Sharp Uniform Bounds for Steady Potential Fluid-Poisson Systems

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Abstract. We consider steady potential hydrodynamic-Poisson system with a dissipation term (viscosity) proportional to a small parameter ν in a two or three dimensional bounded domain. We show here that for any smooth solution of a boundary value problem which satisfies that the speed, denoted by $|\nabla\varphi^{\nu}|$, has an upper coarse bound $\widetilde{K}$, uniform in the parameter ν, then a sharper, correct uniform bound is obtained: the viscous speed $|\nabla\varphi^{\nu}|$ is bounded pointwise, at points x_0 in the interior of the flow domain, by cavitation speed (given by Bernoulli's law at vacuum states) plus a term of $\mathcal{O}(\nu^{\beta})\cdot \operatorname{dist}^{-2}\{x_0,\partial\Omega\}$ that depends on $\widetilde{K}$. The exponent is $\beta=1$ for the standard isentropic gas flow model and $\beta=\frac{1}{2}$ for the potential hydrodynamic Poisson system. Both cases are considered to have a $\gamma-$ pressure law with $1<\gamma<2$ in two space dimensions and $1<\gamma<\frac{3}{2}$ in three space dimensions.

These systems have cavitation speeds which take not necessarily constant values. In fact, for the potential hydrodynamic–Poisson systems, cavitation speed is a function that depends on the potential flow function and on the electric potential as well.

In addition, we consider a two dimensional boundary value problem which has been proved to have a smooth solution whose speed is uniformly bounded. In this case we show that the pointwise sharper bound can be extended to the section of the boundary $\partial\Omega\setminus\partial_3\Omega$, where $\partial_3\Omega$ called the outflow boundary. The exponent β varies between 1 and 1/8 depending on the location of x_0 at the boundary and on the curvature of the boundary at x_0.

In particular our estimates apply to classical viscous approximation to transonic flow models (see [CF], [M1], [Se] and [Sy]).

0. Introduction.

The present paper deals with steady two and three dimensional fluid level model that is an approximation to the equations of inviscid potential flow that changes type, i.e. equations that admits regions of ellipticity and hyperbolicity.

These models also appear in higher hierarchies of macroscopic approximation of particle–charged systems in the modeling of electron–ion plasmas and semiconductor devices where the transport is induced by the superposition of an internal and an externally applied electric field.

The resulting macroscopic approximation yields a fluid level equation coupled with a Poisson equation for the corresponding electric potential. See, for instance, Anile and Pennisi [AP], Azoff [Az], Baccarani and Woderman [BW], Blotekjaer [Bo], Jerome [J],

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Markowich, Ringhofer and Schmeiser [MR], Poupaud [Pp], on justifications for these models.

If no electric field is present, the system reduces to a two-dimensional steady irrotational compressible viscous flow model in a "channel". This is just a special case of the fluid-Poisson systems that yields a model viscous approximations to transonic flow in a channel. Classical references on this model can be found in Courant-Friedricks [CF], Morawetz [M1], Serrin [Se] and Synge [Sy].

We recall that the transonic flow model is given by

$$\operatorname{div}(\rho\nabla\varphi) = 0, \qquad \frac{1}{2}|\nabla\varphi|^2 + i(\rho) = K \tag{0.1}$$

where φ is the potential flow function, $\nabla\varphi$ associated velocity field, and $i(\rho)$ in the Bernoulli equation, represents the enthalpy function and it is usually a power law for the density ρ that satisfies $i(\rho), i'(\rho) > 0$. The constant K, the Bernoulli's constant, needs to be positive.

Existence of physical meaningful solutions to system (0.1) remains an unsolved problem. It is not even known what is the domain and boundary value problem that would yield an entropic weak solution.

Hence, in an attempt to construct entropic solutions to system (0.1), viscous approximation models are usually considered, with the hope that they can be solved and have enough uniform bounds in the viscosity parameter in order to obtain compactness res ults that would yield the existence of entropic solutions to the inviscid system (0.1) by vanishing viscosity methods.

An very good approximation to transonic flow models (that is, steady potential flow) is given by

$$\operatorname{div}(\rho\nabla\varphi) = 0 \tag{0.2}$$

$$\frac{1}{2}|\nabla\varphi|^2 - (K - i(\rho)) = \nu\, g(|\nabla\varphi|)\, \Delta\varphi. \tag{0.3}$$

Equation (0.3) gives an streamline approximation of Bernoulli's law. See as references for this viscous formulations and its justifications Courant and Friedrichs [CF], Serrin [Sn], Synge [Sy], Morawetz [M2]. For a survey on numerical simulations for transonic flow and approximations see Jameson [Ja] and references therein.

In a recent work in collaboration with C.S. Morawetz [GM], we posed and solved a boundary value problem to a class of potential fluid-Poisson systems, which includes system (0.2)-(0.3). There we show the existence of smooth strong solutions that ha ve uniform bounds in the viscosity parameter. More precisely, we show that there is a one parameter family of solutions (φ^ν, ρ^ν) which are infinitely differentiable in the flow domain, with ρ^ν and $|\nabla\varphi^\nu|$ uniformly bounded in ν, and strictly positive for ν fixed.

This is a first and fundamental step in order to achieve a convergence in ν result that would yield a weak solution for the inviscid problem, as the one in (0.1) in standard gas dynamics or the larger class of problems for inviscid hydrodynamic–Poisson systems.

In the case of 2–dimensional transonic flow, convergence of strong solutions of system (0.2)–(0.3) to weak solutions of (0.1) have been outlined by Morawetz in [M1], [M2] using methods of compensated compactness presented by Murat [Mu], Tartar [Tt] and Di Perna [DP], G.Q.Chen [CG], for the initial value problem for the one-dimensional time dependent compressible fluid system of two equations, as in 1–dimensional isentropic gas dynamics with a power pressure law. This problem has been solved by means of artificial viscosity and perturbations, for any positive initial density; see Lions, Perthame, Tadmor [LP] and Lions, Perthame and Souganidis [LS].

All these methods require uniform estimates in the parameters of the approximation. However, the bound for $|\nabla\varphi^\nu|$ obtained in [GM] is too coarse if the domain of the flow is not a rectangle.

Clearly, if we pursue a weak solution (φ^0,ρ^0) of problem (0.1) which is a strong limit of some subsequence of solutions (φ^ν,ρ^ν) of (0.2)–(0.3), then it is to be expected that as $i(\rho^0)$ and $\frac{1}{2}|\nabla\varphi^0|^2$ are numbers between 0 and K, and hence $i(\rho^\nu)$ and $\frac{1}{2}|\nabla\varphi^\nu|^2$ should also be between 0 and K up to, at most, an $\mathcal{O}(\nu^\beta)$-correction.

In fact, we have shown in [GM] that if the speed $|\nabla\varphi^\nu|$ is prescribed in a section of the boundary adjacent to two streamlines boundaries, then $0<i(\rho^\nu)<K$ but

$$0<\frac{1}{2}|\nabla\varphi^\nu|^2\le \sup\frac{1}{2}|\nabla\varphi^\nu|^2\le \widetilde{K}, \tag{0.4}$$

where $\widetilde{K}=(K+C\nu^{1/2})M$, with M a constant that depends only on the flow domain and C depends on the boundary data and the flow domain.

Therefore, it becomes essential to improve the last estimate (0.4) if one wants to study the limiting configuration for either the transonic flow problem or the inviscid limit for the potential hydrodynamic–Poisson model.

The present paper shows that any smooth solutions (ρ^ν,φ^ν) of (0.2)–(0.3), that satisfy estimate (0.4) (potential isentropic gas flow case), it also satisfies

$$0<\frac{1}{2}|\nabla\varphi^\nu(x)|^2\le K+\mathcal{C}\ \frac{\nu}{\left(\mathrm{dist}\{x,\partial\Omega\}\right)^2}, \tag{0.5}$$

for any x in the interior of the 2 or 3 dimensional flow domain Ω, and a growth condition on the enthalpy function $i(\rho)$, to be specified below. In particular, if the enthalpy function is the one associated with a γ–pressure law with $1<\gamma$, then the necessary growth condition is satisfied for $1<\gamma<2$ in the 2–dimensional case and $1<\gamma<\frac{3}{2}$ in the 3–dimensional one. The constant $\mathcal{C}$ depends on $\Omega,\widetilde{K}$ and the growth conditions for the functions $i(\rho)$ and $g(|\nabla\varphi(x)|)$ from (0.3).

In addition, for the 2–dimensional case (where [GM] showed existence of solutions for a boundary value problem associated with (0.2)–(0.3) that satisfied estimate (0.4)) we extend estimate (0.5) to some boundary points x in $\partial\Omega\backslash\partial_3\Omega$, where $\partial_3\Omega$ denotes the section of the boundary of the flow domain Ω where the speed was prescribed. The parameter ν is replaced by ν^β for these boundary estimates and the exponent β depends on the location of x in $\partial\Omega \setminus \partial_3\Omega$, and $\mathcal{C}$ denotes a number that depends on the local curvature of the boundary at the point x and the data of the boundary problem and the coarse bound $\widetilde{K}$. In fact $\mathcal{C}$ is bounded by a function of the Jacobian transformation that corresponds to the conformal map that takes Ω into a rectangle.

We also prove here an estimate similar to (0.5) for the potential fluid-Poisson system presented below. In this case the estimate reads

$$0 < \frac{1}{2}|\nabla\varphi^\nu(x)|^2 + \mathcal{R}(\varphi^\nu) - q\Phi^\nu \ \leq K + \mathcal{C}\ \frac{\nu}{(\mathrm{dist}\{x, \partial\Omega\})^2}, \tag{0.6}$$

also for any x in the interior of the 2 or 3 dimensional flow domain Ω, and the same growth condition on the enthalpy function $i(\rho)$ as in the gas flow case. Here the constant $\mathcal{C}$ depends on $\Omega, \widetilde{K}$, the growth conditions for the functions $i(\rho)$ and $g(|\nabla\varphi(x)|)$ and the bounds on $\mathcal{R}$ and $\Phi\nu$ (this are proven to be ν independent bounds in the 2–dimensional existence theory).

We point out that cavitation speed in isentropic gas flow is the constant value $(2K)^{1/2}$. However, for the hydrodynamic fluid–Poisson system, cavitation speed is not constant any longer . It is the speed at vacuum state given by the model, i.e. $|\nabla\varphi^\nu(x)|$ reaches cavitation speed when takes the value $(\,2(K\mathcal{R}(\varphi^\nu) - q\Phi^\nu)(x))^{1/2}$. Hence, the convergence analysis in the limiting vanishing parameter ν will also need estimate (0.6).

The technique we use to prove (0.5) consists in showing that if the quantity $\omega = \frac{|\nabla\varphi^\nu|^2}{2} - K$ becomes positive then it satisfies an elliptic differential inequality, which depends on ν, that admits a comparison principle and whose positive solutions can be compared with and majorized by positives supersolutions of order $\frac{\mathcal{C}(x)\nu^\beta}{(\mathrm{dist}\{x,\partial_3\Omega\})^2}$ at the point $x \in \overline{\Omega}\backslash\partial_3\Omega$.

It appears that the estimate (0.5) will still hold even if the section $\partial_3\Omega$ of the boundary is taken to infinity, as the conformal map that takes Ω into an infinite strip tends to the identity map at infinity.

We also remark that this estimate deteriorates as x is at $\nu^{\beta/2}$-distance from the boundary section $\partial\Omega_3$, with $\beta = 1, \frac{1}{2}$, suggesting the possible formation of large boundary layers near $\partial\Omega_3$, as expected from viscous approximations in bounded domains.

In the following first section we present the potential fluid-Poisson model and previous results for the boundary value problem, then we outline the results proven in the next sections.

1. Presentation of the problem in the general case

The viscous perturbation model to transonic flow equations $((\Phi = 0))$ or a compressible irrotational steady fluid-Poisson system we consider is given by

$$\mathrm{div}(\rho\nabla\varphi) = 0 \tag{1.1.1}$$

$$\nu\Delta\varphi = -\nu(\ln\rho)_\varphi|\nabla\varphi|^2 = \frac{1}{g(|\nabla\varphi|)}\left(\frac{|\nabla\varphi|^2}{2} + i(\rho) - K + \mathcal{R}(\varphi) - q\Phi\right) \tag{1.1.2}$$

$$\Delta\Phi = \alpha(\rho - C(x)) \tag{1.1.3}$$

in a piecewise smooth domain Ω that can be conformally transformed into a rectangle with a C^2 conformal transformation.

This system with a boundary value problem associated with it has already been introduced in [GM] and we refer to the references therein on the justifications for this model.

We have shown there the existence of solutions (ρ, φ, Φ) in $(C^{1,\alpha}(\overline{\Omega}) \times C^{2,\alpha}(\overline{\Omega}) \times W^{2,p}(\overline{\Omega}))$ for a boundary value problem to be described below, if the function $\mathcal{R}(\varphi)$ satisfies $\|\mathcal{R}^{(i)}\| \leq \mathcal{R}$, $i = 0, 1, 2$, independently of ν and under the the following conditions for the function g, the enthalpy function $i(\rho)$, and compatibility condition on the data:

. The function $g(|\nabla\varphi|)$ is such that the term

$$\mathcal{F}(|\nabla\varphi|^2, Q_B(\rho, \varphi, \Phi)) = \frac{\frac{|\nabla\varphi|^2}{2} + i(\rho) - K + \mathcal{R}(\varphi) - q\Phi}{g(|\nabla\varphi|)|\nabla\varphi|}$$

is bounded in $|\nabla\varphi|$, where $Q_B(\rho, \varphi, \Phi)$ denotes the cavitation speed given by Bernoulli's equation.

. The enthalpy function $i(\rho)$ satisfies $i(\rho), i'(\rho) > 0$ and the growth condition

$$\frac{1}{n-1} i(\rho) - \rho\, i'(\rho) \geq \frac{1}{n-1} k\, i(\rho) \quad \text{for some} \quad k, 0 < k < 1, \tag{1.2}$$

where $n = 2$ or 3 denotes the space dimension.

. For $\mathcal{R}_L < \mathcal{R}(\varphi) < \mathcal{R}_U$, the Bernoulli constant K satisfies the compatibility condition

$$K - \mathcal{R}_U + q\Phi_L(K) > 0 \tag{1.3}$$

where $\Phi_L(K) = \inf_{\partial_1\Omega\cup\partial_3\Omega} \gamma - \alpha \sup_{\overline{\Omega}} |F_x|^2 (\sup_{\overline{\Omega}} C(x) + i^{-1}(K - \mathcal{R}_L + q\Phi_U))$, for $\Phi_L(K) \leq \Phi^\nu \leq \Phi_U$ where $\Phi_L(K, \Phi_U)$ are ν-independent and depend on the domain Ω and the data of the problem.

Remark: Equations (1.1) correspond to dissipative approximation to a compressible flow model that satisfies a γ-law: $i(\rho) = \frac{\gamma}{\gamma-1}\rho^{\gamma-1}$, so condition (1.2) is satisfied with $k = 2 - \gamma$ and $1 < \gamma < 2$, if $n = 2$ the space dimension and $k = 3 - 2\gamma$ and $1 < \gamma < \frac{3}{2}$ if $n = 3$.

The function g imposes a growth condition for the dissipative term under very low or very high speeds. This was found to be (see [GM]) a necessary condition to solve the equations at the viscous level, where they neither cavitate nor stagnate in the 2–dimensional flow domain under consideration.

Nevertheless a posteriori, after obtaining ν-uniform bound for the speed $|\nabla\varphi^\nu|$ for the boundary value problem posed in [GM], we can let g be one for values of speed below that ν-uniform bound and to be like $|\nabla\varphi|^{-2}$ near zero speed. In particular, our viscous approximation is the standard linear viscosity for moderate speeds but with a non-linear correction at very low and very high speeds.

Thus, we shall assume here that g is a monotone increasing function and

$$g(|\nabla\varphi|) = \left(1 + \frac{|\nabla\varphi|^2}{2}\right)^{\alpha/2}, \quad \alpha \geq 1 \tag{1.4}$$

for sufficiently large values of $\frac{|\nabla\varphi|^2}{2}$.

We point out that the compatibility condition (1.3) reduces to $K > 0$ for the approximation to transonic flow model, which is the standard assumption on the Bernoulli's constant. In the case of fluid-Poisson system under consideration, since $i^{-1}(\rho)$ has superlinear growth, then a K verifying (1.3) exists only if the data is chosen adequately.

We take into consideration two possible boundary value problems in the case of 2–dimensional flow model. One of them is the one with data given in [GM], that is prescribing an inflow boundary, two adjacent tangential flow boundaries (i.e. two walls), and in the rest of it, we prescribe positive non-cavitating speed (i.e. the magnitude of velocity field).

The other boundary value problem is the one that corresponds to prescribe an outflow boundary condition (i.e the flow potential φ is constant) on the section where the speed was prescribed in the above case.

In fact, we consider some special 2–D flow domains: Let Ω be as in figure 1 below

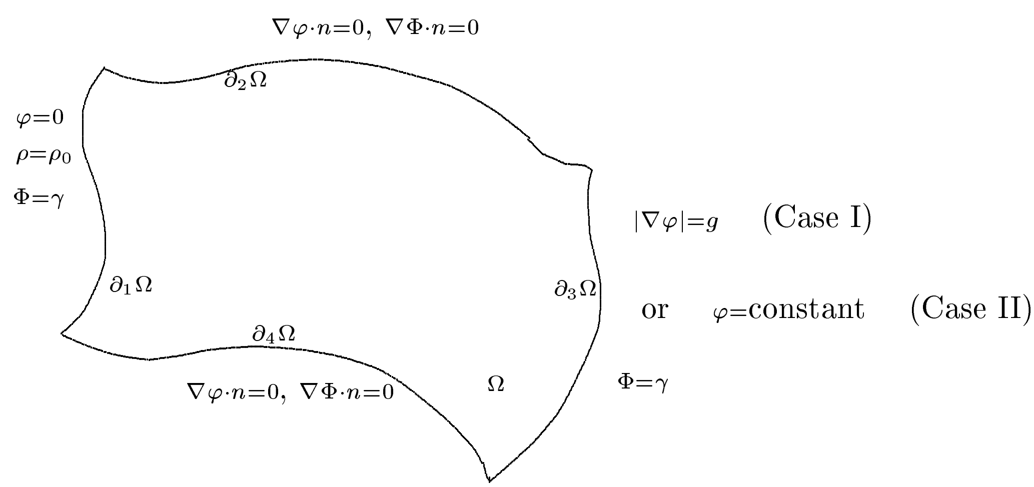

Fig. 1 The flow domain Ω and boundary data

That is , the boundary of Ω is the union of four smooth curves section that meet each other at a right angle. Thus, there is a unique conformal transformation that takes Ω into a rectangle R that keep fixed three points (take any three of the angle points including the two that correspond to the inflow boundary meeting the tangential flow ones.) In addition the conformal map is smooth (C^3).

We denote the boundary sections as follows: $\partial\Omega_1$ the inflow boundary section, $\partial\Omega_2$ and $\partial\Omega_4$ the tangential flow boundary sections, and $\partial\Omega_3$ the remainder part of it.

Hence the two boundary value problems under consideration have both same data on the inflow and tangential flow boundary sections, namely, the potential flow function $\varphi =$ constant on $\partial_1\Omega$, with $(\nabla\varphi \cdot n|_{\partial_1\Omega})(\omega_1) < 0$ and ω_1 is a corner point where $\partial_1\Omega$ meets $\partial_2\Omega$; $\nabla\varphi \cdot n = 0$ on $\partial_\tau\Omega = \partial_2\Omega \cup \partial_4\Omega$. As usual n denotes the outer unit normal.

The density is prescribed at the inflow boundary, i.e. at $\partial_1\Omega$, so that $\rho = r(x)$ on $\partial_1\Omega$ and the electric potential Φ satisfies Dirichlet conditions, i.e. $\Phi = \gamma$ on $\partial_1\Omega \cup \partial_3\Omega$ and $\nabla\Phi \cdot n = 0$ on $\partial_\tau\Omega$.

Thus the first boundary value problem prescribes $|\nabla\varphi| = g(x) > 0$, on $\partial_3\Omega$, and the other one just $\varphi =$ constant on $\partial_3\Omega$ (i.e an outflow boundary if this constant is larger that the one for the inflow boundary.)

We have shown in [GM] that for the first boundary value problem for system (1.1) where $i(\rho)$ satisfies condition (1.2) and g satisfies (1.4), with $\alpha = 1$, there exists a solution $\rho^\nu, \varphi^\nu, \Phi^\nu$ as stated above, such that there is a $\nu_0 = \nu_0(\|g\|_{C^1}, \Omega)$

$$0 < k_\nu < |\nabla\varphi^\nu| \le \widetilde{K}, \quad 0 < l_\nu < \rho^\nu < L^*, \quad \text{for} \quad \nu \le \nu_0 \tag{1.5}$$

and $|\nabla\Phi^\nu|$, $|\Phi^\nu| \le M^*$ all in $\overline{\Omega}$ with $\widetilde{K}$, L^* and M^* independent of ν.

In fact the upper bound $\widetilde{K}$ for the speed is given in terms of cavitation speed and the domain Ω. That is, if $F\colon\ \Omega \to R$ is the conformal map that takes Ω into a rectangle R,

$$\widetilde{K} = \{\sup_{\overline{\Omega}} |K - \Re(\varphi) + q\Phi| + C\,\nu^{1/2}\}\frac{\sup_{\overline{\Omega}} |F_x|}{\inf_{\overline{\Omega}} |F_x|}, \qquad \nu \le \nu_0 \tag{1.6}$$

where $|F_x|$ is the Jacobian of the real valued transformation associated with F, $\nu_0 = \nu_0(k, \|g\|_{C^1}, \|F\|_{C^{1,1}(\overline{\Omega})})$ and $C = C(k, \|g\|_{C^1}, \|F\|_{C^{1,1}(\overline{\Omega})}, K, q, \|\Re\|_{C^{0,1}(\overline{\Omega})}$, boundary data).

Then, estimate (1.5) is not sharp, as it gives that the speed corresponding to the viscous flow is bounded by a term of order $\mathcal{O}(\nu^{1/2})$, away from a factor of cavitation speed. By cavitation speed we mean the value that corresponds to the inviscid speed given by Bernoulli's law, that is setting density $\rho = 0$ in the inviscid model $\nu \equiv 0$.

The factor is $\sup_{\overline{\Omega}} |F_x| \cdot \{\inf_{\overline{\Omega}} |F_x|\}^{-1} = \exp\{\ \mathrm{osc}(\log|F_x|)\}$ and it is related to the geometry of the domain, a sort of measure of how far is the flow domain from a rectangle, since

$$\mathrm{osc}(\log|F_x|) \le \sup_{\Omega} |\nabla(\log|F_x|)| \mathrm{diam}(\overline{\Omega}) \le \sup_{\overline{\Omega}} |D_{ij}F| (\inf|F_x|)^{-1} \mathrm{diam}(\overline{\Omega}).$$

Unless the domain Ω is originally a rectangle (or ν-close to a rectangle) the value of $\widetilde{K}$ is too coarse. As we anticipated in the introduction, we improve the value of $\widetilde{K}$.

The proof of the sharper estimate needs is the existence of an approximation φ^ν solution to (1.1) under conditions (1.2), (1.3) and (1.4), such that $|\nabla\varphi^\nu| < \widetilde{K}$, where $\widetilde{K}$ is a ν-uniform constant that depends on the data and the flow domain Ω (see [GM] for the 2-dimensional case.)

Remark: The existence of 3-dimensional solutions to system (1.1) is an open problem. However, under the assumption of existence of solutions $(\rho^\nu, \varphi^\nu, \Phi^\nu)$ to a 3–dimensional boundary value problem, where conditions (1.2), (1.3) and (1.4) are satisfied and $|\nabla\varphi^\nu|$ admits a ν–uniform bound denoted by $\widetilde{K}$, then $|\nabla\varphi^\nu|$ satisfies a sharper pointwise bound "close" to cavitation speed in the interior of the flow domain.

Hence, the following pointwise estimate will be proven for $|\nabla\varphi^\nu(x)|$ in the interior of the 2 or 3– dimensional domain for any solution of a boundary value problem associated with system (1.1)–(1.4) where estimates (1.5) is satisfied:

$$(1.7) \qquad |\nabla\varphi^\nu(x_0)|^2 \le 2(K - \mathcal{R}(\varphi^\nu(x_0)) + q\Phi^\nu(x_0)) + \frac{C\nu^\beta}{(\mathrm{dist}(x_0, \partial\Omega))^2} \quad \text{for all} \quad \nu \le \nu_1$$

where ν_1 and C both depend on $k^{-1}, \widetilde{K}, K, g, q, \alpha$ the bounds of $\mathcal{R}$ and Φ. The parameter ν_1 depends on $\mathrm{dist}\{x_0, \partial\Omega\}$. The exponent β is 1 for the compressible gas model and 1/2 for the fluid Poisson system, but it is independent of the space dimension.

Furthermore, in the 2–dimensional case, we can extend the pointwise estimate to the boundary region wich includes the inflow and tangential flow regions. Therefore, for $\Omega \subset \mathbb{R}^2$

$$(1.8) \qquad |\nabla\varphi^\nu(x_0)|^2 \le 2(K - \mathcal{R}(\varphi^\nu(x_0)) + q\Phi^\nu(x_0)) + \frac{C\nu^\beta}{(\mathrm{dist}(x_0, \partial_3\Omega))^2} \quad \text{for all} \quad \nu \le \nu_1$$

where $\beta = \frac{1}{2}$ if the point x_0 lies on the tangential boundary $\partial_\tau\Omega = \partial_2\Omega \cup \partial_4\Omega$. $\beta = \frac{1}{8}$ if x_0 lies on the inflow boundary $\partial_1\Omega$ and $\beta = 1$ if $\partial_1\Omega$ is locally flat around x_0 (i.e. $\kappa_x = 0$ in a neighborhood of x_0 relative to $\partial_1\Omega$). In all cases the constants ν_1 and C depend on $k^{-1}, g, \widetilde{K}, K, \mathcal{R}, q, \alpha$, data, κ_{x_0}, with κ_{x_0} the local curvature of $\partial\Omega$ at x_0.

Estimate (1.8) suggests the possible formation of large boundary layers near the outflow boundary $\partial_3\Omega$ at distances less of $\mathcal{O}(\nu^{1/2})$ away from the tangential boundary, and of order $\mathcal{O}(\nu^{1/4})$ near the tangential boundary. This behavior excludes velocity overshoots above cavitation speed for the viscous solutions near shock formation for the limiting configuration, away from $\partial_3\Omega$.

Maybe one of the most relevant aspects of this technique is that it works in 3-dimensions as well. Provided the existence of solutions with ν-independent bounds up to the boundary, these estimates seem not to depend on the conformal map but rather on the local parameterization of the boundary to a flat one. As it was described above, we do not include here boundary estimates in the 3–dimensional case due to lack of knowledge of a boundary value problem that yield existence of solutions with a coarse ν–uniform bound for the speed. However estimate (1.7) holds *under the assumption* of existence of solutions with ν–independent bounds for the speed up to the boundary of the flow domain.

Finally, in the 2–dimensional case the following conclusion holds. Assume existence of smooth solutions for the second boundary value problem presented above where φ^ν is prescribed on $\partial_3\Omega$ as an arbitrary constant above cavitation ratio (i.e. the ratio between cavitation speed and the length of the shortest curve of those that define the tangential flow walls for the domain Ω). Assume that conditions (1.2), (1.3) and (1.4) (i.e. $\partial_3\Omega$ is now an outflow boundary). Then $\varphi^\nu \in C^{2,\alpha}(\overline{\Omega})$ and $0 < k_\nu \le |\nabla\varphi^\nu| < K_\nu$, but $|\nabla\varphi^\nu|$ can not be ν-uniformly bounded.

We divide the rest of the paper into three sections. We prove in section 2 estimate (1.7) in the interior of Ω for the transonic flow model (i.e. $\Phi \equiv 0$, $\mathcal{R} \equiv 0$), most of the relevant features are already here.

The third section extends the results to the full system (1.1), i.e. for potential fluid-Poisson systems.

Finally section 4 extends the estimates to the inflow boundary $\partial_1\Omega$ and the tangential boundary $\partial_\tau\Omega = \partial_2\Omega \cup \partial_4\Omega$.

2. The interior sharper estimate for the approximating model to transonic flow when the speed admits a ν-uniform coarse bound.

Let
$$\omega(x) = \frac{1}{2}|\nabla\varphi^\nu(x)|^2 - K$$
be defined in Ω, where φ^ν is a solution for the boundary value problems associated with system (1.1), where $\Phi \equiv \mathcal{R}(\varphi^\nu) \equiv 0$, as defined in the introduction.

From the existence theory, ω is a $C^{1,\alpha}(\Omega)$ function where the C^0 norm of ω depends on ν.

Our aim is to show in this section that for any point $x_0 \in \Omega$

$$\omega(x_0) = \frac{|\nabla\varphi^\nu(x)|^2}{2} - K \le \frac{\mathcal{C}(k^{-1}, \alpha, \widetilde{K}, K)\,\nu}{(\mathrm{dist}\{x_0, \partial\Omega\})^2}, \quad \nu \le \nu_0 \tag{2.1}$$

where $\nu_0 = \nu_0(\mathrm{dist}\{x_0, \partial\Omega\}, k, \alpha, \widetilde{K}, K)$, α the parameter in the function g from condition (1.4), and

$$|\nabla\varphi^\nu(x)| \le \widetilde{K} \qquad \text{for} \quad \nu \le \nu_2(\widetilde{K}) \tag{2.2}$$

with $\widetilde{K}$ a constant independent of ν.

We first state the following lemma, to be proven in the Appendix. It shows that if $\omega > 0$, then ω satisfies an elliptic differential inequality so that ω can not have an interior maximum. This lemma is an extension to *physical* space from an estimate in the *potential flow–stream function* space for the 2–dimensional problem, introduced first by Morawetz in [M1] and later used in [GM].

This estimate was crucial for the derivation of the ν–uniform bounds in [GM]. It shows that the speed associated with solutions of the viscous isentropic gas flow system (1.1.1)–(1.1.2) (for $\Phi \equiv \mathcal{R} \equiv 0$) can not have an interior maximum that takes values above cavitation speed.

Lemma 2.1. *Let* $\omega = \frac{|\nabla \varphi^\nu|^2}{2} - K$ *where* φ^ν *is a solution of system (1.1.1–1.1.2) with* $\Phi \equiv 0$ *and* $\mathcal{R}(\varphi) \equiv 0$. *Then* ω *satisfies*

$$\Delta\omega \geq \frac{k}{2(n-1)\nu^2 g^2}\omega^2 + \frac{1}{\nu}\frac{\nabla\varphi\nabla\omega}{g}\left[1 - \frac{\alpha}{2}\frac{(\omega + i(\rho))}{(1+\omega+K)}\right] - \frac{2}{(n-1)k}\frac{|\nabla\omega|^2}{(\omega+K)} \tag{2.3}$$

on any open region of Ω *where* $\omega \geq 0$. *Here* α *is the parameter from (1.4) and* $0 < k = k(i(\rho)) < 1$ *from condition (1.2).*

The proof of this lemma does not use the coarse bound $\widetilde{K}$ on the quantity $|\nabla\varphi|$ or ω. It just uses the structure of the equations (1.1.1)–(1.1.2) and conditions (1.2), (1.3) and (1.4). Its proof relies on the invariance of system (1.1) under orthonormal change of coordinates, so that the velocity field $\nabla\varphi^\nu$ is estimated at every point in a system of coordinates that points in the direction of the gradient (i.e. in the direction of the flow). In fact the 2- -dimensional interpretation of this system of coordinates is the *potential flow–stream function* space, used in [M1] and [GM] to give a complete proof of existence of a ν–uniform *coarse* bound for the speed $|\nabla\varphi^\nu|$.

Furthermore, this lemma allows us to prove the sharper bound (2.1), in either two or three space dimensions, whenever ω admits a ν-uniform bound $\widetilde{K}$.

Theorem 2.2. Sharper uniform bound for a solution with a coarse uniform bound. *Let* ω *be* ν*-uniformly bounded in* $\overline{\Omega}$ *by a number* $\widetilde{K}$, *with* Ω *either a* $n = 2$ *or 3 dimensional domain. Then there exist a* ν_0 *such that*

$$\omega(x_0) \leq \frac{\mathcal{C}\,\nu}{(dist\{x_0, \partial\Omega\})^2} \quad for \quad \nu \leq \nu_0 \tag{2.4}$$

where $\mathcal{C}$ *is a constant depending on* k^{-1}, α, $\widetilde{K}$ *and* K *and*
$\nu_0 = \min\left\{(dist\{x_0, \partial\Omega\})^2, m, \nu_2(\widetilde{K})\right\}$, *with the number* m *depending on* k, α, K *and* $\widetilde{K}$.

Proof: Take x_0 in Ω. If $\omega(x_0) \leq 0$ then (2.4) is satisfied. If $\omega(x_0) > 0$, since ω is $C^{1,\alpha}(\overline{\Omega})$, then x_0 belongs to the region where $0 < \omega < \widetilde{K}$, uniformly in ν for a $\nu \leq \nu_2(\widetilde{K})$ in $\overline{\Omega}$. By

Lemma 2.1, on any open region where $\omega > 0$, ω solves a differential inequality (2.3). Thus estimating the right-hand side of (2.3) from below
(2.5)

$$\Delta\omega \geq \frac{k}{2(n-1)\nu^2(1+\omega+K)^\alpha}\omega^2 - \frac{2(\omega+K)^{\frac{1}{2}}|\nabla\omega|}{\nu\,(1+\omega+K)^{\frac{\alpha}{2}}}\left[1-\frac{\alpha}{2}\frac{\omega+i(\rho)}{1+\omega+K}\right] - \frac{2}{(n-1)k}\frac{|\nabla\omega|^2}{(\omega+K)}.$$

Now, since $i(\rho) \leq K$ then the right-hand side of (2.5) dominates the expression
(2.6)

$$\frac{k}{2(n-1)\nu^2(1+\widetilde{K}+K)^\alpha}\omega^2 - \frac{2(\widetilde{K}+K)^{1/2}|\nabla\omega|}{\nu}\left(1+\frac{\alpha}{2}(\widetilde{K}+K)\right) - \frac{2}{(n-1)k}\frac{|\nabla\omega|^2}{K}.$$

Taking $M = \max\{2\widetilde{K}, \widetilde{K}+K, 1\}$, combining (2.5) with (2.6), ω satisfies the differential inequality

$$\Delta\omega \geq \frac{1}{2(n-1)\nu^2}\frac{k}{(2M)^\alpha}\omega^2 - \frac{2\alpha}{\nu}M^{\frac{3}{2}}|\nabla\omega| - \frac{2}{(n-1)k}\frac{|\nabla\omega|^2}{K}, \text{ for } \nu \leq \nu_2(\widetilde{K}), \tag{2.7}$$

on any open region where ω is positive.

Clearly (2.7) yields that ω can not have an strict positive interior maximum in Ω as it gives an immediate contradiction otherwise. However, we want to say more about positive ω.

In fact, we shall prove that any solution ω of (2.7), when evaluated at a point of positivity x_0, is below a specific differentiable function f in a ball $B_R(x_0)$ contained in Ω. Moreover the radius R and the function f can be chosen so that

$$f(x_0) \leq \frac{\mathcal{C}(k^{-1}, \widetilde{K}, K, \alpha)\ \nu}{(\text{dist}\{x_0, \partial\Omega\})^2} \tag{2.8}$$

and such that $f(x_0)$ is the minimum value of f in the ball $B_R(x_0)$. That is to say, $\omega - f$ can not have a positive maximum in $\overline{B_\tau(x_0)}$, with $\tau \leq R$, and f achieves its minimum at x_0.

Hence, this function f must be constructed as an upper barrier function for ω solution of (2.7), with $0 < \omega \leq \widetilde{K}$, in the ball $B_R(x_0)$. Therefore we state the following lemma.

Lemma 2.3. *Let x_0 be in the interior of Ω, a $n = 2$ or 3–dimensional domain. There exists a ball $B_R(x_0) \in \Omega$, a number ν_1 and a differentiable function f^ν in $\overline{B_R(x_0)}$ such that*

$$\min_{\overline{B_r(x_0)}} f^\nu(x) = f^\nu(x_0) \leq \frac{\mathcal{C}\,\nu}{(dist\{x_0, \partial\Omega\})^2} \tag{2.9}$$

where f^ν is an upper barrier function for the solution of the differential inequality (2.7) in $\overline{B_R(x_0)}$, for $\nu \leq \nu_1$ and the constant $\mathcal{C} = \mathcal{C}(k^{-1}, \widetilde{K}, K, \alpha)$, the radius R

$= min\{dist\{x_0, \partial\Omega\}, m\}$ *with* $m = m(k, \widetilde{K}, K, \alpha)$ *and the number* $\nu_1 = R^2$. *That is (dropping the superscript* ν *from* f*),* f^ν *satisfies*

$$\Delta f < \frac{1}{2(n-1)\nu^2}\frac{k}{(2M)^\alpha}f^2 - \frac{|\nabla f|}{\nu}2\alpha M^{\frac{3}{2}} - \frac{2}{(n-1)k}\frac{|\nabla f|^2}{K} \quad \textit{for} \quad \nu \leq \nu_1 \tag{2.10}$$

in the interior of $B_R(x_0)$ *and*

$$f > \widetilde{K} \ \textit{on}\ \partial B_R(x_0). \tag{2.11}$$

We use this lemma to complete the proof of theorem 2.2.

First, since $\omega \leq \widetilde{K}$ in $\overline{\Omega}$ for $\nu_0 \leq \min\{\nu_1, \nu_2\}$, then $\omega - f < \widetilde{K} - \widetilde{K} = 0$ on $\partial B_R(x_0)$ for $\nu \leq \nu_0$, and since $\omega - f$ is bounded in $\partial B_R(x_0)$ then $\omega - f$ achieves its maximum at a point, say $\tilde{x}$, in $\overline{B_R(x_0)}$.

Suppose this maximum is positive, i.e. $(\omega - f)(\tilde{x}) > 0$, $\Delta(\omega - f)(\tilde{x}) \leq 0$ and $\nabla\omega = \nabla f$ at $\tilde{x}$. Then, by continuity, there is a ball $B_r(\tilde{x}) \subset B_R(x_0)$ where $\omega - f > 0$ in $B_r(\tilde{x})$ and, in particular, the differential inequality (2.7) for ω holds in $B_r(\tilde{x})$.

Thus, combining the differential inequalities satisfied by ω (2.7) and by f (2.10) in $B_r(\tilde{x})$ yield

$$\begin{aligned} \Delta\omega - \Delta f - \frac{k}{2(n-1)\,\nu^2(2M)^\alpha}(\omega^2 - f^2) + \frac{2\alpha M^{\frac{3}{2}}}{\nu}(|\nabla\omega| - |\nabla f|) \\ + \frac{2}{(n-1)k\,K}(|\nabla\omega|^2 - |\nabla f|^2) > 0, \quad \text{in } B_r(\tilde{x}) \ \text{ for } \nu \leq \nu_0(\alpha, k, \widetilde{K}, K). \end{aligned} \tag{2.12}$$

So evaluating (2.12) at $\tilde{x}$,

$$\Delta(\omega - f)(\tilde{x}) - \frac{k}{2(n-1)\nu^2(2M)^\alpha}(\omega^2 - f^2)(\tilde{x}) > 0.$$

Since $\omega^2 > f^2$ at $\tilde{x}$ and , by condition (1.2) k is also positive, then $\Delta(\omega - f)(\tilde{x})$ is positive, contradicting that $\omega - f$ has a maximum at $\tilde{x}$.

Therefore $\omega - f$ can not have a positive maximum at any point in $B_R(x_0)$, so $\omega \leq f$ in $B_R(x_0)$. In particular,

$$0 < \omega(x_0) \leq f(x_0) \leq \frac{\mathcal{C}\,\nu}{\left(\text{ dist}\{x_0, \partial\Omega\}\right)^2} \tag{2.13}$$

for $\nu \leq \min\left\{\left(\text{ dist}\{x_0, \partial\Omega\}\right)^2, m(k, \alpha, K, \widetilde{K}), \nu_2(\widetilde{K})\right\}$. The constant m and $\mathcal{C} = \mathcal{C}(k^{-1}, \widetilde{K}, K, \alpha)$ are from (2.9).

Hence, the proof of Theorem (2.2) is now completed.

Proof of Lemma 2.3. We take the function

$$f(x) = g(|x - x_0|) = g(r) = \frac{A\nu}{\overline{R}^2 - r^2} \text{ in } B_R(x_0) \tag{2.14}$$

where $R < \overline{R}$, where $A, \overline{R}$ and R are to be chosen so that (2.10) and (2.11) are satisfied.

First, in order to satisfy (2.11) we need A, R and $\overline{R}$ be chosen such that

$$g(R) = \frac{A\nu}{\overline{R}^2 - R^2} > \widetilde{K}.$$

Thus, taking

$$\overline{R} = R + \frac{\nu}{R} \quad \text{and} \quad \frac{\nu}{R^2} < 1 \ , \tag{2.15}$$

then

$$g(R) = \frac{A\nu}{\overline{R}^2 - R^2} = \frac{A\nu}{2\nu + \frac{\nu^2}{R^2}} > \frac{A}{3} > \widetilde{K} \tag{2.16}$$

whenever $A > 3\widetilde{K}$.

Next, we see that $f(x) = g(|x - x_0|)$ satisfies (2.10) for some choices of R and A:

Since $\Delta f = g''(r) + \frac{1}{r}g'(r)$, denoting $D = \overline{R}^2 - r^2$, with $0 < r < R$, then $D \geq \overline{R}^2 - R^2 = 2\nu + \frac{\nu^2}{R^2} > 2\nu$, so that by computing

$$\begin{aligned} (2.17)\quad &\Delta f - \frac{1}{2(n-1)\nu^2}\frac{k}{(2M)^\alpha}f^2 + \frac{|\nabla f|}{\nu}2\alpha M^{\frac{3}{2}} + \frac{2}{(n-1)k}\frac{|\nabla f|^2}{K} \\ &= \frac{4A\nu r^2}{D^3} - 2\frac{A\nu}{D^2} - \frac{k}{2(n-1)(2M)^\alpha}\frac{A^2}{D^2} + \frac{2Ar}{D^2}2\alpha M^{\frac{3}{2}} + \frac{2}{(n-1)k\,K}\frac{A^2\nu^2}{D^4}r^2 \\ &\leq -\frac{k}{2(n-1)(2M)^\alpha}\frac{A^2}{D^2} + \frac{2A\overline{R}^2}{D^2} + \frac{A\overline{R}}{D^2}2\alpha M^{\frac{3}{2}} + \frac{2A^2}{(n-1)k\,K D^2}\overline{R}^2. \end{aligned}$$

Now, by (2.15) $\overline{R}^2 \leq (2R)^2$, then the right-hand side of the above inequality is majorized by

$$\leq \left(-\frac{k}{2(n-1)\,(2M)^\alpha} + \frac{2}{(n-1)k\,K}R^2\right)\frac{A^2}{D^2} + 8A\frac{R^2}{D^2} + 4\alpha M^{\frac{3}{2}}A\frac{R}{D^2}. \tag{2.18}$$

Taking

$$R^2 < \frac{1}{2}\left(\frac{k\,K}{4}\frac{k}{(2M)^\alpha}\right) = m^2(k, K, \widetilde{K}, \alpha), \tag{2.19}$$

(2.18) is majorized by

$$\frac{A}{D^2}\left(\frac{-k}{2(n-1)(2M)^\alpha}A + 2R^2 + 8\alpha M^{\frac{3}{2}}R\right)$$

where we recall k, from condition (1.2), $0 < k < 1$.

Thus, in order to get both (2.18) and so (2.17) negative, and simultaneously (2.16) satisfied, choose any A such that,

$$A > \max\left\{\frac{n\,2^{\alpha+4}M^\alpha}{k}\left(2R^2 + 8\alpha M^{\frac{3}{2}}R\right);\ 3\widetilde{K}\right\}.$$

Therefore, f satisfies (2.10) and (2.11) with

$$R = \min\left\{\mathrm{dist}\{x_0, \partial\Omega\};\ m\right\}, \tag{2.20}$$

for m defined in (2.19),

$$\nu \le R^2 = \min\left\{\,(\mathrm{dist}\{x_0, \partial\Omega\})^2;\ m^2\right\} \tag{2.21}$$

and

$$\mathcal{C} = \mathcal{C}(k^{-1}, \alpha, \widetilde{K}, K) = \max\left\{\frac{2^{\alpha+4}M^\alpha}{k}\left(2R^2 + 8RM^{\frac{3}{2}}\right);\ 4\widetilde{K}\right\}. \tag{2.22}$$

Hence

$$f(x) = \frac{\mathcal{C}\,\nu}{\left(R + \frac{\nu}{R}\right)^2 - |x - x_0|^2}$$

is a barrier function for the solution of the differential inequality (2.7) in $B_R(x_0)$ for R, ν and $\mathcal{C}$ as in (2.20), (2.21) and (2.22) respectively, and

$$f(x_0) \le \frac{\mathcal{C}(k^{-1}, \alpha, \widetilde{K}, K)\,\nu}{\left(\mathrm{dist}\{x_0, \partial\Omega\}\right)^2},$$

so (2.9) holds.

Lemma 2.3 is now proven.

3. Sharper interior estimates for steady fluid-Poisson systems.

In this section we need to refine the argument presented in the previous one. The estimate takes into consideration the effects of coupling the electric potential and the relaxation term.

Let (ρ, φ, Φ) be a solution of system (1.1) with boundary data prescribed as in section 1, where the function $\mathcal{R}(\varphi)$ satisfied $\|\mathcal{R}^{(i)}\| \le \mathcal{R}$, $i = 0, 1, 2$, independently of ν.

Again, we have shown in [GM], for $\alpha = 1$ in condition (1.4), that $\Phi_L(K) \le \Phi^\nu \le \Phi_U$, $|\nabla\Phi^\nu| \le M^*$ and $0 < k_\nu < |\rho^\nu| < L = i^{-1}(K^*)$, where $\Phi_L(K, \Phi_U)$, Φ_U, M^* and $i^{-1}(K^*)$ are ν-independent and depend on the domain Ω and the data of the problem. In addition if $\mathcal{R}_L < \mathcal{R}(\varphi) < \mathcal{R}_U$ and K the Bernoulli constant satisfy the compatibility condition (1.3), namely

$$K - \mathcal{R}_U + q\Phi_L(K) > 0 \ , \tag{3.1}$$

then it is shown in [GM] that

$$i(\rho) \le K^* = K - \mathcal{R}_L + q\Phi_U. \tag{3.2}$$

The aim of this section is to improve the interior bound for $|\nabla\varphi^\nu|$ under the assumption that $|\nabla\varphi^\nu|$ admits a coarse uniform global bound $\widetilde{K}$ for the boundary value problems proposed in section 1, for some fixed α in condition (1.4).

As in the previous section we use a lemma, which is proved in the appendix showing that the quantity $\omega = \frac{|\nabla\varphi^\nu|^2}{2} - K + \mathcal{R}(\varphi^\nu) - q\Phi^\nu$ satisfies a differential inequality. This will provide a maximum principle for values of ω that are above the cavitation speed (i.e. $\omega > 0$).

Lemma 3.1. *Let $\omega^\nu = \frac{|\nabla\varphi^\nu|^2}{2} - K + \mathcal{R}(\varphi^\nu) - q\Phi^\nu$, where φ^ν and Φ^ν are solutions of system (1.1.1–1.1.3). Let W denote*

$$W = \omega^\nu + K - \mathcal{R}(\varphi^\nu) + q\Phi^\nu = \frac{|\nabla\varphi^\nu|^2}{2} \tag{3.3}$$

then (dropping ν for convenience) ω satisfies the differential inequality

$$\begin{aligned}\Delta\omega \ge\ & \frac{k}{2(n-1)\nu^2}\frac{\omega^2}{g^2} - \frac{2}{(n-1)k}\frac{|\nabla\omega|^2}{(K-\mathcal{R}_U+q\Phi_L)} \\ & - |\nabla\omega|\left\{\frac{4}{(n-1)k}\mathcal{R}(K-\mathcal{R}_U+q\Phi_L)^{-1/2} + \frac{4}{(n-1)k}\frac{qM}{(K-\mathcal{R}_U+q\Phi_L)}\right. \\ & \qquad \left. +\frac{1}{\nu}\frac{|\nabla\varphi|}{g}\left[1-\frac{\alpha}{2}\frac{\omega+i(\rho)}{1+W}\right]\right\} \\ & - \frac{1}{\nu\, g}\left\{\left(\mathcal{R}|\nabla\varphi|^2 + qM|\nabla\varphi|\right)\left(1-\frac{\alpha}{2}\frac{\omega+i(\rho)}{(1+W)}\right) + \mathcal{R}(\omega+i(\rho))\right\} \\ & - \frac{4}{(n-1)k}\left(-\mathcal{R} + \frac{qM}{(2(K-\mathcal{R}_U+q\Phi_L))^{1/2}}\right)^2 - q\alpha i^{-1}(K^*)\end{aligned} \tag{3.4}$$

on any open region of Ω where $\omega > 0$, where α is the parameter from (1.4) and $k = k(i(\rho))$ from (1.2).

We can now prove the following theorem given a sharper bound for ω provided that ω admits a global coarse bound.

Theorem 3.2. Sharper uniform bound for a solution of the hydrodynamic-Poisson system with a coarse bound. *Let* $\omega^\nu = \frac{|\nabla\varphi^\nu|^2}{2} - K + \mathcal{R}(\varphi^\nu) - 2\Phi^\nu$ *be* ν*-uniformly bounded in* $\overline{\Omega}$ *by a number* $\widetilde{K}$ *for any value of* $\nu \le \nu_2 = \nu_2(\widetilde{K})$*, then there exists a* ν_0 *such that*

$$\omega(x_0) \le \frac{C\nu^{1/2}}{\left(dist\{x_0, \partial\Omega\}\right)^2}, \quad \nu \le \nu_0 \tag{3.5}$$

where $C = C(k^{-1}, \alpha, \widetilde{K}, K, \mathcal{R}, \mathcal{R}_U, M, \Phi_L, K^*)$ *and*
$\nu_0 = \min\left\{\left(dist\{x_0, \partial\Omega\}\right)^4, m(k, \alpha, K, \widetilde{K}, \mathcal{R}, \mathcal{R}_U, \mathcal{R}_L, M, \Phi_L, \Phi_U), \nu_2(\widetilde{K})\right\}$.

Proof: Dropping the superscript ν, let $\sup_{\overline{\Omega}} \frac{|\nabla\varphi|^2}{2} \le \widetilde{K}$ uniformly in ν for $\nu \le \nu_2(\widetilde{K})$, then $\omega \le \widetilde{K} + K + \mathcal{R}_U - q\Phi_L$. Also, since $W = \omega + K - \mathcal{R}(\varphi) + q\Phi = \frac{|\nabla\varphi|^2}{2}$, then $1 < 1 + W \le 1 + \widetilde{K}$.

Then, from Lemma 3.1, on any open set where ω is positive , combining conditions (1.3), (1.4) and (3.2) on the differential inequality (3.4), it can be seen that ω satisfies

$$\begin{aligned}
\Delta\omega \ge\; & \frac{k}{2(n-1)\nu^2}\frac{\omega^2}{(1+\widetilde{K})^\alpha} - \frac{2}{(n-1)k}\frac{|\nabla\omega|^2}{(K - \mathcal{R}_U + q\Phi_L)} \\
& - \frac{1}{\nu}|\nabla\omega|\left[\frac{4\nu}{(n-1)k}\left(\frac{qM}{K - \mathcal{R}_U + q\Phi_L} + \mathcal{R}(K - \mathcal{R}_U + q\Phi_L)^{-1/2}\right)\right. \\
& \quad \left. + \widetilde{K}^{1/2}\left(1 + \frac{\alpha}{2}(\widetilde{K} + K^*)\right)\right] \\
& - \frac{1}{\nu}\left\{(\mathcal{R}\widetilde{K}^2 + qM\widetilde{K})\left(1 + \frac{\alpha}{2}(\widetilde{K} + K^*)\right) + (\widetilde{K} + K^*)\mathcal{R}\right\} \\
& - \frac{4}{(n-1)k}\left(-\mathcal{R} + \frac{qM}{(2(K - \mathcal{R}_U + q\Phi_L))^{1/2}}\right)^2 - q\alpha i^{-1}(K^*)
\end{aligned}$$

or, equivalently, in a more compact form,

$$\Delta\omega \ge \frac{k}{2(n-1)(1+\widetilde{K})^\alpha}\frac{\omega^2}{\nu^2} - \frac{2}{(n-1)k}\frac{|\nabla\omega|^2}{(K - \mathcal{R}_U + q\Phi_L)} - \frac{1}{\nu}|\nabla\omega|(\nu A_1 + A_2) - \frac{1}{\nu}(A_3 + \nu A_4), \tag{3.6}$$

on any open set where $\omega > 0$, where $A_i = A_i(k, \alpha, \widetilde{K}, K, \mathcal{R}, \mathcal{R}_L, \mathcal{R}_U, \Phi_L, \Phi_U, K^*)$ are given by

$$
\begin{aligned}
A_1 &= \frac{4}{(n-1)k}\left(\frac{qM}{K-\mathcal{R}_U+q\Phi_L}+\mathcal{R}(K-\mathcal{R}_U+q\Phi_L)\right)^{-1/2}\\
A_2 &= \widetilde{K}^{1/2}\left(1+\frac{\alpha}{2}(\widetilde{K}+K^*)\right)\\
A_3 &= (\mathcal{R}\widetilde{K}^2+qM\widetilde{K})\left(1+\frac{\alpha}{2}(\widetilde{K}+K^*)\right)+(\widetilde{K}+K^*)\,\mathcal{R}\\
A_4 &= \frac{4}{(n-1)k}\left(-\mathcal{R}+\frac{qM}{(2(K-\mathcal{R}_U+q\Phi_L))^{1/2}}\right)^2+q\alpha i^{-1}(K^*)
\end{aligned}
\tag{3.7}
$$

The remainder of the proof of this theorem is identical to the one of Theorem 2.2. Indeed, here also it can be shown that for any $x_0 \in \Omega$ where $\omega(x_0) > 0$, there is a ball of radius r with $\overline{B_r(x_0)} \subset \Omega$, and there is a ν_0, such that ω is below a differentiable positive function f^ν in $\overline{B_r(x_0)}$ for all $\nu \leq \nu_0$ and

$$
f^\nu(x_0) = \frac{\nu^{1/2}\ C(k,\widetilde{K},\alpha,K,\mathcal{R},\mathcal{R}_U,\mathcal{R}_L,M,\Phi_U,\Phi_L,K^*)}{(\ \mathrm{dist}\{x_0,\partial\Omega\})^2}
\tag{3.8}
$$

is the minimum value of f in $\underline{B_r(x_0)}$. That is to say, $\omega - f$ does not have a positive maximum in $\overline{B_r(x_0)}$.

Therefore, the proof of the following lemma completes the proof of Theorem 3.2.

Lemma 3.3. *Let x_0 be in the interior of Ω, then there exists a $\overline{B_r(x_0)} \subset \Omega$, a $\nu_1 > 0$ and a differentiable function f^ν in $\overline{B_r(x_0)}$ such that, for $\nu \leq \nu_1$*

$$
\min_{\overline{B_r(x_0)}} f^\nu(x) = f^\nu(x_0) \leq \frac{C\ \nu^{1/2}}{(\ dist\{x_0,\partial\Omega\})^2}
$$

and f^ν is a barrier function for the solution of the differential inequality (3.6) in $\overline{B_r(x_0)}$. That means (dropping the superscript ν from f)

$$
\Delta f < \frac{k}{2(n-1)(1+\widetilde{K})^\alpha}\frac{f^2}{\nu^2} - \frac{2}{(n-1)k}\frac{|\nabla f|^2}{(K-\mathcal{R}_U+q\Phi_L)} - \frac{1}{\nu}|\nabla\omega|(\nu A_1+A_2)+\frac{1}{\nu}(A_3+\nu A_4)
\tag{3.9}
$$

in the interior of $B_r(x_0)$ and

$$
f > \widetilde{K}+K-\mathcal{R}_U+q\Phi_L \ \ on\ \partial B_r(x_0).
\tag{3.10}
$$

The radius r is given by

$$
r = \min\{\ dist\{x_0,\partial\Omega\}, m(k,\alpha,K,\widetilde{K},\mathcal{R},\mathcal{R}_U,M,\Phi_U,K^*)\}
$$

and the constant C is a $C = C(k^{-1}, \alpha, \widetilde{K}, K, \mathcal{R}, \mathcal{R}_U, M, \Phi_U, K^*)$ *and* $\nu_1 \leq r^4$.

Proof: Again the proof is very similar to the one of Lemma 2.3. However in this case f looses a half power in ν, because the inequality (3.6), or the consequent one (3.9) has the extra term $\frac{1}{\nu}(A_3 + \nu A_4)$ due to the coupling with the Poisson equation and the effect of the relaxation term $R(\varphi)$. Notice that if $\mathcal{R}, \mathcal{R}_L, \mathcal{R}_U, M, \Phi_L, \Phi_u$ and q are set to zero then we recover the conditions of Theorem 2.2 and Lemma 2.3, since $K^* = K$ when $K - \mathcal{R}_U + q\Phi_L = K$ (and (3.9) and (3.10) became (2.10) and (2.11) respectively).

Here, let

$$f(x) = g(|x - x_0|) = g(t) = \frac{A\nu^{1/2}}{\overline{r}^2 - t^2} \tag{3.11}$$

where $\overline{r} = r + \frac{\nu^{1/2}}{r}$, and r to be fixed.

Now, in order to satisfy (3.10) set

$$g(r) = \frac{A\nu^{1/2}}{\overline{r}^2 - r^2} > \widetilde{K} + K - \mathcal{R}_U + q\Phi_L$$

then, A and r must satisfy

$$\frac{A\nu^{1/2}}{2\nu^{1/2} + \frac{\nu}{r^2}} > \frac{A}{2 + \frac{\nu^{1/2}}{r^2}} > \frac{A}{3} > \widetilde{K} + K - \mathcal{R}_U + q\Phi_L$$

which holds whenever

$$\nu^{1/2} < r^2 \tag{3.12}$$

and

$$A > 3(\widetilde{K} + K - \mathcal{R}_U + q\Phi_L) > 0. \tag{3.13}$$

In addition recasting (3.9), we need to show

$$\mathcal{P}(f) \;=\; \Delta f - \mathcal{A}\frac{f^2}{\nu^2} + \mathcal{B}|\nabla f|^2 + \frac{1}{\nu}|\nabla f|\mathcal{C} + \frac{1}{\nu}\mathcal{D} \;< 0 \tag{3.14}$$

where $\mathcal{A} = \frac{k}{2(n-1)(1+\widetilde{K})^{\alpha}}$, $\mathcal{B} = \frac{2}{(n-1)k(K-\mathcal{R}_U+q\Phi_L)}$, $\mathcal{C} = \nu A_1 + A_2$ and $\mathcal{D} = A_3 + \nu A_4$. Setting $D = \overline{r}^2 - t^2$, $0 \leq t \leq r$ as in Lemma 2.3, and computing the operator (3.14) acting on f yields

$$\mathcal{P}(f) \;=\; -2\frac{A\nu^{1/2}}{D^2} + \frac{4A\nu^{1/2}t^2}{D^3} - \mathcal{A}\frac{A^2}{\nu D^2} + \mathcal{B}\frac{A^2\nu}{D^4}t^2 + \frac{\mathcal{C}}{\nu^{1/2}}\frac{A^2}{D^2}t + \frac{1}{\nu}\mathcal{D}$$

which is majorized by

$$\mathcal{P}(f) \ \le\ 4A\nu^{1/2}\frac{r^2}{D^3} - \mathcal{A}\frac{A^2}{\nu D^2} + \mathcal{B}A^2\nu\frac{r^2}{D^4} + \frac{\mathcal{C}}{\nu^{1/2}}\frac{A^2}{D^2}r + \frac{1}{\nu}\mathcal{D}. \tag{3.15}$$

Now since $D = \overline{r}^2 - t^2 \ge \overline{r}^2 - r^2 = 2\nu^{1/2} + \frac{\nu}{r^2} > 2\nu^{1/2}$, then $\frac{2\nu^{1/2}}{D} < 1$, so that the RHS of (3.15) is dominated by

$$4\frac{A}{D^2}r^2 - \frac{\mathcal{A}}{\nu}\frac{A^2}{D^2} + \frac{\mathcal{B}}{4}\frac{A^2}{D^2}r^2 + \frac{\mathcal{C}}{\nu^{1/2}}\frac{A^2}{D^2}r + \frac{1}{\nu}\mathcal{D}. \tag{3.16}$$

To see that there is an A and r such that (3.16) is strictly negative, multiply (3.16) by $\frac{\nu D^2}{A}$ (A will be chosen positive), then

$$\mathcal{P}(f) \ \le\ 4\nu r^2 - \mathcal{A}A + \mathcal{D}\frac{D^2}{A} + \nu^{1/2}(\nu^{1/2}\mathcal{B}r + \mathcal{C})Ar. \tag{3.17}$$

Since $D \le \overline{r}^2 < (2r)^2$, and using condition (3.13) for A, we see that (3.17) is dominated by

$$-\mathcal{A}A + 4\frac{\mathcal{D}r^2}{3\widetilde{K} + K - \mathcal{R}_U + q\Phi_L} + \nu^{1/2}(\nu^{1/2}\mathcal{B}r + \mathcal{C})Ar + 4\nu r^2. \tag{3.18}$$

Now if $\nu^{1/2} < r^2 < 1$, then for any $\nu \le \nu_1$

$$\nu^{1/2}(\nu^{1/2}\mathcal{B}r + \mathcal{C})Ar \le r(\mathcal{B} + \mathcal{C})A,$$

then (3.18) is dominated by

$$\mathcal{P}(f) \ \le (-\mathcal{A} + r(\mathcal{B} + \mathcal{C}))A + \frac{4\mathcal{D}}{3(\widetilde{K} + K - \mathcal{R}_U + q\Phi_L)} + 4. \tag{3.19}$$

Now, we choose r so small that $r \le m$ with

$$m = \min\left\{\frac{\mathcal{A}}{2(\mathcal{B} + \mathcal{C})}, 1\right\}. \tag{3.20}$$

Then (3.19) yields

$$\mathcal{P}(f) \ \le\ -\frac{\mathcal{A}}{2}A + \frac{4\mathcal{D}}{3(\widetilde{K} + K - \mathcal{R}_U + q\Phi_L)} + 4 \ <\ 0 \tag{3.21}$$

whenever

$$A > \frac{8}{\mathcal{A}}\left(\frac{\mathcal{D}}{3(\widetilde{K} + K - \mathcal{R}_U + q\Phi_L)} + 1\right). \tag{3.22}$$

From (3.14) we have $\mathcal{D} = A_3 + \nu A_4 \le A_3 + A_4$, $\mathcal{A} = \frac{k}{2(n-1)(1+\widetilde{K})^\alpha}$, $\mathcal{B} = \frac{2}{(n-1)k}(K - \mathcal{R}_U + q\Phi_L)^{-1}$, and $\mathcal{C} = \nu A_1 + A_2 \le A_1 + A_2$. So using (3.12)–(3.13), the function

$$f(x) = \frac{k^{-1}\, A\, \nu^{1/2}}{\overline{r}^2 - (x - x_0)^2} \qquad \overline{r} = r + \frac{\nu^{\frac{1}{2}}}{r} \tag{3.23}$$

satisfies (3.9) and (3.10). Therefore, we choose

$$A = \max\left\{4(\widetilde{K} + K - \mathcal{R}_U + q\Phi_L), \frac{8}{k}(1+\widetilde{K})^\alpha\left(\frac{A_3 + A_4}{3(\widetilde{K} + K - \mathcal{R}_U + q\Phi_L)} + 1\right)\right\}, \tag{3.24}$$

and

$$r \le \min\{\ \mathrm{dist}\{x_0, \partial\Omega\};\, m\}\,, \tag{3.25}$$

m from (3.20) and,

$$\nu \le \nu_0 = \min\left\{(\ \mathrm{dist}\{x_0, \partial\Omega\})^4;\, m^4, \nu_2\right\}.$$

Thus from the dependence of $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ if we write $C = A$ from (3.24) then

$$\begin{aligned} &C = C(k^{-1}, \alpha, K, \widetilde{K}, \mathcal{R}, \mathcal{R}_U, M, \Phi_L, K^*) \text{ and} \\ &m = m\{k, \alpha, K, \widetilde{K}, \mathcal{R}, \mathcal{R}_U, M, \Phi_L, K^*\} \end{aligned} \tag{3.27}$$

from (3.20), With these choices we get

$$f(x_0) \le \frac{C\ \nu^{1/2}}{(\ \mathrm{dist}\{x_0, \partial\Omega\})^2}, \qquad \text{for } \nu \le \nu_0 \tag{3.28}$$

where

$$\nu_0 = \min\{(\ \mathrm{dist}\{x_0, \partial\Omega\})^4, m, \nu_2\} \tag{3.29}$$

with $\nu_2 = \nu_2(\widetilde{K})$, the one from the coarse bound.

The proof of Lemma 3.3 is now completed, yielding the completion of the proof for Theorem 3.2

4. Sharper estimates at the boundary.

As it has been shown in [GM], if the domain does not have locally "flat" (zero curvature) boundaries, then, the domain Ω is conformally transformed into Ω_F, a rectangle, such that the tangential flow walls are transformed into opposite walls of Ω_F. Since the flow equation maintains its structure under conformal transformations then system (1.1)–(1.4) is transformed into system

$$
\text{(4.1)}\quad
\begin{aligned}
&\text{i)} && x' = F(x), \quad F\colon \Omega \to \Omega_F\\
&\text{ii)} && \Delta'\varphi' = -\nabla' \ln\rho' \nabla\varphi'\\
&\text{iii)} && \Delta'\varphi' = \frac{|F_x^{-1}|^2}{\nu}\,\frac{[(|\nabla'\varphi'|\,|F_x|)^2 - K + i(\rho') + \mathcal{R}(\varphi') - q\Phi']}{g\,(|\nabla\varphi'|\,|F_x|)}\\
&\text{iv)} && \Delta'\Phi' = \alpha|F_x^{-1}|^2(\rho' - C'(x'))
\end{aligned}
$$

where $\Delta'\varphi' = \Delta\varphi|F_x^{-1}|^2$ and $\nabla_{x'}\varphi' = |F_x^{-1}|\nabla\varphi$ and $|F_x|$ and $|F_x^{-1}|$ denote the Jacobian of the associated with the real valued local change of coordinates that correspond to the conformal map that takes Ω into the rectangle Ω_F.

In order to obtain a pointwise estimate for $|\nabla'\varphi'|$ at $x'_0 \in \partial_1\Omega_F$, the inflow boundary, we also use the comparison techniques developed in sections 2 and 3, but now the comparison is done with f^ν in a neighborhood $B_{r(\nu)}(x_0)$, where the radius $r(\nu)$ depends on ν as well. This introduces no new difficulty, and it allows us to construct ν dependent barrier functions for an appropriate differential inequality satisfied by a quantity ω that corresponds to cavitation speed, (i.e. $\omega = (|\nabla'\varphi'|\,|F_x|)^2 - K + \mathcal{R}(\varphi') - q\Phi'$) in neighborhoods $\mathcal{N}(x_0,\nu)$ depending on x_0 and ν.

We first consider the tangential flow boundary, later the inflow boundary. At the boundary section $\partial_3\Omega_F$ the speed is prescribed.

4.1. Estimates at the tangential flow boundaries.

As in previous sections, we start with a lemma, proved in the appendix, showing that the quantity $\omega = \frac{(|\nabla'\varphi'^\nu|\,|F_x|)^2}{2} - K + \mathcal{R}(\varphi'^\nu) - q\Phi'^\nu$ satisfies a differential inequality leading to a maximum principle for values of ω above cavitation speed (i.e. $\omega > 0$).

Lemma 4.1. *Let* $\omega = \frac{(|\nabla'\varphi'|\,|F_x|)^2}{2} - K + \mathcal{R}(\varphi') - q\Phi'$. *Let* $W = \omega + K - \mathcal{R}(\varphi') + q\Phi'$, *then in the neighborhood* $\mathcal{N}'$ *where equations (4.1. i–iv) are valid,* ω *satisfies*

$$
\begin{aligned}
\Delta\omega \geq\ & \frac{1}{2\nu^2}\frac{k\omega^2}{(1+W)^\alpha} - \frac{4}{k\mathcal{M}|F_x|^4}|\nabla\omega|^2 - \frac{|\nabla\omega|}{\nu}\left\{\frac{2W^{1/2}}{|F_x|}\left(\frac{1}{(1+W)^{\alpha/2}} - \frac{\alpha}{2}\frac{\omega+i(\rho)}{(1+W)^{\alpha/2+1}}\right)\right.\\
&\left. +\nu\left[2\left|\nabla|F_x|^{-4}\right| + \frac{8}{k\mathcal{M}}\left(|F_x|^{-4}(\mathcal{R}\mathcal{M}+qM) + (K - \mathcal{R}_U + q\Phi_L + \omega)\left|\nabla|F_x|^{-4}\right|\right)\right]\right\}\\
&- \frac{1}{\nu}\left\{2W^{1/2}\frac{(\omega+K^*)}{(1+W)^{\alpha/2}}\left|\nabla|F_x|^{-4}\right| + \frac{\mathcal{R}(\omega+K^*)}{(1+W)^{\alpha/2}}\right\}\\
\text{(4.2)}\quad &- \frac{4}{k\mathcal{M}}\left|\nabla|F_x|^{-2}\right|^2(\omega + (K - \mathcal{R}_U + q\Phi_L))^2\\
&- \frac{4}{k\mathcal{M}}(\mathcal{R}\mathcal{M}+qM)^2|F_x|^{-4} - q\alpha K^* - \frac{W\Delta|F_x|^2}{|F_x|^2}\\
&- \left|\nabla|F_x|^{-4}\right|\left\{\mathcal{R}\mathcal{M} + qM + \frac{8}{k\mathcal{M}}(\mathcal{R}\mathcal{M}+qM)(\omega + K - \mathcal{R}_U + q\Phi_L)\right\}
\end{aligned}
$$

on any open region of $\Omega_F \cap \mathcal{N}$ *where* $\omega > 0$. *Here* $\mathcal{M} = \sqrt{2}\max_{\overline{\mathcal{N}}} |F_x|^{-4}(K - \mathcal{R}_U + q\Phi_L) > 0$, α *the parameter of the function* g *from condition (1.4) and* $k = k(i(\rho))$ *from condition (1.2),* K^* *comes from (3.2),* K *satisfies* $K - \mathcal{R}_U + q\Phi_L > 0$ *from the compatibility condition (3.1), and* M *here denotes* $\|\Phi^\nu\|_{C^{0,1}(\overline{\Omega})}|F_x|^{-1}$.

Next, as in previous sections we are in conditions to prove the following theorem that yields sharper uniform bounds for the speed on any point of the tangential boundary $\partial_\tau\Omega = \partial_2\Omega \cup \partial_4\Omega$ it already has a global coarse ν-uniform bound in $\overline{\Omega}$.

Remark. It is worth to remark at this point that if the boundary is locally flat in $\mathcal{N}$ then $|F_x| = 1 = |F_x|^{-1}$ and $\nabla|F_x| = 0 = \Delta|F_x|$, so that the differential inequality (a.30) is identical to (a.19). On the other hand, if $\mathcal{R} = \mathcal{R}_U = \mathcal{R}_L = M = \Phi_U = \Phi_L = 0$ then $K = K^*$ then (a.30) corresponds to the differential inequality that is satisfied by the speed above cavitation values (i.e. $\frac{|\nabla\varphi|^2}{2} \geq K$) for the viscous approximation to transonic flow.

Theorem 4.2. Sharper uniform estimates on the tangential flow boundaries. *Let* x_0 *belong to the interior of* $\partial_\tau\Omega = \partial_2\Omega \cup \partial_4\Omega$ *relatively to* $\partial\Omega$, *and let* $\omega^\nu = \frac{|\nabla\varphi^\nu|^2}{2} - K + R(\varphi^\nu) - q\Phi^\nu$, *where* $(\varphi^\nu, \rho^\nu, \Phi^\nu)$ *solves the boundary value problem presented in section 1 associated with (1.1)–(1.4). Let* $\widetilde{K}$ *be an* ν*-uniform upper bound for* $\frac{|\nabla\varphi^\nu|^2}{2}$ *valid for all* $\nu \leq \nu_2(\widetilde{K})$. *Then there exists*
$\nu_0 = \min\left\{(dist\{x_0, \partial\Omega\backslash\partial_\tau\Omega\})^4, m(k, \alpha, \widetilde{K}, K, \mathcal{R}_U, \mathcal{R}_L, \mathcal{R}, M, \Phi_U, \Phi_L), \nu_2(\widetilde{K})\right\}$, *such that*

$$\omega(x_0) \leq \frac{\mathcal{C}\ \nu^{1/2}}{(dist\{x_0, \partial\Omega\backslash\partial_\tau\Omega\})^2} \tag{4.3}$$

where $\mathcal{C} = \mathcal{C}(k^{-1}, \alpha, \widetilde{K}, K, \mathcal{R}, \mathcal{R}_U, M, \Phi_L, K^*, \kappa_{x_0})$ *where* $\kappa_{x_0} = \|F_x\|_{C^1(\mathcal{N}(x_0))}$ *and* $|F_x|$ *denotes the Jacobian of the transformation that rectifies (locally) the boundary at* x_0, *with that* $|F_x| > c > 0$ *uniformly up to the boundary, and* $\mathcal{N}$ *is a neighborhood of* x_0.

Proof: Let x_0 be in the interior of $\partial_\tau\Omega$ relatively to $\partial\Omega$, then let $\mathcal{N}$ be a neighborhood of x_0 and $F|_{\mathcal{N}}$: $\mathcal{N} \to \mathcal{N}'$ the local change of variables defined on $\mathcal{N}$ by the conformal map F that takes Ω into the rectangle Ω_F. Clearly $\mathcal{N}'$ is a neighborhood of $x_0' = F(x_0)$, and the transformed maps $({\varphi'}^\nu, {\rho'}^\nu, {\Phi'}^\nu)$ solve the transformed boundary value problem associated with (4.1) i–iv).

Thus, the quantity ω_F defined as

$$\omega_F = \frac{|\nabla{\varphi'}^\nu|^2|F_x|^2}{2} - K + \mathcal{R}({\varphi'}^\nu) - q{\Phi'}^\nu \leq \widetilde{K} - K + \mathcal{R}_U - q\Phi_L \tag{4.4}$$

and

$$W = \omega_F + K - \mathcal{R}({\varphi'}^\nu) + q{\Phi'}^\nu = \frac{|\nabla{\varphi'}^\nu|^2|F_x|^2}{2} \leq \widetilde{K}. \tag{4.5}$$

Now, since the transformation takes the condition $\nabla\varphi \cdot n(x) = 0$ into $\nabla\varphi' \cdot n'(x') = 0$ for $x' \in \mathcal{N}'$ where now $\mathcal{N}' \cap \Omega_F$ is "flat" (or a section of a segment in our 2-dimensional configuration,) then the quantity ω_F satisfies

$$\nabla\omega_F \cdot n' = \frac{\nabla(|\nabla\varphi'^{\nu}|^2|F_x|^2)}{2} \cdot n' + \mathcal{R}'\nabla\varphi' \cdot n' + q\nabla\Phi' \cdot n' = 0. \tag{4.6}$$

Thus, we may reflect ω_F evenly with respect to $\mathcal{N}' \cap \Omega_F$ to define ω_F^e in $\mathcal{N}'$. From standard elliptic theory and extension of their solutions across "flat" boundaries by even reflections, it follows that ω_F^e is exactly $\frac{(|\nabla'\varphi'^{\nu^e}||F_x^e|)^2}{2} - K + \mathcal{R}(\varphi'^{\nu^e}) - q\Phi'^{\nu^e}$ where $(\varphi'^{\nu^e}, \rho'^{\nu^e}, \Phi'^{\nu^e})$ solve locally the reflected problem associated with (4.1) i–iv) in $\mathcal{N}'$ when reflecting across flat sections of the tangential flow boundary. This is achieved by reflecting $|F_x|$ evenly, and recalling that $\nabla\varphi'^{\nu} \cdot n = 0$ on $\mathcal{N}' \cap \Omega_F$ provide the necessary compatibility condition in order to obtain regular solutions. In addition, the reflected solutions inherit all the bounds and regularity of the original one.

Therefore, $\omega = \omega_F^e$ satisfies a differential inequality (4.2) on any open region where $\omega > 0$ where $W = \omega_F^e + K - \mathcal{R}(\varphi^e) + q\Phi^e$.

Thus, $\omega \le \widetilde{K} + K + \mathcal{R}_U - q\Phi_L$ and $1 \le W \le \widetilde{K}$ in $\mathcal{N}$, so the right-hand side of (4.2) dominates the one given by

$$\begin{aligned}
&\frac{k}{(1+\widetilde{K})^\alpha}\frac{\omega^2}{2\nu^2} - \frac{4}{k\mathcal{M}}\max_{\mathcal{N}}|F_x|^{-4}|\nabla\omega|^2 \\
&- \frac{|\nabla\omega|}{\nu}\left\{\widetilde{K}^{1/2}\max_{\mathcal{N}}|F_x|^{-1}\left(1 + \frac{\alpha}{2}\widetilde{K} - K + \mathcal{R}_U - q\Phi_L + K^*\right)\right. \\
&\left. + \nu\left[\max_{\overline{\mathcal{N}}}|\nabla|F_x|^{-4}| + \frac{8}{k\mathcal{M}}(\max_{\overline{\mathcal{N}}}|F_x|^{-4}(\mathcal{R}\mathcal{M} + qM) + \widetilde{K}\max_{\overline{\mathcal{N}}}|\nabla|F_x|^{-4}|)\right]\right\} \\
&- \frac{1}{\nu}\left(\max_{\overline{\mathcal{N}}}|\nabla|F_x|^{-4}|2\widetilde{K}^{1/2} + \mathcal{R}\right)\left(\widetilde{K} - K + \mathcal{R}_U - q\Phi_L + K^*\right) \\
&- \left\{\frac{4}{k\mathcal{M}}(\max_{\overline{\mathcal{N}}}|\nabla|F_x|^{-2}|^2\widetilde{K}^2 + (\mathcal{R}\mathcal{M} + qM)^2\max_{\mathcal{N}}|F_x|^{-4}) + q\alpha K^* + \widetilde{K}\max_{\overline{\mathcal{N}}}\frac{\Delta|F_x|^2}{|F_x|^2}\right. \\
&\left. + \max_{\overline{\mathcal{N}}}|\nabla|F_x|^{-4}|\left\{(\mathcal{R}\mathcal{M} + qM)\left(1 + \frac{8\widetilde{K}}{k\mathcal{M}}\right)\right\}\right\}.
\end{aligned} \tag{4.7}$$

As in Theorem 3.2 we write expression (4.7) as

$$\frac{k}{(1+\widetilde{K})^\alpha}\frac{\omega^2}{2\nu^2} - \frac{4}{k\mathcal{M}}\max_{\mathcal{N}}|F_x|^{-4}|\nabla\omega|^2 - \frac{1}{\nu}|\nabla\omega|(\nu A_1 + A_2) - \frac{1}{\nu}(A_3 + \nu A_4) \tag{4.8}$$

where

$$
\begin{aligned}
A_1 &= \max_{\overline{\mathcal N}} \left|\nabla |F_x|^{-4}\right| + \frac{8}{k\mathcal M}\big(\max_{\overline{\mathcal N}} |F_x|^{-4}(\mathcal R\mathcal M + qM) + \widetilde K \max_{\overline{\mathcal N}} \left|\nabla |F_x|^{-4}\right|\big),\\
A_2 &= \widetilde K^{1/2}\left(1 + \frac{\alpha}{2}\widetilde K - K + \mathcal R_U - q\Phi_L + K^*\right),\\
A_3 &= \left(\max_{\overline{\mathcal N}} \left|\nabla |F_x|^{-4}\right| 2\widetilde K^{1/2} + \mathcal R\right)\left(\widetilde K + \mathcal R_U - \mathcal R_L + q(-\Phi_L + \Phi_U)\right),\\
A_4 &= \frac{4}{k\mathcal M}\left(\max_{\overline{\mathcal N}} \left|\nabla |F_x|^{-2}\right|^2 \widetilde K^2 + \max_{\overline{\mathcal N}} |F_x|^{-4}(\mathcal R\mathcal M + qM)^2\right) + q\alpha K^*,\\
&\quad + \max_{\overline{\mathcal N}} \frac{\left|\Delta |F_x|^2\right|}{|F_x|^2}\widetilde K + \max_{\overline{\mathcal N}} \left|\nabla |F_x|^{-4}\right| \left((\mathcal R\mathcal M + qM)\left(1 + \frac{8\widetilde K}{k\mathcal M}\right)\right).
\end{aligned} \tag{4.9}
$$

Thus, $\omega = \omega_F^e \le \widetilde K - K + \mathcal R_U - q\Phi_L$ satisfies the differential inequality

$$
\Delta\omega - \mathcal A\frac{\omega^2}{\nu^2} + \mathcal B|\nabla\omega|^2 + \frac{1}{\nu}\mathcal C|\nabla\omega| + \frac{1}{\nu}\mathcal D \ge 0 \tag{4.10}
$$

on any open region of $\mathcal N$ where $\omega > 0$, and

$$
\mathcal A = \frac{k}{2(1+\widetilde K)^\alpha}, \quad \mathcal B = \frac{4}{k\sqrt 2(K - \mathcal R_U + q\Phi_L)}. \tag{4.11}
$$

$$
\mathcal C = \nu A_1 + A_2 \quad \text{and} \quad \mathcal D = A_3 + \nu A_4
$$

where $A_i, i = 1, 4$ defined in (4.9).

Note that the differential inequality has exactly the same structure as the corresponding one for the interior estimate of the fluid Poisson systems (see (3.6)). Then the proof of this theorem is completed by just pointing out two facts:

i) We can construct a barrier function f in order to get control of ω at $x_0' \in \mathcal N'$. This argument is like that of Lemma 3.3, now ν_0, $\mathcal C$ and R depend on $\|F_x\|_{C^1(\overline{\mathcal N})}$ and $\|F_x^{-1}\|_{C^1(\overline{\mathcal N})}$. Thus, the analogue of Lemma 3.3 is now as

Lemma 4.3. *Let $x_0' \in \partial_\tau\Omega_F \cap \mathcal N'$. Then there exists a $\overline{B_r(x_0')} \subset \mathcal N'$, a ν_0 and a differentiable function f^ν in $\overline{B_r(x_0')}$ such that for $\nu \le \nu_0$ with the following properties: first*

$$
\min_{B_r(x_0)} f^\nu(x') = f^\nu(x_0') \le \frac{C\ \nu^{1/2}}{\left(\ dist\{x_0', \partial\Omega_F \backslash \partial_\tau\Omega_F\}\right)^2}; \tag{4.12}
$$

and second f^ν is a barrier function for the solution of the differential inequality (4.10) in $\overline{\mathcal B_r(x_0)}$. That means f^ν is a supersolution of the operator associated with (4.10) and $f > \widetilde K + K - \mathcal R_U + q\Phi_L$ on $\partial B_r(x_0')$. Moreover we may take

$$
r = \min\{\ dist\{x_0', \partial\Omega_F\backslash\partial_\tau\Omega_F\};\ m(k, \alpha, K, \widetilde K, \mathcal R, \mathcal R_U, \mathcal R_L, M, \Phi_U, \Phi_L,\ \|F_x^{-1}\|_{C^1(\overline{\mathcal N'})}\}
$$

and

$$\nu_0 = \min\{r^4; \nu_2(\widetilde{K})\}.$$

Also denoting by $\kappa_{x_0} = \|F_x\|_{C^1(\overline{\mathcal{N}})} = \|F_x^{-1}\|_{C^1(\overline{\mathcal{N}'})}$, *the constant* C *depends on*

$$C = C(k^{-1}, \alpha, K, \widetilde{K}, \mathcal{R}, \mathcal{R}_U, \mathcal{R}_L, M, \Phi_U, \Phi_L, \kappa_{x_0}).$$

Proof. From (3.24) and (3.25) of the proof of Lemma 3.3, it follows that the parameters in Lemma 4.3 are

$$\nu_0 = \min\left\{ \left(\text{ dist}\{x_0', \partial\Omega_F \backslash \partial_\tau \Omega_F\}\right)^4, \left(\frac{\mathcal{A}}{\mathcal{B}+\mathcal{C}}\right)^4, \nu_2(\widetilde{K}) \right\}, \tag{4.13}$$

with $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ from (4.11), and

$$C = \max\left\{ 4(\widetilde{K} + K - \mathcal{R}_U + q\Phi_L), \frac{8}{k}(1+\widetilde{K})^\alpha \left(\frac{(A_3 + A_4)}{3(\widetilde{K} + K - \mathcal{R}_U + q\Phi_L)} + 1 \right) \right\}, \tag{4.14}$$

with A_3 and A_4 from (4.9), and for $\text{dist}\{x_0', \partial\mathcal{N}'\} > \text{ dist}\{x_0', \partial\Omega_F\backslash\partial_\tau\Omega_F\}$

$$r = \min\left\{ \text{ dist}\{x_0', \partial\Omega_F\backslash\partial_\tau\Omega_F\}; m \right\}, \qquad \text{with } m = \min\left\{ \frac{\mathcal{A}}{\mathcal{B}+\mathcal{C}}; 1 \right\}, \tag{4.15}$$

$\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ from (4.11). The rest of the proof is identical to that of Lemma 3.3

ii) The comparison theorem presented in section 2 shows that $\omega_F^e - f$ can not have a positive maximum in $\overline{B_r(x_0)}$. Thus, for $\nu \le \nu_0$, (ν_0 being given by (4.13))

$$\left(\frac{|\nabla\varphi|^2}{2} - K + \mathcal{R}(\varphi) - q\Phi\right)(x_0) = \omega_F(x_0') = \omega_F^e(x_0') \le f(x_0') = \frac{C\ \nu^{1/2}}{\left(\text{ dist}\{x_0', \partial\Omega_F\backslash\partial_\tau\Omega_F\}\right)^2}.$$

Therefore for x_0 in the tangential boundary of the original domain

$$\frac{|\nabla\varphi|^2}{2}(x_0) \le K - \mathcal{R}(\varphi)(x_0) + q\Phi(x_0) + \frac{C\ \nu^{1/2}}{\left(\text{ dist}\{F(x_0), \partial\Omega_F\backslash\partial_\tau\Omega_F\}\right)^2} \tag{4.16}$$

where $\nu \le \nu_0$, ν_0 from (4.13) and C is given by (4.14).

Since $\text{dist}\{F(x_0), \partial\Omega_F\backslash\partial_\tau\Omega_F\} \ge |F_x|^{-1} \text{ dist}\{x_0, \partial\Omega\backslash\partial_\tau\Omega\}$, (4.3) follows for $\mathcal{C} = C|F_x|^{-1}$. The proof of Theorem 4.2 is now complete.

4.2. The inflow boundary region $\partial_1\Omega$.

The boundary condition $\varphi =$ constant on $\partial_1\Omega$ and $(\nabla\varphi \cdot n|_{\partial_1\Omega})(x_0) < 0$ combined with the existence theory yields that $\nabla\varphi \cdot n < 0$ and $\nabla\varphi\cdot\tau = 0$ along $\partial_1\Omega$, where n and τ are the outer unit normal and unit tangent directions to $\partial_1\Omega$.

Similarly on the transformed domain Ω_F, $0 > \nabla\varphi\cdot n = \nabla\varphi'|F_x|\cdot n' = -\varphi'_{x'}|F_x|$ as $n' = (-1, 0)$ on $\partial_1\Omega_F$ and $\varphi'_{y'} = \varphi'_{y'y'} = 0$ on $\partial_1\Omega_F$.

Computing $\omega = \omega_F$ as $\omega = \frac{(|\nabla'\varphi'||F_x|)^2}{2} - K + \mathcal{R}(\varphi') - q\Phi'$, we get (after dropping the primes)

$$\nabla\omega\cdot n = -\varphi_x\varphi_{xx}|F_x|^2 - \mathcal{R}'\cdot\varphi_x + q\Phi_x - \varphi_x^2|F_x|^2\,|\nabla \ln|F_x||$$

with φ_x positive.

We analyze this with the equation

$$\Delta\varphi = \frac{1}{\nu}\frac{\omega + i(\rho)}{g(\omega)}, \tag{4.17}$$

in which $g(\omega) = (1+\omega+K-\mathcal{R}(\varphi)+q\Phi)^{\alpha/2}$ for $\omega > 0$, to set the following boundary condition for ω at the inflow boundary $\partial_1\Omega$

$$\begin{aligned}\nabla\omega\cdot n = &-\frac{2}{\nu}(\omega+K-\mathcal{R}(\varphi)+q\Phi)^{1/2}\left(\frac{\omega+i(\rho)}{g(\omega)} + \nu\mathcal{R}'(\varphi)\right)\\ &+ q\Phi_x + 2(\omega+K-\mathcal{R}(\varphi)+q\Phi)|\nabla \ln|F_x||.\end{aligned} \tag{4.18}$$

Using that $|\nabla\Phi| \le M$, $|\mathcal{R}'| \le \mathcal{R}$, and using the L^∞-bounds for $\mathcal{R}(\varphi)$ and Φ we can see that the normal derivative of ω at the boundary $\partial_1\Omega_F$ on a region where $\omega > 0$ is bounded by

$$\begin{aligned}\nabla\omega\cdot n \le &-\frac{\sqrt{2}}{\nu}(K-\mathcal{R}_U+q\Phi_L)^{1/2}(\omega+i(\rho)-\nu\mathcal{R}) + qM\\ &+ 2(\widetilde{K}+K-\mathcal{R}_L+q\Phi_U)\max_{\partial_1\Omega}|\nabla \ln|F_x||\\ \le\ &\sqrt{2}(K-\mathcal{R}_U+q\Phi_L)^{1/2}\mathcal{R} + qM\\ &+ 2(\widetilde{K}+K-\mathcal{R}_L+q\Phi_U)\max_{\partial_1\Omega}|\nabla\log|F_x||\\ =\ &\widetilde{\mathcal{A}}(\widetilde{K}, K, \mathcal{R}_L, \Phi_U, \|F_x\|_{C^1(\partial_1\Omega)}),\end{aligned} \tag{4.19}$$

since the compatibility condition (3.1) on the data implies $K - \mathcal{R}_L + q\Phi_U > 0$.

Once again a comparison estimate for ω can be obtained at the boundary. However this time it is necessary to use the maximum principle at the boundary.

Theorem 4.4. Sharper uniform estimate at the inflow boundary *Let $x_0 \in \partial_1\Omega$ and let ω^ν be ν-uniformly bounded in $\overline{\Omega}$ by a number $\widetilde{K}$ for, $\nu \le \nu_2(\widetilde{K})$.*

Then, there is a $\nu_0 = \min\left\{m(k,\alpha,\widetilde{K},K,\mathcal{R},\mathcal{R}_u,\mathcal{R}_L,M,\Phi_U,\Phi_L,\|F_x\|_{C^1(\overline{\mathcal{N}})});\nu_2(\widetilde{K})\right\}$ such that

$$\omega(x_0) \le C\,\nu^{1/8}, \quad \nu \le \nu_0 \tag{4.20}$$

with $C = C(k^{-1}, \alpha, \widetilde{K}, K, \mathcal{R}, \mathcal{R}_U, \mathcal{R}_L, M, \Phi_U, \Phi_L, \|F_x\|_{C^1(\overline{\mathcal{N}})})$.

Remark. The exponent 1/8 of ν is obtained from the construction of an upper barrier function described below. We shall see that for "flat" boundaries $\partial_1\Omega$, the exponent can be chosen 1.

Proof of Theorem 4.4. In order to get control on boundary points, we need to construct an upper barrier function f^ν at $x_0' \in \overline{\partial_1\Omega_F}$, such that it is not only a supersolution for the differential inequality (4.10) but also has a strictly larger normal derivative than $\mathcal{A}$ from condition (4.19). The following lemma yields such a barrier function.

First, see that if x_0' is an endpoint of $\partial_1\Omega_F$, then reflecting evenly with respect to the section of $\partial_\tau\Omega_F$ that contains x_0', both the domain and the solution of system 4.1 i)–iv), makes x_0' an interior point of the inflow boundary for the reflected problem (see that the boundary value problem is compatible for such reflection and regularity [G3] and [GM]).

Lemma 4.5. *Let* $x_0' \in \partial_1\Omega_F$ *(if* x_0' *is an endpoint we work with the even extension of the domain and equation), then there exists a* $\nu_0 = \nu_0(\alpha, k, \widetilde{K}, K, \mathcal{R}, \mathcal{R}_L, \mathcal{R}_U, M, \Phi_U, \Phi_L, \|F_x\|_{C^1(\overline{\mathcal{N}})})$, *and, a neighborhood* $\mathcal{N}'(\nu, x_0')$ *of* x_0' *for every* $\nu \le \nu_0$ *and a differential function* f^ν *in* $\overline{\mathcal{N}'(\nu, x_0')}$, $\nu \le \nu_0$, *such that*

$$f^\nu(x_0') \le C\ \nu^{1/8} \quad \nu \le \nu_1 \tag{4.21}$$

and f^ν *is an upper barrier function for the solution* ω *of (4.10) in* $\mathcal{N}'(\nu, x_0') \cap \overline{\Omega}_F$. *That is* f^ν *satisfies (dropping the supraindex* ν*)*

$$\begin{aligned} &i) \quad f \text{ is a supersolution of the differential operator} \\ &\qquad \text{associated with (4.10) in the interior of } \mathcal{N}'(x_0', \nu) \cap \Omega_F\ , \\ &ii) \quad f > \widetilde{K} + K - \mathcal{R}_U + q\Phi_L \text{ in } \Omega_F \cap \partial\mathcal{N}'(x_0', \nu)\ , \\ &iii) \quad \nabla f \cdot n > \tilde{\mathcal{A}}(\widetilde{K}, K, \mathcal{R}_L, \mathcal{R}_U, \Phi_L, \Phi_U, \|F_x\|_{C^1(\overline{\mathcal{N}})}) \text{ in } \partial\Omega_F \cup \mathcal{N}'(x_0', \nu). \end{aligned} \tag{4.22}$$

for $\nu \le \nu_0$ *and* $C = (\alpha, k^{-1}, \widetilde{K}, \mathcal{R}, \mathcal{R}_U, \mathcal{R}_L, M, \Phi_U, \Phi_L, \|F_x\|_{C^1(\overline{\mathcal{N}})})\}$.
Furthermore, $\mathcal{N}'(x_0', \nu) = B_{\frac{\nu^{1/8}}{2}}(x_0')$.

We postpone the proof of this lemma until the end of Theorem 4.4.

We are now in conditions to show that $\omega_F - f$ can not have a positive maximum in $\overline{\mathcal{N}'(x_0', \nu) \cap \Omega_F}$ where x_0' is a point where $\omega(x_0) = \omega_F(x_0') > 0$. Let ν_0 be the number obtained in Lemma 4.5. Since $\omega \le \widetilde{K} + K - \mathcal{R}_U + q\Phi_L$ for $\nu \le \nu_2$, then by (4.22) ii) $\omega_F - f < 0$ on $\Omega_F \cap \partial\mathcal{N}'(x_0', \nu)$ for $\nu \le \nu_0$. In addition, for $\nu \le \nu_0$ combining (4.20) with (4.22) iii) yields that $\nabla(\omega_F - f) \cdot n < 0$ on $\partial\Omega_F \cap \mathcal{N}'(x_0', \nu)$, so that $\omega_F - f$ is increasing inwards across the boundary.

Therefore, $\omega_F - f$ can not have a maximum in $\partial(\Omega_F \cap \mathcal{N}'(x_0', \nu))$, $\nu \le \nu_0$. An identical argument as the one used in Theorem 2.2 to ruled out an interior maximum shows that if ω_F satisfies the inequality (4.9) and f satisfies (4.22) i) in $\Omega_F \cap \mathcal{N}'(x_0', \nu)$ then $\omega_F - f$ can not have a positive maximum in the interior of $\Omega_F \cup \mathcal{N}'(x_0', \nu)$, for $\nu \leq \nu_0$. Therefore $\omega_F - f \leq 0$ in $\overline{\mathcal{N}'(x_0', \nu) \cap \Omega_F}$. In particular

$$\omega(x_0) = \omega_F(x_0') \leq f(x_0') \leq C\nu^{1/8}. \tag{4.23}$$

for $\nu \leq \nu_0$.

The proof of Theorem 4.4 is completed.

Remark. The proof of Lemma 4.5 needs that the neighborhoods $\overline{\mathcal{N}'(x_0', \nu)}$ depend on ν. This condition was not necessary for the interior estimates.

Proof of Lemma 4.5. In order to construct a function f^ν that satisfies (4.22) i), ii) and iii), one could use the one constructed in Lemma 4.3 (settling $\mathcal{R} = M = \Phi_L = \Phi_U = \mathcal{R}_L = \mathcal{R}_U = 0$ the approximation to transonic flow). Such an f^ν would satisfy (4.22) i) and ii), but if f^ν is radial with respect to x_0' and $\nabla f(x_0') = 0$ then $\nabla f \cdot n(x_0') = 0$, so that the condition iii) is not satisfied. However a modification can be done to $f^\nu(r)$, by adding a linear function and reducing the neighborhood to the possible lowest order of ν such that f^ν remains positive in $\overline{\mathcal{N}'_{\nu, x_0'} \cap \partial\Omega_F}$ and satisfies conditions (4.22).

Let $A = A(\mathcal{A}, \mathcal{B}, C, \mathcal{D})$ be the constant of condition (3.24), and R the one from condition (3.20), i.e. $R \leq m = \min\left\{\frac{\mathcal{A}}{\mathcal{B}+\mathcal{C}}, 1\right\}$ where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ are the parameters from (4.11). Note that we are assuming here, without loss of generalization, that $\mathrm{dist}\{x_0', \partial\Omega_F \backslash (\partial_\tau\Omega_F \cup \partial_1\Omega_F)\} = \mathrm{dist}\{x_0', \partial_3\Omega_F\} \geq 1$.

Let $x_0' = (x_1, x_2) \in \partial_1\Omega_F$ and $\mathbf{x} \in B_\tau(x_0')$, we take a positive

$$f^\nu(\mathbf{x}) = \frac{A\nu^{1/2}}{\overline{R}^2 - r^2} + B((\mathbf{x} - x_0') \cdot n + \nu^{1/8}) = g^\nu(r) + B(x_1 - x) + B\nu^{1/8} \tag{4.24.1}$$

$$r = |\mathbf{x} - x_0'| \leq \tau = \frac{\nu^{1/8}}{2}, \quad \text{and} \quad \overline{R} = \tau + \frac{\nu^{1/2}}{\tau}. \tag{4.24.2}$$

Clearly, since $(\mathbf{x} - x_0') \cdot n = \mathbf{x}_1 - x_1$, $\nabla B((\mathbf{x} - x_0') \cdot n + \nu^{1/8}) \cdot n = B$, so

$$\nabla f^\nu \cdot n|_{\partial_1\Omega_F \cap B_\tau(x_0)} = \nabla g^\nu(r) \cdot n + B = \frac{2A\nu^{1/2} r}{\overline{R}^2 - r^2} \cdot n\Big|_{\partial_1\Omega_F \cap B_\tau} + B \tag{4.25}$$

so that condition 4.22 iii) will be satisfied if

$$B > -\frac{2A\nu^{1/2} r}{\overline{R}^2 - r^2} \cdot n\Big|_{\partial_1\Omega_F \cap B_\tau(x_0)} + \tilde{\mathcal{A}} \tag{4.26}$$

where $\tilde{\mathcal{A}}$ is the constant from (4.19).

In order to satisfy 4.22 ii) we first see that if $r = \tau = \frac{\nu^{1/8}}{2}$ then, for $x = \mathbf{x} \cdot n$

$$f^\nu(\mathbf{x})\Big|_{\partial B_\tau} = \Big[g^\nu(r) + B(x_1 - x)\Big]\Big|_{\partial B_\tau(x_0')} + B\nu^{1/8} > g^\nu(\tau) - \frac{B}{2}\nu^{1/8}$$
$$= \frac{A\nu^{1/2}}{(\tau + \frac{\nu^{1/2}}{\tau})^2 - \tau^2} - \frac{B}{2}\nu^{1/4} > \frac{A\nu^{1/2}}{2\nu^{1/2} + \frac{\nu}{\tau^2}} - \frac{B}{2}\nu^{1/8} =$$
$$= \frac{A}{2 + \nu^{1/2}} - \frac{B}{2}\nu^{1/8} > \frac{A}{3} - \frac{B}{2}\nu^{1/8}. \tag{4.27}$$

Hence, 4.22 ii) is equivalent to show that A is large enough and ν small enough such that for $\tau = |\mathbf{x} - x_0'|$

$$f^\nu(\tau) > \frac{A}{3} - \frac{B}{2}\nu^{1/8} > \widetilde{K} + K - \mathcal{R}_U + q\Phi_L. \tag{4.28}$$

so (4.22) (ii) holds for $\mathcal{N}'(x_0', \nu) = B_\tau(x_0') = B_{\frac{\nu^{1/8}}{2}}(x_0')$.

Finally in order to show that f^ν is a supersolution of differential operator $P(\omega)$ given by the differential inequality (4.10), i.e.

$$P(\omega) = \Delta\omega - \frac{\mathcal{A}}{\nu^2}\omega^2 + \mathcal{B}|\nabla\omega|^2 + \frac{\mathcal{C}}{\nu}|\nabla\omega| + \frac{1}{\nu}\mathcal{D} \tag{4.29}$$

where $\mathcal{A}, \mathcal{B}, C$ and $\mathcal{D}$ are given by (4.11), it is enough to show that $P(f^\nu) < 0$ for $\nu \le \tilde{\nu}$, in $B_\tau(x_0')$, $\tilde{\nu}$ to be chosen.

For $\tau = \frac{R}{2}$, using Lemma 4.3 and the given choices of A, R and ν_0 respectively, it follows that $P(g^\nu(r)) < 0$ in $B_\tau(x_0)$ for $\nu \le \nu_0$, ν_0 from (4.13).

Therefore, in order to show that f^ν satisfies $P(f^\nu) < 0$, we compute

$$\begin{aligned} &1) \quad \Delta f^\nu = \Delta(g^\nu + B(x_1 - x) + B\nu^{1/8}) = \Delta g^\nu \\ &2) \quad \nabla f^\nu = \nabla g^\nu - B. \end{aligned} \tag{4.30}$$

$P(f^\nu)$ can be computed as

$$\begin{aligned} P(f^\nu) = \Delta g^\nu - \frac{\mathcal{A}}{\nu^2}[g^\nu + B(x_1 - x) + B\nu^{1/8}]^2 \\ + \mathcal{B}|\nabla g^\nu - B|^2 + \frac{\mathcal{C}}{\nu}|\nabla g^\nu - B| + \frac{1}{\nu}\mathcal{D} \end{aligned} \tag{4.31}$$

and estimated by

$$\begin{aligned} P(f^\nu) \le P(g^\nu) - \frac{\mathcal{A}}{\nu^2}2g^\nu B(x_1 - x + \nu^{1/8}) - \frac{\mathcal{A}}{\nu^2}(B(x_1 - x) + \nu^{1/8})^2 \\ + \mathcal{B}2B|\nabla g^\nu| + \mathcal{B}B^2 + \frac{1}{\nu}(\mathcal{C}B + \mathcal{D}). \end{aligned} \tag{4.32}$$

Then, we need to choose a large enough A and B and small enough ν_0 and show that the right-hand side of (4.32) is strictly negative.

From (4.24) $g^\nu(r) = \frac{A\nu^{1/2}}{\overline{R}^2 - r^2}$ and $r \leq \tau = \frac{\nu^{1/8}}{2}$ with $\nu < \nu_0$ the one from condition (4.13) then,

$$\overline{R}^2 - r^2 = \left(\tau + \frac{\nu^{1/2}}{\tau}\right)^2 - r^2 = (\nu^{1/8} + 2\nu^{3/8})^2 - r^2,$$

then $\frac{\nu^{1/4}}{2} \leq \overline{R}^2 - r^2 \leq 9\nu^{1/4}$, for $\nu < 1$. Then

$$Bg^\nu(x - x_1 + \nu^{1/8})\Big|_{x \in B_{\frac{\nu^{1/8}}{2}}(x_0')} \geq 2B\frac{A\nu^{1/2}}{9\nu^{1/4}}(-|x - x_1| + \nu^{1/8}) > \frac{AB}{9}\frac{(\nu^{3/8})}{2} \tag{4.33}$$

for $|x - x_1| < \frac{\nu^{1/8}}{2}$. Also,

$$|\nabla g^\nu| \leq \frac{A\nu^{1/2}}{\overline{R}^2 - r^2} < \frac{A\nu^{1/2+1/8}}{\nu^{1/4}} = A\nu^{5/8-1/4} = A\nu^{3/8}. \tag{4.34}$$

Therefore, using $P(g^\nu) < 0$ and $|x - x_1| < \frac{\nu^{1/8}}{2}$, (4.32) combined with (4.33) and (4.34) yield the inequality

$$\begin{aligned} P(f^\nu) &< -\frac{\mathcal{A}}{\nu^2}\left[\frac{AB}{9}\nu^{3/8} + B^2\nu^{1/4}\right] + \mathcal{B}BA\nu^{3/8} + \mathcal{B}B^2 + \frac{1}{\nu}(\mathcal{C}B + \mathcal{D}) < \\ &< -\frac{\mathcal{A}}{\nu^{2-1/4}}(AB + B^2) + \mathcal{B}AB\nu^{3/8} + \mathcal{B}B^2 + \frac{1}{\nu}(\mathcal{C}B + \mathcal{D}). \end{aligned} \tag{4.35}$$

So that, $P(f^\nu) < 0$ iff

$$\begin{aligned} &-\mathcal{A}B(A + B) + \mathcal{B}AB\nu^{2+1/8} + \nu^{7/4}\mathcal{B}B^2 + \nu^{3/4}(\mathcal{C}B + \mathcal{D}) < \\ &-\mathcal{A}B(A + B) + \mathcal{B}(AB + B^2)\nu^{7/4} + (\mathcal{C}B + \mathcal{D}) < 0 \end{aligned} \tag{4.36}$$

Therefore since $\mathcal{A}, \mathcal{B}, C$ and $\mathcal{D}$ are given and A is a constant that takes the form of (3.24) and $\nu < \nu_0$ from condition (4.13). Then we must chose a larger A, smaller ν_0 (if necessary) and a positive B such that (4.26), (4.28) and (4.36) hold simultaneously (so that 4.22 i), ii), iii) hold simultaneously, and hence, completing the proof of Lemma 4.5). Thus, from (4.26) B must be large enough such that

$$B > -\frac{2A\nu^{1/2}r}{\overline{R}^2 - r^2} \cdot n\Big|_{\partial_1\Omega_F \cap B_\tau(x_0')} + \tilde{\mathcal{A}}. \tag{4.37}$$

Since $|\nabla g^\nu|_{\partial_1\Omega_F \cap B_\tau(x_0)} > -\frac{2A\nu^{1/2}r}{\overline{R}^2 - r^2} \cdot n|_{\partial_1\Omega_F \cap B_\tau(x_0')}$ and $\tau = \frac{\nu^{1/8}}{2}$, from estimate (4.34) on $|\nabla g^\nu|$ combined with (4.37) yields that B must be large enough so that

$$B > 2A\nu^{3/8} + \tilde{\mathcal{A}}. \tag{4.38}$$

From (4.28),

$$\frac{A}{3} > \widetilde{K} + K - \mathcal{R}_U + q\Phi_L + \frac{B}{2}\nu^{1/8}. \tag{4.39}$$

Setting $A = B$ in (4.36), (4.38) and (4.39), choose B large enough so that

$$B > \max\{2B\nu^{3/8} + \tilde{\mathcal{A}}, 3(\widetilde{K} + K - \mathcal{R}_U + q\Phi_L) + B\frac{3}{2}\nu^{1/8}\} \tag{4.40}$$
$$\text{and} \quad 2B^2 > (2\mathcal{B}B^2\nu^{7/4} + \mathcal{C}B + \mathcal{D})\frac{1}{\mathcal{A}}.$$

Next, let $\tilde{\nu}$ be such that,

$$\tilde{\nu} < \min\left\{4^{-8/3}; \quad 3^{-2}; \quad \left(\frac{4\mathcal{B}}{\mathcal{A}}\right)^{-4/7}; \quad \nu_0\right\} \tag{4.41}$$

with ν_0 of (4.13), then, for $\nu < \tilde{\nu}$, the inequalities (4.38), (4.39) and (4.40) hold if B is large enough such that

$$B > \max\left\{2\tilde{\mathcal{A}}; \quad 6(\widetilde{K} + K - \mathcal{R}_U + q\Phi_L)\right\} \quad \text{and} \quad B^2 > (\mathcal{C}B + \mathcal{D})\frac{1}{4\mathcal{A}}.$$

Clearly taking B large enough as

$$A = B > \max\left\{2\tilde{\mathcal{A}};\ 6(\widetilde{K} + K - \mathcal{R}_U + q\Phi_L);\ \left(\left(\frac{\mathcal{C}}{4\mathcal{A}}\right)^2 + \frac{\mathcal{D}}{4\mathcal{A}}\right)^{1/2} + \frac{\mathcal{C}}{4\mathcal{A}};\ C\right\} \tag{4.42}$$

where $\tilde{\mathcal{A}}$ is from (4.19), $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ from (4.11) and C the one from lemma 3.3. Therefore the function

$$f^\nu(\mathbf{x}) = \frac{A\ \nu^{1/2}}{2\nu^{1/4}(\mathbf{x} - x_0')^2} + B((\mathbf{x} - x_0') \cdot n + \nu^{1/8}) \tag{4.43}$$

where A is given by (4.42), for $\nu < \nu_0 = \tilde{\nu}$ from (4.41), satisfies Lemma 4.5 if $\mathcal{N}'(x_0', \nu) = B_{\frac{\nu^{1/8}}{2}}(x_0')$, where (renaming A by C)

$$\begin{aligned} C &= C(\alpha, k^{-1}, \widetilde{K}, \mathcal{R}, \mathcal{R}_L, \mathcal{R}_U, M, \Phi_U, \Phi_L, \|F_x\|_{C^2(\overline{\mathcal{N}})}) \\ \nu_1 &= \tilde{\nu}(\alpha, k, \widetilde{K}, \mathcal{R}, \mathcal{R}_L, \mathcal{R}_U, M, \Phi_U, \Phi_L, \|F_x\|_{C^2(\overline{\mathcal{N}})}). \end{aligned} \tag{4.44}$$

Therefore (4.21) holds for C and ν_1 given by (4.44). The proof of Lemma 4.5 is now completed.

Remark. (Sharper estimate for locally flat boundaries). For the transonic flow approximation, if $\partial_1\Omega$ is locally flat about x_0 then condition (4.19) simply states that

$$\nabla\omega \cdot n \leq 0. \tag{4.45}$$

In this case it is possible to find an upper barrier function that yields a better approximation to cavitation speed. Taking

$$f^\nu(\mathbf{x}) = \frac{A\nu}{(\mathcal{R} + \frac{\nu}{\mathcal{R}})^2 - (\mathbf{x} - x_1)^2}, \quad \mathbf{x} \in B_{\mathcal{R}}(x_1) \tag{4.46}$$

where $\text{dist}\{x_0, x_1\} = \frac{\mathcal{R}}{2}$, and the point x_1 in the interior of Ω and belonging to the orthogonal line to $\partial_1\Omega$ that passes through x_0, $\mathcal{R}$ satisfies as in Lemma 2.3 that $\nu \leq \nu_1 \leq \mathcal{R}^2$. Thus, since

$$\nabla f^\nu \cdot n = -\frac{g'(r)}{r}(r\cos\theta)$$

with θ measured zero from the inward normal direction at x_0, $\frac{3\pi}{4} < \theta < \frac{5\pi}{4}$, then

$$\nabla f^\nu \cdot n = |g'(r)|\,|\cos\theta| \geq \frac{A\nu r}{2\mathcal{R}}|\cos\theta| \geq A\nu|\cos\theta| \tag{4.47}$$

for all points r on $\partial_1\Omega \cap B_{\mathcal{R}}(x_1)$, as $\frac{\mathcal{R}}{2} < r < \frac{3}{4}\mathcal{R}$ and $\frac{3\pi}{4} < \theta < \frac{5\pi}{4}$, $\nu \leq \nu_1$, the right-hand side of (4.47) remains always positive. So 4.22 iii) is satisfied. In addition if B and $\nu < \nu_1$ where B and ν_1 are the constants from Lemma 2.3, then 4.22 i) and ii) are also satisfied so Lemma 4.5 holds.

Appendix

Proof of Lemma 2.1. Let $n = 2$ or 3, the dimension of the space.

Combining (1.1.1) and (1.1.2) (as $\Phi = 0$, $\mathcal{R} \equiv 0$ and equation 1.1.3 is dropped), we get that (ρ, φ) solve the equation

$$\Delta\varphi = \frac{1}{\nu}\frac{1}{g(|\nabla\varphi|)}\left(\frac{|\nabla\varphi|^2}{2} - K + i(\rho)\right) = \frac{1}{\nu}\frac{1}{g(|\nabla\varphi|)}(\omega + i(\rho)). \tag{a.1}$$

Since we want to find a differential operator for ω, we first compute the Laplacian of $\frac{|\nabla\varphi|^2}{2} - K$, i.e.

$$\begin{aligned}\Delta\left(\frac{1}{2}|\nabla\varphi|^2 - K\right) &= \Delta\frac{1}{2}(\varphi_x^2 + \varphi_y^2) = \text{ div } (\nabla\varphi \cdot \nabla(\varphi_{x_i})) = \sum_{i=1}^{n} |\nabla\varphi_{x_i}|^2 + \nabla\varphi \cdot \Delta(\nabla\varphi) \\ &= \sum_{i,\, j=1}^{n} \varphi_{x_i\, x_j}^2 + \nabla\varphi\nabla(\Delta\varphi) = \|\text{Hess}(\varphi)\|^2 + \nabla\varphi\nabla(\Delta\varphi).\end{aligned} \tag{a.2}$$

Setting $t^2 = 2(\omega + K) = |\nabla\varphi|^2$, the term $\nabla\varphi\nabla(\Delta\varphi)$ can be written as

$$\frac{1}{\nu}\nabla\varphi\nabla\left(\frac{1}{g(t)}(\omega + i(\rho))\right) = \frac{1}{\nu}\nabla\varphi\nabla\left(\frac{1}{g(t)}\right)(\omega + i(\rho)) + \frac{\nabla\varphi}{\nu g(t)}(\nabla\omega + \nabla i(\rho)). \tag{a.3}$$

Since $\nabla\varphi\cdot\nabla i(\rho) = i'(\rho)\rho\nabla\varphi\nabla\ln\rho = -i'(\rho)\,\rho\,\Delta\varphi = -\frac{1}{\nu g(t)}i'(\rho)\,\rho\,(\omega+i(\rho))$. Combining (a.3) with (a.2) and this computation yields

$$\Delta\omega = \sum_{i\, j=1}^{n} \varphi_{x_i\, x_j}^2 - \frac{1}{\nu}\frac{g'(t)}{g^2(t)}\,\nabla\varphi\,\nabla t\,(\omega + i(\rho)) - \frac{i'(\rho)\,\rho}{\nu^2\, g^2(t)}\,(\omega + i(\rho)) + \frac{\nabla\varphi\nabla\omega}{\nu\, g(t)}\ . \tag{a.4}$$

Now we want to show that the right-hand side of (a.4) is bigger than

$$\frac{k}{2(n-1)\nu^2 g^2}\,\omega^2 + \frac{\nabla\varphi\,\nabla\omega}{\nu g}\left[1 - \frac{\alpha}{2}\,\frac{(\omega + i(\rho))}{1 + \omega + K}\right] - \frac{2}{(n-1)k}\,\frac{|\nabla\omega|^2}{(\omega + K)} \tag{a.5}$$

where k is from condition (1.2).

This would involve only algebraic computations (no further differentiation are needed). Since all the quantities under consideration are independent of the choice of orthonormal coordinates, at any given point $\mathbf{x}_0$, we may assume that $\nabla\varphi(\mathbf{x}_0)$ points in the $\mathbf{e}_1$ direction (i.e. $\varphi_y(\mathbf{x}_0) = \varphi_z(\mathbf{x}_0) = 0$). First, the quantity

$$\sum_{i,\, j=1}^{n} \varphi_{x_i\, x_j}^2(\mathbf{x}_0) = \sum_{i,\, j=1}^{n} \varphi_{\mathbf{e}_i\, \mathbf{e}_j}^2(\mathbf{x}_0) \geq \sum_{i=1}^{n} \varphi_{\mathbf{e}_i\, \mathbf{e}_i}^2(\mathbf{x}_0)\ . \tag{a.6}$$

Next, we need to estimate $\sum_{i=1}^n \varphi^2_{\mathbf{e}_i\,\mathbf{e}_i}(\mathbf{x}_0)$ from below in terms of $\Delta\varphi$. Therefore we write, for $n=2$ or 3

$$\begin{aligned}(\Delta\varphi - \varphi_{\mathbf{e}_1\mathbf{e}_1})^2 &= \sum_{i,j=2}^n \varphi_{\mathbf{e}_i\mathbf{e}_i}\varphi_{\mathbf{e}_j\mathbf{e}_j} \leq \\ &\leq \sum_{i,j=2}^n \left(\frac{1}{2}\varphi^2_{\mathbf{e}_i\mathbf{e}_i} + \frac{1}{2}\varphi^2_{\mathbf{e}_j\mathbf{e}_j}\right) \leq (n-1)\sum_{i=2}^n \varphi^2_{\mathbf{e}_i\mathbf{e}_i}.\end{aligned} \tag{a.7}$$

Since $n \geq 2$,

$$(\Delta\varphi)^2 - 2\Delta\varphi\,\varphi_{\mathbf{e}_1\mathbf{e}_1} \leq (\Delta\varphi - \varphi_{\mathbf{e}_1\mathbf{e}_1})^2 \leq (\Delta\varphi - \varphi_{\mathbf{e}_1\mathbf{e}_1})^2 + \varphi^2_{\mathbf{e}_1\mathbf{e}_1} \leq (n-1)\sum_{i=1}^n \varphi^2_{\mathbf{e}_i\mathbf{e}_i}.$$

Therefore, combining (a.6), (a.7) and this last inequality, gives the estimate for the $\|\mathrm{Hess}(\varphi)\|^2$ in terms of $\Delta\omega$ and the second pure derivative of φ in the direction of the gradiant, namely

$$\sum_{i,\,j=1}^n \varphi^2_{x_i\,x_j}(\mathbf{x}_0) \geq \left(\frac{1}{n-1}(\Delta\varphi)^2 - \frac{2}{n-1}\,\varphi_{\mathbf{e}_1\,\mathbf{e}_1}\,\Delta\varphi\right)(\mathbf{x}_0). \tag{a.8}$$

We are now in conditions to estimate $\Delta\omega$ at the point $\mathbf{x}_0$, where $\omega(\mathbf{x}_0) > 0$. Combining (a.1), (a.4) with (a.8), and $2t\nabla t = 2\nabla\omega$,

$$\begin{aligned}\Delta\omega \geq & \frac{1}{n-1}\frac{(\omega+i(\rho))^2}{\nu^2\,g^2(t)} - \frac{\rho\,i'(\rho)}{\nu^2\,g^2(t)}(\omega+i(\rho)) - \frac{2}{n-1}\varphi_{\mathbf{e}_1\,\mathbf{e}_1}\frac{(\omega+i(\rho))}{\nu\,g(t)} \\ & - \frac{1}{\nu}\frac{g'(t)}{g^2(t)}\frac{(\omega+i(\rho))}{2t}\nabla\varphi\,\nabla\omega + \frac{\nabla\varphi\,\nabla\omega}{\nu\,g(t)}.\end{aligned} \tag{a.9}$$

The first two terms of (a.9) can be combined into

$$\begin{aligned}&\frac{(\omega+i(\rho))}{\nu^2\,g^2(t)}\left(\frac{1}{n-1}(\omega+i(\rho)) - \rho\,i'(\rho)\right) \geq \\ &\frac{(\omega+i(\rho))}{\nu^2\,g^2(t)}\frac{1}{n-1}(\omega+ki(\rho)\,) \geq \frac{k}{n-1}\frac{(\omega+i(\rho))^2}{\nu^2\,g^2(t)},\end{aligned} \tag{a.10}$$

with $\frac{1}{n-1}i(\rho) - \rho\,i'(\rho) \geq \frac{1}{n-1}\,k\,i(\rho)$ and $0 < k < 1$, both by condition (1.2).

Next we estimate the third term of the right hand side of (a.9) using Schwartz inequality

$$\frac{2}{n-1}\varphi_{\mathbf{e}_1\,\mathbf{e}_1}\frac{(\omega+i(\rho))}{\nu\,g(t)} \leq \frac{2}{n-1}\,\mathbf{p}\,\varphi^2_{\mathbf{e}_1\,\mathbf{e}_1} + \frac{2}{n-1}\mathbf{p}^{-1}\frac{(\omega+i(\rho))^2}{\nu^2\,g(t)^2}. \tag{a.11}$$

Taking $\mathbf{p}^{-1} = \frac{k}{2}$, k from condition (1.2) and combining (a.9), (a.10) and (a.11) the following estimate holds at every point $\mathbf{x}_0$

$$\text{(a.12)}\quad \Delta\omega \geq \frac{k}{2(n-1)} \frac{(\omega + i(\rho))^2}{\nu^2\, g(t)^2} - \frac{4}{(n-1)k}\varphi^2_{\mathbf{e}_1\,\mathbf{e}_1} + \frac{\nabla\varphi\,\nabla\omega}{\nu}\left(\frac{1}{g(t)} - \frac{g'(t)}{g^2(t)}\frac{(\omega + i(\rho))}{2t}\right).$$

Next we write $\varphi_{\mathbf{e}_1\,\mathbf{e}_1}$ in terms of ω and $\nabla\omega$ at the point $\mathbf{x}_0$ using

$$\nabla\omega = \nabla\left(\frac{|\nabla\varphi|^2}{2} + K\right) = \nabla\varphi \cdot \nabla(\varphi_{x_i}) = \varphi_{\mathbf{e}_1}\varphi_{\mathbf{e}_1\,\mathbf{e}_1},$$

so that

$$\nabla\omega\nabla\varphi = \varphi^2_{\mathbf{e}_1}\varphi_{\mathbf{e}_1\,\mathbf{e}_1}.$$

Therefore $|\nabla\omega \cdot \nabla\varphi|^2 = (\varphi^2_{\mathbf{e}_1}\varphi_{\mathbf{e}_1\,\mathbf{e}_1})^2$ and

$$\text{(a.13)}\qquad \varphi^2_{\mathbf{e}_1\,\mathbf{e}_1}(\mathbf{x}_0) = \frac{|\nabla\omega \cdot \nabla\varphi|^2}{|\nabla\varphi|^4}(\mathbf{x}_0) \leq \frac{|\nabla\omega|^2}{2(\omega + K)}(\mathbf{x}_0).$$

From (a.12) and (a.13)

$$\text{(a.14)}\quad \Delta\omega \geq \frac{k}{2(n-1)\nu^2 g^2}(\omega + i(\rho))^2 + \frac{\nabla\varphi\nabla\omega}{\nu}\left(\frac{1}{g} - \frac{g'}{g^2}\frac{\omega + i(\rho)}{2t}\right) - \frac{2}{(n-1)k}\frac{|\nabla\omega|^2}{(\omega + K)}.$$

Next, we need to compute $\left(\frac{1}{g} - \frac{g'}{g^2}\frac{\omega+i(\rho)}{2t}\right)$ in detail, as we look at the behavior of $g(|\nabla\varphi|)$ for $\omega > 0$ (i.e. $|\nabla\varphi|^2 > 2K$).

Since $g(|\nabla\varphi|) = \left(1 + \frac{|\nabla\varphi|^2}{2}\right)^{\alpha/2} = (1 + \omega + K)^{\alpha/2}$, we can estimate the terms of (a.14) that contain $g(t)$ simply by computing $g'(t) = \frac{\alpha}{2}\left(1 + \frac{t^2}{2}\right)^{\alpha/2 - 1} 2t$, so that

$$\text{(a.15)}\qquad \frac{1}{g} - \frac{g'}{g^2}\frac{\omega + i(\rho)}{2t} = \frac{1}{g}\left(1 - \frac{\alpha}{2}\frac{\omega + i(\rho)}{1 + \omega + K}\right).$$

Finally , combining (a.14) with (a.15) yields the inequality

$$\text{(a.16)}\qquad \Delta\omega \geq \frac{k}{2(n-1)\nu^2 g^2}\omega^2 + \frac{1}{\nu}\frac{\nabla\varphi\nabla\omega}{g}\left[1 - \frac{\alpha}{2}\frac{\omega + i(\rho)}{1 + \omega + K}\right] - \frac{2}{(n-1)k}\frac{|\nabla\omega|^2}{(\omega + K)}.$$

So Lemma 2.1 is now proven.

Proof of Lemma 3.1.

This proof is rather similar to the one of Lemma 2.1. Let $n = 2$ or 3 be the dimension of the space.

As in (a.2) and (a.4) and (a.8) ω satisfies

$$\Delta\omega \geq \frac{1}{(n-1)\nu^2}\frac{1}{g^2(t)}(\omega+i(\rho))^2 - \frac{2}{(n-1)\nu}\frac{\varphi_{\mathbf{e}_1\mathbf{e}_1}}{g(t)}(\omega+i(\rho)) + \nabla\varphi\nabla\left(\frac{1}{\nu g(t)}(\omega+i(\rho))\right) + \Delta(\mathcal{R}(\varphi)-q\Phi) \tag{a.17}$$

where $t^2 = |\nabla\varphi|^2 = 2(\omega+K-\mathcal{R}(\varphi)+q\Phi)$. Since $g(t) = \left(1+\frac{|\nabla\varphi|^2}{2}\right)^{\alpha/2}$ for $\omega > 0$ then we write $g(t) = (1+\omega+K-\mathcal{R}(\varphi)+q\Phi)^{\alpha/2} = (1+W)^{\alpha/2}$.

Recalling that $C(x)$ is positive and, by (3.2) $\rho \leq i^{-1}(K^*)$

$$\begin{aligned}\Delta(\mathcal{R}(\varphi)-q\Phi) &= \mathcal{R}''\Delta\varphi - q\alpha(\rho - C(x)) \\ &= \mathcal{R}''\frac{1}{\nu g(t)}(\omega+i(\rho)) - q\alpha\rho + q\alpha C(x) \geq \\ &\quad -\mathcal{R}\frac{(\omega+i(\rho))}{\nu g(t)} - q\alpha i^{-1}(K^*).\end{aligned} \tag{a.18}$$

Then, combining (a.18) with equivalent estimates to (a.10) and (a.11), the right-hand side of (a.17) dominates

$$\frac{1}{2(n-1)\nu^2}\frac{k}{n\,g^2(t)}(\omega+i(\rho))^2 - \frac{4}{(n-1)k}\varphi^2_{\mathbf{e}_1\mathbf{e}_1} + \frac{\nabla\varphi\nabla\omega}{\nu g(t)} + \frac{\nabla\varphi\nabla g^{-1}(t)}{\nu}(\omega+i(\rho)) + \mathcal{R}_L\frac{(\omega+i(\rho))}{g(t)} - q\alpha i^{-1}(K^*). \tag{a.19}$$

Next, the second and third term are estimated as in Lemma 2.1. In a local coordinate system where $\varphi_{\mathbf{e}_i} = 0, i > 1$,

$$\nabla\omega = \varphi_{\mathbf{e}_1}\nabla\varphi_{\mathbf{e}_1} + \mathcal{R}'(\varphi)\nabla\varphi - q\nabla\Phi.$$

Computing $\varphi_{\mathbf{e}_1\mathbf{e}_1}$ as in (a.13)

$$\varphi_{\mathbf{e}_1\mathbf{e}_1}\varphi^2_{\mathbf{e}_1} = \nabla\omega\nabla\varphi - \mathcal{R}'(\varphi)|\nabla\varphi|^2 + q\nabla\Phi\nabla\varphi. \tag{a.20}$$

Thus, for $\omega \geq 0$, $|\nabla\varphi| = \sqrt{2}(\omega+K-\mathcal{R}(\varphi)+q\Phi)^{1/2} \geq \sqrt{2}(K-\mathcal{R}_U+q\Phi_L)^{1/2} > 0$ and

$$-\frac{4}{(n-1)k}\varphi^2_{\mathbf{e}_1\mathbf{e}_1} \geq -\frac{4}{(n-1)k}\left\{\frac{|\nabla\omega|}{\sqrt{2}(K-\mathcal{R}_U+q\Phi_L)^{1/2}} - \mathcal{R} + \frac{qM}{\sqrt{2}(K-\mathcal{R}_U+q\Phi_L)^{1/2}}\right\}^2. \tag{a.21}$$

As in (a.15) compute

$$
\begin{aligned}
&\frac{\nabla\varphi\nabla\omega}{\nabla g(t)}+\frac{\nabla\varphi\nabla g^{-1}(t)}{\nu}(\omega+i(\rho))\\
&\qquad=\frac{\nabla\varphi\nabla\omega}{\nu}\left(\frac{1}{g(t)}+\frac{g'(t)}{g^2(t)}\frac{\omega+i(\rho)}{2t}\right)\\
&\qquad=\frac{1}{\nu}\frac{\nabla\varphi\nabla\omega}{g}\left(1-\frac{\alpha}{2}\frac{\omega+i(\rho)}{1+W}\right)\\
&\qquad\quad+\frac{1}{\nu}\frac{-\mathcal{R}'|\nabla\varphi|^2+q\nabla\Phi\nabla\varphi}{g(t)}\left[1-\frac{\alpha}{2}\frac{\omega+i(\rho)}{1+W}\right].
\end{aligned}
\tag{a.22}
$$

Therefore, combining (a.17) with (a.19), (a.20), (a.21) and (a.22) the following inequality holds

$$
\begin{aligned}
\Delta\omega \geq\ & \frac{1}{\nu^2}\frac{k}{2(n-1)\nu^2 g^2(W)}\omega^2\\
&-\frac{2}{(n-1)k}\frac{|\nabla\omega|^2}{(K-\mathcal{R}_U+q\Phi_L)}-\frac{4}{(n-1)k}\frac{qM|\nabla\omega|}{K-\mathcal{R}_U+q\Phi_L}\\
&-\frac{4}{(n-1)k}\left\{\frac{\mathcal{R}\,\nabla\omega}{(2(K-\mathcal{R}_U+q\Phi_L))^{1/2}}+\left(-\mathcal{R}+\frac{qM}{(2(K-\mathcal{R}_U+q\Phi_L))^{1/2}}\right)^2\right\}\\
&\frac{1}{\nu\, g}\left[\nabla\varphi\nabla\omega-\mathcal{R}|\nabla\varphi|^2-qM|\nabla\varphi|\right]\left(1-\frac{\alpha}{2}\frac{\omega+i(\rho)}{1+W}\right)-\frac{1}{\nu}\mathcal{R}\frac{(\omega+i(\rho))}{g}\\
&-q\alpha i^{-1}(K^*),
\end{aligned}
\tag{a.23}
$$

so recombining it yields Lemma 3.1.

Proof of Lemma 4.1.

Again this proof is similar to the previous lemmas in this appendix, but it contains the modification due to the local transformation that "flattens" the boundary. For convenience the drop the subscripts ν and primes.

For $\omega=\frac{(|\nabla\varphi|\,|F_x|)^2}{2}-K+\mathcal{R}(\varphi)-q\Phi$ as in (a.2) and (a.4) and (a.17) compute $\Delta(\omega\cdot|F_x|^{-2})$ as on one hand

$$
\Delta(\omega|F_x|^{-2})=\Delta\omega|F_x|^{-2}+2\nabla\omega\nabla|F_x|^{-2}+\omega\Delta|F|^{-2}
\tag{a.24}
$$

and, on the other hand, is

$$
\Delta(\omega|F_x|^{-2})=\Delta\left(\frac{|\nabla\varphi|^2}{2}+(-K+\mathcal{R}(\varphi)-q\Phi)|F_x|^{-2}\right).
\tag{a.25}
$$

As in the two previous lemmas, we estimate

$$\Delta(\omega|F_x|^{-2}) \geq (\Delta\varphi)^2 - 2\varphi_{\mathbf{e}_1\mathbf{e}_1}\Delta\varphi + \nabla\varphi\nabla\Delta\varphi + \Delta((-K+\mathcal{R}(\varphi)-q\Phi)|F_x|^{-2}). \tag{a.26}$$

Combining estimate (a.24) with estimate (a.26)

$$\begin{aligned} \Delta\omega \geq{}& \frac{1}{\nu^2}\frac{[\omega+i(\rho)]^2}{g^2(t)} - \frac{2}{\nu}\varphi_{\mathbf{e}_1\mathbf{e}_1}\frac{(\omega+i(\rho))}{g} + \nabla\varphi\nabla(\Delta\varphi) \\ &- 2\nabla\omega\frac{\nabla|F_x|^{-2}}{|F_x|^2} - \omega\frac{\Delta|F_x|^{-2}}{|F_x|^2} + (-K+\mathcal{R}(\varphi)-q\Phi)\frac{\Delta|F_x|^{-2}}{|F_x|^2} \\ &+ 2(\nabla\mathcal{R}(\varphi)-q\nabla\Phi)\frac{\nabla|F_x|^{-2}}{|F_x|^2} + \left(\mathcal{R}''\frac{1}{\nu}\frac{(\omega+i(\rho))}{g(t)} + q\alpha i^{-1}(K^*)\right). \end{aligned} \tag{a.27}$$

Using equation 4.1 iii) to estimate $\nabla\varphi\nabla(\Delta\varphi)$,

$$\begin{aligned} \nabla\varphi\nabla(\Delta\varphi) &= \nabla\varphi\cdot\nabla\left(\Delta\varphi|F_x|^2\cdot\frac{1}{|F_x|^2}\right) \\ &= \nabla\varphi\nabla(\Delta\varphi|F_x|^2) + \nabla\varphi\cdot\Delta\varphi 2\nabla(|F_x|^{-2}) \\ &= \frac{\nabla\varphi}{\nu}\nabla\left(\frac{\omega+i(\rho)}{g(t)}\right) + 2\frac{\nabla\varphi}{\nu}\left(\frac{\omega+i(\rho)}{g(t)}\right)\nabla|F_x|^{-2}. \end{aligned} \tag{a.28}$$

As in Lemma 2.1, the first term of (a.25) is estimated as in (a.3), and since

$$\nabla\varphi\cdot\nabla i(\rho) = -\frac{1}{\nu g(t)}(\omega+i(\rho))i'(\rho)\rho, \tag{a.29}$$

combining (a.27) with (a.28) and (a.29) along with the condition on the enthalpy function for $n=2$–space dimensions $i(\rho)-i'(\rho)\rho = k\,i(\rho)$ with $0<k<1$, $i(\rho)\leq K^* = K-\mathcal{R}_L+q\Phi_U$ the following estimate holds

$$\begin{aligned} \Delta\omega \geq{}& \frac{1}{\nu^2}\frac{k}{2g^2(t)}(\omega+i(\rho))^2 - \frac{4}{k}\varphi^2_{\mathbf{e}_1\mathbf{e}_1} + \frac{\nabla\varphi}{\nu}\frac{\nabla\omega}{g(t)} + \frac{\nabla\varphi}{\nu}\nabla g^{-1}(t)(\omega+i(\rho)) \\ &+ \frac{\mathcal{R}''}{\nu}\frac{\omega+i(\rho)}{g(t)} - q\alpha K^* \\ &+ \left(\frac{2\nabla\varphi|F_x|}{\nu}\frac{(\omega+i(\rho))}{g(t)}|F_x|^{-1} + 2\nabla\omega + \mathcal{R}'\nabla\varphi - q\nabla\Phi\right)\frac{\nabla\ln|F_x|}{|F_x|^4} \\ &- W\frac{\Delta|F_x|^2}{|F_x|^2}. \end{aligned} \tag{a.30}$$

Estimating $\varphi_{\mathbf{e}_1\mathbf{e}_1}$ as in (a.21), taking into consideration that $\nabla(\omega|F_x|^{-2}) = \varphi_{\mathbf{e}_1}\varphi_{\mathbf{e}_1\mathbf{e}_1}$ $+(\mathcal{R}'(\varphi)\varphi_x - q\nabla\Phi)|F_x|^{-2}+$ $(K+\mathcal{R}(\varphi)-q\Phi)\nabla|F_x|^{-2}$ in a local coordinate system where $\varphi_{\mathbf{e}_2}=0$, then for $\omega\geq 0$

$$|\nabla\varphi| = \sqrt{2}|F_x|^{-2}(\omega+K-\mathcal{R}(\varphi)+q\Phi)^{1/2} \geq \sqrt{2}\max_{\mathcal{N}}|F_x|^{-2}(K-\mathcal{R}_U+q\Phi_L)^{1/2}, \tag{a.31}$$

so that

$$(\text{a.32})\; -\frac{4}{k}\varphi^2_{\mathbf{e}_1\,\mathbf{e}_1} \geq -\frac{4}{k}\frac{1}{\sqrt{2}\max_{\mathcal{N}}|F_x|^{-4}(K-\mathcal{R}_U+q\Phi_L)}$$
$$\{|\nabla(W|F_x|^{-2})| + (\mathcal{R}\sqrt{2}\max_{\mathcal{N}}|F_x|^{-4}(K-\mathcal{R}_U+q\Phi_L) - qM)|F_x|^{-2}$$
$$+ (K-\mathcal{R}_U+q\Phi_L)\nabla|F_x|^{-2}\}^2.$$

Let $\mathcal{M} = \sqrt{2}\max_{\mathcal{N}}|F_x|^{-4}(K-\mathcal{R}_U+q\Phi_L)$ then combining (2.30) with (a.31) and (a.32) and $g(W) = (1+W)^{\alpha/2}$, if $\omega > 0$

$$(\text{a.33})\quad \Delta\omega \geq \frac{k}{2\nu^2 g^2}\omega^2 - \frac{2W^{1/2}}{|F_x|}\frac{|\nabla\omega|}{\nu}\left(\frac{1}{(1+W)^{\alpha/2}} - \frac{\alpha}{2}\frac{\omega+i(\rho)}{(1+W)^{\alpha/2+1}}\right)$$
$$-\frac{4}{k\mathcal{M}}\Big\{\left(|\nabla\omega||F_x|^{-2} + \omega|\nabla|F_x|^{-2}|\right)^2$$
$$+2\left[|\nabla\omega|\,|F_x|^{-2} + \omega|\nabla F_x|^{-2}\right]$$
$$\left[(\mathcal{R}\mathcal{M}+qM)|F_x|^{-2} + (K-\mathcal{R}_U+q\Phi_L)|\nabla|F_x|^{-2}|\right]$$
$$+\left[(\mathcal{R}\mathcal{M}+qM)|F_x|^{-2} + (K-\mathcal{R}_U+q\Phi_L)|\nabla|F_x|^{-2}|\right]^2\Big\}$$
$$-\frac{\mathcal{R}}{\nu}\frac{\omega+K^*}{(1+W)^{\alpha/2}} - q\alpha K^* - W\frac{\Delta|F_x|^2}{|F_x|^2}$$
$$-\left(2\frac{W^{1/2}}{\nu}\frac{\omega+K^*}{(1+W)^{\alpha/2}}|F_x|^{-1} + 2\nabla\omega + \mathcal{R}\,M + qM\right)|\nabla|F_x|^{-4}|.$$

Finally, reorganize the terms that contain $|\nabla\omega|$ and put them together as

$$(\text{a.34})\quad \Delta\omega \geq \frac{1}{\nu^2}\frac{k}{2(1+W)^{\alpha}}\omega^2 - \frac{4}{k\mathcal{M}|F_x|^4}|\nabla\omega|^2$$
$$-\frac{|\nabla\omega|}{\nu}\Big\{\frac{2W^{1/2}}{|F_x|}\left(\frac{1}{(1+W)^{\alpha/2}} - \frac{\alpha}{2}\frac{(\omega+i(\rho))}{(1+W)^{\alpha/2+1}}\right)$$
$$+\nu\Big[\frac{8\omega|\nabla|F_x|^{-4}|}{k\mathcal{M}} + \frac{8}{k\mathcal{M}}|F_x|^{-4}(\mathcal{R}\mathcal{M}+qM)$$
$$+2|\nabla|F_x|^{-4}|\frac{8}{k\mathcal{M}}(K-\mathcal{R}_U+q\Phi_L)|\nabla|F_x|^{-4}|\Big]\Big\}$$
$$-|\nabla|F_x|^{-4}|\Big\{\frac{8\omega}{k\mathcal{M}}(\mathcal{R}\mathcal{M}+qM)\frac{8}{k\mathcal{M}}(\mathcal{R}\mathcal{M}+qM)(K-\mathcal{R}_U+q\Phi_L)$$
$$+\mathcal{R}\mathcal{M}+qM+\frac{2}{\nu}\frac{W^{1/2}}{(1+W)^{\alpha/2}}(\omega+K^*)\Big\}$$
$$-\frac{4}{k\mathcal{M}}|\nabla|F_x^{-2}|\,|^2(\omega+(K-\mathcal{R}_U+q\Phi_L))^2$$
$$-\frac{4}{k\mathcal{M}}(\mathcal{R}\mathcal{M}+qM)^2|F_x|^{-4} - \frac{\mathcal{R}}{\nu}\frac{\omega+K^*}{(1+W)^{\alpha/2}} - q\alpha K^* - W\frac{\Delta|F_x|^2}{|F_x|^2}.$$

Recombining, (a.34) we can obtain inequality (4.2).

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