unique classical solution $\varphi(t,k)$ *for* $t > 0$. *Moreover, there exists a unique function* $f(t,v) \geq 0$ *such that*

$$f \in L^\infty(0,\infty;L^1(\mathbb{R}^3)) \qquad \textit{and} \qquad \varphi(t,v) = \int_{\mathbb{R}^3} f(t,v)\exp(-i(k\cdot v))\,dv.$$

Therefore, the function $f(t,v)$ *defines a unique solution of* (3.2) *satisfying the initial condition* $f(0,v) = f_0(v)$.

Remark 3.4 The uniqueness is related to the explicit form that we have for $\varphi(t,k)$ in terms of a series. More details can be found in section 13 of [6].

4 Linearized equation

First of all, let us write $\mathcal{L}_+(\varphi,\varphi)$ in a different way. Equation (3.5) can be written as

$$\mathcal{L}_+(\varphi,\varphi) = \frac{1}{4\pi}\int_{S^2} \mathcal{F}\left(\frac{|k|n-k}{2}\right)\,dn$$

where

$$\mathcal{F}(u) = \varphi(-zu)\varphi(k+zu) \ \text{ and } \ z = \frac{1+e}{2}\ .$$

The identity

$$\frac{1}{4\pi}\int_{S^2} \mathcal{F}\left(\frac{|k|n-k}{2}\right)\,dn = \frac{1}{2\pi|k|}\int_{\mathbb{R}^3} \delta(2\xi\cdot k + |\xi|^2)\mathcal{F}\left(\frac{\xi}{2}\right)\,d\xi\ . \tag{4.1}$$

can be verified by completing the square in the delta Dirac function, taking polar coordinates $\xi = rn$ and performing the change of variables $r^2 = s$. Meanwhile, applying directly polar coordinates $\xi = rn$ and changing variables $r(r+2k\cdot n) = s+2k\cdot n$ we have

$$\begin{aligned}
\int_{\mathbb{R}^3} \delta(2\xi\cdot k + |\xi|^2)\mathcal{F}\left(\frac{\xi}{2}\right)\,d\xi &= \int_{S^2}\int_0^\infty \delta(r(r+2k\cdot n))\mathcal{F}\left(\frac{rn}{2}\right) r^2\,dr\,dn \\
&= 2\int_{S^2} |(k\cdot n)_-|\mathcal{F}(-(k\cdot n)n)\,dn \\
&= \int_{S^2} |k\cdot n|\mathcal{F}(-(k\cdot n)n)\,dn
\end{aligned}$$

Therefore,

$$\frac{1}{4\pi}\int_{S^2} \mathcal{F}\left(\frac{|k|n-k}{2}\right)\,dn = \frac{1}{2\pi|k|}\int_{S^2} |k\cdot n|\mathcal{F}(-(k\cdot n)n)\,dn\ .$$

Using this formula in (3.5), we have proved the following identity.

Lemma 4.1 *Equation* (3.5) *can be written as*

$$\frac{\partial\varphi}{\partial t} = \frac{1}{2\pi}\int_{S^2}\Big\{\varphi(z(k\cdot n)n)\varphi(k - z(k\cdot n)n) - \varphi(0)\varphi(k)\Big\}\frac{|k\cdot n|}{|k|}\,dn = \mathcal{L}(\varphi,\varphi) \tag{4.2}$$

where $z = \frac{1+e}{2}$.

Previous formula for $\mathcal{L}(\varphi,\varphi)$ implies the following property of the Fourier transformed operator,

$$\begin{aligned}
\mathcal{L}(\exp(i(k\cdot u)),\exp(i(k\cdot u))) &= 0 && \text{for any } u\in\mathbb{R}^3 \\
\mathcal{L}(\exp(i(k\cdot u))\varphi,\exp(i(k\cdot u))\varphi) &= \exp(i(k\cdot u))\mathcal{L}(\varphi,\varphi) && \text{for any } u\in\mathbb{R}^3 \quad (4.3)
\end{aligned}$$

Next, let us study equation (4.2) near the fixed point $\varphi_* = 1$. Because of property (4.3) the behavior of the solutions near any other $\exp(i(k\cdot u))$ for any $u \in \mathbb{R}^3$, $|u| \neq 0$, is quite similar to the case $|u| = 0$. In fact, (4.3) is the equivalent of the translational invariance of equation (2.9). Let us remark that linearizing the Fourier transformed equation around the state $\varphi_* = 1$ is like linearizing our original equation around the Delta Dirac distribution.

Thus, let us consider perturbations of $\varphi_* = 1$ in equation (4.2) of the form

$$\varphi(t,k) = 1 + \psi(t,k)\ ; \qquad \|\psi\| \ll 1$$

with $\|\cdot\|$ is the continuous norm. We obtain the linearized equation

$$\frac{\partial \psi}{\partial t} = \mathcal{L}_2\psi \ = \ \frac{1}{2\pi}\int_{S^2}\frac{|k\cdot n|}{|k|}\Big[\psi(z(k\cdot n)n) + \psi(k - z(k\cdot n)n) - \psi(0) - \psi(k)\Big]$$

We can prove the following result.

Theorem 4.2 *Polynomial eigenfunctions and corresponding eigenvalues of $\mathcal{L}_2$ are given by*

$$\begin{cases} \mathcal{L}_2\varphi_{nlm}(k) = -\lambda_{nl}\varphi_{nlm}(k) \\ \varphi_{nlm}(k) = |k|^{2n+l}Y_{lm}\left(\frac{k}{|k|}\right) \\ n = 0,1,\dots\ ; \quad l = 0,1,\dots\ ; \quad m = -l,\dots,l \end{cases}$$

where $Y_{lm}(u)$ are the spherical harmonics with

$$\begin{aligned}\lambda_{nl} \ &= \ 1 + \delta_{n0}\delta_{l0} - 2\Big\{z^{2n+l}\int_0^1 \alpha^{2n+l+1}P_l(\alpha)\,d\alpha + \\ &\quad + \frac{1}{z(2-z)}\int_{1-z}^1 \alpha^{2n+l+1}P_l\left(\frac{1-z+\alpha^2}{(2-z)\alpha}\right)\,d\alpha\Big\},\end{aligned}$$

where P_l are the Legendre polynomials. Moreover, $\lambda_{nl} > 0$ except two values

$$\lambda_{00} = \lambda_{01} = 0 \tag{4.4}$$

In particular,

$$\begin{aligned}\lambda_n = \lambda_{n0} \ &= \ 1 + \delta_{n0} - \frac{1}{n+1}\left[z^{2n} + \frac{1-(1-z)^{2(n+1)}}{1-(1-z)^2}\right] = \\ &= \ 1 + \delta_{n0} - \frac{1}{n+1}\left[z^{2n} + \sum_{k=0}^{n}(1-z)^{2k}\right].\end{aligned} \tag{4.5}$$

We refer to [6, 7] for all the details of the derivation since the proof is identical with minor changes. Also we refer to [17] for properties of the spherical harmonics.

Remark 4.3 As a consequence, when e is constant, $0 < e < 1$ the spectrum of the linearized operator consists of two zero eigenvalues (4.4), they corresponds to the two conservation laws (mass and momentum), and a sequence of isolate points on the negative part of the real axis. It is clear that $\lambda_{nl} \to 1$ as $(2n+l) \to \infty$, and in particular we have $\lambda_n \to 1$ as $n \to \infty$. Note that the spectrum is given by the values $-\lambda_{nl}$.

In particular,

$$\lambda_{n1} = 1 - \frac{(1-a)^{2n+1}}{n+2} - \frac{1}{1+a}\left\{\frac{a}{n+1}\frac{1-a^{2(n+1)}}{1-a^2} + \frac{1}{n+2}\frac{1-a^{2(n+2)}}{1-a^2}\right\}$$

$n \geq 0$, where $a = 1 - z$. Thus, $\lambda_{01} = 0$ for any value $0 \leq a \leq 1/2$. Finally, we recover in the case $z = 1$ well-known formula for the eigenvalues of the linearized classical Boltzmann equation for Maxwell-molecules given by

$$\lambda_{nl} = 1 + \delta_{n0}\delta_{l0} - 4\int_0^1 \alpha^{2n+l+1} P_l(\alpha)\, d\alpha$$

The main difference from the elastic case ($z = 1$) is that we obtain a small eigenvalue

$$\lambda_{10} = z(1-z)$$

as a result of perturbation of the value $\lambda_{10} = 0$ for $z = 1$, that is, the elastic case.

5 Isotropic equation and its self-similar solutions

Let us come back for a while to the original Boltzmann equation (2.9). We can try to look for solutions of self-similar type, that is,

$$f(t,v) = \rho\theta(t)^{-3/2} g(\theta(t)^{-1/2}(v-u)) \tag{5.1}$$

where $\rho, u, \theta(t)$ are the mass, momentum and temperature of f. These states are called homogeneous cooling states.

Therefore, g must satisfy

$$\int_{\mathbb{R}^3} g\, dw = 1\ ; \quad \int_{\mathbb{R}^3} wg\, dw = 0\ ; \quad \int_{\mathbb{R}^3} |w|^2 g\, dw = 3$$

Computing the derivative of $\theta(t)$ and taking into account (3.3) for $\psi = |v-u|^2$ we find, after some computations, that

$$\frac{d}{dt}\int_{\mathbb{R}^3} |v-u|^2 f\, dv = 3\rho\theta'(t) = -3\tilde{B}\frac{1-e^2}{4}\rho^2\theta(t)^{3/2}.$$

Simplifying we deduce

$$\theta'(t) = -\frac{1-e^2}{4}\rho\tilde{B}\theta(t)^{3/2}. \tag{5.2}$$

Substituting (5.1) into (2.9) we deduce the following equation for g:

$$-\frac{3}{2}\rho\theta'(t)\theta(t)^{-5/2}\ \mathrm{div}_w(wg) = \rho^2\tilde{B}\theta(t)^{-1}\tilde{Q}(g,g) \tag{5.3}$$

where

$$\tilde{Q}(g,g) = \int_{\mathbb{R}^3}\left[\frac{1}{e(\theta(t))}g(w^*)g(v^*) - g(v)g(w)\right]dw\ .$$

Substituting (5.2) into (5.3) and simplifying we have

$$\tilde{Q}(g,g) - \frac{3}{8}(1-e^2)\ \mathrm{div}_w(wg) = 0 \tag{5.4}$$

Since $\theta(t)$ depends on t the only possibility for (2.9) to have a solution of the form (5.1) is that e does not depend on $\theta(t)$, therefore e constant. This point clarifies some discussion in [15]. When e is constant we can have solutions of the form (5.1) since in (5.4) we have a closed equation for g and then we solve the equation for $\theta(t)$.

The rest of this section is devoted to prove rigorously that these states exist and to study conditions for their existence.

First, let us consider the equation for isotropic solutions, that is, solutions of the form $\varphi(t,\eta)$ where $\eta = \frac{|k|^2}{2}$. We are going to have

$$\begin{array}{lcl} |k_-|^2 = z^2\frac{1-\zeta}{2}|k|^2\ , & & \zeta = \frac{k\cdot n}{|k|}\ , \qquad z = \frac{1+e}{2}\ , \\ k\cdot k_- = \frac{z}{2}(1-\zeta)|k|^2 & \text{and} & |k_+|^2 = |k|^2\left[1 - \frac{1-\zeta}{2}z(2-z)\right]\ . \end{array}$$

Now, we can evaluate equation (3.5) in spherical coordinates with polar axis directed along k and we obtain

$$\frac{\partial}{\partial t}\varphi\left(\frac{|k|^2}{2}\right) = \frac{1}{2}\int_{-1}^{1}\left\{\varphi\left(z^2\frac{|k|^2}{2}\frac{1-\zeta}{2}\right)\varphi\left(\frac{|k|^2}{2}\left(1-\beta\frac{1-\zeta}{2}\right)\right) - \varphi(0)\varphi\left(\frac{|k|^2}{2}\right)\right\}d\zeta$$

where $\beta = z(2-z)$. Fixing $\eta = \frac{|k|^2}{2}$ and $s = \frac{1-\zeta}{2}$ we have

$$\frac{\partial\varphi}{\partial t} = \int_0^1\left\{\varphi(z^2 s\eta)\varphi((1-\beta s)\eta) - \varphi(0)\varphi(\eta)\right\}ds \tag{5.5}$$

where $\beta = 1 - \frac{(1-e)^2}{4}$. Now, the temperature $\theta(t)$ is given by

$$\theta(t) = -\varphi'_\eta(t,\eta)\Big|_{\eta=0}$$

provided $\varphi(t,0) = 1$.

So if $\beta =$ constan, then equation (5.5) is invariant under dilations $\eta \to \alpha\eta$ with $\alpha \in \mathbb{R}^+$. Therefore, similar to the elastic case $\beta = 1$ [6, 7], we can prove that the equation (5.5) admits self-similar solutions

$$\varphi(t,\eta) = \tilde{\varphi}(\exp(-\mu t)\eta)$$

with $\mu \in \mathbb{R}$ to be chosen and where $\tilde{\varphi}$ satisfies the equation

$$\mu\eta\frac{d\tilde{\varphi}}{d\eta} + \int_0^1 \left\{\tilde{\varphi}(z^2 s\eta)\tilde{\varphi}((1-\beta s)\eta) - \tilde{\varphi}(0)\tilde{\varphi}(\eta)\right\} ds = 0$$

Looking for series solutions of the form

$$\tilde{\varphi}(\eta) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\tilde{\varphi}_n\eta^n \ ; \qquad \tilde{\varphi}_0 = 1 \tag{5.6}$$

we obtain a recurrent formula for $\tilde{\varphi}_n$, $n \geq 1$;

$$\tilde{\varphi}_n[-\mu n + \lambda_n] = \sum_{k=1}^{n-1} H(k, n-k)\tilde{\varphi}_k\tilde{\varphi}_{n-k}$$

where the right-hand side is 0 for $n = 1$, and

$$\begin{aligned} \lambda_n = \lambda_{n0} &= \int_0^1 [1 - z^{2n}s^n - (1-\beta s)^n]\, ds \ ; \qquad && n \geq 1 \\ H(k, n-k) &= z^{2k}\binom{n}{k}\int_0^1 s^k(1-\beta s)^{n-k}\, ds \ , \qquad && 1 \leq k \leq n-1 \end{aligned} \tag{5.7}$$

As we shall see later, the only interesting case for possible applications is the case $\mu = \lambda_1 = z(1-z)$.

Since we can choose an arbitrary value $\tilde{\varphi}_1 = \theta_0 > 0$, then $\tilde{\varphi}_n = \theta_0^n u_n$ where u_n is defined recursively by

$$\begin{aligned} u_n &= (\lambda_n - n\lambda_1)^{-1}\sum_{k=1}^{n-1} H(k, n-k)u_k u_{n-k} \ , \qquad n \geq 2 \\ u_1 &= 1 \end{aligned} \tag{5.8}$$

The convergence of the Taylor series (**??**) on the whole axis $\eta \in \mathbb{R}$ can be proved similarly to [6] provided $\lambda_n \neq n\lambda_1$ for all $n \geq 2$. Let us remark that

$$\beta = z(2-z) = z + (1-z)z \geq z > z^2,$$

and thus,

$$\begin{aligned} \sum_{k=1}^{n-1} H(k, n-k) &= \int_0^1 \sum_{k=1}^{n-1}\binom{n}{k} z^{2k}s^k(1-\beta s)^{n-k}\, ds \\ &\leq \int_0^1 [1 - (\beta s)^n - (1-\beta s)^n]\, ds < \lambda_n \end{aligned}$$

Therefore, taking into account that $\lambda_n \to 1$ as $n \to \infty$ (Remark 4.3) and considering

$$b = \max_{n\geq 2}\frac{\lambda_n}{|\lambda_n - n\lambda_1|}$$

one can easily prove by induction that u_n satisfies $|u_n| \leq b^{n-1}$, $n \geq 1$ and then we have the convergence. We have proved the following result

Theorem 5.1 *The equation* (3.5) *has the self similar solution*

$$\varphi(t,k) = \tilde{\varphi}\left(\frac{|k|^2}{2}\theta(t)\right) \ ; \qquad \theta(t) = \theta(0)\exp(-\mu t)$$

where $\mu = \frac{1-e^2}{4} = z(1-z) = \lambda_1$ *and* $u_0 = u_1 = 1$

$$\tilde{\varphi}(\eta) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} u_n \eta^n$$

with u_n *given by* (5.8)*, provided* $\lambda_n \neq n\lambda_1$ *for all* $n \geq 2$.

As a consequence we obtain the following corollary for equation (2.9) by changing to the original time variable (3.1).

Corollary 5.2 *The equation* (2.9) *with constant* e *has the self-similar solution*

$$f(t,v) = \rho\theta(t)^{-3/2} g(\theta(t)^{-1/2}(v-u))$$

where ρ, u *are the density and momentum and* $\theta(t)$ *is the temperature that verifies*

$$\theta'(t) = -\frac{1-e^2}{4}\rho\tilde{B}\theta(t)^{3/2}$$

provided $n\lambda_1 \neq \lambda_n$ *for all* $n \geq 2$.

Remark 5.3 Theorem 5.1 is not valid for those values of $0 < e < 1$ such that $\lambda_n(e) = n\frac{1-e^2}{4}$ for some $n \geq 2$. This condition may be violated for a numerable number of values of e, $\{e_n\}$ with $\{e_n\} \to 1$ as $n \to \infty$. This comes from the explicit formula for λ_n.

Remark 5.4 The coefficients u_n have the same sign as the moments $\langle |v|^{2n} \rangle$ of the distribution function (see next section). It follows from (5.8) that for any $0 < e < 1$ there exists a number $N = N(e)$ such that $u_N < 0$. Therefore the self-similar solution "distribution function" can not be non-negative. Anyway as we shall see in next section this self-similar solution controls the asymptotic behavior in some cases.

6 General solution and moment equations

We recall here a connection between the moments of $f(t,v)$ and the derivatives of $\varphi(t,k)$. For simplicity, we consider only isotropic functions and by now we omit the dependence on time. Since $\varphi = \varphi(\eta)$, $\eta = \frac{|k|^2}{2}$ and $f = f(|v|)$ then

$$\varphi(\eta) = 4\pi \int_0^{\infty} f(r)\frac{\sin\sqrt{\eta}r}{\sqrt{\eta}r} r^2\, dr$$

Then, we formally obtain

$$\varphi\left(\frac{|k|^2}{2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{|k|^2}{2}\right)^n \varphi_n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} m_n |k|^{2n}$$

where m_n is the nth moment of f and

$$\varphi_n = \frac{n!}{(2n+1)!} 4\pi \int_0^\infty r^{2(1+n)} f(r)\, dr.$$

Thus, we have

$$\varphi(\eta) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \varphi_n \eta^n \; ; \qquad \varphi_n = (-1)^n \varphi^{n)}(0)$$

where $\varphi^{n)}(0)$ is the nth-derivative of φ at 0 and finally

$$\varphi^{n)}(0) = \frac{(-1)^n 2^n n!}{(2n+1)!} m_n \; , \qquad n \geq 0 \; . \tag{6.1}$$

Once we have seen the relation between the derivatives at 0 of φ and the moments of f, let us obtain the moment equations. We consider the equation (5.5) and look for solutions in the form

$$\varphi(t, \eta) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \varphi_n(t) \eta^n \; .$$

Assuming that

$$\varphi(0, \eta) = \varphi_0(\eta) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \varphi_n(0) \eta^n \; .$$

By substituting into (5.5) we obtain the following recursive definition for $\varphi_n(t)$

$$\varphi_0 = 1 \; , \qquad \frac{d\varphi_1}{dt} + \lambda_1 \varphi_1 = 0 \; ,$$

$$\frac{d\varphi_n}{dt} + \lambda_n \varphi_n = \sum_{k=0}^{n-1} H(k, n-k) \varphi_k \varphi_{n-k} \; , \qquad n \geq 2 \; .$$

where the notation in (5.7) is used. Thus, using (6.1) we have

$$\varphi_1(t) = \varphi_1(0) \exp(-\Lambda_1(t)) = \theta(t) \; , \qquad \varphi_1(0) = \theta(0) \; ,$$

$$\begin{aligned} \varphi_n(t) = \varphi_n(0) \exp(-\Lambda_n(t)) + \\ \sum_{k=1}^{n-1} \int_0^t \varphi_k(\tau) \varphi_{n-k}(\tau) [H(k, n-k)](\tau) \exp\{-(\Lambda_n(t) - \Lambda_n(\tau))\}\, d\tau \end{aligned} \tag{6.2}$$

where $\Lambda_n(t)$ is given by

$$\Lambda_n(t) = \int_0^t \lambda_n(e(\theta(s)))\, ds \; , \qquad n \geq 1 \; .$$

Remember that our original time variable is different. But we will work with this time scale and we will write the final results in the original time (3.1).

To make formula (6.2) explicit we need to find $\theta(t)$. $\theta(t)$ solves

$$\theta'(t) = -\lambda_1 \theta = -\frac{1-e^2(\theta)}{4}\theta \;, \qquad \theta(0) = \theta_0 \;.$$

We have two cases.

6.1 Constant restitution coefficient

In this case, we obtain $\{\lambda_n\}$ are constants, and then $\varphi_1 = \theta_0 \exp(-\lambda_1 t)$. For $\varphi_2(t)$ we find

$$\varphi_2(t) = \left[\varphi_2(0) - \frac{\theta_0^2}{\lambda_2 - 2\lambda_1}\right] \exp(-\lambda_2 t) + \frac{\theta(t)^2}{\lambda_2 - 2\lambda_1} \;.$$

Remark that the second term is a contribution of the similarity solution (5.7), theorem 5.1. The general formula for $\varphi_n(t)$ looks like

$$\varphi_n(t) = A_n \exp(-n\lambda_1 t) + \cdots + B_n \exp(-\lambda_n t) \;. \tag{6.3}$$

The terms in the expression in dots are linear combination of functions of the type

$$\exp\left\{-t\sum_{k=1}^{n} r_k \lambda_k\right\} \qquad \text{with } \sum_{k=1}^{n} k r_k = n \text{ and } r_k \geq 0 \;.$$

Since $\lambda_n > 0$ we deduce that $\varphi_n(t) \to 0$ as $t \to \infty$ for any $n \geq 1$. As a first consequence, we obtain that for fixed constant restitution coefficient e the asymptotic behavior $t \to \infty$ is given by $\varphi_0 = 1$ and in terms of the distribution function by the delta Dirac centered at zero. In terms of the model this is correct because we do not have any external energy source to compensate the loss of energy through collisions. Therefore, for e fixed we have

$$f(t,v) \to \delta_o(v) \quad \text{as } t \to \infty$$

in the weak topology of measures.

The self-similar solution includes only the first term in (6.3). These terms dominate as $t \to \infty$ if and only if

$$\lambda_1 < \frac{1}{n} \min \left\{\sum_{k=1}^{n} r_k \lambda_k \;;\; \sum_{k=1}^{n} k r_k = n \text{ and } r_k \geq 0\right\} \tag{6.4}$$

provided there are no resonances, i.e.,

$$\lambda_m \neq \sum_{k=1}^{m-1} r_k \lambda_k \text{ with } \sum_{k=1}^{m-1} k r_k = m \text{ and } r_k \geq 0 \;.$$

Otherwise we obtain additional terms in (6.3). The resonances are absent in the elastic case $e = 1$ [6] and they appear only for special values of e.

Finally, let us clarify that the self-similar solution can dominate in the limit $t \to \infty$ when $\lambda_1 \to 0$ ($e \to 1$) due to (6.4). So, finally in the asymptotics $t \to \infty$, $\lambda_1 \to 0$ we obtain

$$\varphi_n(t) = \theta(t)^n \left\{1 + O(\lambda_1) + O(\exp\{-(\lambda_n - n\lambda_1)\})\right\}$$

and $\theta(t) = \theta_0 \exp(-\lambda_1 t)$.

Hence, the self-similar solution in corollary 5.2 describes the large time asymptotics of $\varphi_n(t)$ only if $\lambda_n > n\lambda_1$. This condition is satisfied for large n only if $\lambda_1 \ll \lambda_n$. In such a case e is close to 1 and then the self-similar solution is close to the Maxwellian distribution with accuracy $O(\lambda_1)$. Therefore the leading asymptotic term of $\varphi(t, x)$ reads as

$$\varphi(t, x) \simeq \exp(-x\theta(t)) \ ; \qquad \theta(t) = \theta_0 \exp(-\lambda_1 t)$$

as $\lambda_1 \to 0$, $t \to \infty$, $\lambda_1 t = \text{constant}$.

We can now return to the initial time scale (3.1) and we have that

$$\frac{d\theta}{dt} = -\lambda_1 \theta$$

corresponds to

$$\frac{d\theta}{dt} = -\tilde{B}\frac{1 - e^2}{4}\theta(t)^{3/2}$$

and thus,

$$\theta(t) = \frac{\theta_0}{(1 + \tilde{B}\frac{1-e^2}{8}\sqrt{\theta_0}\, t)^2} \ , \qquad t \geq 0$$

and

$$\varphi(t, k) \simeq \exp\left(-\frac{|k|^2}{2}\theta(t)\right) \quad \text{as} \quad t \to \infty,\ e \to 1,\ (1 - e^2)t = \text{constant}$$

or equivalently

$$f(t, v) \simeq (2\pi\theta(t))^{-3/2} \exp\left(-\frac{|v|^2}{2\theta(t)}\right)$$

as $t \to \infty$, $e \to 1$, $(1 - e^2)t = \text{constant}$.

6.2 Non-constant restitution coefficient

Let us now assume that $e(\theta)$ and that the behavior of $e(\theta)$ for small energy collisions is more elastic. Consider that $e(\theta)$ is a decreasing continuous function of θ that tends to some $0 < e_0 < 1$ as $\theta \to \infty$ and that $e(0) = 1$. We will assume that

$$\lim_{\theta \to 0} \frac{1 - e(\theta)}{C\theta^{\alpha/2}} = 1 \tag{6.5}$$

with $0 < \alpha < 1$. If we compute the equation for $\theta(t)$ we obtain

$$\frac{d\theta}{dt} = -\frac{1-e^2(\theta)}{4}\tilde{B}\theta(t)^{3/2} \tag{6.6}$$

therefore, at the first stage of relaxation $\theta(t)$ decreases with the same rate as other moments $\varphi_n(t)$, $n \geq 2$. The separation of time scales begins when the temperature $\theta(t)$ is so small that $\theta(t) \ll 1$ and (6.5) plays an important role. Then, when $\theta(t)$ is small, equation (6.6) reads as

$$\frac{d\theta}{dt} \simeq \bar{B}\theta(t)^{(\alpha+3)/2}$$

and hence

$$\varphi(t,k) \simeq \exp\left(-\frac{|k|^2}{2}\theta(t)\right) \quad \text{for small } \ \theta \ .$$

Also, we have another possibility, that is, to have $e_0 \to 1$ as we did for the constant case. Doing this we obtain that

$$\varphi(t,k) \simeq \exp\left(-\frac{|k|^2}{2}\theta(t)\right) \quad \text{as } \ t \to \infty, \ e_0 \to 1$$

in such a way that $t(1-e_0^2) = \text{constant}$, where

$$\frac{d\theta}{dt} = -\tilde{B}\frac{1-e^2(\theta)}{4}\theta^{3/2} = -\tilde{B}h(\theta)\theta^{3/2}$$

and

$$f(t,v) \simeq (2\pi\theta(t))^{-3/2}\exp\left(-\frac{|v|^2}{2\theta(t)}\right)$$

$e_0 \to 1$, $t \to \infty$, $(1-e_0^2)t = \text{constant}$.

7 Euler equations. Hydrodynamic limit.

On the basis of the previous considerations, we assume the Maxwellian form of the distribution function, provided that the mean free path

$$\epsilon = \frac{1}{B}$$

is small enough. Then, we obtain formally the following dissipative Euler equations for density $\rho(t,x)$, bulk velocity $u(t,x)$ and temperature $\theta(t,x)$ as

$$\begin{aligned}
\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho u) &= 0 \\
\frac{\partial u}{\partial t} + (u\cdot\nabla)u + \frac{1}{\rho}\nabla p &= 0 \qquad\qquad (7.1) \\
\frac{\partial\theta}{\partial t} + (u\cdot\nabla)\theta + \frac{2}{3}\theta\operatorname{div}u &= -\frac{1}{\epsilon}\frac{1-e(\theta)^2}{4}\rho\tilde{G}(\rho)\theta^{3/2}
\end{aligned}$$

where $p = \rho\theta$ and $\tilde{G} \geq 1$ related to the function G in (2.4). The equations differ from the usual gas-dynamics Euler equations because of the dissipative term in the equation for the temperature.

Remark 7.1 The system (7.1) is formally obtained from our pseudo-maxwellian model Boltzmann-Enskog equation. It is clear however that the Enskog shift does not play any role for $\epsilon \to 0$. Moreover, exactly the same dissipative Euler equations can be obtained directly from the initial inelastic hard sphere kinetic equation (2.3) provided an appropriate value of the constant S in our approximation (2.7) is chosen. Thus, the material of this section relates not only to our model, but to the dissipative Euler equations derived directly from the initial inelastic hard sphere model without pseudo-maxwellian simplifications (2.3).

Remark 7.2 The equations (7.1) with constant e were briefly considered before by several authors (see [1, 8]). Our aim is to stress some qualitative differences between constant and temperature dependent restitution coefficients

7.1 Constant restitution coefficient

If e is constant our Maxwellian form for the distribution function only makes sense for $\frac{1-e^2}{4}$ small. Rigorously speaking, the set of equations (7.1) represents formal asymptotics of the inelastic scattering Boltzmann equation as $\frac{1-e^2}{4} \to 0$, $\epsilon \to 0$ in such a way that $\frac{1-e^2}{4\epsilon} = \lambda$ remains constant. In this case, the third equation reads as

$$\frac{\partial \theta}{\partial t} + (u \cdot \nabla)\theta + \frac{2}{3}\theta \mathrm{div} u \;=\; -\lambda\rho\tilde{G}(\rho)\theta^{3/2} \tag{7.2}$$

and the functions ρ, u and θ are well-defined on the time interval $[0, t^*]$ where the set of equations (7.1) has a unique solution. This is the only case, small inelasticity of the order of the Knudsen number, in which the hydrodynamic approximation makes sense.

7.2 Non-constant restitution coefficient

In this case $\lambda_1(\theta) = \frac{1-e(\theta)^2}{4}$ tends to 0 as $\theta \to 0$, is a continuous increasing function, and tends to a constant $\frac{1-e_o^2}{4}$ as $\theta \to \infty$. In this case, the hydrodynamic description is valid in two situations:

1. As in the constant case, take $e_o \to 1$, $\epsilon \to 0$ in such a way that $\frac{1-e_o}{\epsilon} = \lambda$ constant, then one obtain as third equation

   $$\frac{\partial \theta}{\partial t} + (u \cdot \nabla)\theta + \frac{2}{3}\theta \mathrm{div} u \;=\; -\lambda\rho\tilde{G}(\rho)h(\theta)\theta^{3/2}$$

   where $h(\theta) = 1 - e(\theta)$. For instance, taking $h(\theta) = C\theta^\alpha$ near 0 and later on constant.

2. The second possibility is for small temperature. We considered that equation (6.6) makes sense only for $\theta \to 0$, $\epsilon \to 0$ in such a way that $\frac{\theta}{\epsilon} = \lambda$ constant. Then, we consider $p = \frac{1}{\epsilon}\rho\theta$ and thus we obtain the set of equations

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho u) &= 0 \\
\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \frac{\epsilon}{\rho}\nabla p &= 0 \\
\frac{\partial p}{\partial t} + (u \cdot \nabla)p + \frac{5}{3}p\mathrm{div}u &= -c\tilde{G}(\rho)p^2
\end{aligned} \tag{7.3}$$

for the case $e(\theta) \simeq 1 - c\sqrt{\theta}$ as $\theta \to 0$. Thus, the formal asymptotics for small temperature is given by conservation of mass and a Burgers-type equation for the momentum with a small term proportional to the pressure.

8 Small perturbations of the homogeneous solution

We consider the dissipative Euler system given by (7.1) with $\tilde{G}(\rho) = 1$. We analyze, as in the previous section the constant and the non-constant restitution coefficient separately.

8.1 Constant restitution coefficient

Taking e constant, then the temperature equation takes the form of (7.2). Clearly, there are spatially homogeneous solutions with constant density and velocity and temperature of the type $\theta_o(t) = Ct^{-2}$. Due to scaling arguments, without loss of generality, we may choose the solution $\rho_o = 1$, $u_o = 0$ and $\theta_o(t) = (\frac{\lambda}{2}t)^{-2}$ as the solution to be perturbed. Therefore, setting

$$\rho = \rho_o + \tilde{\rho} \quad , \quad u = u_o + \tilde{u} \quad , \quad \theta = \theta_o + \tilde{\theta},$$

linearizing the system (7.1) around this state and dropping the tildes, we obtain the following linear system for (ρ, u, θ)

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \mathrm{div}u &= 0 \\
\frac{\partial u}{\partial t} + \nabla\theta + \theta_o(t)\nabla\rho &= 0 \\
\frac{\partial \theta}{\partial t} + \frac{2}{3}\theta_o(t)\mathrm{div}u + \lambda\left[\rho\theta_o^{3/2} + \frac{3}{2}\theta\theta_o(t)^{1/2}\right] &= 0
\end{aligned}$$

Therefore, looking for solutions which are long wave perturbations of the homogeneous one, we search for solutions of the form

$$\rho = \rho(t)\exp(i(k \cdot x)) \quad , \quad u = u(t)\exp(i(k \cdot x)) \quad , \quad \theta = \theta(t)\exp(i(k \cdot x)),$$

Thus, (ρ, u, θ) must satisfy the system of ODE's given by

$$\begin{aligned} \frac{\partial \rho}{\partial t} + i(k \cdot u) &= 0 \\ \frac{\partial u}{\partial t} + ik(\theta + \theta_o(t)\rho) &= 0 \\ \frac{\partial \theta}{\partial t} + \frac{2}{3} i(k \cdot u)\theta_o(t) + \lambda \theta_o^{1/2}\left[\rho\theta_o + \frac{3}{2}\theta\right] &= 0 \end{aligned} \tag{8.1}$$

Finally, we look for explicit solutions of (8.1) of the form

$$\rho(t) = \rho_1 t^{\gamma} \quad , \quad u(t) = u_1 t^{\gamma-1} \quad , \quad \theta(t) = \theta_1 t^{\gamma-2},$$

with constant (ρ_1, u_1, θ_1). A simple computation shows that this is a solution if the following linear algebraic equations are satisfied,

$$\begin{aligned} \gamma\rho_1 + i(k \cdot u_1) &= 0 \\ (\gamma - 1)u_1 + ik\left(\theta_1 + \frac{4}{\lambda^2}\rho_1\right) &= 0 \\ (\gamma - 2)\theta_1 + \frac{8}{3\lambda^2} i(k \cdot u_1) + \frac{8}{\lambda^2}\rho_1 + 3\theta_1 &= 0 \end{aligned}$$

This system has non-trivial solutions provided its determinant is zero, i. e.,

$$\gamma\left\{(\gamma - 1)(\gamma - 2 + a) + b|k|^2\right\} + c|k|^2\left\{(\gamma - 2 + a) - d\right\} = 0$$

with some constants a, b, c, d. Searching for roots of the polynomial in γ as a functions of k and λ we find that for $|k|$ small (small long wave perturbations) there is a root close to 1. Precisely, there exists $\gamma = 1 + O(|k|^2)$ root of this polynomial. Therefore, we have found a solution of the system (8.1) that diverges as $t \to \infty$, so the system is unstable with respect to small long wave perturbations. The instability was mentioned in [8] without proof.

8.2 Non-constant restitution coefficient

Here, we consider only the case in which the restitution coefficient is $e(\theta) = 1 - c\sqrt{\theta}$ for small temperature. Then, the stability analysis is performed for the system (7.3) with $G(\rho) = 1$ in terms of density, velocity and pressure.

After the rescaling given by $t = \frac{2}{c}\tilde{t}$ and $x = \frac{2}{c}\tilde{x}$, system (7.3) reduces to the case $c = 2$. Omitting the tildes and denoting by

$$\psi(t, x) = \frac{1}{p(t, x)}$$

system (7.3) becomes

$$\frac{\partial\rho}{\partial t} + \mathrm{div}(\rho u) \;=\; 0$$

$$\rho\frac{\partial u}{\partial t} + \rho(u\cdot\nabla)u + \epsilon\nabla\frac{1}{\psi} \;=\; 0$$

$$\frac{\partial\psi}{\partial t} + (u\cdot\nabla)\psi - \frac{5}{3}\psi\,\mathrm{div}u \;=\; 1$$

Now, considering small perturbations of the equilibrium solutions

$$\rho = 1+\tilde{\rho} \quad , \quad u = \tilde{u} \quad , \quad \psi = t + \tilde{\psi},$$

and assuming for brevity that the space dependence of the perturbation is only on the first component of x and u, that we denote the same for simplicity, omitting the tildes yields the linear system

$$\begin{aligned} \rho_t + u_x &= 0 \\ u_t - \frac{\epsilon}{t^2}\psi_x &= 0 \\ \psi_t - \frac{5}{3}t\; u_x &= 0 \end{aligned} \tag{8.2}$$

From the last two equations we have

$$\frac{3}{5}\left(t^{-1}\psi_t\right)_t \;=\; u_{xt} \;=\; \frac{\epsilon}{t^2}\psi_{xx}. \tag{8.3}$$

Next, we search for solutions of (8.3) of the form

$$\psi(t,x) \;=\; y(t)\exp(i(k\cdot x))$$

then, $y(t)$ must satisfy the equation

$$\frac{3}{5}\left(t^{-1}y_t\right)_t \;=\; -\frac{\epsilon}{t^2}k^2 y.$$

or equivalently,

$$ty'' - y' \;=\; -\frac{5}{3}\epsilon k^2 y. \tag{8.4}$$

A long time asymptotics is given by the function $y_\infty = \exp(\alpha\sqrt{t})$, where α is found by plugging into (8.4) and matching it, obtaining

$$\frac{\alpha^2}{4} = -\frac{5}{3}\epsilon k^2.$$

Therefore, the linear waves solutions of system (8.3) are asymptotically given by

$$\psi(t,x) = \exp\left(ik(x - 2\xi_o\sqrt{\epsilon t})\right)$$

where $\xi_o = \sqrt{5/3}$ on the background of the spatially homogeneous solution, that is, substituting ψ in (8.2) and solving for ρ and u. Thus, in this case we obtain linear stability of the homogeneous solutions which is another difference with respect to the constant case.

Finally, we remark that the asymptotic phase velocity of the waves is given by

$$C(t) = 2\xi_o \frac{d}{dt}\sqrt{\epsilon t} = \sqrt{\frac{5}{3}\frac{\epsilon}{t}} = \sqrt{\frac{5}{3}\theta_o(t)}$$

and coincides with the time dependent speed of sound in the usual Euler gas.

Acknowledgments

The first author thanks TICAM for kind hospitality and financial support during his visit in November-December 1998. The second author is supported by the DGESIC MEC-Spain Perfeccionamiento de Doctores en el Extranjero fellowship. Partially supported by Russian Basic Research Foundation (grant Boltzmann), TMR European Union ERB FMBX-CT97-0157 and DGES (Spain) PB98-1281. The third author is supported by NSF under grant DMS 9623037.

References

[1] Bellomo, N., Esteban, M., Lachowicz, M., *Nonlinear kinetic equations with dissipative collisions*, Appl. Math. Letter, **8**, 47-52 (1995).

[2] Benedetto, D., Caglioti, E., *The collapse phenomenon in one dimensional inelastic point particle system*,to appear in Physica D.

[3] Benedetto, D., Caglioti, E., Carrillo, J. A., Pulvirenti, M., *A non-maxwellian steady distribution for one-dimensional granular media*, J. Stat. Phys., **91**, 979-990 (1998).

[4] Benedetto, D., Caglioti, E., Pulvirenti, M., *A Kinetic Equation for Granular Media*, Math. Mod. and Num. An., **31**, 615-641 (1997).

[5] Bizon, C., Shattuck, Swift, J. B., M. D., McCormick, W. D., Swinney, H. L., *Patterns in 3D vertically oscillated granular layers: simulation and experiment*, Phys. Rev. Letters, **80**, 57-60 (1998).

[6] Bobylev, A. V., *Exact solutions of the nonlinear Boltzmann equation and the theory of relaxation of a maxwellian gas*, Teoret. Mat. Fiz., **60**, 280-310 (1984).

[7] Bobylev, A. V., *The theory of the nonlinear spatially uniform Boltzmann equation for Maxwell molecules*, Sov. Sci. Rev. C. Math. Phys., **7**, 111-233 (1988).

[8] Brey, J. J., Dufty, J. W., Santos, A., *Dissipative dynamics for hard spheres*, J. Stat. Phys., **87**, 1051-1068 (1997).

[9] Cercignani, C., *Recent developments in the mechanics of granular materials*, Fisica matematica e ingegneria delle strutture, 119-132, Pitagora Editrice, Bologna 1995.

[10] Cercignani, C., Illner, R., Pulvirenti, M., *"The mathematical theory of dilute gases"*, Springer series in Applied Mathematical Sciences, vol 106, Springer-Verlag 1994.

[11] Constantin, P., Grossman, E., Mungan, M., , *Inelastic collisions of three particles on a line as a two dimensional billiard*, Physica D, **83**, 409-420 (1995).

[12] Esteban, M. J., Perthame, B., *On the modified Enskog equation for elastic and inelastic collisions. Models with spin*, Ann. Inst. H. Poincar Anal. Non Linaire, **8**, 289-308 (1991).

[13] Gidaspow, D. *Multiphase Flow and Fluidization, Continuum and Kinetic Theory Descriptions*, Academic Press, 1994.

[14] Goldman, D., Shattuck, M. D., Bizon, C., McCormick, W. D., Swift, J. B., Swinney, H. L., *Absence of inelastic collapse in a realistic three ball model*, Phys. Rev. E, **57**, 4831-4833 (1998).

[15] Goldshtein, A., Shapiro, M., *Mechanics of collisional motion of granular materials. Part I. General hydrodynamic equations*, J. Fluid Mech., **282**, 75-114 (1995).

[16] Jenkins, J. T., Richman, M. W., *Grad's 13-moment system for a dense gas of inelastic spheres*, Arch. Rat. Mech. Anal., **87**, 355-377 (1985).

[17] Mathews, J., Walker, R., *Mathematical methods of physics*, Addison-Wesley 1970.

[18] Sela, N., Goldhirsch, I., *Hydrodynamic equations for rapid flows of smooth inelastic spheres, to Burnett order*, J. Fluid Mech., **361**, 41-74 (1998).