Arch. Rational Mech. Anal. 156 (2001) 183–203
Digital Object Identifier (DOI) 10.1007/s002050100114

Positive Solutions to Singular Second and Third Order Differential Equations for Quantum Fluids

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Communicated by C. M. Dafermos

Abstract

We analyze a quantum trajectory model given by a steady-state hydrodynamic system for quantum fluids with positive constant temperature in bounded domains for arbitrary large data. The momentum equation can be written as a dispersive third-order equation for the particle density where viscous effects are incorporated. The phenomena that admit positivity of the solutions are studied. The cases, one space dimensional dispersive or non-dispersive, viscous or non-viscous, are thoroughly analyzed with respect to positivity and existence or non-existence of solutions, all depending on the constitutive relation for the pressure law. We distinguish between isothermal (linear) and isentropic (power law) pressure functions of the density. It is proved that in the dispersive, non-viscous model, a classical positive solution only exists for "small" (positive) particle current densities, both for the isentropic and isothermal case. Uniqueness is also shown in the isentropic subsonic case, when the pressure law is strictly convex. However, we prove that no weak isentropic solution can exist for "large" current densities. The dispersive, viscous problem admits a classical positive solution for all current densities, both for the isentropic and isothermal case, with an "ultra-diffusion" condition.

The proofs are based on a reformulation of the equations as a singular elliptic second-order problem and on a variant of the Stampacchia truncation technique. Some of the results are extended to general third-order equations in any space dimension.

1. Introduction

The present paper is concerned with hydrodynamic models for quantum fluids in bounded domains. The evolution of the quantum fluid is governed by the conservation laws of mass and momentum for the particle density n and the particle current density J:

$$\frac{\partial n}{\partial t} + \operatorname{div} J = 0, \tag{1.1}$$

$$\frac{\partial J}{\partial t} + \operatorname{div}\Big(\frac{J \otimes J}{n} + P\Big) - nF = W - \nu B, \tag{1.2}$$

where $P = (P_{ik})$ denotes the pressure term, F the sum of the (external) forces, W the momentum relaxation term, and νB the viscous term with viscosity $\nu \geqq 0$. The tensor product $J \otimes J$ is given by the components $J_i J_k$ with $i, k = 1, \dots, d$.

We consider an isothermal or isentropic quantum fluid of charged particles. Then, the pressure tensor is assumed to be of the form $P = (Tp(n)\delta_{ik})$ where δ_{ik} is the Kronecker symbol. The pressure function p is given by the relation $p(n) = n$ in the *isothermal* case and $p(n) = n^\alpha$ with $\alpha > 1$ in the *isentropic* case, and T is a (scaled) temperature constant. We assume that the force F is the gradient of the sum of the electrostatic potential V and the quantum Bohm potential

$$Q = \delta^2 \frac{1}{\sqrt{n}} \Delta\sqrt{n},$$

$\delta > 0$ being the scaled Planck constant. Equations (1.1), (1.2) are coupled to Poisson's equation for the electrostatic potential,

$$\lambda^2 \Delta V = n - C, \tag{1.3}$$

where $\lambda > 0$ denotes the scaled Debye length, and $C = C(x)$ models fixed charged background ions. The relaxation term is given by $W = -J/\tau$, $\tau = \tau(x) > 0$ being the relaxation time. The choice of the viscous term νB will be defined below.

With these assumptions, the stationary quantum hydrodynamic equations with viscosity can be formulated as

$$\operatorname{div} J = 0, \tag{1.4}$$

$$\operatorname{div}\Big(\frac{J \otimes J}{n}\Big) + T\nabla p(n) - n\nabla V - \delta^2 n \nabla\Big(\frac{\Delta\sqrt{n}}{\sqrt{n}}\Big) = -\frac{J}{\tau} - \nu B. \tag{1.5}$$

The equations (1.3)–(1.5) are solved in a bounded domain $\Omega \subset \mathbb{R}^d$ ($d \geqq 1$) occupied by the fluid. Equations (1.4), (1.5) are in scaled form; we refer to [19] for the choice of the scaling.

In the case $\delta = 0$ and $\nu = 0$, we get the classical hydrodynamic equations which are considered, for instance, by DEGOND & MARKOWICH [6] and MARKOWICH [26] in two and three space dimensions for subsonic flow, and by GAMBA [8] in one space dimension for transonic flow. The two-dimensional viscous hydrodynamic equations $\delta = 0$ and $\nu > 0$ are studied, in the potential flow formulation, by GAMBA & MORAWETZ [9,12].

The quantum hydrodynamic equations for $\delta > 0$ and $\nu = 0$ arise in semiconductor modeling where it has been used for analyzing the flow of electrons in quantum semiconductor devices, like resonant tunneling diode models of GARDNER & RINGHOFER [13,14]. Recent quantum chemistry calculations using Quantum Trajectories Methods of LOPREORE & WYATT [25] have been proposed to study resonant scattering with one-dimensional double barrier potentials [27] in order to obtain properties of transmission probabilities. In addition, these quantum trajectory models have been used in the modeling of collinear chemical reactions [31] and in models for photo-dissociation of molecules by SALES-MAYOR *et al.* [29]. Very similar model equations have been employed in other areas of physics, e.g., in superfluidity [24] and in superconductivity [7].

We refer to [5, 13–16] for a justification and derivation of the quantum hydrodynamic equations.

Mathematically, these models have been studied by JEROME & ZHANG [32] and GYI & JÜNGEL [18] in one space dimension and by JÜNGEL [20] in several space dimensions. The existence of strong solutions of the boundary-value problem (with Dirichlet and/or Neumann boundary conditions) could be proved under a smallness condition on the data (current density or applied voltage) which corresponds essentially to a subsonic condition of the underlying classical hydrodynamic problem. However, no results are available for "large" data.

Viscous or diffusive terms in the quantum hydrodynamic equations are recently derived by ARNOLD *et al.* [1] from the Wigner-Fokker-Planck equation via a moment method, and by GARDNER & RINGHOFER [14] from a Wigner-relaxation model, via a Chapman-Enskog expansion method based on scaling arguments.

Our goal is to show that if a special class of viscosities is considered, a positive solution of the problem exists for all current densities.

Consider the one-dimensional equations (1.3)–(1.5) with the viscous term $B = n(\beta(n))_{xx}$:

$$\left(\frac{J^2}{n} + Tp(n)\right)_x - nV_x - \delta^2 n\left(\frac{(\sqrt{n})_{xx}}{\sqrt{n}}\right)_x = -\frac{J}{\tau(x)} - \nu n(\beta(n))_{xx}, \tag{1.6}$$

$$\lambda^2 V_{xx} = n - C(x), \tag{1.7}$$

in the interval $\Omega = (0, 1)$ for prescribed $J > 0$, subject to the boundary conditions

$$n(0) = n_0, \quad n(1) = n_1, \quad V(0) = V_0, \quad V_x(0) = -E_0, \tag{1.8}$$

$$\delta^2 \frac{(\sqrt{n})_{xx}(0)}{\sqrt{n_0}} - \nu\beta'(n_0)n_x(0) = \frac{J^2}{2n_0^2} + Th(n_0) - V_0 + K, \tag{1.9}$$

where $h(s)$ is the *enthalpy* function defined by $h'(s) = p'(s)/s$, $s > 0$, and $h(1) = 0$, and $K > 0$ is a constant whose value is given below (see (1.14)). In the isothermal case, the enthalpy is $h(s) = \log(s)$ $(s \geqq 0)$; in the isentropic case one gets $h(s) = \frac{\alpha}{\alpha-1}(s^{\alpha-1} - 1)$ $(s > 0)$, with $\alpha > 1$.

Notice that we need three boundary conditions for n, since (1.6) is of third order. We do not prescribe the potential V at both $x = 0$ and $x = 1$ (but only at $x = 0$), since the current density is given. Then, we can compute the applied voltage $V(1) - V(0)$ from the solution of the differential equations. This yields a well-defined current-voltage characteristic. The third condition (1.9) can be interpreted as a boundary condition for the quantum Fermi potential (or quantum velocity potential; cf. [20]).

One of the main assumptions of this paper is that we choose the viscous term as follows:

$$\beta(n) = -\frac{1}{\gamma-1}\frac{1}{n^{(\gamma-1)/2}} \quad \text{with } \gamma > 4. \tag{1.10}$$

This assumption is made by enforcing the minimal growth condition to the viscous term so that the system admits positive solutions for all values of the current densities. Thus, the main objective of this paper is to understand the phenomena that admit existence and positivity of the solutions to the *third-order* boundary value problem (1.6)–(1.9) for $\nu = 0$ and $\nu > 0$.

It is well known that the positivity of solutions to higher-order equations is a delicate problem since maximum principle arguments generally do not apply. We refer to [2, 3, 21, 28] for recent studies of the positivity of solutions to *fourth-order* elliptic equations where similar difficulties arise. It turns out that for the problem (1.6)–(1.9), the *ultra-diffusive* term given by β prevents the solution from cavitating and provides the growth necessary to obtain a uniform bound in the dispersive parameter δ. In this sense, this term corresponds to the *hyper-diffusive* corrections sometimes used for stable numerical approximations to the Navier-Stokes equations.

More precisely, let us consider the following cases:

Case $\delta = 0$, $\nu > 0$. This problem has been studied by GAMBA [8] with $B = n_{xx}$. It is shown that for isentropic pressure functions p, there exists a positive solution to the one-dimensional equations (1.6)–(1.8). Moreover, the lower and upper bounds for this solution do not depend on the viscosity ν, and it is possible to perform rigorously the limit $\nu \to 0$ and to obtain entropic inviscid solutions.

Case $\delta > 0$, $\nu > 0$. Consider isentropic or isothermal pressure functions. We prove that for any given $J > 0$, there exists a classical solution (n, V) to (1.6)–(1.9) (Theorem 2.1). Furthermore, the particle density satisfies

$$0 < m(\nu) \leqq n(x) \leqq M \quad \text{for all } x \in \Omega. \tag{1.11}$$

The constants $m(\nu)$ and M do not depend on $\delta > 0$, but $m(\nu)$ depends on ν such that $m(\nu) \to 0$ as $\nu \to 0$. In order to get an explicit positive lower bound for n, the *ultra-diffusive* term $\nu n \beta(n)_{xx}$ is necessary. This term is used to control the convective term J^2/n.

Case $\delta > 0$, $\nu = 0$. Without viscosity, we only get the existence of "subsonic" solutions. We call a solution (n, V) to (1.6)–(1.8) "subsonic" if the density n is positive and satisfies the condition

$$J/n < \sqrt{Tp'(n)} \qquad \text{in } \Omega.$$

More precisely, for isothermal pressure functions or sufficiently large $E_0 > 0$ in the case of isentropic pressure, there exists a constant $J_0 > 0$ such that for all $0 < J \leqq J_0$ there exists a classical solution (n, V) to (1.6)–(1.9) with positive lower and upper bounds for n not depending on δ (Corollary 4.1). This solution is "subsonic". Moreover, for isentropic pressure functions $p(n) = n^\alpha$ with $\alpha > 2$, there exists a constant $J_1 > 0$ such that for all $J \geqq J_1$ the problem (1.6)–(1.9) cannot have a weak solution with positive n (Corollary 4.1). Finally, we prove uniqueness for isentropic pressure functions $p(n) = n^\alpha$ with $\alpha \geqq 2$, and sufficiently large $T > 0$ and $E_0 > 0$, in the class of "subsonic" solutions (Theorem 5.1).

We remark that, once uniform δ-bounds are obtained, it is possible to study the asymptotic limits corresponding to small dispersion [10].

In the final section of this paper we show details how to extend our mathematical techniques to some general multi-dimensional quantum trajectory models involving third-order boundary value problems of the type

$$\begin{aligned} \nabla(A(u)\Delta u) = {}& \mu\nabla(F(u)) + \nabla(G(u)) \\ & + \nu\nabla(\tilde{B}(u)(1\cdot\nabla)u) \text{ in } \Omega \subset \mathbb{R}^d, \\ \mathcal{B}u = {}& u_0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.12}$$

where $(1\cdot\nabla)u = \sum_j \partial_j u$ and $\mu > 0$. Depending on the boundary operator $\mathcal{B}u = u_0$ and boundary geometry of the domain, it is possible to analyze the boundary value problem (1.12) for nonlinear functions A, $\tilde{B}$, F and G satisfying some growth conditions (see Section 6). This problem is singular since $F(u) = u^{-a}$ with $a > 0$ is admissible.

We notice that third-order equations are also used in the modeling of long water waves in channels with small depths (Korteweg-de-Vries equation; see, e.g., [22]) and of light waves guided in an optical fiber (third-order Schrödinger equation; see [23]). In fact, we can prove for this problem that in the case $\nu = 0$ there exists a solution for sufficiently small $\mu > 0$, whereas there is no solution for sufficiently large $\mu > 0$ (Theorem 6.2). If $\nu > 0$, then a solution exists for all $\mu > 0$ (Theorem 6.1).

Notice that the quantum hydrodynamic equation (1.5) can be written via the change of variable $u = \sqrt{n}$ in the form

$$\begin{aligned} \delta^2\nabla(u^{-1}\Delta u) = {}& \nabla\Big(\frac{|J|^2}{2u^4}\Big) + T\nabla h(u^2) - \nabla V + \frac{J}{\tau u^2} + \nu\frac{B}{u^2} \\ & + u^{-4}\big((\nabla\otimes J) - (J\otimes\nabla)\big)J \\ & - 2u^{-5}\big(|J|^2\nabla u - (J\cdot\nabla u)J\big). \end{aligned} \tag{1.13}$$

In this situation, the current density is generally not constant. Notice that the last two terms vanish in the one-dimensional case. The study of the above general third-order problem is an important step in the existence analysis of the *multi-dimensional* quantum hydrodynamic equations, to be studied in a separate project (see [11]).

For the proofs of the above results we combine the techniques developed in [8,12,18,20]. The main idea is to integrate (1.6) once in order to get an elliptic *singular* second-order problem for which comparison principle arguments apply. The explicit lower and upper bounds for the particle density are obtained by using a variant of the Stampacchia truncation method [30]. For the existence results we employ the Leray-Schauder fixed point theorem.

Next, in order to derive the system of second-order equations to be analyzed, we rewrite (1.6) as

$$n\left[\frac{J^2}{2n^2}+Th(n)-V-\delta^2\frac{(\sqrt{n})_{xx}}{\sqrt{n}}+J\int_0^x\frac{ds}{\tau n}+\nu\beta(n)_x\right]_x=0.$$

This implies, if $n>0$,

$$\frac{J^2}{2n^2}+Th(n)-V-\delta^2\frac{(\sqrt{n})_{xx}}{\sqrt{n}}+J\int_0^x\frac{ds}{\tau n}+\nu\beta(n)_x=-K,$$

where K is a constant which comes from (1.9). Observing that

$$\beta(n)=-n^{-(\gamma-1)/2}/(\gamma-1)$$

and setting $w=\sqrt{n}$ gives

$$\delta^2 w_{xx}=\frac{J^2}{2w^3}+Twh(w^2)-Vw+Kw+Jw\int_0^x\frac{ds}{\tau w^2}+\nu\frac{w_x}{w^\gamma}, \tag{1.14}$$

$$\lambda^2 V_{xx}=w^2-C(x). \tag{1.15}$$

These equations have to be solved in $\Omega=(0,1)$ subject to the boundary conditions

$$w(0)=w_0,\quad w(1)=w_1,\quad V(0)=V_0,\quad V_x(0)=-E_0, \tag{1.16}$$

where $w_0=\sqrt{n_0}$ and $w_1=\sqrt{n_1}$. For the constant K we choose

$$K\stackrel{\text{def}}{=}V_0+\max(-E_0,0)+\lambda^{-2}M^2,\ \text{where} \tag{1.17}$$

$$M\stackrel{\text{def}}{=}\max(w_0,w_1,M_0), \tag{1.18}$$

and M_0 is such that $h(M_0^2)\geqq 0$. The constant K is taken in such a way that $-V(x)+K\geqq 0$ holds (see Section 2). Notice that every solution (w,V) of (1.14)–(1.16) satisfying $w>0$ in Ω gives a solution (n,V) to (1.6), (1.7) subject to the boundary conditions (1.8), (1.9).

In this paper we impose the following assumptions:

(H1) $h\in C^1(0,\infty)$ and p' (defined by $p'(s)=sh'(s)$, $s>0$) are non-decreasing, and h satisfies

$$\lim_{s\to\infty}h(s)>0,\qquad \lim_{s\to 0+}h(s)<0,\qquad \lim_{s\to 0+}\sqrt{s}\,h(s)>-\infty. \tag{1.19}$$

The viscous term β is given by (1.10).

(H2) $C\in L^2(\Omega)$, $C\geqq 0$ in Ω; $\tau\in L^\infty(\Omega)$, $\tau(x)\geqq\tau_0>0$ in Ω.

(H3) $J,w_0,w_1,\delta,\lambda,T>0$; $\nu\geqq 0$; $V_0,E_0\in\mathbb{R}$.

The isothermal enthalpy ($h(s)=\log(s)$, $s>0$) and the isentropic enthalpies ($h(s)=\frac{\alpha}{\alpha-1}(s^{\alpha-1}-1)$, $s\geqq 0$, with $\alpha>1$) are included in (H1).

The outline of the paper is as follows. In Section 2 we prove the existence of solutions to (1.14)–(1.16) for $\nu>0$. Section 3 is devoted to the existence analysis for the case $\nu=0$. The non-existence of solutions to (1.14)–(1.16) for $\nu=0$ is shown in Section 4. In Section 5 the uniqueness of "subsonic" solutions is proved. Finally, in Section 6 we extend our methods to some third-order equations in several space dimensions.

2. Existence of dispersive and viscous solutions ($\delta > 0$ and $\nu > 0$) for all prescribed current J

In this section we prove the following theorem:

Theorem 2.1. *Let the hypotheses* (H1)–(H3) *hold and let* $\nu > 0$. *Then, for any* $J > 0$, *there exists a classical solution* $(w, V) \in (C^2(\overline{\Omega}))^2$ *to* (1.14)–(1.16) *satisfying*

$$0 < m(\nu) \leqq w(x) \leqq M, \quad \textit{for all } \; x \in \Omega.$$

Remark 2.1. The constant $m(\nu)$ is defined by

$$m(\nu) = \min(w_0, w_1, m_1, m_2),$$

where

$$\begin{aligned} h(4m_1^2) &\leqq 0, \\ m_2 &\leqq \left(\frac{1}{2^{\gamma+1}} \frac{\nu}{J^2/2 + J/\tau_0 + \max(0, K - k)} \right)^{1/(\gamma-4)}, \\ k &= V_0 - \max(E_0, 0) - \lambda^{-2} \|C\|_{L^1(\Omega)}. \end{aligned}$$

The constant M is defined in (1.18).

In order to prove Theorem 2.1, define the function

$$r(x) = \varepsilon(2 - x), \quad x \in [0, 1], \quad 0 < \varepsilon < \min(1, M/2), \tag{2.1}$$

and consider the truncated problem

$$\begin{aligned} \delta^2 w_{xx} &= \frac{J^2 w}{2t_\varepsilon(w)^4} + Twh(w^2) - Vw + Kw \\ &\quad + Jw \int_0^x \frac{ds}{\tau t_\varepsilon(w)^2} + \nu \frac{(t_r(w_M))_x w}{t_r(w_M)^{\gamma+1}}, \end{aligned} \tag{2.2}$$

$$\lambda^2 V_{xx} = w^2 - C(x) \;\text{ in }\; \Omega, \tag{2.3}$$

where $t_\varepsilon(w) = \max(\varepsilon, w)$ and $t_r(w_M) = \max\big(r(\cdot), \min(M, w(\cdot))\big)$. We can show the existence of solutions to (2.2), (2.3), (1.16) for any $J > 0$ and any $\nu \geqq 0$.

Proposition 2.1. *Let the assumptions* (H1)–(H3) *hold and let* $\nu \geqq 0$ *and* $\varepsilon > 0$. *Then, for any* $J > 0$, *there exists a solution* $(w, V) \in (H^2(\Omega))^2$ *to* (2.2), (2.3), (1.16) *satisfying* $0 \leqq w(x) \leqq M$ *in* Ω.

For the proof of Proposition 2.1, we consider the approximate problem

$$\begin{aligned} \delta^2 w_{xx} &= \frac{J^2 w^+}{2t_\varepsilon(w)^4} + Tw^+h(w^2) - Vw^+ + Kw^+ \\ &\quad + Jw^+ \int_0^x \frac{ds}{\tau t_\varepsilon(w)^2} + \nu \frac{(t_r(w_M))_x w^+}{t_r(w_M)^{\gamma+1}}, \end{aligned} \tag{2.4}$$

$$\lambda^2 V_{xx} = w_M^2 - C(x) \;\text{ in }\; \Omega, \tag{2.5}$$

where $w^+ = \max(0, w)$, $w_M = \min(M, w)$. Let (w, V) be a weak solution to (2.4), (2.5), (1.16). Then we have the following *a priori* estimates:

Lemma 2.1 (L^∞ estimates). *The following inequalities hold for all* $x \in \Omega$*:*

$$0 \leqq w(x) \leqq M, \qquad k \leqq V(x) \leqq K, \tag{2.6}$$

where

$$k = V_0 - \max(E_0, 0) - \lambda^{-2}\|C\|_{L^1(\Omega)}.$$

Proof. The problem (2.5), (1.16) is equivalent to

$$V(x) = V_0 - E_0 x + \lambda^{-2} \int_0^x \int_0^y (w(z)_M^2 - C(z))\,dz\,dy, \tag{2.7}$$

for $x \in [0, 1]$, which implies

$$V_0 - \max(E_0, 0) - \lambda^{-2}\|C\|_{L^1(\Omega)} \leqq V(x) \leqq V_0 + \max(-E_0, 0) + \lambda^{-2} M^2.$$

This shows the second chain of inequalities of (2.6).

Using $w^- = \min(0, w)$ as test function in (2.4), we get immediately $w \geqq 0$ in Ω. Finally, with the test function $(w - M)^+ = \max(0, w - M)$ in (2.4), we obtain

$$\begin{aligned}\int_\Omega \big((w - M)_x^+\big)^2 dx \leqq \int_\Omega (w - M)^+ w \Big[-\frac{J^2}{2t_\varepsilon(w)^4} - Th(M^2) + (V - K) \\ - J \int_0^x \frac{ds}{\tau t_\varepsilon(w)^2} \Big] dx \\ - \nu \int_\Omega \frac{(t_r(w_M))_x w (w - M)^+}{t_r(w_M)^{\gamma+1}}\,dx,\end{aligned}$$

taking into account the monotonicity of h. The first integral on the right-hand side is non-positive since $V \leqq K$ and $h(M^2) \geqq h(M_0^2) \geqq 0$. Therefore

$$\int_\Omega \big((w - M)_x^+\big)^2 dx \leqq -\nu \int_\Omega \frac{(t_r(w_M))_x w (w - M)^+}{t_r(w_M)^{\gamma+1}} dx \leqq 0.$$

Hence $w \leqq M$ in Ω.

Lemma 2.2 (H^1 estimates). *There exist constants* $c_1, c_2 > 0$ *depending only on given data and on* δ, ε *and* M *(but not on* w *and* V*) such that*

$$\|w\|_{H^1(\Omega)} \leqq c_1, \quad \|V\|_{H^1(\Omega)} \leqq c_2.$$

Proof. The second assertion follows from Lemma 2.1 and

$$V_x(x) = -E_0 + \lambda^{-2} \int_0^x (w(y)_M^2 - C(y))\,dy.$$

The first assertion follows from Lemma 2.1 and (2.4), after employing the test function $w - w_D$, where $w_D(x) = (1-x)w_0 + xw_1$. Indeed, we have

$$\delta^2 \int_\Omega w_x^2\,dx = \delta^2 \int_\Omega w_x w_{Dx}\,dx - \nu \int_\Omega \frac{(t_r(w_M))_x w(w-w_D)}{t_r(w_M)^{\gamma+1}}\,dx - \int_\Omega (w-w_D)w\left[\frac{J^2}{2t_\varepsilon(w)^4} + Th(w^2) - V + K + J\int_0^x \frac{ds}{\tau t_\varepsilon(w)^2}\right]dx.$$

Using Young's inequality for the first two integrals on the right-hand side and Lemma 2.1 for the last integral, we easily get

$$\frac{\delta^2}{2}\int_\Omega w_x^2 \leqq c.$$

Lemma 2.3 (H^2 estimate). *There exists a constant* $c_3 > 0$ *not depending on* w *such that*

$$\|w\|_{H^2(\Omega)} \leqq c_3.$$

Proof. The lemma follows immediately from (2.4), Lemma 2.2 and the embedding $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$.

Proof of Proposition 2.1. We apply the Leray-Schauder fixed point theorem (see, e.g., [17]). Let $u \in H^1(\Omega)$ and let $V \in H^2(\Omega)$ be the unique solution of

$$\lambda^2 V_{xx} = u_M^2 - C(x) \quad \text{in } \Omega, \qquad V(0) = V_0,\ V_x(0) = -E_0.$$

Let $w \in H^2(\Omega)$ be the unique solution of

$$\delta^2 w_{xx} = \sigma\left[\frac{J^2 u^+}{2t_\varepsilon(u)^4} + Tu^+h(u^2) - Vu^+ + Ku^+ + Ju^+\int_0^x \frac{ds}{\tau t_\varepsilon(u)^2}\right] + \sigma\nu\frac{(t_r(u_M))_x u^+}{t_r(u_M)^{\gamma+1}},$$

$$w(0) = \sigma w_0,\ w(1) = \sigma w_1, \text{ where } \sigma \in [0,1].$$

This defines the fixed-point operator $S : H^1(\Omega)\times[0,1] \to H^1(\Omega)$, $(u,\sigma) \mapsto w$. It holds $S(u,0) = 0$ for all $u \in H^1(\Omega)$. Similarly as in the proofs of Lemmas 2.1–2.3 we can show that there exists a constant $c > 0$ independent of w and σ such that

$$\|w\|_{H^2(\Omega)} \leqq c$$

for all $w \in H^1(\Omega)$ satisfying $S(w,\sigma) = w$. Standard arguments show that S is continuous and compact, if we note that the embedding $H^2(\Omega) \hookrightarrow H^1(\Omega)$ is compact. Thus, the fixed-point theorem applies.

For the proof of Theorem 2.1 we only have to show that w is strictly positive in Ω.

Proof of Theorem 2.1. Taking $(w-r)^- \in H_0^1(\Omega)$ (see (2.1) for the definition of r) as test function in (2.4) gives:

$$
\begin{aligned}
\delta^2 &\int_\Omega \big((w-r)_x^-\big)^2\,dx \\
&= \int_\Omega (-(w-r)^-)w\bigg[\frac{J^2}{2t_\varepsilon(w)^4} + Th(w^2) - V + K \\
&\qquad + J\int_0^x \frac{ds}{\tau t_\varepsilon(w)^2} + \frac{\nu(t_r(w))_x}{t_r(w)^{\gamma+1}}\bigg]dx \\
&\leqq \int_\Omega (-(w-r)^-)w\bigg[\frac{J^2}{2\varepsilon^4} + Th(r^2) - k + K + \frac{J}{\tau_0\varepsilon^2} + \frac{\nu r_x}{r^{\gamma+1}}\bigg]dx \\
&\leqq \int_\Omega (-(w-r)^-)w\bigg[\frac{J^2}{2\varepsilon^4} + Th(r^2) - k + K + \frac{J}{\tau_0\varepsilon^2} - \frac{\nu\varepsilon}{(2\varepsilon)^{\gamma+1}}\bigg]dx,
\end{aligned}
$$

using the monotonicity of h and Lemma 2.1. Therefore

$$
\begin{aligned}
\delta^2 &\int_\Omega \big((w-r)_x^-\big)^2\,dx \\
&\leqq \int_\Omega (-(w-r)^-)w\bigg[\frac{J^2}{2\varepsilon^4} + Th(4\varepsilon^2) - k + K + \frac{J}{\tau_0\varepsilon^2} - \frac{\nu}{2^{\gamma+1}\varepsilon^\gamma}\bigg]dx.
\end{aligned}
$$

We claim that for sufficiently small $\varepsilon > 0$, the expression in the brackets is non-positive, which implies $w(x) \geqq r(x) \geqq \varepsilon > 0$ in Ω, i.e., we get the assertion of the theorem after taking $m(\nu) = \varepsilon$. Now choose $\varepsilon \in (0,1)$ such that (see (H1))

$$
\varepsilon \leqq \left(\frac{1}{2^{\gamma+1}}\,\frac{\nu}{J^2/2 + J/\tau_0 + \max(K-k,0)}\right)^{1/(\gamma-4)} \quad \text{and} \quad h(4\varepsilon^2) \leqq 0.
$$

Notice that $\gamma > 4$. Then, since $\varepsilon \leqq 1$,

$$
\begin{aligned}
&\frac{J^2}{2\varepsilon^4} + Th(4\varepsilon^2) + K - k + \frac{J}{\tau_0\varepsilon^2} - \frac{\nu}{2^{\gamma+1}\varepsilon^\gamma} \\
&\qquad \leqq \frac{1}{\varepsilon^4}\left(\frac{J^2}{2} + \frac{J}{\tau_0} + \max(0, K-k) - \frac{\nu}{2^{\gamma+1}\varepsilon^{\gamma-4}}\right) \\
&\qquad \leqq 0.
\end{aligned}
$$

3. Existence of subsonic, isothermal, dispersive solutions ($\delta > 0$ and $\nu = 0$)

In the case of vanishing viscosity we can only expect to show existence of solutions for sufficiently small $J > 0$, corresponding to a subsonic condition for the hydrodynamic equations ($\delta = 0$).

In this section we need the following assumption. Assume that there exists $m_0 > 0$ such that

$$\frac{1}{2}Tp'(m_0^2) + Th(m_0^2) + \frac{1}{\tau_0}\sqrt{Tp'(m_0^2)} + K - k \leqq 0. \tag{3.1}$$

Since $K - k = -E_0 + \lambda^{-2}(M^2 + \|C\|_{L^1(\Omega)})$, this assumption is satisfied if, for instance,

(i) $\lim_{s\to 0+} h(s) = -\infty$, or
(ii) $E_0 > 0$ is sufficiently large.

Indeed, in case (i), the condition (3.1) becomes true for sufficiently small $m_0 > 0$; we observe that p' is non-decreasing such that $p'(m_0^2)$ cannot be arbitrarily large near zero. For instance, the isothermal enthalpy $h(s) = \log(s)$ satisfies (i). In case (ii) we can choose, for instance, $m_0 > 0$ such that $h(m_0^2) \leqq 0$, and then take $E_0 > 0$ large enough such that

$$E_0 \geqq \frac{1}{2}Tp'(m_0^2) + \frac{1}{\tau_0}\sqrt{Tp'(m_0^2)} + \lambda^{-2}(M^2 + \|C\|_{L^1(\Omega)}). \tag{3.2}$$

Now, define

$$m = \min\{w_0, w_1, m_0\}. \tag{3.3}$$

Theorem 3.1. *Let the assumptions* (H1)–(H3) *and* (3.1) *hold and let* $\nu = 0$. *Furthermore, let* $J > 0$ *be such that*

$$J \leqq J_0 \overset{\text{def}}{=} m^2\sqrt{Tp'(m^2)}. \tag{3.4}$$

Then there exists a classical solution $(w, V) \in (C^2(\overline{\Omega}))^2$ *to* (1.14)–(1.16) *satisfying*

$$0 < m \leqq w(x) \leqq M \ \ in \ \ \Omega,$$

where m *and* M *are defined in* (3.3) *and* (1.18) *respectively.*

Remark 3.1. The condition (3.4) can be interpreted as a "subsonic condition" since it implies that the velocity J/n satisfies

$$\frac{J}{n} = \frac{J}{w^2} \leqq \frac{J}{m^2} \leqq \sqrt{Tp'(m^2)} \leqq \sqrt{Tp'(n)}.$$

Proof. By Proposition 2.1, there exists a solution (w, V) to the truncated problem (2.4), (2.5), (1.16) with $\varepsilon = m > 0$ (and $\nu = 0$). It remains to show that $w \geqq m$ in

Ω. Using $(w-m)^-$ as test function gives

$$
\begin{aligned}
&\delta^2 \int_\Omega \big((w-m)_x^-\big)^2\,dx \\
&\quad = \int_\Omega (-(w-m)^-)w\left[\frac{J^2}{2t_m(w)^4} + Th(w^2) - V + K + J\int_0^x \frac{ds}{\tau t_m(w)^2}\right]dx \\
&\quad \leqq \int_\Omega (-(w-m)^-)w\left[\frac{J_0^2}{2m^4} + Th(m^2) - k + K + \frac{J_0}{\tau_0 m^2}\right]dx \\
&\quad = \int_\Omega (-(w-m)^-)w\left[\frac{1}{2}Tp'(m^2) + Th(m^2) - k + K + \frac{1}{\tau_0}\sqrt{Tp'(m^2)}\right]dx \\
&\quad \leqq \int_\Omega (-(w-m)^-)w\left[\frac{1}{2}Tp'(m_0^2) + Th(m_0^2) - k + K + \frac{1}{\tau_0}\sqrt{Tp'(m_0^2)}\right]dx \\
&\quad \leqq 0,
\end{aligned}
$$

in view of the definition (3.1) of m_0. This implies that $w \geqq m$ in Ω.

Next, we prove that every weak solution is necessarily strictly positive:

Proposition 3.1. *Let $(w, V) \in (H^1(\Omega))^2$ be a weak solution to* (1.14)–(1.16) *with $\nu \geqq 0$. Then there exists $m > 0$ such that $w(x) \geqq m > 0$ for all $x \in \Omega$.*

Proof. By the definition of weak solution, each term of the right-hand side of (1.14) (with $\nu = 0$) belongs to $L^1(\Omega)$. In particular $1/w^3 \in L^1(\Omega)$, so that $w_{xx} \in L^1(\Omega)$, which implies $w \in W^{2,1}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$. Now, suppose that there exists $x_0 \in \Omega$ such that $w(x_0) = 0$. Then

$$
w(x) = (x - x_0)\int_0^1 w_x(\theta x + (1-\theta)x_0)\,d\theta, \qquad x \in \Omega,
$$

and

$$
\int_0^1 \frac{dx}{|w(x)|^3} \geqq \int_0^1 \left(\int_0^1 |w_x(s)|\,ds\right)^{-3} |x - x_0|^{-3}\,dx = \infty,
$$

contradicting the integrability of $1/w^3$. Hence $w > 0$ in Ω, and since w is continuous in $\overline{\Omega}$, there exists $m > 0$ such that $w(x) \geqq m$ for all $x \in \Omega$.

4. Non-existence of isentropic dispersive solutions ($\delta > 0$ and $\nu = 0$) for large data

We show that, under some condition on h, a weak solution to (1.14)–(1.16) cannot exist if $J > 0$ is large enough. For this, we have to prove that w is bounded from above independently of J.

Lemma 4.1. *Let (w, V) be any weak solution to* (1.14)–(1.16) *with $\nu = 0$ and with any $K \in \mathbb{R}$ and any $J > 0$. Furthermore, let*

$$
h(s) \geqq c_0(s^{\alpha-1} - 1) \ \ \textit{for} \ \ s \geqq 0, \quad \textit{with} \ \ \alpha > 2. \tag{4.1}
$$

Then there exists $M_1 > 0$, independent of J, such that $w(x) \leqq M_1$ for all $x \in \Omega$.

Remark 4.1. The bound M_1 does not depend on K if $K > 0$.

Proof. Take $(w-\Lambda)^+$ with $\Lambda \geqq \max(w_0, w_1)$ as test function in (1.11) to get

$$\begin{aligned}\delta^2 \int_\Omega (w-\Lambda)_x^{+2}\,dx &= \int_\Omega (w-\Lambda)^+ w\Big[-\frac{J^2}{2w^4} - J\int_0^x \frac{ds}{\tau w^2} - h(w^2) - K\Big]\,dx \\ &\quad + \int_\Omega (w-\Lambda)^+ wV\,dx.\end{aligned} \tag{4.2}$$

The main difficulty is to estimate the last integral. From (2.7) it follows that

$$V(x) \leqq V_0 + \max(-E_0, 0) + \lambda^{-2}\int_\Omega w^2\,dx.$$

Since

$$\begin{aligned} w^2 = (w+\Lambda)(w-\Lambda) + \Lambda^2 &\leqq (w+\Lambda)(w-\Lambda)^+ + \Lambda^2 \\ &\leqq 2w(w-\Lambda)^+ + \Lambda^2, \end{aligned}$$

we get

$$V(x) \leqq V_0 + \max(-E_0, 0) + 2\lambda^{-2}\int_\Omega w(w-\Lambda)^+\,dx + \Lambda^2\lambda^{-2}$$

and, setting $V_1 \stackrel{\text{def}}{=} V_0 + \max(-E_0, 0)$,

$$\begin{aligned}\int_\Omega (w-\Lambda)^+ wV\,dx &\leqq \int_\Omega (w-\Lambda)^+ w(V_1 + \Lambda^2\lambda^{-2})\,dx \\ &\quad + 2\lambda^{-2}\Big(\int_\Omega (w-\Lambda)^+ w\,dx\Big)^2 \\ &\leqq V_1\int_\Omega (w-\Lambda)^+ w\,dx + \lambda^{-2}\int_\Omega (w-\Lambda)^+ w^3\,dx \\ &\quad + 2\lambda^{-2}\int_\Omega (w-\Lambda)^{+2} w^2\,dx \\ &\leqq V_1\int_\Omega (w-\Lambda)^+ w\,dx + 3\lambda^{-2}\int_\Omega (w-\Lambda)^+ w^3\,dx,\end{aligned}$$

since $\Lambda \leqq w$ on $\{x : w(x) \geqq \Lambda\}$ and $(w-\Lambda)^+ \leqq w$. Therefore we get from (4.2)

$$\begin{aligned}\delta^2 \int_\Omega (w-\Lambda)_x^{+2}\,dx \\ \leqq \int_\Omega (w-\Lambda)^+ w[V_1 + 3\lambda^{-2}w^2 - K - c_0(w^{2\alpha-2} - 1)]\,dx.\end{aligned}$$

Since $2\alpha - 2 > 2$, there exists $M_1 > 0$ such that

$$c_0 M_1^{2\alpha-2} - 3\lambda^{-2}M_1^2 - V_1 + K - c_0 \geqq 0, \tag{4.3}$$

which implies, after taking $\Lambda = M_1$, that $w \leqq M_1$ in Ω.

Theorem 4.1. *Let* (H1)–(H3) *and* (4.1) *hold, let* $\nu = 0$, *and let* $K \in \mathbb{R}$ *be arbitrarily given. Then there exists* $J_1 > 0$ *such that, if* $J \geqq J_1$, *then the problem* (1.14)–(1.16) *cannot have a weak solution.*

Remark 4.2. The constant $J_1 > 0$ is defined by

$$\frac{J_1^2}{2M_1^3} = 8\delta^2(\max(w_0, w_1) + 1) - Th_0 + M_1(|V_0| + |E_0| + \lambda^{-2}M_1^2 - \min(0, K)), \tag{4.4}$$

where $h_0 = \inf\{sh(s^2) : 0 < s \leqq M_1\} > -\infty$ and $M_1 > 0$ is defined by (4.3).

Proof. Suppose that there exists a weak solution (w, V) to (1.14)–(1.16) for all $J > 0$ for some $K \in \mathbb{R}$. By Lemma 4.1, $w \leqq M_1$ in Ω. Moreover, by Proposition 3.1, $w > 0$ and $w \in H^1(\Omega) \hookrightarrow C^0(\overline{\Omega})$ and $w > 0$ in Ω. Therefore $w_{xx} \in C^0(\overline{\Omega})$.

Thus w is a classical solution. Let $J \geqq J_1$, where J_1 is defined in (4.4). Then

$$\begin{aligned}\delta^2 w_{xx} &\geqq \frac{J^2}{2M_1^3} + Th_0 - w(\sup_\Omega V - \min(0, K)) \\ &\geqq \frac{J_1^2}{2M_1^3} + Th_0 - M_1(|V_0| + |E_0| + \lambda^{-2}M_1^2 - \min(0, K)) \\ &= 8\delta^2(\max(w_0, w_1) + 1).\end{aligned}$$

Introduce $q(x) = 4(\max(w_0, w_1) + 1)(x - \frac{1}{2})^2 - 1$, $x \in [0, 1]$. Then

$$\begin{aligned} q(0) = q(1) &= \max(w_0, w_1), \\ q_{xx}(x) &= 8(\max(w_0, w_1) + 1), \end{aligned}$$

for $x \in \Omega$, which implies

$$w - q \leqq 0 \text{ on } \partial\Omega, \quad (w - q)_{xx} \geqq 0 \text{ in } \Omega.$$

The maximum principle gives $w - q \leqq 0$ in $\overline{\Omega}$. In particular, $w(\frac{1}{2}) \leqq q(\frac{1}{2}) = -1 < 0$, which contradicts the positivity of w.

Corollary 4.1. *Let* (H1)–(H3) *hold and let* $\nu = 0$.

(i) *Let* (3.1) *hold. Then there exists* $J_0 > 0$ *such that for all* $J \leqq J_0$ *there exists a solution* $(n, V) \in (C^3(\overline{\Omega}))^2$ *to* (1.6)–(1.9) *with strictly positive* n.
(ii) *Let* (4.1) *hold. Then there exists* $J_1 > 0$ *such that for all* $J \geqq J_1$, *the problem* (1.6)–(1.9) *cannot have a weak solution.*

Proof. The first part follows from Theorem 3.1 after setting $n = w^2$ and differentiating (1.11) with $\nu = 0$. The second part follows from Theorem 4.1 since the constant $K \in \mathbb{R}$ is fixed in (1.9). Thus the problems (1.6)–(1.9) and (1.14)–(1.16) are equivalent and the non-existence of solutions to (1.14)–(1.16) implies the non-existence of solutions to (1.6)–(1.9).

5. Uniqueness of subsonic, isentropic, dispersive solutions ($\delta > 0$ and $\nu = 0$)

We can prove the uniqueness of "subsonic" solutions to (1.6)–(1.9). Our main result is the following theorem.

Theorem 5.1. *Let* (H1)–(H3) *hold and let* $\nu = 0$ *and* $1/\tau \equiv 0$. *Moreover, let* h' *be non-decreasing and let* $\varepsilon > 0$. *Then there exist* $E_0 > 0$ *and* $T_0 > 0$ *such that for all* $T \geqq T_0$ *there is uniqueness of weak solutions to* (1.14)–(1.16) (*and to* (1.6)–(1.9)) *in the class of positive densities satisfying the "subsonic" condition*

$$J/w(x)^2 \leqq \sqrt{(1-\varepsilon)Tp'(w(x)^2)} \qquad \textit{for all } x \in \Omega. \tag{5.1}$$

For instance, isentropic enthalpies $h(s) = \frac{\alpha}{\alpha-1}(s^{\alpha-1} - 1)$ with $\alpha \geqq 2$ satisfy the condition on h' of Theorem 5.1.

Proof. Let $m_0 > 0$ be such that $h(m_0^2) \leqq 0$. Choose $T \geqq T_0$, where $T_0 > 0$ is defined below. Then let $E_0 > 0$ be chosen such that the inequality (3.2) holds. Then the assumptions of Theorem 3.1 are satisfied. Now let (w_1, V_1) be the solution to (1.14)–(1.16) constructed in Theorem 3.1, i.e., $w_1 \geqq m$ holds in Ω, and $m > 0$ does not depend on $T > 0$. Moreover, there exists $\varepsilon > 0$ such that (5.1) holds for w_1. Finally, let (w_2, V_2) be a second weak solution to (1.14)–(1.16) satisfying (5.1). Then w_2 is positive in Ω, by Proposition 3.1, and w_1, w_2 are classical solutions of (1.11).

Introduce $\chi(w) = J^2/2w^4 + (1-\varepsilon)Th(w^2)$. Then, proceeding similarly to [4], we write

$$-\delta^2\frac{w_{1,xx}}{w_1} + \delta^2\frac{w_{2,xx}}{w_2} = -\frac{\chi(w_1)}{w_1} + \frac{\chi(w_2)}{w_2} - \varepsilon T(h(w_1^2) - h(w_2^2)) + V_1 - V_2.$$

Multiplying this equation by $w_1^2 - w_2^2$, integrating over Ω and integrating by parts, we obtain, after elementary computations,

$$\begin{aligned} \int_\Omega (w_1^2 + w_2^2)(\ln w_1 - \ln w_2)_x^2\, dx \\ &= \int_\Omega \left[\left(w_{1,x} - \frac{w_1}{w_2}w_{2,x}\right)^2 + \left(w_{2,x} - \frac{w_2}{w_1}w_{1,x}\right)^2\right] dx \\ &= -\int_\Omega \left(\frac{\chi(w_1)}{w_1} - \frac{\chi(w_2)}{w_2}\right)(w_1^2 - w_2^2)\, dx \\ &\quad - \varepsilon T\int_\Omega (h(w_1^2) - h(w_2^2))(w_1^2 - w_2^2)\, dx \\ &\quad + \int_\Omega (V_1 - V_2)(w_1^2 - w_2^2)\, dx. \end{aligned} \tag{5.2}$$

We first estimate the second integral on the right-hand side, using the assumption that h' is non-decreasing:

$$\begin{aligned}
-\varepsilon T \int_\Omega (h(w_1^2) - h(w_2^2))(w_1^2 - w_2^2)\,dx \\
&= -\varepsilon T \int_\Omega \int_0^1 h'(\theta w_1^2 + (1-\theta)w_2^2)(w_1^2 - w_2^2)^2 d\theta\,dx \\
&\leqq -\varepsilon c T \int_\Omega (w_1^2 - w_2^2)^2\,dx,
\end{aligned}$$

where

$$c = \int_0^1 h'(\theta m^2)\,d\theta > 0$$

is independent of $T > 0$.

Now multiply the difference of (1.15) for V_1 and V_2 by $V_1 - V_2$, integrate over Ω and integrate by parts to get

$$\begin{aligned}
\int_\Omega (V_1 - V_2)(w_1^2 - w_2^2)\,dx \\
&= -\lambda^2 \int_\Omega (V_1 - V_2)_x^2\,dx + \lambda^2 (V_1 - V_2)(1)(V_1 - V_2)_x(1) \\
&\leqq -\lambda^2 \int_\Omega (V_1 - V_2)_x^2\,dx + \lambda^{-2}\Big(\int_0^1 (w_1^2 - w_2^2)\,dx\Big)^2 \\
&\leqq -\lambda^2 \int_\Omega (V_1 - V_2)_x^2\,dx + \lambda^{-2}\int_0^1 (w_1^2 - w_2^2)^2\,dx,
\end{aligned}$$

where we used the expression (2.6) for V_1 and V_2 and Jensen's inequality.

Finally, we estimate the first integral on the right-hand side of (5.2). The function $s \mapsto \chi(s)/s$ is non-decreasing if and only if

$$\frac{d}{ds}\frac{\chi(s)}{s} = -\frac{2J^2}{s^5} + 2(1-\varepsilon)Tsh'(s^2) = \frac{2}{s}\Big(-\frac{J^2}{s^4} + T(1-\varepsilon)p'(s^2)\Big) \geqq 0.$$

In view of condition (5.1) this implies that

$$-\int_\Omega \Big(\frac{\chi(w_1)}{w_1} - \frac{\chi(w_2)}{w_2}\Big)(w_1^2 - w_2^2)\,dx \leqq 0.$$

Hence, we obtain from (5.2)

$$\begin{aligned}
\int_\Omega (w_1^2 + w_2^2)(\ln w_1 - \ln w_2)_x^2\,dx \\
&\leqq -\lambda^2 \int_\Omega (V_1 - V_2)_x^2\,dx + (\lambda^{-2} - \varepsilon c T)\int_0^1 (w_1^2 - w_2^2)^2\,dx \\
&\leqq 0,
\end{aligned}$$

choosing $T \geqq T_0$, where $T_0 \overset{\text{def}}{=} 1/\lambda^2 \varepsilon c$. Hence $w_1 = w_2$ and $V_1 = V_2$ in Ω.

6. Third-order equations in several dimensions

In this section we show how the methods of the previous sections can be extended to a large class of third-order equations in any space dimension. This is the first important step for performing the existence analysis of the multi-dimensional quantum hydrodynamic equations. Since the proofs are similar to those of the previous sections, we only sketch the proofs. A rigorous proof of an existence result for the quantum trajectory models in 2 or 3 dimensions will require an additional detailed description of the flow domain and boundary conditions. Such study is part of a future project.

Let $\Omega \subset \mathbb{R}^d$ ($d \geqq 1$) be a bounded "channel like" domain and consider

$$\nabla(A(u)\Delta u) = \mu\nabla(F(u)) + \nabla(G(u)) + \nu\nabla(B(u)(1\cdot\nabla)u) \quad \text{in } \Omega, \tag{6.1}$$
$$\mathcal{B}u = u_D \quad \text{on } \partial\Omega, \tag{6.2}$$

where $\nu \geqq 0$ and $\mu > 0$. Recall that $(1\cdot\nabla)u = \sum_j \partial_j u$.

We must assume that the boundary conditions are such that the integration of (6.1) along integral curves is well defined; for instance, a two- or three-dimensional domain, homeomorphic to bounded cylindrical domains (as a section of a duct) with two opposite walls, denoted by $\mathcal{D}$, where Dirichlet data is prescribed for the density (as contact boundaries), and the remainder of the boundary, denoted by $\mathcal{N}$, are "insulating" walls corresponding to homogeneous Neumann conditions (as insulating boundaries). Such a two-dimensional configuration was applied and studied in the existence of positive densities for steady potential viscous fluid-Poisson systems in [12].

Therefore the boundary operator becomes $\mathcal{B}u = u\mid_{\mathcal{D}} \cup \nabla u\cdot\eta\mid_{\mathcal{N}}$, where η is the outer normal vector to the boundary of the domain $\partial\Omega = \mathcal{D}\cup\mathcal{N}$. Thus $u_D = f(x)$ prescribed in $\mathcal{D}$ and $u_D = 0$ in $\mathcal{N}$. We point out that if the boundary surfaces $\mathcal{D}$ and $\mathcal{N}$ intersect orthogonally, the standard reflection techniques for second-order equations, with the given boundary conditions, apply and yield optimal regularity for the solutions of the equation.

Now, integrating (6.1) along integral curves we obtain

$$\Delta u = \mu f(u) + g(u) + Ka(u) + \nu b(u)(1\cdot\nabla)u, \tag{6.3}$$

where $K \in \mathbb{R}$ is a constant and

$$f(u) = \frac{F(u)}{A(u)}, \quad g(u) = \frac{G(u)}{A(u)}, \quad a(u) = \frac{1}{A(u)}, \quad b(u) = \frac{B(u)}{A(u)}.$$

Any solution $u \in H^2(\Omega)$ to (6.3), (6.2) solves the problem (6.1), (6.2) and vice versa. For simplicity, we suppose that $\mathcal{D} = \partial\Omega$. We assume that

$$\partial\Omega \in C^{1,1},\ u_D \in H^2(\Omega)\cap L^\infty(\Omega)\ ;\ u_D \geqq u_0 > 0 \text{ on } \partial\Omega\ ; \tag{6.4}$$
$$a, b, f, g \in C(0,\infty),\ a, g \text{ are non-decreasing},\ a, b \text{ are non-negative}\ ; \tag{6.5}$$
$$g(0+) < 0\ ;\ g(+\infty) > 0\ ;\ \lim_{s\to 0+} a(s)/sb(s) = 0\ ; \tag{6.6}$$
$$\inf_{0<s<M} f(s) > 0 \quad \forall M > 0\ ;\ \lim_{s\to 0+} f(s)/sb(s) = 0. \tag{6.7}$$

Admissible functions are, for instance,

$$f(u) = u^{-\alpha}, \quad g(u) = u^{\beta} - 1, \quad a(u) = u^{\gamma}, \quad b(u) = u^{-\delta}$$

with $\alpha, \beta, \gamma > 0$, $\gamma > \delta - 1$, and $\delta > 1 + \alpha$.

Theorem 6.1. *Let $\nu > 0$ and $K > 0$. Then there exists a solution $u \in H^2(\Omega)$ to* (6.1), (6.2) *for all $\mu > 0$.*

Remark 6.1. The assumption $K > 0$ can be weakened by choosing appropriate assumptions on the functions a, f, and g.

Proof. Since Ω is bounded, there exists $R > 0$ such that Ω is contained in the ball $B_R(0)$ of radius R and center 0. Introduce the comparison function $\phi(x) = \phi(x_1, \dots, x_d) = \frac{\varepsilon}{R}(2R - x_1)$ and $0 < \varepsilon \leqq \min(u_0, M/3)$. Set $t_\phi(u_M) = \max(\phi(\cdot), \min(M, u(\cdot)))$ and consider the truncated problem

$$\Delta u = \mu \frac{f(t_\phi(u_M))}{t_\phi(u)} u^+ + \frac{g(t_\phi(u_M))}{t_\phi(u)} u^+ + K \frac{a(t_\phi(u_M))}{t_\phi(u_M)} u^+ + \nu \frac{b(t_\phi(u_M))}{t_\phi(u_M)} u^+ (1 \cdot \nabla)(t_\phi(u_M)) \quad \text{in } \Omega, \tag{6.8}$$

$$u = u_D \quad \text{on } \partial\Omega, \tag{6.9}$$

where $u^+ = \max(0, u)$. Using the methods of the proof of Proposition 2.1, we easily get the existence of a solution $u \in H^2(\Omega)$ to (6.8),(6.9) for any $\nu \geqq 0$ (here we use that $Ka(s) \geqq 0$). It remains to show that $\varepsilon \leqq u(x) \leqq M$ for $x \in \Omega$.

First we observe that by using u^- as test function in (6.8), we immediately conclude that $u(x) \geqq 0$ in Ω. In order to prove that $u \leqq M$ in Ω for some $M > 0$ we use $(u - M)^+$ with $M \geqq \|u_D\|_{L^\infty(\Omega)}$ as test function in (6.8). Since there exists a constant $M > 0$ such that $g(M) \geqq 0$, we can show as in the proof of Lemma 2.1 that $(u - M)^+ = 0$ in Ω.

For the lower bound we use $(u - \phi)^-$ as test function in (6.8), observing that $\varepsilon \leqq \phi(x) \leqq 3\varepsilon$ in Ω:

$$\begin{aligned}
&\int_\Omega |\nabla(u - \phi)^-|^2 \, dx \\
&\quad = \int_\Omega \big(-(u-\phi)^-\big) u \Big[\mu \frac{f(t_\phi(u))}{t_\phi(u)} + \frac{g(t_\phi(u))}{t_\phi(u)} + K \frac{a(t_\phi(u))}{t_\phi(u)} \\
&\qquad + \nu \frac{b(t_\phi(u))}{t_\phi(u)} (1 \cdot \nabla)(t_\phi(u))\Big] dx \\
&\quad = \int_\Omega \big(-(u-\phi)^-\big) u \Big[\mu \frac{f(\phi)}{\phi} + \frac{g(\phi)}{\phi} + K \frac{a(\phi)}{\phi} + \nu \frac{b(\phi)}{\phi} (1 \cdot \nabla)\phi\Big] dx \\
&\quad \leqq \int_\Omega \big(-(u-\phi)^-\big) u b(\phi) \Big[\mu \frac{f(\phi)}{\phi b(\phi)} + \frac{g(\phi)}{\phi b(\phi)} + \frac{Ka(\phi)}{\phi b(\phi)} - \frac{\nu}{3R}\Big] dx \\
&\quad \leqq 0,
\end{aligned}$$

by choosing ε small enough such that the inequalities $g(\phi) \leqq 0$, $f(\phi)/\phi b(\phi) \leqq \nu/6\mu R$, and $a(\phi)/\phi b(\phi) \leqq \nu/6KR$ are satisfied in Ω. This is possible in view of the conditions (6.6), (6.7). We conclude that $u \geqq \phi \geqq \varepsilon$ in Ω.

Theorem 6.2. *Let* $\nu = 0$.

(i) *Let* $g(0+) < 0$. *Furthermore, assume that* $K > 0$ *and* $\lim_{s\to 0+} a(s)/s = 0$. *Then there exists* $\mu_0 > 0$ *such that for all* $0 < \mu \leqq \mu_0$ *there exists a solution* $u \in H^2(\Omega)$ *to* (6.1), (6.2) *satisfying*

$$\Delta u = \mu f(u_D) + g(u_D) + Ka(u_D) \quad \text{on } \partial\Omega. \tag{6.10}$$

(ii) *Let* $g(0+) > -\infty$. *Then there exists* $\mu_1 > 0$ *such that for all* $\mu \geqq \mu_1$ *there is no weak solution to* (6.1), (6.2), (6.10).

Proof. For the first part of the theorem we only have to show that the solution u to (6.8), (6.9) is strictly positive for sufficiently small $\mu > 0$. Take $\varepsilon = m > 0$ and $\delta > 0$ such that $m \leqq u_0$, $g(\phi)/\phi \leqq -\delta < 0$, and $a(\phi)/\phi \leqq \delta/2K$ (if $K > 0$) in Ω. The existence of m and δ is ensured by condition (6.6) and the assumption on a. Let $f_m = \sup_{m<s<3m} f(s)/s$. Then $f_m > 0$ in view of assumption (6.7). Choose $0 < \mu_0 \leqq \delta/2f_m$ and let $0 < \mu \leqq \mu_0$. Using $(u-m)^-$ as test function in (6.7) yields

$$\begin{aligned}
&\int_\Omega |\nabla(u-m)^-|^2\,dx \\
&\quad = \int_\Omega \big(-(u-m)^-\big)u\Big[\mu\frac{f(t_\phi(u))}{t_\phi(u)} + \frac{g(t_\phi(u))}{t_\phi(u)} + K\frac{a(t_\phi(u))}{t_\phi(u)}\Big]dx \\
&\quad \leqq \int_\Omega \big(-(u-m)^-\big)u\Big[\mu_0\frac{f(\phi)}{\phi} + \frac{g(\phi)}{\phi} + K\frac{a(\phi)}{\phi}\Big]dx \\
&\quad \leqq \int_\Omega \big(-(u-m)^-\big)u\big[\mu_0 f_m - \delta/2\big]dx \\
&\quad \leqq 0.
\end{aligned}$$

Thus $u \geqq m > 0$ in Ω.

To prove the second part of the theorem, let $u \in H^1(\Omega)$ be a solution to (6.2), (6.3) with $u \leqq M$ in Ω and $K \in \mathbb{R}$. Using the positivity of f, we can easily see that the constant M does not depend on μ. Let $x_0 \in \Omega$. Since Ω is open, there exists a ball $B_r(x_0)$ of radius r with center x_0 contained in Ω. Set $f_0 = \inf\{f(s) : 0 < s < M\} > 0$ and $a_0 = \inf\{a(s) : 0 < s < M\} \geqq 0$ and choose $\mu_1 > 0$ such that $L \stackrel{\text{def}}{=} \mu_1 f_0 + g(0+) + \max(0, Ka_0) \geqq 2d(M+1)/r^2$ and $L \geqq M$. Notice that we assumed $g(0+) > -\infty$. We show that $u(x_0) < 0$ which is a contradiction to the non-negativity of u. The following holds (in the sense of distributions):

$$\begin{aligned}
\Delta u &\geqq \mu f_0 + g(0+) + \max(0, Ka_0) \geqq \mu_1 f_0 + g(0+) + \max(0, Ka_0) \\
&= L \ \text{ in } \ \Omega.
\end{aligned}$$

Now define

$$q(x) = \frac{L}{2d}|x - x_0|^2 - 1 \quad \text{for } x \in B_r(x_0).$$

Then

$$\Delta q = L \quad \text{in } B_r(x_0), \qquad q = r^2 L/2d - 1 \geqq 0 \quad \text{on } \partial B_r(x_0).$$

This implies

$$\Delta(u-q) \geqq 0 \quad \text{in } B_r(x_0), \quad u - q \leqq M - r^2 L/2d + 1 \leqq 0 \ \text{ on } \partial B_r(x_0).$$

By the maximum principle, we conclude that $u - q \leqq 0$ in $B_r(x_0)$. In particular, $u(x_0) \leqq q(x_0) = -1 < 0$, which is a contradiction.

Acknowledgements. We acknowledge support from the DAAD-NSF Program. IRENE GAMBA is supported by NSF under grant DMS 9971779 and by TARP under grant 003658-0459-1999. ANSGAR JÜNGEL was partially supported by the Gerhard-Hess Program of the Deutsche Forschungsgemeinschaft, grant number JU 359/3-1, by the TMR Project "Asymptotic Methods in Kinetic Theory", grant number ERB FMBX CT97 0157, and by the Erwin-Schrödinger-Institut in Vienna (Austria) where a part of this research was carried out.

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(*Accepted July 1, 2000*)
Published online February 14, 2001 –