Stationary solutions for the Boltzmann-Vlasov-Poisson equations; hydrodynamic limit in a half line

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Abstract

In this paper we analyze the stationary Boltzmann-Vlasov-Poisson model. We start by investigating the well posedness in a slab and we establish a priori bounds uniformly with respect to the size of the slab. We identify the hydrodynamic limit for small relaxation times and finally we compute this regime in a half line.

Keywords: Transport equations, Plasma physics models.

AMS classification: 82D10, 78A35, 35Q99.

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1 Introduction

We consider a density $f$ of charged particles interacting both through their self-consistent electric field $E$ and collisions. We analyze here the model

$$v\partial_x f + E(x)\partial_v f = Q(f), \quad (x, v) \in [0, L] \times \mathbb{R},$$

$$\frac{d}{dx} E = \int_{\mathbb{R}} f(x, v) \, dv, \quad x \in [0, L],$$

where $Q$ is the relaxation operator defined by

$$Qf(x, v) = \frac{1}{\tau} \left( M(v) \int_{\mathbb{R}} f(x, w) \, dw - f(x, v) \right), \quad M(v) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2},$$

and $\tau > 0$ is the relaxation time. We supplement these equations with boundary conditions by prescribing the incoming particle densities at $x \in \{0, L\}$

$$f(x = 0, v > 0) = g_0(v), \quad f(x = L, v < 0) = g_L(v).$$

We impose also the values of the electric potential or of the electric field on the boundary. This model arises in charge transport phenomena, with application in semiconductor theory and plasma physics [?]. The existence for the above problem has been studied in three dimensional bounded domains, cf. [?]. The crucial point here was to observe that $M_\phi(x, v) = \exp(-\frac{v^2}{2} - \phi(x))$ with $E = -\frac{d\phi}{dx}$ solves (??) and to use the maximum principle in order to estimate the solutions of (??) with incoming particle densities comparable with $M_\phi$. Explicit solutions for the linear problem (??), (??) with constant electric field have been constructed in [?] and used in [?] for solving the Milne problem. We refer to [?] for mathematical results on the time depending system. In the present work we consider even more general data with finite kinetic energy and entropy, see [?] for a first work on this topic under such hypotheses. One of the key points is to write the balance for the total energy and entropy involving an entropy production term. Multiplying (??) by $\frac{v^2}{2} + \phi(x) + 1 + \ln f$ yields

$$v\partial_x \left( \frac{v^2}{2} + \phi(x) + \ln f \right)f + E \partial_v \left( \frac{v^2}{2} + \phi(x) + \ln f \right)f = \frac{1}{\tau} (\rho(x) M(v) - f) \times \left( \frac{\ln f(x, v)}{\rho(x) M(v)} + \phi(x) + 1 + \ln \frac{\rho(x)}{\sqrt{2\pi}} \right).$$
Integrating the above equality with respect to \((x, v) \in [0, L[ \times \mathbb{R})\) we obtain bounds for the outgoing energy and entropy and also for the production term \(\frac{1}{\tau} \int_0^L \int_{\mathbb{R}} (f(x, v) - \rho(x) M(v)) \ln \frac{f(x,v)}{\rho(x)M(v)} \, dv \, dx\). We are looking for solutions in the half plan \((x, v) \in \mathbb{R}^+ \times \mathbb{R}\). A standard procedure for this is to construct solutions \((f_L, E_L)\) for any \(L > 0\), to estimate them uniformly with respect to \(L\) and finally to let \(L \to +\infty\).

Of course, the behavior of the solutions \((f_L, E_L)_{L>0}\) strongly depends on the choice of the boundary conditions. At the left boundary \(x = 0\) it is natural to prescribe the incoming particle density \(g_0\). Since there are no incoming particles at \(+\infty\) we require that \(g_L = 0\) for any \(L > 0\). It remains to determine the "good" conditions to be imposed at \(x = L \to +\infty\) on the electric field. For justifying our choice let us neglect the collisions (we expect that the same behavior holds in the collisional model). Integrating the Vlasov equation with respect to \(v \in \mathbb{R}\) we deduce that the current density \(j = \int_{\mathbb{R}} v f \, dv\) is constant with respect to \(x\) (the same occurs when collisions are taken into account). Since \(-\frac{d^2}{dx^2} \phi = \rho(x) \geq 0\), the electric potential is concave. Let us see what happens if \(\phi\) is bounded from above. A straightforward computation involving the characteristics of the Vlasov equation yields

\[
j = \int_{v>v_0} vg_0(v) \, dv, \quad v_0 = \sqrt{2(\phi_{\max} - \phi(0))}.
\]

This comes from the fact that all the characteristics starting from \(x = 0\) with \(0 < v < v_0\) have not enough kinetic energy for passing the potential barrier \(\phi_{\max}\) and thus they do not create any electric current. If \(g_0 > 0\) we deduce that \(j(x) > 0\) at any point \(x \in \mathbb{R}^+\) and in particular at \(x = +\infty\), which is not reasonable. If the potential is not bounded from above (which implies that \(\phi\) is non decreasing) the current vanishes \(j = 0\). We impose the boundary condition \(E(L) = -\frac{d\phi}{dx}(L) = 0\).

The concavity of \(\phi\) implies that \(\frac{d}{dx} \phi \geq 0\) and therefore \(\phi\) is non decreasing. Actually we will see that the potential goes to \(+\infty\) as \(x\) tend to \(+\infty\). Our main result is

**Theorem 1.1** Assume that \(g_0\) is a non negative function satisfying

\[
\int_{v>0} \left(1 + \frac{v^2}{2} + |\ln g_0(v)|\right) g_0(v) \, dv < +\infty.
\]
Then for any $\tau > 0$, $L > 0$ there is a weak solution of the Boltzmann-Vlasov-Poisson problem (??), (??), (??) with the boundary conditions $f(x = 0, v > 0) = g_0(v)$, $f(x = L, v < 0) = 0$, $E(L) = 0$. Moreover for any $\tau_0 > 0$ there is a constant $C = C(\tau_0, g_0)$ such that

$$
\|(1 + v^2)f\|_{L^1([0, L] \times \mathbb{R})} + \|E\|_{L^2([0, L])} + \left\|\int_{\mathbb{R}} v^2 f(\cdot, v) \, dv\right\|_{L^\infty([0, L])} + \|E\|_{L^\infty([0, L])} \leq C,
$$

for any $0 < \tau \leq \tau_0$, $L > 0$.

The paper is organized as follows. Section 2 is devoted to the study of the linear Boltzmann-Vlasov problem. The existence of weak solution follows by standard iterative procedure. A delicate point is to estimate uniformly the $L^1$ norms of the approximating sequence. This can be achieved by considering first smooth electric fields and performing a detailed analysis of the characteristics. The same approach is carried out for estimating uniformly the total kinetic energy and entropy and therefore, by compactness method one gets the existence of weak solution for continuous electric fields. In Section 3 we establish the existence of solution for the Boltzmann-Vlasov-Poisson problem. The construction follows by fixed point method. We obtain also uniform bounds for the electric field and kinetic energy. In the last section we perform asymptotic analysis: first we keep fixed the size $L$ and pass to the limit for small relaxation times $\tau \searrow 0$; second we analyze the behavior for large sizes $L \to +\infty$. Finally we obtain the analytic expression for the limit regime.

## 2 The Boltzmann-Vlasov problem

In this section we study the existence and uniqueness of mild solution (or solution by characteristics) for the Boltzmann-Vlasov problem

$$
v \partial_x f + E \partial_v f = \frac{1}{\tau}(\rho(x) M(v) - f(x, v)), \quad (x, v) \in [0, L] \times \mathbb{R},
$$

$$
f(x = 0, v > 0) = g_0(v), \quad f(x = L, v < 0) = g_L(v),
$$
where $E$ is a Lipschitz continuous function. We recall briefly the notion of mild solution for the Vlasov problem with source terms $S \in L^1([0, L[ \times \mathbb{R})$. For any $(x, v) \in ([0, L[ \times \mathbb{R}^+)) \cup ([0, L[ \times \mathbb{R}^-)$ we denote by $(X(s; x, v), V(s; x, v))$ the characteristic of the transport operator $v \partial_x + E \partial_v$

$$\frac{dX}{ds} = V(s; x, v), \quad \frac{dV}{ds} = E(s, X(s; x, v)), \quad s_{\text{in}}(x, v) < s < s_{\text{out}}(x, v),$$

verifying the conditions $X(s = 0; x, v) = x$, $V(s = 0; x, v) = v$. Here the notations $s_{\text{in}}/s_{\text{out}}(x, v)$ stand for the entry/exit times of the characteristic $(X(\cdot; x, v), V(\cdot; x, v))$ with respect to $[0, L[$. Let us introduce the critical velocities

$$v_0 = \left(2 \max_{x \in [0, L]} \phi(x) - \phi(0)\right)^{\frac{1}{2}}, \quad v_L = -\left(2 \max_{x \in [0, L]} \phi(x) - \phi(L)\right)^{\frac{1}{2}},$$

where $\phi$ is a potential for $E$, i.e., $E = -\frac{d}{dx} \phi$. Using the conservation of the total energy $\frac{1}{2} v^2 + \phi(x)$ along the characteristics we prove as in $[?]$ (see also $[?]$)

**Proposition 2.1** Assume that $E \in W^{1, \infty}([0, L[)$ and denote by $\phi$ a primitive of $-E$.

1) For any $0 < v < v_0$ there are $x_0 \in [0, L[$, $0 < s_0 \leq s_{\text{out}}(0, v) \leq +\infty$ such that

$$0 < X(s; 0, v) < x_0, \quad V(s; 0, v) > 0, \quad 0 < s < s_0, \lim_{s \nearrow s_0} (X, V)(s; 0, v) = (x_0, 0).$$

Moreover, if $\phi'(x_0) \neq 0$ then

$$s_0 = \frac{s_{\text{out}}(0, v)}{2} < +\infty, \quad (X, V)(s; 0, v) = (X, -V)(2s_0 - s; 0, v), \quad \forall s \in [0, 2s_0].$$

In particular $X(s_{\text{out}}(0, v); 0, v) = 0$.

2) For any $v > v_0$ we have $s_{\text{out}}(0, v) < +\infty, V(s; 0, v) > 0, \forall 0 \leq s \leq s_{\text{out}}(0, v)$ and $X(s_{\text{out}}(0, v); 0, v) = L$.

3) For any $v_L < v < 0$ there are $x_L \in [0, L[$, $0 < s_L \leq s_{\text{out}}(L, v) \leq +\infty$ such that

$$L > X(s; L, v) > x_L, \quad V(s; L, v) < 0, \quad 0 < s < s_L, \lim_{s \nearrow s_L} (X, V)(s; L, v) = (x_L, 0).$$

Moreover, if $\phi'(x_L) \neq 0$ then

$$s_L = \frac{s_{\text{out}}(L, v)}{2} < +\infty, \quad (X, V)(s; L, v) = (X, -V)(2s_L - s; L, v), \forall s \in [0, 2s_L].$$
In particular $X(s_{\text{out}}(L,v); L,v) = L$.

4) For any $v < v_L$ we have $s_{\text{out}}(L,v) < +\infty$, $V(s; L, v) < 0$, $\forall \ 0 \leq s \leq s_{\text{out}}(L,v)$ and $X(s_{\text{out}}(L,v); L,v) = 0$.

When $E$ is non decreasing (saying that $\phi$ is concave) we deduce that

$$-\infty < s_{\text{in}}(x, v) \leq s_{\text{out}}(x, v) < +\infty, \ a.e. \ (x, v) \in ([0, L[ \times \mathbb{R}^+) \cup ([0, L] \times \mathbb{R}^-).$$

Take $S : [0, L[ \times \mathbb{R} \to \mathbb{R}$ a measurable function, $\alpha > 0$ and let us consider the problem

$$\alpha f + v \partial_x f + E \partial_v f = S(x, v), \ (x, v) \in [0, L[ \times \mathbb{R}, \quad (7)$$

$$f(x = 0, v > 0) = g_0(v), \ f(x = L, v < 0) = g_L(v). \quad (8)$$

Formally we have

$$\frac{d}{ds}\{e^{\alpha s}f(X(s; x, v), V(s; x, v))\} = e^{\alpha s}S(X(s; x, v), V(s; x, v)), \ s_{\text{in}}(x, v) < s < s_{\text{out}}(x, v).$$

**Definition 2.1** Assume that $g_0 : \mathbb{R}^+ \to \mathbb{R}$, $g_L : \mathbb{R}^- \to \mathbb{R}$, $S : [0, L[ \times \mathbb{R} \to \mathbb{R}$ are measurable functions, $\alpha > 0$. The mild solution of (7), (8) is given by

$$f(x, v) = e^{\alpha s_{\text{in}}(x, v)}g_\beta(V(s_{\text{in}}(x, v); x, v)) + \int_{s_{\text{in}}(x, v)}^0 e^{\alpha \tau}S(X(\tau; x, v), V(\tau; x, v)) d\tau,$$

with $\beta = 0$ if $X(s_{\text{in}}(x, v); x, v) = 0$ and $\beta = L$ if $X(s_{\text{in}}(x, v); x, v) = L$.

The main properties of the mild solution are summarized below

**Theorem 2.1**

1) If $g_0, g_L, S \geq 0$ then $f \geq 0$;

2) If $g_0, g_L, S$ are bounded, then $f$ is bounded and

$$\|f\|_{L^\infty([0, L[ \times \mathbb{R})} \leq \max \left\{ \|g_0\|_{L^\infty(\mathbb{R}^+)}, \|g_L\|_{L^\infty(\mathbb{R}^-)}, \frac{1}{\alpha} \|S\|_{L^\infty([0, L[ \times \mathbb{R})} \right\};$$

3) Assume that $v_{g_0} \in L^1(\mathbb{R}^+)$, $v_{g_L} \in L^1(\mathbb{R}^-)$, $S \in L^1([0, L[ \times \mathbb{R})$. Then $f \in L^1([0, L[ \times \mathbb{R})$ and for any bounded test function $\psi$ we have

$$\int_0^L \int_\mathbb{R} f(x, v)\psi(x, v) \, dv \, dx = \int_{v > 0} v_{g_0}(v) \int_0^{s_{\text{out}}(0, v)} e^{-\alpha s}\psi(X(s; 0, v), V(s; 0, v)) \, ds \, dv \quad (9)$$

$$- \int_{v < 0} v_{g_L}(v) \int_{s_{\text{out}}(L, v)}^0 e^{-\alpha s}\psi(X(s; L, v), V(s; L, v)) \, ds \, dv$$

$$+ \int_0^L \int \psi(X(s; x, v), V(s; x, v)) \, ds \, dv \, dx.$$
In particular

\[ \|f\|_{L^1([0,L],[0])} \leq \frac{1}{\alpha} \left( \int_{v>0} |v g_0(v)|\,dv + \int_{v<0} |v g_L(v)|\,dv + \int_0^L \int |S(x,v)|\,dv\,dx \right); \]

4) If \( v g_0 \in L^1(\mathbb{R}^+), v g_L \in L^1(\mathbb{R}^-), S \in L^1([0,L] \times \mathbb{R}) \) then \( f \) has traces \( f(0,\cdot) \in L^1(\mathbb{R}^-; |v|\,dv), f(L,\cdot) \in L^1(\mathbb{R}^+; v\,dv) \) and it is a weak solution (i.e., in distribution sense) for (7), (8)

\[ \int_0^L \int R f(\alpha \theta - v \partial_x \theta - E \partial_v \theta)\,dv\,dx + \int_{v>0} v f(L,v)\theta(L,v)\,dv - \int_{v<0} v f(0,v)\theta(0,v)\,dv \]
\[ = \int_{v>0} v g_0(v)\theta(0,v)\,dv - \int_{v<0} v g_L(v)\theta(L,v)\,dv \]
\[ + \int_0^L \int R S(x,v)\theta(x,v)\,dv\,dx, \quad (10) \]

for any test function \( \theta \in C^1_c([0,L] \times \mathbb{R}). \) In particular we have

\[ \alpha \|f\|_{L^1([0,L],[0])} + \|vf(0,\cdot)\|_{L^1(\mathbb{R}^-)} + \|vf(L,\cdot)\|_{L^1(\mathbb{R}^+)} \leq \|v g_0\|_{L^1(\mathbb{R}^+)} + \|v g_L\|_{L^1(\mathbb{R}^-)} \]
\[ + \|S\|_{L^1([0,L] \times \mathbb{R})}. \]

We recall also a very easy lemma (see [?] for a more general result)

**Lemma 2.1** Assume that \( E \) is a Lipschitz continuous function. Then for any characteristic we have

\[ |V(s_1) - V(s_2)| \leq 2 \left( 2L \|E\|_{L^\infty([0,L])} \right)^{1/2}, \quad s_{in} \leq s_1 \leq s_2 \leq s_{out}. \]

**Proof.** Suppose that \( V(s_1), V(s_2) \) have the same sign, for example \( 0 \leq V(s_1) \leq V(s_2). \) Denoting by \( \phi \) a primitive of \(-E\) we have by the conservation of the energy along the characteristic

\[ \frac{1}{2} V(s_1)^2 + \phi(X(s_1)) = \frac{1}{2} V(s_2)^2 + \phi(X(s_2)), \]

implying that \( V(s_2) = (V(s_1)^2 + a)^{1/2} \) with \( a = 2(\phi(X(s_1)) - \phi(X(s_2))) \in [0, 2L\|E\|_{L^\infty}]. \)

It is easily seen that

\[ V(s_2) - V(s_1) = \frac{a}{(V(s_1)^2 + a)^{1/2} + V(s_1)} \leq a^{1/2} \leq 2(2L\|E\|_{L^\infty})^{1/2}. \]
If $V(s_1), V(s_2)$ have opposite signs, there is $s_3 \in [s_1, s_2]$ such that $V(s_3) = 0$ and thus $|V(s_1) - V(s_2)| = |V(s_1) - V(s_3)| + |V(s_3) - V(s_2)| \leq 2a^{1/2}$. 

In order to construct a solution for (??), (??) we proceed by iterations. Assume that $g_0, g_L \geq 0$ and consider $f^0$ the mild solution of

$$\frac{1}{\tau} f^0(x, v) + v \partial_x f^0 + E \partial_v f^0 = 0, \quad (x, v) \in ]0, L] \times \mathbb{R},$$

$$f^0(0, v > 0) = g_0(v), \quad f^0(L, v < 0) = g_L(v).$$

By the first point of Theorem ?? we know that $f^0 \geq 0$. For any $n \in \mathbb{N}$, given $f^n \geq 0$ we denote by $f^{n+1}$ the mild solution of

$$\frac{1}{\tau} f^{n+1}(x, v) + v \partial_x f^{n+1} + E \partial_v f^{n+1} = \frac{1}{\tau} \rho^n(x) M(v), \quad (x, v) \in ]0, L] \times \mathbb{R},$$

$$f^{n+1}(0, v > 0) = g_0(v), \quad f^{n+1}(L, v < 0) = g_L(v),$$

where $\rho^n(x) = \int_\mathbb{R} f^n(x, v) \, dv$. We check easily that $0 \leq f^0 \leq f^1 \leq \ldots \leq f^n \leq f^{n+1} \leq \ldots$ We are looking now for a uniform bound of the $L^1$ norms $\int_0^L \int_\mathbb{R} f^n(x, v) \, dv \, dx$.

**Proposition 2.2** Assume that $g_0, g_L \geq 0$ such that

$$G := \int v g_0(v) \, dv - \int v g_L(v) \, dv < +\infty.$$

Then there is a constant depending on $\tau, L$ and $\|E\|_{L^\infty([0, L])}$ such that

$$\int_0^L \int_\mathbb{R} f^n(x, v) \, dv \, dx \leq C_1(\tau, L, \|E\|_{L^\infty}) G, \quad \forall n.$$

**Proof.** By the third point of Theorem ?? we know that $f^0 \in L^1([0, L] \times \mathbb{R})$ and $\int_0^L \int_\mathbb{R} f^0(x, v) \, dv \, dx \leq \tau G$. Moreover if $f^n$ belongs to $L^1([0, L] \times \mathbb{R})$ for some $n \geq 0$, then $f^{n+1}$ is also a $L^1$ function

$$\int_0^L \int_\mathbb{R} f^{n+1}(x, v) \, dv \, dx \leq \tau G + \int_0^L \int_\mathbb{R} \rho^n(x) M(v) \, dv \, dx = \tau G + \int_0^L \int_\mathbb{R} f^n(x, v) \, dv \, dx.$$

Therefore $(f^n)_n \subset L^1([0, L] \times \mathbb{R})$. Applying (??) with $\psi = 1$ yields

$$\int_0^L \int_\mathbb{R} f^{n+1}(x, v) \, dv \, dx = \int v g_0(v) \int_0^L e^{-\frac{v}{\tau}} \, ds \, dv - \int v g_L(v) \int_0^L e^{-\frac{v}{\tau}} \, ds \, dv$$

$$+ \frac{1}{\tau} \int_\mathbb{R} \rho^n(x) M(v) \int_0^L e^{-\frac{v}{\tau}} \, ds \, dv \, dx$$

$$\leq \tau G + \int_0^L \rho^n(x) h(x) \, dx \leq \tau G + \int_0^L \rho^{n+1}(x) h(x) \, dx,$$  \quad (11)
with \( h(x) = \frac{1}{\tau} \int_{\mathbb{R}} M(v) \int_{0}^{s_{\text{out}}(x,v)} e^{-\frac{s}{\tau}} \, ds \, dv \) for any \( x \in [0, L] \). Take \( R = 4(2L \| E \|_{L^\infty})^{1/2} \) and write \( h(x) = \int_{|v| \leq R} ... dv + \int_{|v| > R} ... dv = h_1(x) + h_2(x), \, x \in [0, L] \). Using the inequality \( \int_{0}^{s_{\text{out}}(x,v)} e^{-\frac{s}{\tau}} \, ds \leq \tau \) we obtain \( h_1(x) \leq \int_{\mathbb{R}} M(v) 1_{\{|v| \leq R\}} \, dv, \, x \in [0, L] \). For estimating \( h_2 \) we use Lemma ??; observe that for any \((s, x, v)\) such that \(|v| > R, s \in [0, s_{\text{out}}(x, v)]\) we have

\[
|V(s; x, v)| \geq |v| - \frac{R}{2} > \frac{|v|}{2},
\]

and thus \( s_{\text{out}}(x, v) \leq \frac{2L}{|v|} \). We obtain for any \( x \in [0, L] \)

\[
h_2(x) \leq \int_{\mathbb{R}} M(v) \left(1 - e^{-\frac{2L}{|v|}}\right) 1_{\{|v| > R\}} \, dv.
\]

Combining the estimates for \( h_1, h_2 \) yields

\[
h(x) \leq 1 - \int_{\mathbb{R}} M(v) e^{-\frac{2L}{|v|}} 1_{\{|v| > R\}} \, dv = \gamma(\tau, L, \|E\|_{L^\infty}) < 1, \, x \in [0, L].
\]

Coming back in (??) we deduce that

\[
\int_{0}^{L} \int_{\mathbb{R}} f^{n+1}(x, v) \, dv \, dx \leq \frac{\tau G}{1 - \gamma}, \, \forall n.
\]

Notice that the coefficient \( \gamma = \gamma(\tau, L, \|E\|_{L^\infty}) \) tends towards 1 when \( \tau \searrow 0 \), \( L \to +\infty \) or \( \|E\|_{L^\infty} \to +\infty \).

**Definition 2.2** We say that \( f \) is a mild solution for the Boltzmann-Vlasov problem (??), (??) if \( f \) belongs to \( L^1([0, L] \times \mathbb{R}) \) and it is a mild solution of (??), (??) with \( \alpha = 1/\tau \) and the source term \( S(x, v) = \rho(x)M(v)/\tau \), where \( \rho = \int_{\mathbb{R}} f \, dv \).

Using now the monotone convergence theorem we have \( \lim_{n \to +\infty} f^n = f \) in \( L^1([0, L] \times \mathbb{R}) \), \( \lim_{n \to +\infty} \rho^n = \int_{\mathbb{R}} f \, dv =: \rho \) in \( L^1([0, L]) \) and we deduce easily that \( f \) is a mild solution for (??), (??). Moreover, by Theorem ?? we deduce that \( f \) has traces \( f(0, \cdot) \in L^1(\mathbb{R}^-; |v| \, dv), \, f(L, \cdot) \in L^1(\mathbb{R}^+; v \, dv) \) and satisfies the weak formulation (??) with \( S(x, v) = \rho(x)M(v)/\tau \). Actually the mild solution of the Boltzmann-Vlasov problem is unique. Indeed take \( f \in L^1([0, L] \times \mathbb{R}) \) a mild solution of (??) with \( f(0, v > 0) = f(L, v < 0) = 0 \). Taking \( \psi = \text{sgn} f \) in (??) we obtain

\[
\int_{0}^{L} \int_{\mathbb{R}} |f(x, v)| \, dv \, dx \leq \int_{0}^{L} |\rho(x)| h(x) \, dx \leq \gamma(\tau, L, \|E\|_{L^\infty}) \int_{0}^{L} \int_{\mathbb{R}} |f(x, v)| \, dv \, dx.
\]
Since $\gamma < 1$ we deduce that $f = 0$. We have proved the result

**Proposition 2.3** Assume that $E \in W^{1,\infty}([0, L])$ and $g_0, g_L \geq 0$ such that

$$G = \int_{v > 0} vg_0(v) \, dv - \int_{v < 0} vg_L(v) \, dv < +\infty.$$  

Then there is a unique mild solution of the Boltzmann-Vlasov problem (??), (??).

**Corollary 2.1** Assume that $g_0, g_L \geq 0$ such that $G < +\infty$ and consider $(E_n)_n \subset W^{1,\infty}([0, L])$ a bounded sequence in $L^\infty([0, L])$. Therefore the sequence of the mild solutions $f_n$ for the Boltzmann-Vlasov problem corresponding to the fields $E_n$ is bounded in $L^1([0, L] \times \mathbb{R})$.

**Proof.** Observe that the function $\gamma(\tau, L, e) = 1 - \int_{\mathbb{R}} M(v) e^{-\frac{2Lv}{|v|^2}} 1_{\{|v| > 4\sqrt{\frac{L}{\tau}}\}} \, dv$ is strictly increasing with respect to $e \in \mathbb{R}^+$ and $\gamma(\tau, L, e) < 1, \forall (\tau, L, e)$. Therefore we have for any $n$

$$\int_0^L \int_{\mathbb{R}} f_n(x, v) \, dv \, dx \leq \frac{\tau G}{1 - \gamma(\tau, L, \|E_n\|_{L^\infty})} \leq \frac{\tau G}{1 - \gamma(\tau, L, e_\infty)},$$

where $e_\infty = \sup_{n \in \mathbb{N}} \|E_n\|_{L^\infty}$.

We estimate now the current of the solution constructed above. Consider $j_\pm(x) := \int_{\mathbb{R}} v_\pm f(x, v) \, dv \in [0, +\infty]$, where $v_\pm = \max(0, \pm v)$.

**Proposition 2.4** Assume that $E \in W^{1,\infty}([0, L])$ and $g_0, g_L \geq 0$ such that $G < +\infty$. We denote by $f \in L^1([0, L] \times \mathbb{R})$ the unique mild solution of the Boltzmann-Vlasov problem. Then

1) $j_\pm$ belong to $L^\infty([0, L])$ and

$$\max\{\|j_-\|_{L^\infty([0, L])}, \|j_+\|_{L^\infty([0, L])}\} \leq G + \frac{1}{\tau} \|f\|_{L^1([0, L] \times \mathbb{R})};$$

2) The total current $j = j_+ - j_-$ is constant with respect to $x$ and $|j| \leq G$. In particular we have

$$\int_{v > 0} vf(L, v) \, dv - \int_{v < 0} vf(0, v) \, dv = G;$$

(12)

3) If $g_L = 0$ then $j \in [0, G]$.  

10
Proof. 1) Applying (??) with $\psi(x,v) = v_\pm \theta(x)$ where $\theta \in L^1([0,L])$, $\theta \geq 0$ yields

$$\int_0^L j_\pm(x) \theta(x) \, dx \leq \int_{v>0} v g_0(v) \int_0^{s_{\text{out}}(0,v)} V_\pm(s;0,v) \theta(X(s;0,v)) \, ds \, dv$$

$$- \int_{v<0} v g_L(v) \int_0^{s_{\text{out}}(L,v)} V_\pm(s;L,v) \theta(X(s;L,v)) \, ds \, dv$$

$$+ \frac{1}{\tau} \int_0^L \int_{\mathbb{R}} \rho(x) M(v) \int_0^{s_{\text{out}}(x,v)} V_\pm(s,x,v) \theta(X(s;x,v)) \, ds \, dv \, dx.$$ 

By Proposition ?? it is easily seen that for any $(x,v) \in ([0,L] \times \mathbb{R})$ the following holds:

$$\int_0^{s_{\text{out}}(x,v)} V_\pm(s;x,v) \theta(X(s;x,v)) \, ds \leq \int_0^L \theta(u) \, du,$$

and therefore we obtain

$$\int_0^L j_\pm(x) \theta(x) \, dx \leq G \|\theta\|_{L^1([0,L])} + \frac{1}{\tau} \|f\|_{L^1([0,L] \times \mathbb{R})} \|\theta\|_{L^1([0,L])},$$

implying that

$$\|j_\pm\|_{L^\infty([0,L])} \leq G + \frac{1}{\tau} \|f\|_{L^1([0,L] \times \mathbb{R})}.$$

2) By the previous point we know that $j = j_+ - j_-$ belongs to $L^\infty([0,L])$. Applying (??) with a test function $\theta = \theta(x)$, $\theta \in C^1_c([0,L])$ one gets

$$- \int_0^L j(x) \frac{d}{dx} \theta(x) \, dx = \frac{1}{\tau} \int_0^L \theta(x) \int_{\mathbb{R}} \{\rho(x) M(v) - f(x,v)\} \, dv \, dx = 0,$$

saying that $\frac{d}{dx} j = 0$. Taking $\theta = 1$ yields

$$\int_{v>0} v f(L,v) \, dv - \int_{v<0} v f(0,v) \, dv = G,$$

and thus we deduce

$$-G \leq \int_{v<0} v f(0,v) \, dv \leq j(0) = j(L) \leq \int_{v>0} v f(L,v) \, dv \leq G.$$ 

3) Assume now that $g_L = 0$. We have for any $x \in [0,L]$

$$G \geq j(x) = j(L) = \int_{\mathbb{R}} v f(L,v) \, dv = \int_{v>0} v f(L,v) \, dv \geq 0.$$

The Proposition ?? establishes the existence and uniqueness
of the mild solution for smooth electric fields \( E \in W^{1,\infty}(]0, L[) \). We investigate now the more general situation of bounded electric fields \( E \in L^\infty(]0, L[) \). We claim that in this case there is a unique weak solution of the Boltzmann-Vlasov problem.

**Definition 2.3** Assume that \( g_0 \in L^1(\mathbb{R}^+; v \, dv) \), \( g_L \in L^1(\mathbb{R}^-; |v| \, dv) \), \( E \in L^\infty(]0, L[) \). We say that \( f \in L^1(]0, L[ \times \mathbb{R}) \) is a weak solution of ??, ?? if it has traces \( f(0, \cdot) \in L^1(\mathbb{R}^-; |v| \, dv) \), \( f(L, \cdot) \in L^1(\mathbb{R}^+; v \, dv) \) and

\[
\int_0^L \int_{\mathbb{R}} f \left( \frac{1}{\tau} - v \partial_x \theta - E(x) \partial_v \theta \right) \, dv \, dx + \int_{v>0} f(L, v) \theta(L, v) \, dv - \int_{v<0} f(0, v) \theta(0, v) \, dv \\
= \int_{v>0} v g_0(v) \theta(0, v) \, dv - \int_{v<0} v g_L(v) \theta(L, v) \, dv \\
+ \frac{1}{\tau} \int_0^L \int_{\mathbb{R}} \rho(x) M(v) \theta(x, v) \, dv \, dx,
\]

(13)

for any test function \( \theta \in C^1_c([0, L] \times \mathbb{R}) \) where \( \rho(\cdot) = \int_{\mathbb{R}} f(\cdot, v) \, dv \).

**Proposition 2.5** Assume that \( g_0 \in L^1(\mathbb{R}^+; v \, dv) \), \( g_L \in L^1(\mathbb{R}^-; |v| \, dv) \), \( E \in L^\infty(]0, L[) \). Then there is at most one weak solution of the problem ??, ??.

**Proof.** Consider \( f_1, f_2 \in L^1(]0, L[ \times \mathbb{R}) \) two weak solutions of ??, ?? and let \( f = f_1 - f_2 \in L^1(]0, L[ \times \mathbb{R}) \), \( f(0, \cdot) = f_1(0, \cdot) - f_2(0, \cdot) \in L^1(\mathbb{R}^-; |v| \, dv) \), \( f(L, \cdot) = f_1(L, \cdot) - f_2(L, \cdot) \in L^1(\mathbb{R}^+; v \, dv) \). We will prove that \( f = 0 \), \( f(0, \cdot) = 0 \), \( f(L, \cdot) = 0 \). We have in the distribution sense

\[
v \partial_x f + E \partial_v f = \frac{1}{\tau} (\rho(x) M(v) - f(x, v)) = Q(f),
\]

\[
f(0, v > 0) = f(L, v < 0) = 0,
\]

where \( \rho = \int_{\mathbb{R}} f \, dv \). We deduce that

\[
v \partial_x |f| + E \partial_v |f| = \frac{1}{\tau} (\rho(x) M(v) - f(x, v)) \text{sgn} f,
\]

and therefore we obtain

\[
\int_{v>0} \int_{\mathbb{R}} |f(L, v)| \, dv - \int_{v<0} \int_{\mathbb{R}} |f(0, v)| \, dv + \frac{1}{\tau} \int_0^L \int_{\mathbb{R}} (f(x, v) - \rho(x) M(v)) \text{sgn} f \, dv \, dx = 0.
\]
Following the idea in [?] we can write
\[
\int_{\mathbb{R}} (f(x, v) - \rho(x) M(v)) \text{sgn} f \, dv = \int_{\mathbb{R}} (f - \rho M)(\text{sgn} f - \text{sgn}(\rho M)) \, dv \geq 0,
\]
and we deduce that
\[
\int_{v > 0} v |f(L, v)| \, dv - \int_{v < 0} v |f(0, v)| \, dv + \frac{1}{\tau} \int_{0}^{L} \int_{\mathbb{R}} (f - \rho M)(\text{sgn} f - \text{sgn}(\rho M)) \, dv \, dx = 0,
\]
implying that \( f(L, v > 0) = f(0, v < 0) = 0 \) and \( \text{sgn} f(x, v) = \text{sgn} \rho(x), \quad (x, v) \in ]0, L[ \times \mathbb{R} \). Therefore the sign of \( f \) is constant with respect to \( v \) and one gets
\[
v \partial_{x} |f| + E \partial_{v} |f| = \frac{1}{\tau} \left( M(v) \int_{\mathbb{R}} |f(x, w)| \, dw - |f(x, v)| \right) = Q(|f|). \tag{14}
\]
Take \( \phi \) a primitive of \(-E\). Since \( v \partial_{x}(v^{2}/2 + \phi(x)) + E \partial_{v}(v^{2}/2 + \phi(x)) = 0 \) we have in distribution sense
\[
v \partial_{x} \left( \frac{v^{2}}{2} + \phi \right) |f| + E \partial_{v} \left( \frac{v^{2}}{2} + \phi \right) |f| = \left( \frac{v^{2}}{2} + \phi \right) Q(|f|), \tag{15}\]
\[
\left( \frac{v^{2}}{2} + \phi \right) |f| (0, v) = \left( \frac{v^{2}}{2} + \phi \right) |f| (L, v) = 0, \quad v \in \mathbb{R}. \tag{16}\]
Observe also that
\[
v \partial_{x} (|f| \ln |f|) + E \partial_{v} (|f| \ln |f|) = (1 + \ln |f|)Q(|f|). \tag{17}\]
Combining (??), (??) yields
\[
(v \partial_{x} + E \partial_{v}) \left( \frac{v^{2}}{2} + \phi + \ln |f| \right) |f| = Q(|f|) \left( 1 + \frac{v^{2}}{2} + \phi + \ln |f| \right). \tag{18}\]
Observe now that
\[
1 + \frac{v^{2}}{2} + \phi + \ln |f| = \ln \left( \frac{|f|}{M|\rho|} \right) + \ln \left( \frac{|\rho|}{\sqrt{2\pi}} \right) + 1 + \phi.
\]
Notice also that we have \( |\rho| = \int_{\mathbb{R}} |f| \, dv \) and therefore
\[
\int_{\mathbb{R}} \left( 1 + \frac{v^{2}}{2} + \phi + \ln |f| \right) Q(|f|) \, dv = -\frac{1}{\tau} \int_{\mathbb{R}} (|f| - M|\rho|) \ln \left( \frac{|f|}{M|\rho|} \right) \, dv. \tag{19}\]
\( \phi \)From (??), (??), (??) it is easily seen that
\[
\int_{0}^{L} \int_{\mathbb{R}} (|f| - M|\rho|) \ln \left( \frac{|f|}{M|\rho|} \right) \, dv \, dx = 0,
\]
implying that \( |f(x,v)| = r(x)M(v) \) for some non negative function \( r \) satisfying \( r(0) = r(L) = 0 \). By (??) we have \( \frac{d}{dx} r - E(x)r(x) = 0, \ x \in [0, L] \) and finally \( r = 0 \) saying that \( f = 0 \).

The existence of weak solution for the Boltzmann-Vlasov problem follows by regularization of the electric field. In order to assure weak compactness in \( L^1 \) we estimate \( f|\ln f|, \rho|\ln \rho| \). We use the following lemma, based on standard arguments due to Carleman.

**Lemma 2.2** Assume that \( F = F(z) \) satisfies \( F \geq 0, (z^2/2 + |\ln F|)F \in L^1(\mathbb{R}; d\mu) \) for some non negative borelian measure \( \mu \). Then for all \( k > 0 \) we have

\[
F|\ln F| \leq F \ln F + kz^2 F + \frac{4}{e} e^{-\frac{k}{4} z^2}, \ z \in \mathbb{R},
\]

and

\[
\int_{\mathbb{R}} F(z)|\ln F(z)| \, d\mu \leq \int_{\mathbb{R}} F(z) \ln F(z) \, d\mu + k \int_{\mathbb{R}} z^2 F(z) \, d\mu + C_k(\mu),
\]

with \( C_k(\mu) = \frac{4}{e} \int_{\mathbb{R}} e^{-\frac{k}{4} z^2} \, d\mu \).

**Proof.** Since \( F|\ln F| = F \ln F + 2F(\ln F)_- \), it is sufficient to estimate \( F(\ln F)_- \). For any \( k > 0 \) we have

\[
F(\ln F)_- = -F \ln F \cdot 1_{\{0 < F < e^{-\frac{k}{4} z^2}\}} - F \ln F \cdot 1_{\{e^{-\frac{k}{4} z^2} \leq F < 1\}} \leq C e^{-\frac{k}{4} z^2} + k \frac{z^2}{2} F, \ \forall \ z \in \mathbb{R},
\]

where \( C = \sup_{0 < y < 1} \{-\sqrt{y} \ln y\} \). We check immediately that \( C = 2/e \) and therefore

\[
F(z)|\ln F(z)| \leq F(z) \ln F(z) + kz^2 F(z) + \frac{4}{e} e^{-\frac{k}{4} z^2}.
\]

The second inequality follows easily by integration with respect to the measure \( \mu \).

**Proposition 2.6** Assume that \( E \) belongs to \( W^{1,\infty}(0, L) \) is non decreasing and \( E(L) = 0 \). Let \( g_0 \geq 0 \) such that

\[
G + \tilde{G} := \int_{v > 0} v g_0(v) \, dv + \int_{v > 0} v \left( \frac{v^2}{2} + |\ln g_0(v)| \right) g_0(v) \, dv < +\infty.
\]
Denote by $f$ the unique mild solution of the Boltzmann-Vlasov equation with the boundary conditions $g_0$ and $g_L = 0$. Then we have
\[
\int_{v>0} v^2 + |\ln f(L,v)| f(L,v) \, dv - \int_{v<0} v^2 + |\ln f(0,v)| f(0,v) \, dv \leq 2\tilde{G} + 2C,
\]
(20)
\[
\left| \int_0^L E(x) \, dx \right| + \frac{1}{\tau} \int_0^L \int_\mathbb{R} (f(x,v) - \rho(x)M(v)) \ln \frac{f(x,v)}{\rho(x)M(v)} \, dv \, dx \leq \tilde{G} + C,
\]
(21)
where $C = \frac{4}{\tau} \int_{\mathbb{R}} |v| e^{-\frac{v^2}{16}} \, dv$, $\rho = \int_{\mathbb{R}} f \, dv$, $j = \int_{\mathbb{R}} v f \, dv$.

**Proof.** By Theorem ?? we know that the mild solution $f$ is also weak solution of the Boltzmann-Vlasov problem. As in the proof of Proposition ?? one gets in distribution sense
\[
(v \partial_x + E(x) \partial_v) \left( \ln f + \frac{v^2}{2} + \phi \right) = \frac{1}{\tau} (\rho(x)M(v) - f(x,v)) \left( 1 + \ln f + \frac{v^2}{2} + \phi \right),
\]
where $\phi$ is a primitive of $-E$. Using (??) (with $f$ replaced by $|f|$) yields
\[
\int_{v>0} v^2 + \ln f \, f(L,v) \, dv - \int_{v<0} v^2 + \ln f \, f(0,v) \, dv + \phi(L)j(L) - \phi(0)j(0)
\]
\[
+ \frac{1}{\tau} \int_0^L \int_\mathbb{R} (f - \rho M) \ln \frac{f}{\rho M} \, dv \, dx
\]
\[
= \int_{v>0} v^2 + \ln g_0 \, g_0(v) \, dv.
\]
(22)

By Proposition ?? we know that $j(0) = j(L) \geq 0$. Since $E$ is non decreasing and $E(L) = 0$ we have $\frac{d}{dx} \phi = -E(x) \geq 0$ and thus $\phi(0) \leq \phi(L)$. It follows that $\phi(L)j(L) - \phi(0)j(0) \geq 0$. Applying Lemma ?? with $k = 1/4$ and the measures $d\mu_{\pm} = v_\pm \, dv$ we obtain
\[
\int_{v>0} \frac{v^2}{4} + |\ln f| \, f(L,v) \, dv - \int_{v<0} \frac{v^2}{4} + |\ln f| \, f(0,v) \, dv
\]
\[
\leq \int_{v>0} \frac{v^2}{2} + \ln f \, f(L,v) \, dv - \int_{v<0} \frac{v^2}{2} + \ln f \, f(0,v) \, dv + C.
\]
(23)

Our conclusions follow immediately combining (??), (??).

Notice that the above estimates are uniform with respect to $\tau > 0, L > 0$ and $\|E\|_{L^\infty([0,L])}$. 15
Similar estimates hold in the case of general fields \( E \in W^{1,\infty}(0, L] \) (not necessarily non decreasing with \( E(L) = 0 \)) and incoming densities \( g_0, g_L \geq 0 \), but these bounds will depend on \( L \) and \( \|E\|_{L^{\infty}(0,L]} \) (use the inequalities \(|j| \leq G, |\phi(0) - \phi(L)| \leq L \|E\|_{L^{\infty}}\)).

Following the idea in the proof of Proposition ?? one gets more estimates (depending on \( L \)). We use the lemma

**Lemma 2.3** Assume that \( f = f(x, v) \) is a non negative function satisfying

\[
\int_0^L \int_{\mathbb{R}} \left( 1 + \frac{v^2}{2} + |\ln f(x, v)| \right) f(x, v) \, dv \, dx < +\infty,
\]

and denote by \( \rho \) the charge density \( \rho = \int_{\mathbb{R}} f(\cdot, v) \, dv \). Then we have

\[
\rho(x) \ln \frac{\rho(x)}{\sqrt{2\pi}} \leq \int_{\mathbb{R}} \left( \frac{v^2}{2} + \ln f(x, v) \right) f(x, v) \, dv,
\]

(24)

\[
\int_0^L \rho(x) \, dx \leq \int_0^L \rho(x) \, dx + kL^2 \int_0^L f(x, v) \, dv dx + C_k(L),
\]

(25)

where \( C_k(L) = \frac{4}{e} \int_0^L e^{-kx^2/4} \, dx \).

**Proof.** Consider the convex function \( \varphi : [0, +\infty] \to \mathbb{R}, \varphi(s) = s \ln s, s > 0, \varphi(0) = 0 \) and the measure \( d\nu = M(v) \, dv \). By applying the Jensen inequality

\[
\varphi \left( \int_{\mathbb{R}} g(v) \, dv \right) \leq \int_{\mathbb{R}} \varphi(g(v)) \, dv,
\]

with the function \( g(\cdot) = f(x, \cdot)/M(\cdot) \) one gets

\[
\rho(x) \ln \rho(x) \leq \int_{\mathbb{R}} (\ln \sqrt{2\pi} + \frac{v^2}{2} + \ln f(x, v)) f(x, v) \, dv,
\]

implying (??). The inequality (??) follows by applying Lemma ?? with \( d\mu = 1_{\{0 < x < L\}} \, dx \).

**Proposition 2.7** Assume that \( E \) belongs to \( W^{1,\infty}(0, L], g_0, g_L \geq 0 \) such that

\[
G + \tilde{G} = \int_{v>0} \left( 1 + \frac{v^2}{2} + |\ln g_0| \right) g_0 \, dv - \int_{v<0} \left( 1 + \frac{v^2}{2} + |\ln g_L| \right) g_L \, dv < +\infty.
\]

16
Denote by $f$ the unique mild solution of the Boltzmann-Vlasov problem (12), (13). Then there are constants depending on $\tau, L, \|E\|_{L^\infty([0,L])}$ such that

$$
\int_0^L \int_\mathbb{R} \left( \frac{v^2}{2} + |\ln f(x,v)| \right) f(x,v) \, dv \, dx \leq C_2(\tau, L, \|E\|_{L^\infty}) \left( 1 + G + \tilde{G} \right),
$$

(26)

$$
\int_0^L \rho(x) |\ln \rho(x)| \, dx \leq C_3(\tau, L, \|E\|_{L^\infty}) \left( 1 + G + \tilde{G} \right).
$$

(27)

**Proof.** Consider the function $r(x,v) = \left( \frac{v^2}{2} + \phi + \ln f \right) f(x,v)$ where $\phi$ is a primitive of $-E$, for example $\phi(x) = -\int_0^x E(y) \, dy$. We have $\|\phi\|_{L^\infty([0,L])} \leq L \|E\|_{L^\infty([0,L])}$. As previous we can write

$$
\frac{1}{\tau} r + v \partial_x r + E \partial_v r + \frac{1}{\tau} \rho M \left( \frac{f}{\rho M} - 1 - \ln \frac{f}{\rho M} \right) = \frac{1}{\tau} \rho M \left( \phi + \ln \frac{\rho}{\sqrt{2\pi}} \right).
$$

Using (12) with the function $\psi = 1, \alpha = \frac{1}{\tau}$ and observing that $\rho M \left( \frac{f}{\rho M} - 1 - \ln \frac{f}{\rho M} \right) \geq 0$ we obtain

$$
\int_0^L \int_\mathbb{R} \left( \frac{v^2}{2} + \phi + \ln f \right) f \, dv \, dx \leq \int_0^L \int_\mathbb{R} g_0 \int_{s_{out}(0,v)} e^{-\frac{s}{\tau}} \, ds \, dv
$$

$$
- \int_0^L \int_\mathbb{R} g_0 \int_{s_{out}(v,L)} e^{-\frac{s}{\tau}} \, ds \, dv
$$

$$
+ \int_0^L \rho(x) \left( \phi(x) + \ln \frac{\rho(x)}{\sqrt{2\pi}} \right) h(x) \, dx,
$$

(28)

where $h(x) = \frac{1}{\tau} \int_\mathbb{R} M(v) \int_{s_{out}(x,v)} e^{-\frac{s}{\tau}} \, ds \, dv$. We have already checked that $0 \leq h \leq \gamma(\tau, L, \|E\|_{L^\infty}) < 1$. By Lemma (13) we have

$$
\rho(x) \left( \phi(x) + \ln \frac{\rho(x)}{\sqrt{2\pi}} \right) \leq \rho(x) \phi(x) + \int_\mathbb{R} \left( \frac{v^2}{2} + \ln f \right) f \, dv
$$

$$
= \int_\mathbb{R} \left( \frac{v^2}{2} + \phi(x) + \ln f \right) f(x,v) \, dv.
$$

(29)

Combining (12), (13) yields

$$
\int_0^L \int_\mathbb{R} r(x,v) \, dv \, dx \leq \tau \tilde{G} + \tau \|\phi\|_{L^\infty([0,L])} G + \int_0^L \int_\mathbb{R} r(x,v) h(x) \, dv \, dx,
$$

implying that

$$
\int_0^L \int_\mathbb{R} \left( \frac{v^2}{2} + \ln f \right) f \, dv \, dx \leq \tilde{C}(\tau, L, \|E\|_{L^\infty}) + \int_0^L \int_\mathbb{R} \left( \frac{v^2}{2} + \ln f \right) f h \, dv \, dx,
$$

17
where \( \tilde{C}(\tau, L, \|E\|_{L^\infty}) = \tau \tilde{G} + \tau L \|E\|_{L^\infty} G + 2L \|E\|_{L^\infty} \frac{\tilde{G}}{1-\gamma} \) (we use here the estimates \( \|\phi\|_{L^\infty([0, L])} \leq L \|E\|_{L^\infty([0, L])}, \|f\|_{L^1([0, L])} \leq \tau G/(1-\gamma) \)). Take \( k > 0 \) small enough such that \( 1 - 2k > \gamma \). By Lemma ?? we deduce that

\[
(1 - 2k) \int_0^L \int_{\mathbb{R}} \left( \frac{v^2}{2} + |\ln f| \right) f \, dv \, dx \leq \int_0^L \int_{\mathbb{R}} \left( 1 - 2k \right) \frac{v^2}{2} + |\ln f| f(x, v) \, dv \, dx
\]

\[
\leq \int_0^L \int_{\mathbb{R}} \left( \frac{v^2}{2} + \ln f \right) f(x, v) \, dv \, dx
\]

\[
\leq \tilde{C} + LC_k + \int_0^L \int_{\mathbb{R}} \left( \frac{v^2}{2} + |\ln f| \right) f h \, dv \, dx
\]

\[
\leq \tilde{C} + LC_k + \gamma \int_0^L \int_{\mathbb{R}} \left( \frac{v^2}{2} + |\ln f| \right) f \, dv \, dx,
\]

where \( C_k = \frac{4}{e} \int_{\mathbb{R}} e^{-\frac{k}{4} v^2} \, dv \). Finally one gets

\[
\int_0^L \int_{\mathbb{R}} \left( \frac{v^2}{2} + |\ln f(x, v)| \right) f \, dv \, dx \leq \frac{\tilde{C} + LC_k}{1 - 2k - \gamma}.
\]

The inequality (??) follows immediately from (??) (take for example \( k = 1 \), (??), (??)).

The estimates of Propositions ??, ?? allow us to prove the existence of weak solution for the Boltzmann-Vlasov problem with bounded electric field. For simplicity we consider here only continuous fields. This will be enough for our purposes, since the fields satisfying the Poisson equation (??) are continuous. The proof in the case of bounded fields is a little bit more complicated.

**Theorem 2.2** Assume that \( E \in C([0, L]), g_0, g_L \geq 0 \) such that

\[
G + \tilde{G} = \int_{v>0} v \left( 1 + \frac{v^2}{2} + |\ln g_0| \right) g_0 \, dv - \int_{v<0} v \left( 1 + \frac{v^2}{2} + |\ln g_L| \right) g_L \, dv < +\infty.
\]

Then there is a unique weak solution of the Boltzmann-Vlasov problem (??), (??).

**Proof.** The uniqueness was checked in Proposition ??). For the existence part consider \( (E_n)_n \subset W^{1,\infty}([0, L]) \) a sequence of smooth fields converging towards \( E \) in \( C([0, L]) \). For any \( n \) denote by \( f_n \) the unique mild solution of (??), (??) with the field \( E_n \). By Corollary ?? and the Propositions ??, ??, ?? we have

\[
\sup_{n \in \mathbb{N}} \int_0^L \rho_n(x) |\ln \rho_n(x)| \, dx \leq C_3(\tau, L, \|E\|_{L^\infty})(1 + G + \tilde{G}),
\]
\[ \sup_{n \in \mathbb{N}} \left\{ \int_{v>0} \left( 1 + \frac{v^2}{2} + |\ln f_n| \right) f_n(L,v) \, dv \right\} \leq C_4(\tau, L, \|E\|_{L^\infty}) (1 + G + \tilde{G}), \]

\[ \sup_{n \in \mathbb{N}} \int_0^L \int_{\mathbb{R}} \left( 1 + \frac{v^2}{2} + |\ln f| \right) f(x,v) \, dv \, dx \leq C_5(\tau, L, \|E\|_{L^\infty}) (1 + G + \tilde{G}), \]

for some constants \( C_3, C_4, C_5 \) depending on \( \tau, L \) and \( \|E\|_{L^\infty} \). By Dunford-Pettis theorem we have (after extraction eventually)

\[ \lim_{n \to +\infty} v f_n(0, \cdot) = v f_0(\cdot) \text{ weakly in } L^1(\mathbb{R}^-), \]

\[ \lim_{n \to +\infty} v f_n(L, \cdot) = v f_L(\cdot) \text{ weakly in } L^1(\mathbb{R}^+), \]

\[ \lim_{n \to +\infty} f_n = f \text{ weakly in } L^1([0, L[ \times \mathbb{R}^+), \]

\[ \lim_{n \to +\infty} \rho_n = \int_{\mathbb{R}} f \, dv =: \rho \text{ weakly in } L^1([0, L[.} \]

We check easily that \( f_0, f_L, f \geq 0 \) and

\[ \int_{v>0} \left( 1 + \frac{v^2}{2} + |\ln f_L| \right) f_L(v) \, dv - \int_{v<0} \left( 1 + \frac{v^2}{2} + |\ln f_0| \right) f_0(v) \, dv \]

\[ + \int_0^L \int_{\mathbb{R}} \left( 1 + \frac{v^2}{2} + |\ln f| \right) f \, dv \, dx + \int_0^L \rho \, |\ln \rho| \, dx \leq C_6(\tau, L, \|E\|_{L^\infty}) (1 + G + \tilde{G}). \]

Since \( f_n \) are also weak solutions, we have for any \( n \)

\[ \int_0^L \int_{\mathbb{R}} f_n \left( \frac{1}{\tau} - v \partial_x \theta - E_n \partial_x \theta \right) \, dv \, dx + \int_{v>0} v (f_n \theta)(L, v) \, dv - \int_{v<0} v (f_n \theta)(0, v) \, dv \]

\[ = \int_{v>0} v g_0(v) \theta(0, v) \, dv - \int_{v<0} v g_L(v) \theta(L, v) \, dv + \frac{1}{\tau} \int_0^L \int_{\mathbb{R}} \rho_n(x) M(v) \theta(x, v) \, dv \, dx, \]

for any test function \( \theta \in C^1_c([0, L] \times \mathbb{R}). \) Combining the strong convergence of \( (E_n)_n \) in \( C([0, L]) \) with the weak convergence of \( (f_n)_n, (vf_n(0, \cdot))_n, (vf_n(L, \cdot))_n \) in \( L^1([0, L[ \times \mathbb{R}^+), L^1(\mathbb{R}^-), L^1(\mathbb{R}^+) \) respectively, we deduce that \( f \) is a weak solution of (??), (??) corresponding to the field \( E \), with traces \( f_0, f_L. \)
3 The Boltzmann-Vlasov-Poisson problem

In this section we prove the Theorem ??.

We construct weak solutions \((f, E)\) of the Boltzmann-Vlasov-Poisson system, i.e., \(f\) is a weak solution of the Boltzmann-Vlasov problem and \(E\) is a classical solution of the Poisson problem. We use a standard fixed point procedure. We obtain uniform bounds with respect to \(\tau, L\) for the \(L^\infty\) norms of the kinetic energy and the electric field. Assume that \(g_0\) is a non-negative function satisfying

\[
G + \tilde{G} = \int_{v>0} v \left( 1 + \frac{v^2}{2} + |\ln g_0| \right) g_0(v) \, dv < +\infty.
\]

Take \(\tau, L, A > 0\) and consider \(\mathcal{D}(\tau, L, A)\) the set of continuous functions \(E\) satisfying \(E(L) = 0, \frac{d}{dx} E \geq 0, \|E\|_{L^\infty([0, L])} \leq A\) and

\[
\int_0^L E'(x) (\ln E'(x))_+ \, dx \leq \int_0^L E'(x) (\ln E'(x))_- \, dx + C_6(\tau, L, A)(1 + G + \tilde{G}),
\]

(31)

(see (??) for the definition of \(C_6\)). Applying Lemma ?? with \(d\mu = 1_{\{0 < x < L\}} \, dx\) and \(k = 1\) yields

\[
\int_0^L E'(x) \ln E'(x) \, dx \leq C_6(\tau, L, A)(1 + G + \tilde{G}) + 4A + \frac{4}{e} L < +\infty,
\]

(32)

and therefore (??) can be written

\[
\int_0^L E'(x) \ln E'(x) \, dx \leq C_6(\tau, L, A)(1 + G + \tilde{G}), \quad \forall \ E \in \mathcal{D}(\tau, L, A).
\]

It is easily seen that \(\mathcal{D}(\tau, L, A)\) is convex and compact with respect to the topology of \(C([0, L])\) (use Arzela-Ascoli theorem and (??)). For any \(E \in \mathcal{D}(\tau, L, A)\) we define our fixed point application by

\[
\mathcal{F} E(x) = \max \left\{ -A, -\int_x^L \int_{\mathbb{R}} f(y, v) \, dv \, dy \right\}, \quad x \in [0, L],
\]

where \(f\) is the weak solution of the Boltzmann-Vlasov equation associated to the field \(E\) and the boundary conditions \(g_0, g_L = 0\). Observe that \(\mathcal{F} E(L) = 0, \frac{d}{dx}(\mathcal{F} E) \geq 0, \|\mathcal{F} E\|_{L^\infty([0, L])} \leq A\) and

\[
\int_0^L \frac{d}{dx} \mathcal{F} E \ln \left( \frac{d}{dx} \mathcal{F} E \right) \, dx \leq \int_0^L \frac{d}{dx} \mathcal{F} E \left\{ \ln \frac{d}{dx} \mathcal{F} E \right\} \, dx \leq \int_0^L \rho(x) |\ln \rho| \, dx \leq C_6(\tau, L, A)(1 + G + \tilde{G}),
\]

(32)
saying that $\mathcal{D}(\tau, L, A)$ is left invariant by $\mathcal{F}$. In order to apply the Schauder fixed point theorem it remains to check the continuity of $\mathcal{F}$ with respect to the topology of $C([0, L])$. Take $(E_n)_n \subset \mathcal{D}(\tau, L, A)$ a convergent sequence in $C([0, L])$ towards $E \in \mathcal{D}(\tau, L, A)$. We denote by $f_n, f$ the weak solutions of the Boltzmann-Vlasov problems corresponding to $E_n, E$ respectively. As in the proof of Theorem ?? we deduce that $\lim_{n \to +\infty} f_n = f$ weakly in $L^1([0, L[ \times \mathbb{R})$ (the uniqueness of the weak solution is crucial here), implying the pointwise convergence of $(\mathcal{F}E_n)_n$ towards $\mathcal{F}E$. Since $(\mathcal{F}E_n)_n$ belongs to $\mathcal{D}(\tau, L, A)$ which is a compact set of $C([0, L])$, finally one gets that $\lim_{n \to +\infty} \mathcal{F}E_n = \mathcal{F}E$ in $C([0, L])$. Therefore the application $\mathcal{F}$ has a fixed point $E \in \mathcal{D}(\tau, L, A)$ meaning that

$$v \partial_x f + E(x)\partial_v f = \frac{1}{\tau} (\rho(x)M(v) - f(x, v)), \quad (x, v) \in [0, L[ \times \mathbb{R},$$

(33)

$$f(x = 0, v > 0) = g_0(v), \quad f(x = L, v < 0) = 0,$$

(34)

$$E(x) = \max \left\{-A, -\int_x^L \int_{\mathbb{R}} f(y, v) \, dv \, dy\right\}, \quad x \in [0, L].$$

(35)

The idea is to prove that for $A$ large enough the solution $f$ of the Boltzmann-Vlasov problem associated to the fixed point $E$ satisfies $\|f\|_{L^1([0, L[ \times \mathbb{R})} \leq A$ and therefore (??) gives exactly the solution of the Poisson equation (??) with the boundary condition $E(L) = 0$. Take $A > 0$ such that $A^2 > 2(G + 3\tilde{G} + 2C)$ where $C = \frac{4}{e} \int_{\mathbb{R}} |v| e^{-\frac{v^2}{16}} \, dv$ and consider $x_A = \inf\{x \in [0, L] : \int_x^L \int_{\mathbb{R}} f(y, v) \, dv \, dy \leq A\}$. Obviously we have $0 \leq x_A < L$ and

$$\{x \in [0, L] : \int_x^L \int_{\mathbb{R}} f(y, v) \, dv \, dy \leq A\} = [x_A, L].$$

(36)

Assume that $x_A > 0$. With the notation $k(x) = \int_{\mathbb{R}} v^2 f(x, v) \, dv$ we have

$$\frac{d}{dx} k(x) - \frac{1}{2} \frac{d}{dx} E(x)^2 = \frac{j}{\tau}, \quad x \in ]x_A, L[,$$

(37)

and

$$\frac{d}{dx} k(x) + A \rho(x) = \frac{j}{\tau}, \quad x \in ]0, x_A[.$$
By Proposition ?? we know that $j \geq 0$ and since $E(\cdot)^2$ is non increasing we deduce that $k(\cdot)$ is non increasing on $[0, L]$. In particular by (??) et (??) we obtain
\[ k(x) \leq k(0) \leq \frac{1}{2} \int_{\mathbb{R}} |v|(1 + v^2) f(0, v) \, dv \leq G + 3\tilde{G} + 2C. \] (39)

Integrating (??) over $[x_A, L]$ yields
\[ k(L) - k(x_A) + \frac{1}{2} E(x_A)^2 = -\frac{j}{\tau}(L - x_A) \leq 0, \]
and therefore
\[ \left( \int_{x_A}^{L} \int_{\mathbb{R}} f(y, v) \, dv \, dy \right)^2 = E(x_A)^2 \leq 2(G + 3\tilde{G} + 2C) < A^2, \]
which contradicts (??). Thus we have $x_A = 0$ and
\[ 0 \geq E(x) = -\int_{x}^{L} \int_{\mathbb{R}} f(y, v) \, dv \, dy \geq -\left(2(G + 3\tilde{G} + 2C)\right)^{1/2}, \quad x \in [0, L]. \] (40)

Multiplying now (??) by $v^2$ one gets
\[ \frac{d}{dx} \int_{\mathbb{R}} v^3 f(x, v) \, dv = 2E(x)j = \frac{1}{\tau}(\rho(x) - k(x)). \] (41)

Integrating over $[0, L]$ we deduce by (??) that
\[ 2\tau j(\phi(L) - \phi(0)) + \int_{0}^{L} k(x) \, dx \leq \int_{0}^{L} \rho(x) \, dx + \tau \int_{\mathbb{R}} v^3 (f(0, v) - f(L, v)) \, dv \]
\[ \leq (2(G + 3\tilde{G} + 2C))^{1/2} + 2\tau \tilde{G} := \tilde{G}(\tau). \] (42)

Observe by (??) that the function $x \to k(x) + \frac{j}{2}x$ is non increasing and thus $\int_{0}^{x} k(y) \, dy$, for any $x \in [0, L]$ implying that
\[ k(x) \leq \frac{\tilde{G}(\tau)}{x}, \quad x \in [0, L], \quad \frac{j}{\tau} \leq \frac{2}{L^2} \tilde{G}(\tau). \] (43)

By (??) we have $k(x) - \frac{1}{2}E(x)^2 \geq k(L) \geq 0, \quad x \in [0, L]$, and thus $E(x)^2 \leq 2k(x)$ implying that
\[ \frac{1}{2} \int_{0}^{L} E(x)^2 \, dx \leq \tilde{G}(\tau), \] (44)
\[ E(x)^2 \leq 2k(x) \leq \frac{2}{x} \tilde{G}(\tau), \quad x \in [0, L], \] (45)
\[ \int_{x}^{L} \rho(y) \, dy = -E(x) \leq \left( \frac{2}{x} \tilde{G}(\tau) \right)^{1/2}, \quad x \in [0, L]. \] (46)
4 Hydrodynamic limit in a half line

We intend to identify the limit of the Boltzmann-Vlasov-Poisson model for small relaxation times \( \tau \) and large domains \([0, L]\). First we keep fixed the size \( L \) and we perform the asymptotic analysis for \( \tau \searrow 0 \). Secondly we pass to the limit when \( L \) goes to \(+\infty\). We denote by \( \mathcal{M}_1^1(X) \) the set of non negative bounded Radon measures on \( X \). We recall some definitions and compactness properties in measure spaces (see [?] for more details).

**Definition 4.1** Let \((\mu_n)_{n \in \mathbb{N}}\) be a sequence in \( \mathcal{M}_1^1(X) \). We say that

1) \((\mu_n)_{n \in \mathbb{N}}\) converges vaguely to \( \mu \) iff

\[
\lim_{n \to +\infty} \int_X \varphi \, d\mu_n = \int_X \varphi \, d\mu.
\]

for any continuous function with compact support \( \varphi \in C^0_c(X) \) (actually the convergence holds for any function \( \varphi \in C(X) \) vanishing at infinity i.e., \( \lim_{|x| \to +\infty, x \in X} \varphi(x) = 0 \) );

2) \((\mu_n)_{n \in \mathbb{N}}\) converges tightly to \( \mu \) iff \( (??) \) holds for any continuous and bounded function \( \varphi \in C^0(X) \cap L^\infty(X) \).

**Proposition 4.1**

1) Let \((\mu_n)_{n \in \mathbb{N}}\) be a sequence in \( \mathcal{M}_1^1(X) \) verifying \( \sup_n \mu_n(X) < +\infty \) and such that for any \( \eta > 0 \) there exists a compact set \( K_\eta \subset X \) satisfying \( \sup_n \mu_n(X \cap \overline{C}K_\eta) \leq \eta \).

Then \((\mu_n)_{n \in \mathbb{N}}\) is relatively compact for the tight topology.

2) Let \((\mu_n)_{n \in \mathbb{N}}\) be a sequence in \( \mathcal{M}_1^1(X) \) which converges vaguely to \( \mu \). We assume also that \( \lim_{n \to +\infty} \mu_n(X) = \mu(X) \). Then \((\mu_n)_{n \in \mathbb{N}}\) converges to \( \mu \) tightly.

Let \( L > 0 \) and consider \((\varepsilon_n)_n\) a sequence of positive numbers converging towards 0. We denote by \((f_n, E_n)\) a weak solution of the Boltzmann-Vlasov-Poisson problem in \([0, L[ \times \mathbb{R}\) with the relaxation time \( \tau = \varepsilon_n \) and the boundary conditions \( f_n(0, v > 0) = g_0(v), f_n(L, v < 0) = 0, E_n(L) = 0 \), cf. Section ??. We assume that \( g_0 \) is a non
negative function verifying

\[ G + \tilde{G} = \int_{\mathbb{R}} v \left( 1 + \frac{v^2}{2} + |\ln g_0| \right) g_0(v) \, dv < +\infty. \]

By (??), (??), (??), (??), (??) we know that there is a constant \( C_7 \) depending only on \( G, \tilde{G} \) such that for any \( n \)

\[ \| (1 + v^2) f_n \|_{L^1([0,L] \times \mathbb{R})} + \| F_n \|_{L^2([0,L])} + \left\| \int_{\mathbb{R}} v^2 T_n(\cdot, v) \, dv \right\|_{L^\infty([0,L])} + \| E_n \|_{L^\infty([0,L])} \leq C_7. \]

After extraction eventually we can assume that

\[ \lim_{n \to +\infty} (1, v^2) f_n = (1, v^2) df, \text{ vaguely in } \mathcal{M}_1^1([0, L] \times \mathbb{R})^2, \]

\[ \lim_{n \to +\infty} \rho_n = d\rho, \text{ vaguely in } \mathcal{M}_1^1([0, L]), \]

\[ \lim_{n \to +\infty} E_n = E, \text{ strongly in } L^1([0, L]) \text{ and weakly } \ast \text{ in } L^\infty([0, L]). \]

For the strong convergence of \((E_n)\) in \( L^1([0, L]) \) observe that \((\frac{dE_n}{dx})_n = (\rho_n)_n\) is bounded in \( L^1([0, L]) \). Obviously, since [0, L] is compact, \((\rho_n)_n\) converges tightly. Using the bounds for the total kinetic energy we deduce that \( \lim_{n \to +\infty} \int_0^L \int_{\mathbb{R}} f_n \, dv \, dx = \int_0^L \int_{\mathbb{R}} df \) and by Proposition ?? we obtain that the sequence \((f_n)_n\) converges tightly.

For any test function \( \theta \in C_c([0, L] \times \mathbb{R}) \) we have

\[ -\varepsilon_n \int_0^L \int_{\mathbb{R}} (v \partial_x \theta + E_n(x) \partial_v \theta) f_n(x, v) \, dv \, dx = \int_0^L \left( \int_{\mathbb{R}} M(v) \theta(x, v) \, dv \right) \rho_n(x) \, dx \]

\[ - \int_0^L \int_{\mathbb{R}} \theta(x, v) f_n(x, v) \, dv \, dx, \]

and after passing to the limit for \( n \to +\infty \) one gets

\[ df = M(v) \, dv \, d\rho. \quad (48) \]

By (??) we know that \( \left( \frac{\bar{v}_n}{\varepsilon_n} \right)_n \) is a bounded sequence and thus, after extraction we can assume \( \lim_{n \to +\infty} \frac{\bar{v}_n}{\varepsilon_n} = J \geq 0 \). Actually the limit \( J \) depends on \( L \) and satisfies

\[ 0 \leq J \leq \frac{2}{L^2} \tilde{G}(\sup_{n \in \mathbb{N}} \varepsilon_n). \quad (49) \]
We claim that \((v^2 f_n)_n\) converges tightly. Indeed, take \(\chi \in C^0_c(\mathbb{R})\) such that \(\chi(u) = 1, |u| \leq 1, \chi(u) = 0, |u| \geq 2, 0 \leq \chi \leq 1\) and denote by \(\chi_R\) the function \(\chi_R(\cdot) = \chi\left(\frac{\cdot}{R}\right)\), \(\forall \, R > 0\). Multiplying the Vlasov equation by \(v^2(1 - \chi_R(v))\) and integrating with respect to \((x,v) \in ]0, L[ \times \mathbb{R}\) yields
\[
\int_0^L \int_{|v| > 2R} v^2 f_n \, dv \, dx \leq \left( \int_{|v| > R} v^2 M(v) \, dv \right) \int_0^L \rho_n(x) \, dx + \varepsilon_n \int_{v > R} v^3 g_0(v) \, dv
+ \varepsilon_n \|E_n\|_{L^\infty([0,L])}(2 + \sup_{u \in \mathbb{R}} |u\chi'(u)|) \int_0^L \int_{|v| > R} |v| f_n \, dv \, dx.
\]
Combining with the inequalities
\[
\int_0^L \int_{|v| > R} |v| f_n \, dv \, dx \leq \frac{1}{R} \int_0^L \int_{|v| > R} v^2 f_n \, dv \, dx \leq \frac{C_7}{R},
\]
we deduce that \(\lim_{R \to +\infty} \sup_{n \in \mathbb{N}} \int_{|v| > 2R} v^2 f_n \, dv \, dx = 0\) and therefore by Proposition ?? the sequence \((v^2 f_n)_n\) converges tightly towards \(v^2 df\). Recall that for any \(n\) we have \(\frac{d}{dx} k_n - \frac{1}{2} \frac{d}{dx} E_n(x)^2 = -\frac{\varepsilon_n}{k_n}\) where \(k_n = \int_{\mathbb{R}} v^2 f_n \, dv\). For any test function \(\theta \in C^1_c([0, L])\) we obtain
\[
- \int_0^L \int_{\mathbb{R}} v^2 \theta'(x) f_n \, dv \, dx + \frac{1}{2} \int_0^L E_n(x)^2 \theta'(x) \, dx = -\frac{j_n}{\varepsilon_n} \int_0^L \theta(x) \, dx,
\]
and after passing to the limit for \(n \to +\infty\) one gets
\[
- \int_0^L \int_{\mathbb{R}} v^2 \theta'(x) \, df + \frac{1}{2} \int_0^L E(x)^2 \theta'(x) \, dx = -J \int_0^L \theta(x) \, dx.
\]
By using (??) we have \(\int_0^L \int_{\mathbb{R}} v^2 \theta'(x) \, df = \int_0^L \theta'(x) \, d\rho\) and thus
\[
- \int_0^L \theta'(x) \, d\rho + \frac{1}{2} \int_0^L E(x)^2 \theta'(x) \, dx = -J \int_0^L \theta(x) \, dx,
\]
saying that \(d\rho\) is absolutely continuous with respect to the Lebesgue measure and that its density \(\rho\) is given by
\[
\rho(x) = \frac{1}{2} E(x)^2 + (L - x)J + C, \, x \in [0, L], \quad (50)
\]
for some constant \(C\) depending on \(L\). Multiplying the Poisson equation by a test function \(\theta \in C^1_c([0, L])\) yields
\[
- \int_0^L E_n(x) \theta'(x) \, dx = \int_0^L \rho_n(x) \theta(x) \, dx, \, \forall \, n,
\]
and after passing to the limit for \( n \to +\infty \) one gets
\[
- \int_0^L E(x) \theta'(x) \, dx = \int_0^L \rho(x) \theta(x) \, dx,
\]
saying that \( \frac{dE}{dx} = \rho \) and \( E(L) = 0 \). We claim that \( \rho \) is non increasing and bounded on \([0, L]\) uniformly with respect to \( L > 0 \). For this observe that the functions
\[
k_n := \int_{\mathbb{R}} v^2 f_n \, dv
\]
are non increasing and bounded on \([0, L]\) uniformly with respect to \( L > 0 \), cf. (??). Thus after extraction we have \( \lim_{n \to +\infty} k_n = k \) weakly * in \( L^\infty([0, L]) \) where \( k \) is a non increasing bounded function on \([0, L]\) uniformly with respect to \( L > 0 \). For any \( \theta \in C^1_c([0, L]) \) we can write
\[
\int_0^L \theta(x) \rho(x) \, dx = \frac{1}{2} \int_{\mathbb{R}} \theta(x) v^2 f_n \, dv \, dx
\]
implies that \( \rho = k \). By (??) we can write for any \( x \in ]0, L[ \)
\[
\int_{x}^{L} \rho(y) \, dy = \int_{x}^{L} k(y) \, dy = \lim_{n \to +\infty} \int_{x}^{L} k_n(y) \, dy \leq \frac{\tilde{G}(\sup_{m} \varepsilon_m)}{x}(L - x).
\]
Since the function \( \rho \) is continuous we deduce that
\[
0 \leq C = \rho(L) = \lim_{x \to L} \frac{\int_{x}^{L} \rho(y) \, dy}{L - x} \leq \frac{\tilde{G}(\sup_{m} \varepsilon_m)}{L}. \tag{51}
\]
We investigate now the boundary conditions satisfied by the function \( k \). By (??) we know that \( \frac{d}{dx} k_n - \frac{1}{2} \frac{d}{dx} E_n(x)^2 = -\frac{j_n}{\varepsilon_n} \) in \( \mathcal{D}'([0, L]) \) and therefore
\[
k_n(y) - k_n(0) - \frac{1}{2} E_n(y)^2 + \frac{1}{2} E_n(0)^2 = -\frac{j_n}{\varepsilon_n} y, \quad y \in [0, L].
\]
By (??) observe that
\[
|E_n(y)^2 - E_n(0)^2| \leq 2(2(G + 3\tilde{G} + 2C))^{\frac{1}{2}} \int_0^y \rho_n(z) \, dz.
\]
We obtain for any \( x \in ]0, L[ \)
\[
\left| \frac{1}{x} \int_0^x k_n(y) \, dy - k_n(0) \right| \leq \left| \frac{1}{2x} \int_0^x (E_n(y)^2 - E_n(0)^2) \, dy \right| + \frac{1}{2} \frac{j_n}{\varepsilon_n} y \leq (2(G + 3\tilde{G} + 2C))^{\frac{1}{2}} \int_0^x \rho_n(y) \, dy + \frac{1}{2} \frac{j_n}{\varepsilon_n}.
Observe that \( \int_0^x \rho_n(y) \, dy \leq \int_0^1 \rho_n(y) \chi_v(y) \, dy \) and therefore we have for any \( x \in [0, \frac{1}{2}] \)
\[
\lim_{n \to +\infty} \int_0^x \rho_n(y) \, dy \leq \lim_{n \to +\infty} \int_0^1 \rho_n(y) \chi_v(y) \, dy = \int_0^1 \rho(y) \chi_v(y) \, dy \leq \int_0^{2x} \rho(y) \, dy.
\]
After passing to the limit for \( n \to +\infty \) one gets for any \( x \in [0, \frac{1}{2}] \)
\[
\left| \frac{1}{x} \int_0^x k(y) \, dy - \lim_{n \to +\infty} k_n(0) \right| \leq (2(G + 3\tilde{G} + 2C))^{\frac{1}{2}} \int_0^{2x} \rho(y) \, dy + \frac{x J}{2}.
\]
Since \( k \) is continuous we deduce after passing \( x \searrow 0 \) that \( k(0) = \lim_{x \searrow 0} k(x) = \lim_{x \searrow 0} \rho(x) = \lim_{n \to +\infty} k_n(0) \). The limit of the sequence \( (k_n(0))_n \) can be computed at least when we know that
\[
\frac{1}{\varepsilon_n} (f_n - \rho_n M(v) - j_n v M(v)) = o(\varepsilon_n). \tag{52}
\]

**Generally this condition is satisfied since we have**
\[
f_n - \rho_n M(v) - \varepsilon_n J v M(v) = O(\varepsilon_n^2),
\]
and \( \lim_{n \to +\infty} \frac{j_n}{\varepsilon_n} = J \). We assume that (52) holds. Multiplying the Boltzmann-Vlasov equation by \( |v| \) one gets
\[
\frac{d}{dx} \tilde{k}_n - E_n(x) \int_\mathbb{R}^2 |v| f_n \, dv = -\frac{1}{\varepsilon_n} \int_\mathbb{R} |v| (f_n - \rho_n M(v) - j_n v M(v)) \, dv, \tag{53}
\]
where \( \tilde{k}_n = \int_\mathbb{R}^2 v |v| f_n \, dv \). By (52) we have
\[
\left| \int_0^L \int_\mathbb{R} |v| f_n \, dv \, dx \right| = \left| \int_0^L \int_\mathbb{R} |v| (f_n - \rho_n M(v)) \, dv \, dx \right| \leq \int_0^L \int_\mathbb{R} |f_n - \rho_n M(v)| \, dv \, dx
\[
\leq \left( \int_0^L \int_\mathbb{R} (f_n - \rho_n M) \ln \frac{f_n}{\rho_n M} \, dv \, dx \right)^{\frac{1}{2}} \left( \int_0^L \int_\mathbb{R} \left| \frac{f_n - \rho_n M}{\ln \frac{f_n}{\rho_n M}} \right| \, dv \, dx \right)^{\frac{1}{2}}
\[
\leq (\varepsilon_n (\tilde{G} + C))^{\frac{1}{2}} (2\|f_n\|_{L^1(0,L|\mathbb{R}|)}^2)^{\frac{1}{2}}. \tag{54}
\]
In the last estimate we have used the inequality
\[
\frac{|x - y|}{\ln x - \ln y} \leq \max\{x, y\} \leq x + y, \quad \forall \ x, y > 0.
\]

27
Multiplying (??) by $L - x$ and integrating by parts one gets
\[
\lim_{n \to +\infty} \left( \tilde{k}_n(0) - \frac{1}{L} \int_0^L v|J_n(x,v)| \, dv \, dx \right) = 0.
\]
Notice that \( \lim_{n \to +\infty} \int_0^L v|f_n(x,v)| \, dv \, dx = \int_0^L \int \nu v|M(x)\rho(x)| \, dv \, dx = 0 \) and thus we deduce that
\[
0 = \lim_{n \to +\infty} \tilde{k}_n(0) = \int_{v>0} v^2 g_0(v) \, dv - \lim_{n \to +\infty} \int_{v<0} v^2 f_n(0,v) \, dv.
\]
Finally we obtain that \( \lim_{n \to +\infty} k_n(0) = 2 \int_{v>0} v^2 g_0(v) \, dv \). Therefore we have proved that for any \( L > 0 \) the limit functions \( \rho = \rho_L, \ E = E_L \) satisfy
\[
\rho_L(x) = \frac{d}{dx} E_L = \frac{1}{2} E_L(x)^2 + (L - x)J_L + C_L, \quad x \in ]0,L[,
\]
\[
E_L(L) = 0, \quad \frac{d}{dx} E_L(0) = 2 \int_{v>0} v^2 g_0(v) \, dv,
\]
where the constants \( J_L, C_L \) verify (??), (??). We denote by \( \tilde{E}_L \) the functions \( \tilde{E}_L(x) = E_L(x), \ x \in [0,L] \) and \( \tilde{E}_L(x) = 0, \ x > L \). Since \( \tilde{E}_L(L) > 0, \ (\frac{d}{dx} \tilde{E}_L)_{L>0}, \ (\frac{d^2}{dx^2} \tilde{E}_L)_{L>0} \) are uniformly bounded we can extract a sequence \( (L_p)_p \) diverging towards \( +\infty \) such that
\[
\lim_{p \to +\infty} (\tilde{E}_p(x), \frac{d}{dx} \tilde{E}_p(x)) = (E, \frac{d}{dx} E) \quad \text{uniformly on compact sets of } [0, +\infty[\text{ where } E_p := E_{L_p}, \ \forall \ p. \ Take \ \theta \in C^1_c([0,+\infty[. For \ p \ large \ enough \ we \ have
\[
-\theta(0) \tilde{E}_p(0) - \int_0^{+\infty} \theta'(x) \tilde{E}_p(x) \, dx = \frac{1}{2} \int_0^{+\infty} \theta(x) \tilde{E}_p(x)^2 \, dx
\]
\[
+ \int_0^R ((L_p - x)J_{L_p} + C_{L_p}) \theta(x) \, dx,
\]
where \( R \) is a positive number such that \( \text{supp } \theta \subset [0,R] \). By passing to the limit with respect to \( p \to +\infty \) one gets
\[
-\theta(0) E(0) - \int_0^{+\infty} \theta'(x) E(x) \, dx = \frac{1}{2} \int_0^{+\infty} \theta(x) E(x)^2 \, dx,
\]
saying that
\[
\frac{d}{dx} E = \frac{1}{2} E(x)^2, \quad x > 0.
\]
Observing also that \( E'(0) = \lim_{p \to +\infty} \tilde{E}_p'(0) = 2 \int_{v>0} v^2 g_0(v) \, dv \) we deduce that
\[
E(x) = \frac{2E(0)}{2 - x E(0)}, \quad \rho(x) = \frac{2E(0)^2}{(2 - x E(0))^2}, \quad \phi(x) = 2 \ln \left( 1 - x \frac{E(0)}{2} \right),
\]
where \( E(0) = -2 \left( \int_{v>0} v^2 g_0(v) \, dv \right)^{1/2} \).
References


Let me go back to the existence proof for the eq.

\[v \partial_x f + E \partial_v f = \frac{1}{\tau} < f > M - f\]

on \((0, L) \times \mathbb{R}\). For further purpose, let me consider the Maxwellian \(M\) with temperature \(\theta\). Our proof works with an iterative argument looking at

\[\frac{1}{\tau} f_{n+1} + v \partial_x f_{n+1} + E \partial_v f_{n+1} = \frac{1}{\tau} < f_n > M\]

with incoming boundary conditions \(g_0, g_L\). Multiplying by \(v_\pm\) we obtain an estimate on the energy (see our first draft)

\[0 \leq \int_{\mathbb{R}} v^2 f_{n+1} \, dv \leq \int_{-\infty}^{0} v^2 g_0 \, dv + \int_{0}^{\infty} v^2 g_L \, dv + (\|E\|_\infty + C(\theta)) \int_{0}^{L} \int_{\mathbb{R}} f_n \, dv \, dx\]

Using the \(L^1\) estimate already obtained, we conclude that the kinetic energy is bounded. We denote by \(C\) the corresponding bound that depends on \(L, \tau, \theta, E\) and the (weighted) \(L^1\) norm of the data.

Then, we compute

\[< f_{n+1} > (x) = \int_{\mathbb{R}} f_{n+1}(x,v) \, dv = \int_{|v| \leq R} \ldots dv + \int_{|v| \geq R} \ldots dv \leq 2R||f_{n+1}||_\infty + \frac{1}{R^2} \int_{\mathbb{R}} v^2 f_{n+1} \, dv.\]

Optimize wrt \(R\):

\[< f_{n+1} > (x) \leq 3C^{2/3}||f_{n+1}||_\infty^{1/3}.\]

Now, the formula along the characteristics yields

\[f_{n+1}(x,v) \leq ||f_{inc}||_\infty e^{-t_{\text{out}}(x,v)/\tau} + || < f_n > (x) ||_\infty \frac{1}{\sqrt{2\pi\theta}} (1 - e^{-t_{\text{out}}(x,v)/\tau}).\]

Therefore if the induction hypothesis \(\frac{1}{\sqrt{2\pi\theta}} || < f_{n_1} > (x) ||_\infty \leq K = ||f_{inc}||_\infty\) then we also have \(f_{n+1} \leq K\). Hence the question is: Is the set \(\{0 \leq \rho \leq \sqrt{2\pi\theta} K\}\) left invariant by \(\rho_n \mapsto \rho_{n+1}\)?
Owing to the previous estimates the question can be recast as finding $K > 0$ such that $3C^{2/3}K^{1/3} \leq \sqrt{2\pi\theta} K$ that is

$$27C/(2\pi\theta)^{3/2} \leq K.$$ 

This condition is a bit bizarre since it relates the $L^\infty$ norm (that is $K$) to the (weighted) $L^1$ norm of the data. (Actually, we pick $K \geq \|f_{inc}\|_{\infty}$ verifying the condition.)

To conclude: if this last condition is satisfied, then the solution is bounded, and its $v$-average too.