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THE MILNE PROBLEM FOR HIGH FIELD KINETIC EQUATIONS*

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Abstract. Half space problems of the linear Boltzmann equation with a constant driving force are considered. Such problems model boundary layers between kinetic zones and fluid zones described by a high field limit of the Boltzmann equation. Existence, uniqueness, and asymptotic behavior of solutions are studied for positive and negative driving forces. In the positive case, the force field accelerates the particles, and we show that the solution of the half space problem is determined only by the inflow data. In contrast, for negative forces, the behavior at infinity has to be prescribed in order to insure uniqueness. Due to the nonvanishing forces, the problem does not possess any entropy. The existence and uniqueness issues are dealt with by supersolution techniques, while the asymptotic behavior is analyzed by semiexplicit integration of the equations along the characteristics. In the case of relaxation time approximation, a fast numerical method for computing the asymptotic state method is presented and tested.

Key words. kinetic theory of plasmas, nonequilibrium statistical mechanics, boundary layer problems in kinetic equations, numerical methods for kinetic equations

AMS subject classifications. 82B, 82C, 82D

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1. Introduction. Macroscopic fluid models are usually obtained from kinetic equations in collision dominated situations. Diffusion scalings are used when the equilibrium states (for which collisions are transparent) carry no current. Depending on the specific collision phenomena taken into account, asymptotics methods based on scaling assumptions lead to various diffusion models like the drift-diffusion [29], the energy-transport [8, 18], or the spherical harmonics expansion (SHE) model [32, 31, 4, 21, 17, 16].

When the driving forces are strong enough that their effect is of the same order of magnitude as collisional effects, another scaling, called high field scaling, has to be used. For the linear Boltzmann operator, the limit equation has been formally shown by Poupaud [30] to be a linear convection equation with the convective term depending on the force field. When the force field is the gradient of a potential coupled through a mean field approximation, a nonlinear system is obtained with a first order correction corresponding to augmented diffusion and transport [12, 13].

Kinetic high field models and associated macroscopic models have been considered in [13, 12, 34, 30, 5]. Recent comparisons between kinetic multiscale domain decomposition and the Monte Carlo method were presented in [20, 1, 10].

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However, up to now, no analysis of the kinetic boundary layer problem to find the correct boundary conditions for the fluid approximation has been performed. Such an analysis is also required if one wants to solve the matching problem for kinetic and macroscopic equations. Here, an interface region between the two equations has to be considered. The matching problem has to be solved, for example, for domain decomposition approaches simultaneously solving kinetic and macroscopic equations in different regions of the computational domain.

Boundary and interface regions are described by a transition layer where a stationary kinetic equation is solved. A standard assumption is that the layer has a slab symmetry, that is, the particle distribution is constant on surfaces parallel to the interface. Rigorous analysis of boundary value problems for linear transport kinetic equations in the absence of forces, known as the half space problem, and its corresponding limiting behavior in a strong collisional regime and long time scaling linear, as the length of the transition layer is comparable to the reference collision frequency, known as the Milne problem, was initiated [6] by means of spectral methods and semigroup theory.

For charged transport models, the force field gradient of the electrostatic potential is bounded along flat boundaries where the potential either is prescribed or is a solution of the corresponding mean field equation. In both cases, the force field will become a constant in the rescaled layer. In in the drift-diffusion regime due to weak force field forces, the rescaled force field vanishes. The corresponding half space and Milne problem was studied in [28], and computations for the corresponding fluid kinetic interface procedure for numerical implementations of hybrid methods were due to [35, 24].

For the case of strong force field regimes, one expects a slab symmetry whenever the curvature of the interface is small compared to the reciprocal of the mean free path and when the force field is normal to the interface. Consequently, the space coordinate reduces to, say, x the distance to the boundary or interface. After scaling it like $\frac{x}{\epsilon}$, where ϵ is the order magnitude of the mean free path, one has to solve a kinetic half space problem.

These strong force field scalings are characterized by nonstatistical equilibrium states $P = P(v)$; that is, they are $L^1_k(\mathcal{R}_v)$ space homogeneous solutions to the layer problem, with nonvanishing mean or first moment, which depend on the force field and on the Maxwellian in the kernel of the collision operator and the scattering function. This problem was treated by Trugman and Taylor [34] for the relaxation operator, and by Poupaud [30] for the general linear operator in three dimensions.

The first part of this paper treats the existence and uniqueness results for half space problems corresponding to strong force field scaling, both for positive and negative forces, and describes their corresponding asymptotic behavior, since from a practical point of view, the objects of great interest in obtaining boundary or matching conditions are the asymptotic states and the outgoing distribution (Albedo operator). The only assumption is that the boundary incoming data is a positive L^1_k bounded by a multiple of the state P.

In the case of positive forces, the force field accelerate the particles. Here we show that the solution of the half space problem is determined only by the inflow data. In fact, we prove that the unique solution $f(x, v)$ of the half space problem satisfies the condition that f_∞/P belongs to $L^\infty(\mathcal{R}^+_x \times \mathcal{R}_v)$. In addition $\lim_{x\to\infty} f(x,\cdot)/P$ converges to a proper factor n_∞. This factor is uniquely determined by the quotient of the mean of the solution, which is space independent, and the mean of the nonstatistical equilibrium state P, and thus it depends on the boundary data. This result indicates that under such strong forced scaling, the kinetic equation will admit an asymptotic non-Maxwellian homogeneous stationary state; that is, under *strong acceleration*, the unscaled original kinetic solution should take a local stationary state which does not correspond to statistical equilibrium, whose asymptotic limit is a singularly perturbed augmented transport-diffusion converging to convective transport.

In contrast, for the case of strong negative forces, particles are slowed down, and under the same conditions for existence, one needs to prescribe the behavior of f at the right end of the layer in order to get uniqueness of the half space problem. In fact, we prove that for *any* given constant parameter n_∞, there exists a unique positive solution f_∞/P, belonging to $L^\infty(\mathcal{R}_x^+ \times \mathcal{R}_v)$, to the stationary problem in the rescaled layer, such that $\lim_{x\to\infty} f(x,\cdot)/P = n_\infty$. This essentially indicates that the behavior at infinity does not depend on the inflow boundary data.

A way to see the difference between the positive and negative force is that, in the former, the characteristic curves passing at $x = 0$ for the first order layer equation grow to $+\infty$ for $v > 0$ as $x \to \infty$, and come from $-\infty$ for $v < 0$. However, for the negative forced equation, the characteristic curves passing at $x = 0$ for $v > 0$ will turn back to intersect the axis $x = 0$ for $v < 0$. In particular for this second case, one may prescribe the behavior at infinity.

This anisotropic nature of the problem has as a consequence the lack of a natural entropy functional that controls the decay in space, such as it is possible to obtain in the low field scaling case. This motivates us to introduce new analytical methods based on comparison techniques by super- and subsolutions, namely, a maximum principle for solutions to kinetic stationary boundary value half space problems, basically introduced by Poupaud in [29] in order to treat boundary value problems for the stationary Vlasov–Maxwell system.

We recall that in the low field scaling case, the characteristic curves passing at $x = 0$ for the first order layer equation are all constant straight lines $v = v_o$ for all v_o, that is, all parallel to the x-axis. In particular, it has been shown that a corresponding boundary layer problem has a solution to the Milne problem given by an asymptotic behavior approaching a Maxwellian state, independent of forces [28]. In this case the boundary layer problem is similar to the one for a kinetic equation in the absence of forces, as treated in [6]. In both cases a diffusion limit arises, which may have a weak drift proportional to the field, corresponding to low field scaling.

In the second part of the paper, we describe a numerical procedure which computes n_∞, depending on the initial data, for the case of positive forces and a relaxation collision operator. It uses a classical Chapman–Enskog–type expansion to approximate the solution. We obtain a force field modified Marshak condition, which is a higher order correction to prescription of incoming fluxes. Our calculation recovers the classical Marshak condition for diffusion approximations as the force fields tends to zero. The method is seen to converge very fast numerically. It seems to give accurate results when compared to the available explicit solutions in some special cases. For approaches to the numerical solution of the standard half space problem in gas dynamics and semiconductor equations, we refer the reader to [2, 14, 22, 33], and for a mathematical investigation, to [3, 15, 23]. We expect a future implementation of very efficient hybrid computational schemes that will be able to link nonstatistical equilibrium scales by their anisotropic diffusion convective limits, as well as to solve the coupling of convective regions to diffusion regions by transition layer or interfaces, as is steadily observed in strongly doped device simulation under hot-electron regimes.

The paper is organized as follows. In section 2 we present the strong force field equations. Section 3 contains an analytical investigation of the half space problem for both the negative and positive forces. In both cases, existence and uniqueness results with the asymptotic behavior at infinity are investigated. In section 4 the numerical procedure and some numerical results are presented in the case of relaxation operators.

2. High field kinetic equations. The drift-collision balance regime. We consider the semiclassical linear Boltzmann equation in dimensionless variables for an electron gas for a semiconducting material in the parabolic band approximation, with a strong force field scaling

$$\eta \partial_t f + v \cdot \nabla_z f + \frac{\eta}{\epsilon} E(z,t) \cdot \nabla_v f = \frac{1}{\epsilon} Q(f) \tag{2.1}$$

with $z, v \in \mathcal{R}^3$. The general linear collision operator under consideration is

$$Q(f) = \int s(v,v')[M(v)f(v') - M(v')f(v)]\, dv' = Q^+(f) - \sigma(v) f. \tag{2.2}$$

The scattering function $s(v,v')$ is symmetric and satisfies

$$0 < s_0 \leq s(v,v') \leq s_1 < +\infty \quad \text{and} \quad s(v,v') = s(v',v), \tag{2.3}$$

and σ denotes the collision frequency

$$\sigma(v) = \int s(v,v') M(v')\, dv', \tag{2.4}$$

whereas

$$Q^+(f) = \int s(v,v') f(v') M(v)\, dv' \tag{2.5}$$

is the gain operator. Throughout this paper the notation $\langle h \rangle$ stands for $\int h(v)\, dv$.

Here we use the standard notation $M(v)$ for the centered, reduced Maxwellian $M = (2\pi)^{-\frac{3}{2}} \exp(-\frac{v^2}{2})$.

As a motivation for this problem, we look at semiconductor modeling as the main example of a transport phenomenon that exhibits stationary nonequilibrium statistical states. Usually, the vector field $E = E(z,t) = -\nabla_z \Phi$ denotes the scaled electric force, which is determined by a Poisson equation for the potential Φ:

$$\nabla_z \cdot (\nabla_z \Phi) = \gamma \left(\frac{1}{\eta^d} \int_{\mathcal{R}^3} f dv - C(z) \right),$$

where γ is the inverse to the scaled Debye length of the device and $C(z)$, which denotes the ion background, is bounded, measurable, and largely varying. It is worth mentioning that strong force field scalings are present due to space inhomogeneities, such as short base channel devices, under strong forward bias that produces a region of positive charges inside the channel (i.e., $\gamma^{-1} \gg 0$). Such an effect is known as hot-electron transport. Under these assumptions on $C(z)$, classical potential theory implies that the solution of the Poisson equation in a bounded channel-like region yields a continuous bounded force field $E(z)$.

Because of this effect, assume dimensionless parameters η and γ both of order $O(1)$ and that ϵ, the scaled mean free path, is small. This scaling assumption corresponds to the drift-collision balance scaling introduced in [30, 13]. Such a scaling is realized, for instance, in the modeling of silicon doped diodes with $0.4\,\mu$m channel [20, 1, 10] under potential bias of 1eV. These simulations exhibit the formation of transition layers in the drain junction, with a clear jump from a close to convective state in the channel region to a diffusion equilibrium at the contact. In addition, inside the channel, there is a clear region where the numerical probability distribution function, the solution to the approximated kinetic Poisson system, takes a definite state away from statistical equilibrium. Such a configuration corresponds to a relatively strong forced field scale with respect to collisions against a background. This may be the case for other collisional plasma physics applications under strong force fields.

The problem we want to study is related to the solution and its asymptotic behavior in a given layer of length ϵ. This layer is inversely proportional to the drift-collision scale associated with the reciprocal of the scaled mean free path of $o(\epsilon)$ for the kinetic problem under such a regime.

From now we focus on the problem of having a force field $E(x,t)$ given in a transition layer or boundary with a slab symmetry; that is, the particle distribution is constant on surfaces parallel to the interface. For the case of strong force field regimes, one expects such a slab symmetry whenever the curvature of the interface is small compared to the reciprocal of the mean free path and when the force field is normal to the interface.

In order to obtain the boundary or interface layer equations in a slab geometry, fix a point $\hat{z}$ on the boundary, assume that the electric force is orthogonal to the interface, and rescale as usual the space coordinate in the layer normal to the boundary with the mean free path ϵ, introducing the new coordinate x orthogonal to the boundary:

$$x = \frac{(\hat{z} - z) \cdot n}{\epsilon}.$$

Here, n denotes the outer normal to the boundary or interface. This transformation yields the new coordinates $(x, \hat{z})$ instead of z in the slab layer. To $O(1)$ one obtains, after applying the transformation to (2.1),

$$v \cdot n \partial_x f + \eta E \cdot \nabla_v f, = Q(f),$$

where, as $\epsilon \to 0$, the variable $x \in [0, \infty)$, and the field $E = E(x = 0, \hat{z}, t)$ does not depend on x and thus is constant. This problem has to be supplied with the ingoing function at the boundary, i.e., at $x = 0$; that is, $f(0, v)$, $v \cdot n > 0$, with n the outer normal to the boundary at $x = 0$. In order to have the force field E constant it is enough that the potential Φ is regular enough so that $\nabla\Phi$ is bounded at the slab boundary.

To simplify the problem, we assume from now on that the z_1-coordinate points in the direction of the normal, so that that $E = (E_1, 0, 0)$ and that $\tau = 1$, $\eta = 1$. Then the above reduces to the following one-dimensional problem:

$$v_1 \partial_{z_1} f + E_1 \partial_{v_1} f = Q^+(f) - \sigma(v) f,$$

with $x \in [0, \infty)$, $v_1 \in \mathcal{R}$, $f = f(z_1, v_1)$. Then $M(v)$ in the definition of Q^+ and σ reduces to the one-dimensional Maxwellian $M(v_1) = (2\pi)^{-\frac{1}{2}} \exp(-\frac{v_1^2}{2})$ for $v_1 \in \mathcal{R}$.

From now on, we use the notation $(x, v) \in \mathcal{R}^+ \times \mathcal{R}$ rather than (z_1, v_1). The Milne problem takes the following form:

$$\begin{aligned} v \partial_x f + E \partial_v f = Q(f) = Q^+(f) - \sigma(v) f, \\ \varphi(0, v) = k(v), \quad v > 0, \end{aligned} \tag{2.6}$$

with $x \in [0, \infty)$, $v \in \mathcal{R}$, and an ingoing positive function k satisfying the conditions stated below.

Before announcing the main theorems, we define the homogeneous solution $P_{\sigma,E}(v)$ as the unique function which satisfies

$$E\partial_v P = Q(P) = Q^+(P) - \sigma(v)P, \qquad \text{with } \langle P\rangle = 1, \tag{2.7}$$

using the notation

$$\langle P\rangle = \int P(v)dv. \tag{2.8}$$

In addition, for any integrable solution f of (2.6),

$$j = \langle vf\rangle \quad \text{is } x\text{-independent.} \tag{2.9}$$

The proof of this statement is trivial for any integrable solution.

Solvability of problem (2.7) in $L^\infty \cap L^1$ can be found in Trugman and Taylor [34] for the relaxation-type operator in one dimension. It has also been discussed in Frosali, Van der Mee, and Paveri-Fontana [19]. The most general result has been obtained by Poupaud [30], who finds solutions to (2.7) in L^1 for general linear collision operators in higher dimensions, depending on the integrability of the collision frequency. In addition he shows that the solution function P is unique and positive. Recently, this result has been generalized to the collision operators with Pauli-exclusion terms [7].

For completeness we recall the Poupaud solution representation to problem (2.7), obtained via spectral analysis [30] of the following linear integral operator:

$$\begin{aligned} P_E(v) &= L_E(Q^+(P))(v) \\ &= \int_0^\infty \exp\left(-\int_0^\tau \sigma(v-\mu E)d\,\mu\right)\int_{\mathcal{R}} s(v-\tau E, w)P_E(w)d\,w\, M(v-\tau E)d\,\tau \end{aligned} \tag{2.10}$$

for $E \neq 0$ such that $\langle P\rangle = 1$. The operator $L_E : L^1 \to L^1_\sigma$ is the inversion operator to $E\cdot\nabla_v + \sigma(v)$, defined by

$$L_E(f)(v) = \int_0^\infty \exp\left(-\int_0^\tau \sigma(v-\mu E)d\,\mu\right) f(v-\tau E)d\,\tau. \tag{2.11}$$

Poupaud proves that the integral equation (2.10) has a unique integrable (L^1) positive solution if and only if

$$\int_0^\infty \sigma(v+\mu E)d\,\mu = +\infty \qquad \text{a.e.} \tag{2.12}$$

In addition, the solution satisfies the property

$$P_E(v) = P_{-E}(-v). \tag{2.13}$$

It is clear that in our case, by (2.3), the scattering function $s(v, v')$ is bounded above and below by positive constants, so that the collision frequency $\sigma(v)$ function as defined in (2.4) satisfies the infinite integrability compatibility condition (2.12).

Moreover, the unique solution P to problem (2.7) has all moments bounded. Indeed by (2.2) and (2.3) the following moment recursion inequality holds:

$$\langle v^k P\rangle \le \frac{s_1}{s_0}\langle v^k M\rangle + E\langle v^{k-1}P\rangle. \tag{2.14}$$

In the particular case of a relaxation collision operator, when $s(v',v) = \tau^{-1}$,

$$Q(f)(v) = \frac{1}{\tau}\left(M(v)\int_{\mathcal{R}^3} f(v')dv' - f(v)\right) = \frac{\langle f\rangle M - f}{\tau}, \tag{2.15}$$

with τ the relaxation time, one obtains an explicit formula for the dominant P state, as the right-hand side of (2.10) is computable. Setting $\tau = 1$, without loss of generality, the probability distribution function P, a solution to (2.7) with the collisional form (2.15), is explicitly given, as originally computed in [19], by

$$P_E(v) = \frac{1}{2E}\exp\left(\frac{-\lambda}{E}\right) erfc\left(\lambda\sqrt{\frac{1}{2}}\right) \tag{2.16}$$

with $E > 0$, $\lambda = \frac{2}{E} - v$, and $erfc(x) = \frac{1}{2} + \frac{1}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$.

In addition, moments are explicitly computed by a recursion formula [13], and, in the one-dimensional case, the first three satisfy

$$\langle vP_E\rangle = E,$$
$$\langle v^2 P_E\rangle = 1 + 2E^2,$$
$$\langle v^3 P_E\rangle = 3E + 6E^3.$$

The main result for the first part of the paper, *the Milne problem for strong force fields*, is stated as follows.

THEOREM 1 (positive force field). *Let $E > 0$ be a given positive real number. Let $P_{\sigma,E}$ (we shall also use the short notation P) be the solution of the space homogeneous equation* (2.7). *Assume that $0 \le k(v) \le KP(v)$ for some constant K. Then,* (2.6) *has a unique positive solution such that $f/P \in L^\infty(\mathcal{R}_x^+ \times \mathcal{R}_v)$. Moreover, there exists a constant $n_\infty = \frac{\langle vf\rangle}{\langle vP\rangle}$ such that*

$$\lim_{x\to+\infty} f(x,v) = n_\infty P(v) \quad \textit{pointwise.} \tag{2.17}$$

THEOREM 2 (negative force field). *Let $E < 0$ be a given negative real number. Let $P_{\sigma,E}$ (we shall also use the short notation P) be the solution of the space homogeneous equation* (2.7). *Assume that $0 \le k(v) \le KP(v)$ for some constant K. Then, for any given $n_\infty \in \mathcal{R}^+$, there exists a unique positive solution f_{n_∞} of* (2.6) *such that $f_{n_\infty}/P \in L^\infty(\mathcal{R}_x^+ \times \mathcal{R}_v)$ and*

$$\lim_{x\to+\infty} f_{n_\infty}(x,v) = n_\infty P(v) \quad \textit{pointwise.}$$

In both cases the integrability of f follows from the integrability of P.

3. Analysis of the Milne problem.

3.1. Properties independent of $sgn(E)$. We first start by showing that the current carried by the homogeneous solution P, that is $\langle vP\rangle$, has the same sign as E. Namely, we claim the following.

LEMMA 1. *The solution P of problem* (2.7) *with the linear collisional form* (2.2) *satisfies $E\langle vP\rangle > 0$ and $0 < E\langle v^3 P\rangle < K < \infty$ if $E \ne 0$.*

Proof. In the case of relaxation the statement is trivial since, by (2.17), $E\langle vP\rangle = E^2$ and $E\langle v^3 P\rangle = E^2 + 3E^4$.

For the general linear case, the collision operator is self-adjoint in the weighted space $L^2_M = \{f \in L^1_{loc}, \int_{\mathcal{R}} f^2 M^{-1}\,dv < +\infty\}$, so that

$$\langle gQ(f)M^{-1}\rangle = \langle fQ(g)M^{-1}\rangle. \tag{3.1}$$

Since, by symmetrization,

$$\int Q(f)g\,dv = -\frac{1}{2}\int sMM'\left(\frac{f'}{M'} - \frac{f}{M}\right)(g' - g)\,dv\,dv, \tag{3.2}$$

it follows that for all monotone increasing H

$$\int Q(f)H\left(\frac{f}{M}\right)dv \le 0 \tag{3.3}$$

and

$$\int Q(f)H\left(\frac{f}{M}\right)dv = 0 \quad \text{if and only if} \quad f(v) = cM(v)$$

for any constant c.

Now, taking $H(\tau) = \ln\tau$, we obtain

$$\int Q(P)\ln\left(\frac{P}{M}\right)dv = \int Q(P)\left(\ln P + \frac{v^2}{2}\right)dv \le 0. \tag{3.4}$$

In addition, by (2.7), $E\frac{\partial P}{\partial v} = Q(P)$; then

$$0 \ge \int Q(P)\left(\ln P + \frac{v^2}{2}\right)dv = \int E\frac{\partial P}{\partial v}\left(\ln P + \frac{v^2}{2}\right)dv.$$

Since by integrability of P and $P\ln P$ the identity

$$\int \ln P\frac{\partial P}{\partial v}dv = \int_R \frac{\partial}{\partial v}(P\ln P - P)dv = 0$$

holds, then

$$E\int \frac{v^2}{2}\,\frac{\partial P}{\partial v}\,dv \le 0.$$

Thus, integrating by parts yields the first inequality

$$E\int vP\,dv \ge 0. \tag{3.5}$$

Next, we show that (3.5) cannot be zero if E is not zero. Indeed, if $E\int vP\,dv = 0$, then, by (3.4), $\int Q(P)\ln\frac{P}{M}\,dv = 0$, which implies $P = c\,M$, and thus P is a multiple of the Maxwellian. Therefore $E\frac{\partial M}{\partial v} = 0$ and $E \neq 0$, which is a contradiction since $\partial M/\partial v$ does not vanish.

Finally, the finiteness of moments for all orders follows from the moments recursion formula (2.14). ∎

THEOREM 3 (existence). *Let E be a given real number. Let P be the solution of the space homogeneous equation* (2.7). *Assume that $K_1P(v) \le k(v) \le K_2P(v)$ for* *some positive constants* K_1 *and* K_2*. Then there exist two solutions* $(\underline{f}, \overline{f})$ *of* (2.6) *called minimal and maximal solutions such that*

$$K_1 P(v) \leq \underline{f}(x,v) \leq \overline{f}(x,v) \leq K_2 P(x,v)$$

and such that any solution f *of* (2.6)*, such that* $K_1 P(v) \leq f(x,v) \leq K_2 P(v)$*, is trapped between* $\underline{f}$ *and* $\overline{f}$:

$$\underline{f}(x,v) \leq f(x,v) \leq \overline{f}(x,v).$$

To construct a solution on the half real line $\mathcal{R}^+$, we first solve the problem on the interval $[0,L]$ and then let L tend to $+\infty$. To this end, we consider the problem

$$\begin{cases} v\varphi_x + E\varphi_v = Q(\varphi)(x,v), \\ \varphi(0,v) = k_1(v) \text{ for } v > 0, \\ \varphi(L,v) = k_2(v) \text{ for } v < 0. \end{cases} \tag{3.6}$$

LEMMA 2. *Assume that* $K_1 P(v) \leq k_{1,2}(v) \leq K_2 P(v)$*. Then* (3.6) *admits a unique solution* φ *such that* $\varphi(x,v)/P(v) \in L^\infty([0,L] \times \mathcal{R}_v)$*. Moreover,*

$$K_1 P(v) \leq \varphi(x,v) \leq K_2 P(x,v).$$

Proof. To prove the existence of a solution, we consider the mapping T_L defined by $f = T_L(g)$, where f is the unique solution of

$$\begin{cases} \sigma(v) f + v f_x + E f_v = Q^+(g)(x,v), \\ f(0,v) = k_1(v) \text{ for } v > 0, \\ f(L,v) = k_2(v) \text{ for } v < 0. \end{cases} \tag{3.7}$$

The function f exists by virtue of [29] and is unique since $\sigma \geq s_0 > 0$. Moreover, the maximum principle insures that $f \geq K_1 P(v)$ if $g \geq K_1 P(v)$ (and $f \leq K_2 P(v)$ if $g \leq K_2 P(v)$). Starting from $f_1(x,v) = K_1 P(v)$, we proceed as in [29] and define $f^n = T_L f^{n-1}$ and set $\varphi = \lim_{n \to +\infty} \frac{1}{n} \sum_{l=1}^n f_l$. It is then clear that $K_1 P \leq f_n \leq K_2 P$ and φ is a solution of (3.7) which satisfies $K_1 P \leq \varphi \leq K_2 P$.

The uniqueness follows by an entropy argument developed in [9]. For the sake of completeness, we detail this argument. We set h to be the difference between two solutions. Then h is a solution of

$$\begin{cases} v h_x + E h_v = Q(h), \\ h(0,v) = 0 \text{ for } v > 0, \\ h(L,v) = 0 \text{ for } v < 0. \end{cases}$$

Using the inequality $\int Q(h) sgn(h)\, dv \leq 0$ and the fact that equality holds if and only if the sign of $h(x,v)$ does not depend on v, we obtain

$$\int_0^{+\infty} v|h(L,v)|\, dv - \int_{-\infty}^0 v|h(0,v)|\, dv = \int_0^L \int_{\mathcal{R}} Q(h) sgn(h)\, dv\, dx,$$

which implies that $h(0,v) = 0$ and $h(L,v) = 0$ for $v \in \mathcal{R}$ and that the sign of $h(x,v)$ does not depend on v. Setting $H = |h|$, since the collisional form is linear, then $Q^+(h) sgn(h) = Q^+(H)$. Therefore,

$$\begin{cases} \sigma(v) H + v H_x + E H_v = Q^+(H), \\ H(0,v) = 0 \text{ for } v \in \mathcal{R}, \\ H(L,v) = 0 \text{ for } v \in \mathcal{R}. \end{cases}$$

This implies that $H = 0$ after an integration along the characteristics.

Proof of existence Theorem 3. The maximal and minimal solutions are respectively obtained by solving problem (3.6), with $k_1 = k$ and $k_2 = K_2 P$ (for the maximal solution) and $k_1 = k$ and $k_2 = K_1 P$ (for the minimal solution). Indeed, define $f^+ = T_L^+(g)$ and $f^- = T_L^-(g)$ as the unique solutions of

$$\begin{cases} \sigma(v) f^+ + v f_x^+ + E f_v^+ = Q^+(g)(x,v), \\ f^+(0,v) = k(v) \text{ for } v > 0, \\ f^+(L,v) = K_2 P(v) \text{ for } v < 0, \end{cases} \tag{3.8}$$

$$\begin{cases} \sigma(v) f^- + v f_x^- + E f_v^- = Q^+(g)(x,v), \\ f^-(0,v) = k(v) \text{ for } v > 0, \\ f^-(L,v) = K_1 P(v) \text{ for } v < 0. \end{cases} \tag{3.9}$$

Then the maximal and minimal solutions are defined by $\overline{f}_L = T_L^+(\overline{f}_L)$ and $\underline{f}_L = T_L^-(\underline{f}_L)$, and so

$$K_1 P(v) \le \underline{f}_L \le \overline{f}_L \le K_2 P(v).$$

Moreover, $\overline{f}_L = \lim_{n\to+\infty} (T_L^+)^n (K_2 P)$ (the sequence $((T_L^+)^n(K_2P))$ being pointwise decreasing) and $\underline{f}_L = \lim_{n\to+\infty}(T_L^-)^n(K_1 P)$ (the sequence being pointwise increasing). The above constructed sequences satisfy the following monotonicity properties.

LEMMA 3. *If* $L_1 \le L_2$, *then* $\overline{f}_{L_1} \ge \overline{f}_{L_2}$ *and* $\underline{f}_{L_1} \le \underline{f}_{L_2}$ *on* $[0, L_1] \times \mathcal{R}$.

Proof. Let $L_1 \le L_2$ and $H = \overline{f}_{L_1} - \overline{f}_{L_2}$ on $[0, L_1] \times \mathcal{R}_v$; then H^m is the solution of

$$\begin{cases} v H_x + E H_v = Q(H), \\ H(0,v) = 0 \text{ for } v > 0, \\ H(L_1, v) = K_2 P(v) - \overline{f}_{L_2}(L_1, v) \ge 0 \quad \text{for } v > 0. \end{cases}$$

Therefore $H \ge 0$ by virtue of Lemma 2. The inequality for $\underline{f}_L$ is obtained analogously. ☐

Let us now pass to the limit $L \to +\infty$. For this purpose, we notice that $K_1 P(v) \le \underline{f}_L \le \overline{f}_L \le K_2 P(v)$ and that $\underline{f}_L$ is increasing with respect to L, while $\overline{f}_L$ is decreasing with respect to L. The pointwise limits $\underline{f}$ and $\overline{f}$ of $\underline{f}_L$ and $\overline{f}_L$ as L tends to $+\infty$ are obviously solutions of the problem (2.6) and satisfy

$$K_1 P \le \underline{f} \le \overline{f} \le K_2 P.$$

The only thing left to show now is that any solution $f \in [K_1 P, K_2 P]$ of (2.6) is trapped between $\underline{f}$ and $\overline{f}$. To this aim, set $g = f - T_L^-(f)$. Then g is the solution of

$$\begin{cases} v g_x + E g_v + \sigma g = 0, & \\ g(0,v) = 0, & v > 0, \\ g(L,v) = f(L,v) - K_1 P(v) \ge 0, & v < 0, \end{cases}$$

which implies $g \ge 0$. Hence $T_L^-(f) \le f$. The maximum principle insures that $T_L^\pm(g_1) \le T_L^\pm(g_2)$ whenever $g_1 \le g_2$. Therefore $(T_L^-)^m(f) \le f$. However, $(T_L^-)^m(f) \ge (T_L^-)^m(K_1 P)$. Since $\underline{f}_L = \lim_{m\to+\infty}(T_L^-)^m(K_1P)$, we deduce from the above inequality that $f \ge \underline{f}_L$ on $[0, L] \times \mathcal{R}$, which leads to $f \ge \underline{f}$. The inequality $f \le \overline{f}$ is obtained analogously. The proof of Theorem 3 is now complete. ☐

Next, we study the uniqueness and the asymptotic behavior for the solutions.

3.2. The Milne problem for strong positive forces. The aim of this section is to complete the proof of Theorem 1. First, we show *uniqueness*, that is, $\overline{f} = \underline{f}$ for arbitrary K_2. This proof, which is rather short, uses the asymptotic behavior to be shown next. However, we leave the asymptotic behavior for last, since its proof does not require uniqueness of the solutions.

THEOREM 4 (uniqueness for the case of strong positive forces). *Assume that $E > 0$ and that $0 \leq k \leq KP$. Then $\overline{f}$ and $\underline{f}$ coincide.*

First we prove the following proposition.

PROPOSITION 1. *Let $h = \overline{f} - \underline{f}$. Then $\partial_x h \geq 0$, and there exists $\alpha \geq 0$ such that*

$$\lim_{x\to+\infty} h(x,v) = \alpha P(v).$$

Proof. Take the function $h_{L,a}(x,v) = \overline{f}_{L+a}(x+a,v) - \underline{f}_{L+a}(x+a,v) - \overline{f}_L(x,v) + \underline{f}_L(x,v)$ for $a > 0$. Then $h_{L,a}$ satisfies

$$v\partial_x h_{L,a} + E\partial_v h_{L,a} = Q(h_{L,a})$$

and $h_{L,a}(0,v) \geq 0$ for $v > 0$, while $h_{L,a}(L,v) = 0$ for $v < 0$. Therefore $h_{L,a} \geq 0$ uniformly in L, which implies by passing to the limit $L \to +\infty$ that $h(x+a,v) - h(x,v) \geq 0$. Since a is arbitrary, then $\partial_x h \geq 0$. $\square$

Proof of Theorem 4. Let $h = \overline{f} - \underline{f}$. By construction, $h \geq 0$.

However, because of the boundary condition at $x = 0$, $h(0,v) = 0$ for $v > 0$, and consequently the associated first moment j, which is x-independent, is nonpositive since $j = \langle vh \rangle = \int_{-\infty}^0 vh \leq 0$.

On the other hand, by Proposition 1, on the one hand, $\lim_{x\to+\infty} h(x,v) = \alpha P(v)$ for $\alpha \geq 0$, and on the other hand, by Lemma 1, $\alpha\langle vP \rangle \geq 0$.

Now, since $j = \langle vh \rangle$ is x-independent due to is limit at infinity, $0 \geq j = \alpha\langle vP \rangle \geq 0$. This is only possible if $\alpha = 0$.

Therefore, h is nonnegative, and by Proposition 1, increasing with respect to x and tends to zero as x tends to $+\infty$. Then h is identically equal to zero for all $x \geq 0$, for all v. $\square$

Asymptotic behavior at ∞: Completion of the proof of Theorem 1. Without loss of generality, we renormalize the solution of (2.6) with respect to the constant K of the data. This is equivalent to treating the case $K = 1$. Therefore f solves

$$\begin{cases} \dfrac{v}{E} f_x + f_v + \dfrac{\sigma(v)}{E} f = \dfrac{1}{E} Q^+(f), \\ f(0,v) = k(v) \leq P(v). \end{cases} \tag{3.10}$$

By (3.10), $0 \leq f(x,v) \leq P(v)$ for all v, and its first moment $j = \int_{\mathcal{R}} vf(x,v)\,dv$ is independent of x.

The strategy of the proof works as follows. First we shall prove a *key statement* in Theorem 5, which shows that if the first moment of the solution f of (3.10) is a proper fraction of the first moment of the homogeneous solution P, say by a factor $0 \leq \lambda < 1$, then the spatial asymptotic behavior of f at infinity is given by exactly λP, which is the expected behavior for any solution of the initial value problem (3.10) at infinity. This result is equivalent to an a priori estimate, which means control on the spatial variation of the solutions by control on the variation of its first moment.

Second, we shall see that, in fact, the first moment of *any* solution to problem (3.10) is always a proper fraction of the first moment of the homogeneous solution of the problem; that is, $\lambda P(v)$ for some $0 \leq \lambda < 1$.

Combining both results means that the spatial asymptotic behavior at infinite for f is actually $\lambda P(v)$. That is, the *quotient* between the first moments of the solution f and the homogeneous solution P and between the spatial asymptotic behavior solution of f and the homogeneous solution P, to problems (3.10) and (2.7), respectively, are both the same. In a sense, this is like a Harnack inequality for the kinetic problem.

In fact, these key estimates follow from Lemma 1, which states that if E is positive, then the first moment of the homogeneous solution P is positive. Thus we can make sense of a proper fraction of the first moment of the homogeneous state for a strong force scaling as well as all estimates that follow.

THEOREM 5. *If* $j = \int vf\,dv = \lambda \int vP \geq 0$, *with* $\lambda \in [0,1)$, *then*

$$\lim_{x\to+\infty} f(x,v) = \lambda P(v) = \frac{j}{\langle vP\rangle} P(v). \tag{3.11}$$

The proof of this theorem requires additional partial results that we write as lemmas and corollaries.

LEMMA 4 (initial control for the gain operator). *Assume* $\int vf\,dv = \langle vf\rangle = \lambda\langle vP\rangle > 0$ *for* $0 \leq \lambda < 1$. *Then*

$$Q^+(f) \leq \mu_0 Q^+(P) \quad \textit{for all} \;\; x \geq 0, \tag{3.12}$$

where

$$0 < \mu_0 = 1 - \frac{s_0}{s_1}\frac{1-\lambda}{2v_0}\langle vP\rangle < 1, \tag{3.13}$$

where v_0 *satisfies*

$$0 < \int_{v_0}^{\infty} vP\,dv \leq \frac{1-\lambda}{2}\langle vP\rangle\,, \tag{3.14}$$

and the quotient $\frac{s_0}{s_1}$, *as defined in* (2.3), *measures the oscillation of the scattering rate function.*

Proof. From the existence result, it follows that $0 \leq f \leq P$. Since Q^+ is a positive linear operator,

(3.15)
$$Q^+(f)(v) = Q^+(P) - Q^+(P-f) \leq Q^+(P) - M(v)\int_0^{+\infty} s(v',v)\,(P-f)\,dv'.$$

Then, it is enough to prove that

$$M(v)\int_0^{+\infty} s(v',v)\,(P-f)\,dv' > \beta\, Q^+(P)(v) \quad \text{for some} \;\; \beta < 1. \tag{3.16}$$

In order to estimate (3.16), we use the hypothesis on the first moment of f and P, that is, if

$$\langle vf\rangle = \int_{-\infty}^{+\infty} vf\,dv = \lambda\int_{-\infty}^{+\infty} vP\,dv = \lambda\langle vP\rangle,$$

which yields the first moment flux estimate

$$\int_0^{+\infty} vf\,dv = -\int_{-\infty}^0 vf\,dv + \lambda\langle vP\rangle \leq -\int_{-\infty}^0 vP\,dv + \lambda\langle vP\rangle;$$

that subtracted from

$$\int_0^\infty vP = -\int_{-\infty}^0 vP + \langle vP\rangle$$

leads to the lower bound estimate for the first moment fraction of the difference between the stationary and homogeneous solution

$$\int_0^{+\infty} v(P-f)\,dv \ge (1-\lambda)\langle vP\rangle = \alpha. \tag{3.17}$$

The integrability of the first moment of the homogeneous solution P, and the fact that $\lambda < 1$, imply that there exists a $v_0 > 0$ such that

$$\int_{v_0}^{+\infty} vP\,dv \le \left(\frac{1-\lambda}{2}\right)\langle vP\rangle = \frac{\alpha}{2}\,, \tag{3.18}$$

so that, since $vf > 0$ for $v \ge v_0 > 0$, also

$$\int_{v_0}^{+\infty} v(P-f)\,dv \le \frac{\alpha}{2}\,. \tag{3.19}$$

Next, subtracting inequality (3.19) from inequality (3.17) leads to

$$\int_0^{v_0} v(P-f)\,dv \ge \frac{1-\lambda}{2}\langle vP\rangle. \tag{3.20}$$

Now, recalling that the scattering rate function $s = s(v', v)$ is bounded by $0 < s_0 \le s(v', v) \le s_1 < \infty$, multiplying and dividing the integrand by $s = s(v', v)$ yield a first moment flux fraction difference estimate by a fraction difference for the gain operator

$$\int_0^{v_0} v(P-f) = \int_0^{v_0} v\frac{s}{s}(P-f)\,dv \le \frac{v_0}{s_0}\int_0^{v_0} s(v', v)(P-f)\,dv, \tag{3.21}$$

which combined with inequality (3.20) yields the following lower bound for the right-hand side of (3.21):

$$\int_0^{v_0} s(v', v)(P-f)\,dv \ge \frac{(1-\lambda)}{2v_0}s_0\langle vP\rangle = \frac{\alpha}{2}\frac{s_0}{v_0}\,. \tag{3.22}$$

In addition, since $\langle P\rangle = 1$, then $s_1 \ge \int_{-\infty}^{+\infty} s(v', v)P(v)\,dv \ge s_0$. Thus the right-hand side of (3.22) can be estimated as

$$\frac{\alpha}{2}\frac{s_0}{v_0} \ge \frac{\alpha}{2v_0}\frac{s_0}{s_1}\int_{-\infty}^{+\infty} s(v', v)P(v')\,dv' \;= \frac{\alpha}{2v_0}\frac{s_0}{s_1}Q^+(P), \tag{3.23}$$

where the fraction $\frac{s_0}{s_1} < 1$.

Since $P - f > 0$, inequalities (3.22) and (3.23) lead to

(3.24)

$$M(v)\int_0^{+\infty} s(v', v)(P-f)\,dv \ge M(v)\int_0^{v_0} s(v', v)(P-f)\,dv > \frac{\alpha}{2v_0}\frac{s_0}{s_1}Q^+(P)(v),$$

which yields the inequality (3.16) with $\beta = \frac{\alpha}{2v_0}\frac{s_0}{s_1}$.

Therefore (3.12) holds with

$$0 < \mu_0 = 1 - \frac{\alpha}{2v_0}\frac{s_0}{s_1} < 1, \qquad \alpha = (1-\lambda)\langle vP \rangle \, , \tag{3.25}$$

where v_0 is such that $\int_{v_0}^{+\infty} vP\,dv \le \frac{\alpha}{2}$. □

Remark. The choice of v_0 actually depends on the fact that the first moment of P is finite, that is, on the integrability properties of homogeneous solution P and its corresponding behavior at infinity, and not necessarily on the explicit form of P. This implies that these results can be extended to more general cases, as long as the first moment of P is strictly positive and the corresponding collision frequency is bounded below by a strictly positive constant and above by infinity.

LEMMA 5 (local control of f). *Let $x_k > 0$ such that*

$$Q^+(f) \le \mu_k Q^+(P) \tag{3.26}$$

for any $(x,v) \in D_k = \{(x,v),\, x \ge x_k,\, v \le \sqrt{2E(x-x_k)}\}$; then

$$f \le \mu_k P \quad \text{on } D_k. \tag{3.27}$$

Proof. Recall that $E > 0$. Now for any pair $(x', v') \in D_k$, then $x' \ge x_k$ and $x' = \frac{v'^2}{2E} + x''$, where $x'' > x_k$. Now let (x', v') be fixed (and so is x''), and consider the function $g(v) = f\big(\frac{v^2}{2E} + x'', v\big)$. The argument of the right-hand side of the previous equation lies in D_k, and we have

$$E\frac{\partial g}{\partial v} + \sigma(v) g = Q^+(f)\left(\frac{v^2}{2E} + x'', v\right), \tag{3.28}$$

so that $g(v) \to 0$ as $v \to -\infty$ (because $f(x,v) \le P$).

Now since, by assumption, $Q^+(f) \le \mu_k Q^+(P)$ in D_k, then subtracting the differential inequality from the homogeneous equation satisfied by P, multiplied by μ_0, the difference $g - \mu_k P$ satisfies the differential inequality with the condition in velocity at $-\infty$

$$\begin{aligned} &E\frac{\partial}{\partial v}(g - \mu_k P) + \sigma(v)(g - \mu_k P) \le 0, \\ &\lim_{v \to -\infty} g(v) - \mu_k P(v) = 0. \end{aligned} \tag{3.29}$$

Since $E > 0$, it implies $g \le \mu_k P$. In particular, taking $v = v'$, we get

$$f(x', v') \le \mu_k P(v') \ \text{ on } \ D_k. \tag{3.30}$$

The proof is completed. □

The strategy in order to show (3.11), the expected behavior at infinity for f, consists of constructing pairs (μ_{k+1}, x_{k+1}) for which the control of the gain operator of f by that of P is improved (see (3.26)), and so by Lemma 5, the control of f by P (see (3.27)) is also improved by the same factor μ_{k+1} in such a way that the limit of the sequence $\{\mu_k\}$ is equal to $\lambda \ge 0$, while the limit for $\{x_k\}$ tends to $+\infty$.

The construction of such sequences of pairs entails the following iterative procedure. First, construct iteratively the sequence (μ_k, x_k) starting from $x_0 = 0$ and μ_0 given by Lemma 4, for as long as $f \le \mu_k P$ and $0 \le \lambda < \mu_k$ for $k \ge 0$.

Second, find a pair (μ_{k+1}, x_{k+1}) such that (3.26) holds, that is, $x_{k+1} > x_k$ and $Q^+(f) \leq \mu_{k+1} Q^+(P)$ on D_{k+1}, where $\mu_{k+1} \leq \mu_k$, where the selection of (μ_{k+1}, x_{k+1}) depends on μ_k and λ in a control way so $\lim_{k\to\infty} \mu_k = \lambda$ as $x_k \to \infty$.

Finally, from Lemma 5, it follows that $f \leq \mu_{k+1} P$ on D_{k+1}, for $0 \leq \lambda < \mu_{k+1} < \mu_k < 1$.

This next lemma proves this second step.

LEMMA 6. *Let f satisfy the conditions of Lemmas 4 and 5 for a given pair (μ_k, x_k), and the corresponding D_k for $\lambda < \mu_k$, for $k \geq 0$. Then there exists a (μ_{k+1}, x_{k+1}) such that*

$$Q^+(f) \leq \mu_{k+1} Q^+(P), \qquad 0 < \mu_{k+1} < \mu_k < 1 \ \text{ in } D_k, \tag{3.31}$$

with $x_{k+1} \geq x_k$. Moreover, μ_{k+1} can be chosen so that

$$(\mu_k - \lambda) < C(\mu_k - \mu_{k+1})^2 \ , \qquad \text{with } C = 2\frac{s_1}{s_0}\frac{\langle v^3 P\rangle^3}{\langle v^P\rangle^{1/2}}. \tag{3.32}$$

Before proving Lemma 6, we state a corollary which follows immediately from Lemmas 5 and 6.

COROLLARY 1. *Let D_{k+1} be defined as in Lemma 5; then*

$$f \leq \mu_{k+1} P, \qquad 0 < \mu_{k+1} < \mu_k < 1\,, \qquad \text{on } \ D_{k+1}. \tag{3.33}$$

Proof of Lemma 6. In order to prove (3.31) due to the linearity of the collision form, write

$$Q^+(f) = \mu_k Q^+(P) - Q^+(\mu_k P - f). \tag{3.34}$$

Since $f \leq \mu_k P$ in D_k, then

$$M(v) \int_{-\infty}^{0} s(v,v')\,(\mu_k P(v') - f(x,v'))\,dv' \geq 0 \qquad \text{for } x \geq x_k,$$

with $x_k = \frac{v_k^2}{2E}$. Thus

$$Q^+(\mu_k P - f) \geq M(v) \int_0^\infty s(v,v')\,(\mu_k P(v') - f(x,v'))\,dv', \qquad x \geq x_k. \tag{3.35}$$

Our goal is to see that the right-hand side of (3.35) is bounded by a proper fraction of $\mu_k Q^+(P)$, that is,

$$M(v) \int_0^\infty s(v,v')\,(\mu_k P(v') - f(x,v'))\,dv' \geq \mu_{k+1} Q^+(P) \tag{3.36}$$

for $\mu_{k+1} < \mu_k$ and $x \geq x_{k+1} \geq x_k$, where x_{k+1} is to be determined.

Now, we know from Lemma 5 that $f \leq \mu_k P$ on D_k and that, by assumption, $\langle vf\rangle = \lambda\langle vP\rangle$. Then, as in Lemma 4, since the set $\{x \geq x_k\} \times \{v \leq 0\}$ is in D_k, this implies

$$-\int_{-\infty}^{0} vf\,dv \leq -\mu_k \int_{-\infty}^{0} vP\,dv = -\mu_k\langle vP\rangle + \mu_k \int_0^{+\infty} vP$$

for $x \geq x_k$, which yields

$$\int_0^{+\infty} v(\mu_k P - f)\, dv \geq (\mu_k - \lambda)\langle vP\rangle = \alpha_k \quad \text{for} \quad x \geq x_k. \tag{3.37}$$

Next, we need to choose the set D_{k+1}, which means choosing v_{k+1} and the corresponding x_{k+1} such that we can control them, and $Q^+(f) \leq \mu_{k+1}, Q^+(P)$ for $x_{k+1} > x_k$ and for some $\mu_{k+1} < \mu_k$.

In order to see this fact, first since we can do this construction, as done in Lemma 4, for as long as $\mu_k > \lambda$, and by the integrability properties of the first moment of P, one can choose v_{k+1} such that

$$v_{k+1} \int_{v_{k+1}}^{+\infty} P\, dv \leq \int_{v_{k+1}}^{+\infty} vP\, dv \leq \frac{1}{2}(\mu_k - \lambda)\langle vP\rangle. \tag{3.38}$$

On the other hand, as a consequence of (2.14) and Lemma 1, the third moment of P is bounded, and

$$\int_{v_{k+1}}^{+\infty} vP\, dv \leq \frac{1}{v_{k+1}^2} \int_0^{\infty} v^3 P(v)\, dv \leq \frac{\langle v^3 P\rangle}{v_{k+1}^2} \quad \text{for all } v_{k+1} \geq 0\,.$$

Then, choose v_{k+1} large enough such that both (3.38) and

$$v_{k+1} \leq \frac{C}{\sqrt{\mu_k - \lambda}}, \quad \text{with} \quad C = \frac{\langle v^3 P\rangle}{\sqrt{2\langle vP\rangle}}, \tag{3.39}$$

are satisfied.

Hence, taking $x_{k+1} = \frac{v_{k+1}^2}{2E} + x_k$, it is clear that

$$\{(x, v) : x \geq x_{k+1},\ v \leq v_{k+1}\} \subset D_k, \quad \text{so that} \quad f(x, v) \leq \mu_k P(v). \tag{3.40}$$

Next, rewrite integral estimate (3.37) as

$$\int_0^{v_{k+1}} v(\mu_k P - f)\, dv + \int_{v_{k+1}}^{+\infty} v(\mu_k P - f)\, dv \geq (\mu_k - \lambda)\langle vP\rangle. \tag{3.41}$$

Also, since $\mu_k < 1$ and $f > 0$, from the estimate from below in (3.38) it follows that

$$\int_{v_{k+1}}^{\infty} (\mu_k P - f) \leq \int_{v_{k+1}}^{\infty} P \leq \frac{1}{2v_{k+1}}(\mu_k - \lambda)\langle vP\rangle; \tag{3.42}$$

then, combining (3.41) and (3.42) yields

$$v_{k+1} \int_0^{v_{k+1}} (\mu_k P - f) \geq \int_0^{v_{k+1}} v(\mu_k P - f) \geq \left(1 - \frac{1}{2v_{k+1}}\right)(\mu_k - \lambda)\langle vP\rangle. \tag{3.43}$$

Finally, this last estimate (3.43) leads to the one involving a fraction of the gain operator on the difference $\mu_k P - f$, as follows. First, recalling $0 < s_0 \leq s(v', v) \leq s_1$ and s_1 finite,

$$\int_0^{v_{k+1}} s(v, v')\,(\mu_k P(v') - f(x, v'))\, dv' \geq \left(1 - \frac{1}{2v_{k+1}}\right)\frac{s_0}{v_{k+1}}(\mu_k - \lambda)\langle vP\rangle. \tag{3.44}$$

Second, since $f \leq P$ and $\mu_k < 1$, then

$$\int_{v_{k+1}}^{\infty} s(v',v)\,(\mu_k P(v') - f(x,v'))\,dv' \geq -\frac{s_1}{v_{k+1}}(1-\mu_k)\int_{v_{k+1}}^{\infty} v' P(v')dv' \geq -\frac{s_1}{v_{k+1}}(1-\mu_k)\frac{1}{2v_{k+1}}(\mu_k-\lambda)\langle vP\rangle \tag{3.45}$$

for any $x > x_{k+1}$.

Therefore, gathering (3.44) and (3.45), we obtain the following lower estimate for equation (3.35):

$$\begin{aligned} Q^+(\mu_k P - f) &\geq M(v)\int_0^{+\infty} s(v',v)\,(\mu_k P(v') - f(x,v'))\,dv' \\ &\geq M(v)\left[\left(1-\frac{1}{2v_{k+1}}\right)s_0 - \frac{1}{2v_{k+1}}s_1(1-\mu_k)\right]\frac{\langle vP\rangle}{v_{k+1}}(\mu_k-\lambda) \\ &= M(v)\left[s_0 - \frac{1}{2v_{k+1}}(s_0+s_1) + \frac{s_1}{2}\frac{\mu_k}{v_{k+1}}\right]\frac{\langle vP\rangle}{v_{k+1}}(\mu_k-\lambda) \\ &= M(v)\left(1-\frac{1}{2v_{k+1}}\left[\frac{s_0+s_1}{s_0}\right] + \frac{1}{2v_{k+1}}\frac{s_1}{s_0}\mu_k\right)s_0\frac{\langle vP\rangle}{v_{k+1}}(\mu_k-\lambda). \end{aligned} \tag{3.46}$$

Now, we can choose v_{k+1} even larger than the choices in (3.38) and (3.40) such that $\frac{1}{2v_{k+1}}[\frac{s_1}{s_0}\mu_k - \frac{s_0+s_1}{s_0}] < \frac{1}{2}$, and thus (3.46) leads to

$$Q^+(\mu_k P - f) \geq \frac{s_0}{2}\frac{\langle vP\rangle}{v_{k+1}}(\mu_k-\lambda)M(v) \geq \frac{1}{2}\frac{s_0}{s_1}\frac{\langle vP\rangle}{v_{k+1}}(\mu_k-\lambda)Q^+(P), \tag{3.47}$$

which, after combination with (3.34), leads to

$$Q^+(f) \leq \mu_{k+1}Q^+(P), \tag{3.48}$$

where

$$\mu_{k+1} = \left(\mu_k - \frac{1}{2}\frac{s_0}{s_1}\frac{\langle vP\rangle}{v_{k+1}}(\mu_k-\lambda)\right). \tag{3.49}$$

Finally, from (3.39), v_{k+1} is such that $v_{k+1} \leq C(\mu_k-\lambda)^{-1/2}$; then combining this with (3.49), we get

$$(\mu_k-\lambda) < C(\mu_k-\mu_{k+1})^2 \quad \text{with } C = 2\frac{s_1}{s_0}\frac{\langle v^3P\rangle}{\langle v^P\rangle^{1/2}}. \tag{3.50}$$

Hence, (3.32) holds as well, and thus the proof of Lemma 6 is now completed. □

We can now complete the proof of Theorem 5.

Proof of Theorem 5. For as long as $\mu_k > \lambda$, proceed constructing the sequence $\{x_k\}$ as in Lemma 6. If $\mu_k \leq \lambda$, set $x_{k+1} = x_k$. In particular, since $\mu_k - \mu_{k+1} \to 0$, as k is large, inequality (3.32) implies

$$\lim_{k\to\infty} \mu_k - \lambda = 0 \quad \text{with} \quad \lim_{k\to\infty} x_k = +\infty,$$

which implies

$$\limsup_{k\to\infty} f(x_k,v) \leq \lim_{k\to\infty} \mu_k P(v) \leq \lambda P(v). \tag{3.51}$$

Conversely, applying Lemma 6 and Corollary 1 to $P - f$, since we have assumed $0 \leq \lambda < 1$, then $0 \leq \langle vP - vf \rangle \leq (1-\lambda)\langle vP \rangle$, and also

$$\limsup_{k\to\infty} (P - f)(x_x, v) \leq (1-\lambda)P(v) \leq 0,$$

or equivalently,

$$\liminf_{k\to\infty} f(x_k, v) \geq \lambda P(v). \tag{3.52}$$

Finally, from the construction of the sequence, for μ_k either larger or smaller than λ, (3.51) and (3.52) imply that

$$\lim_{x\to\infty} f(x, v) = \lambda P(v),$$

so (3.11) holds. The proof of Theorem 5 is now completed. □

Finally, in order to complete the proof of the Theorem 1, we define $n_\infty = \frac{\langle vf \rangle}{\langle vP \rangle}$. Then we need to show that n_∞ is always a nonnegative proper fraction, since this has been an assumption in Theorem 5.

THEOREM 6. *If* $\langle vf \rangle = n_\infty \langle vP \rangle$, *then*

$$0 \leq n_\infty \leq 1. \tag{3.53}$$

Proof. First, we recall from the existence construction, if the boundary data is $0 < k(v) \leq P(v)$, for $v > 0$, then $0 < f < P$ for all $x \geq 0$ and all v.

Now, argue by contradiction. If $n_\infty < 0$, take $g = f - n_\infty P$. Clearly $\langle vg \rangle = \langle v(f - n_\infty P) \rangle = 0$.

Therefore, applying Theorem 5 to g with $\lambda = 0$,

$$\lim_{x\to\infty} g(x, v) \leq 0$$

or equivalently,

$$\lim_{x\to\infty} f(x, v) \leq n_\infty P < 0,$$

contradicting $f > 0$ for all (x, v), $x \geq 0$.

Similarly, if $n_\infty > 1$, then take $g = n_\infty P - f$. Then, $g(x, v) < n_\infty P(v)$ and $\langle vg \rangle = 0$. Hence,

$$\lim_{x\to\infty} (n_\infty P - f) \leq 0, \tag{3.54}$$

or equivalently,

$$n_\infty P(v) \leq \lim_{x\to\infty} f(x, v) \leq P(v),$$

which implies $n_\infty \leq 1$, contradicting the assumption. Then (3.53) holds, so Theorem 6 is proven. □

Completion of Theorem 1. If $n_\infty < 1$, then, from Theorems 5 and 6,

$$\lim_{x\to\infty} f(x, v) = K n_\infty P(v),$$

where n_∞ is the fraction of the first moment of $\frac{f}{K}$ with respect to the first moment of P. And, if $n_\infty = 1$ or, equivalently, $\langle v(K\,P - f)\rangle = 0$, one gets, as in (3.54), $0 \leq \lim_{x\to\infty}(K\,P - f) \leq 0$, since, by the existence, also $0 \leq K\,P - f$. Hence

$$\lim_{x\to\infty} f(x,v) = K\,P(v).$$

The proof of Theorem 1 is now completed. ∎

As a corollary to Theorem 1 we have the following.

COROLLARY 2. *Assume that* $E > 0$ *and that* $k(v) = n_\infty P(v)$. *Then the unique solution of* (2.6) *is* $n_\infty P(v)$.

3.3. The Milne problem for strong negative forces. The aim of this subsection is the proof of Theorem 2. In the negative electric field case, we have proven that the upper and lower solutions coincide. This means that *the* solution of (2.6) can be obtained by solving a truncated problem (3.6) with $k_1 = k$ and k_2 arbitrary, the limit as L tends to $+\infty$ being only dependent on k. For positive electric fields, this will not be the case, and the solution *does* depend on the boundary condition k_2.

Proof of Theorem 2. We first proceed with the construction of a solution with the given asymptotic behavior; that is, we construct a solution f of (2.6) which behaves like $n_\infty P_{\sigma,E}(v)$ as x tends to $+\infty$. It is natural to consider the truncated problem

$$\begin{cases} v\partial_x f_L + E\partial_v f_L = Q(f_L)(x,v), \\ f_L(0,v) = k(v) \text{ for } v > 0, \\ f_L(L,v) = n_\infty P(v) \text{ for } v < 0. \end{cases} \tag{3.55}$$

Since $0 \leq k(v) \leq KP(v)$, the maximum principle insures that $0 \leq f_L \leq K_2 P$, where $K_2 = \max(n_\infty, K)$. Therefore, up to the extraction of a subsequence, f_L converges in L^∞_{loc} weak star towards a solution f of (2.6). Of course, since the convergence is only local in x, we cannot say anything at the moment about the asymptotic behavior of f. This is the purpose of the next step.

To analyze the asymptotic behavior, we consider the following truncated solutions:

$$\begin{cases} v\partial_x f^1_L + E\partial_v f^1_L = Q(f^1_L)(x,v), \\ f^1_L(0,v) = 0 \text{ for } v > 0, \\ f^1_L(L,v) = n_\infty P(v) \text{ for } v < 0, \end{cases} \tag{3.56}$$

$$\begin{cases} v\partial_x f^2_L + E\partial_v f^2_L = Q(f^2_L)(x,v), \\ f^2_L(0,v) = KP(v) \text{ for } v > 0, \\ f^2_L(L,v) = n_\infty P(v) \text{ for } v < 0. \end{cases} \tag{3.57}$$

Obviously, $f^1_L \leq f_L \leq f^2_L$. Considering the limits f^1 and f^2 of f^1_L and f^2_L, we have

$$f^1 \leq f \leq f^2.$$

Moreover,

$$f^2 = KP + \left(1 - \frac{K}{n\ \infty}\right) f^1. \tag{3.58}$$

Besides, Proposition 1 insures that f^1 is increasing with respect to x. This implies the existence of α such that

$$\lim_{x\to+\infty} f_1(x,v) = \alpha P_{\sigma,E}(v).$$

It is enough to prove that $\alpha = n_\infty$, because (3.58) implies that f_2 also converges towards $n_\infty P$. Since f is sandwiched between f_1 and f_2, this implies that

$$\lim_{x\to+\infty} f(x,v) = n_\infty P_{\sigma,E}(v).$$

Let us now prove that $\alpha = n_\infty$. To this aim, we invert the x-axis direction by setting

$$g_L^1(x,v) = f_L^1(L-x,-v).$$

This function satisfies the equation

$$\begin{cases} v\partial_x g_L^1 - E\partial_v g_L^1 = \widehat{Q}(g_L^1)(x,v), \\ g_L^1(0,v) = n_\infty P_{\widehat{\sigma},-E} \text{ for } v>0, \\ g_L^1(L,v) = 0 \text{ for } v<0, \end{cases} \tag{3.59}$$

where

$$\widehat{Q}(g)(v) = \int \widehat{\sigma}(v,v')(Mf' - M'f)dv', \qquad \widehat{\sigma}(v,v') = \sigma(-v,-v'),$$

and where we have noticed that

$$P_{\widehat{\sigma},-E}(v) = P_{\sigma,E}(-v).$$

With this transformation, we have replaced the electric field E by $-E$, and we are back to the positive force case. We know from Corollary 2 that the limit of $g_L^1(x,v)$ as L tends to $+\infty$ is nothing but $n_\infty P_{\widehat{\sigma},-E}(v)$. Therefore, we can pass to the limit in the current and get

$$\lim_{L\to+\infty} \langle v g_L^1\rangle = n_\infty \langle v P_{\widehat{\sigma},-E}\rangle = -n_\infty\langle v P_{\sigma,E}\rangle.$$

On the other hand, $\langle v g_L\rangle = -\langle v f_L^1\rangle$, which leads to

$$\lim_{L\to+\infty} \langle v g_L^1\rangle = -\langle v f^1\rangle = -\alpha\langle v P_{\sigma,E}\rangle.$$

As a consequence, $\alpha = n_\infty$, which is the desired result.

The proof of uniqueness is identical to that of Lemma 2 where the truncated case is considered. The details are left to the reader.

The proof of Theorem 2 is completed. ◻

4. A numerical method. As explained in the introduction, for the purpose of finding boundary or transition conditions we are interested only in the asymptotic state n_∞ and in the reflected density $f(0,v)$, $v<0$, of (2.6). In this section we will describe an approximation procedure to compute these values in the case of a relaxation collision operator.

Part of the motivation for studying this problem is that a direct discretization method to solve the half space problem is, in general, costly. The idea behind the method presented here is to solve the macroscopic equations associated with (2.6) and its adjoint equation and to use a Chapman–Enskog–type expansion as an approximate solution, which in the case of relaxation is an exact calculation where diffusion and transport coefficients depend explicitly on the force field E, via the moments of the distribution P, as shown in (2.17).

4.1. Computation of the asymptotic states. We consider the half space equation (2.6) in the relaxation case:

$$v\partial_x f + E\partial_v f = Q(f) = \langle f\rangle M - f, \qquad f(0,v) = k(v), \quad v > 0, \tag{4.1}$$

with $x \in [0,\infty)$ and $E > 0$ a constant. Due to Theorem 1 we have a unique solution f with $\lim_{x\to\infty} f(x,v) = n_\infty P(v)$, where P is the solution of (2.7). Our aim is to determine an accurate and efficient approximation of n_∞.

In addition to (4.1) we consider the corresponding adjoint equation using the weighted inner product $\langle fgP^{-1}\rangle$. It is given by

$$-v\partial_x g - P\partial_v(EgP^{-1}) = Q(gP^{-1}M)M^{-1}P. \tag{4.2}$$

Boundary conditions are

$$g(0,v) = 0, \quad v < 0.$$

A change of variables $v \to -v$ gives the equivalent equation

$$v\partial_x g + E\tilde{P}\partial_v(g\tilde{P}^{-1}) = Q(g\tilde{P}^{-1}M)M^{-1}\tilde{P}, \qquad g(0,v) = 0, \quad v > 0, \tag{4.3}$$

with $\tilde{P}(v) = P(-v)$.

This system is a particular case of the nonhomogeneous problem

$$v\partial_x g + E\tilde{P}\partial_v(g\tilde{P}^{-1}) = Q(g\tilde{P}^{-1}M)M^{-1}\tilde{P}, \qquad g(0,v) = k(v), \quad v > 0. \tag{4.4}$$

For this problem, we can prove the following theorem.

THEOREM 7. (i) *If $E < 0$ and $|k(v)| \le K\tilde{P}(v)$ for some constant K, then* (4.4) *has a unique solution such that $g/\tilde{P} \in L^\infty(\mathcal{R}_x^+ \times \mathcal{R}_v)$. This solution satisfies $|g(x,v)| \le K\tilde{P}(v)$.*

(ii) *If $E > 0$ and $|k(v)| \le K\tilde{P}(v)$ for some constant K, then, for any $j \in \mathcal{R}$, there exists a unique solution g of* (4.4) *such that $g\tilde{P} \in L^\infty(\mathcal{R}_x^+ \times \mathcal{R}_v)$ and $\int_{\mathcal{R}} vg(x,v)\,dv = j$. This unique solution is also characterized by the condition*

$$\lim_{x\to+\infty} g(x,v) = \frac{j}{\int v\tilde{P}\,dv}\,\tilde{P},$$

and g determines $n_\infty = \langle vf\rangle/\langle vP\rangle$ by

$$n_\infty = \int_{v>0} vk(v)g(0,v)P^{-1}(v)dv, \tag{4.5}$$

which is approximated by

$$n_\infty = \frac{\langle vk\rangle_+}{\langle vP\rangle_+} + \frac{\langle vP\rangle_-}{\langle vP\rangle\left\langle \frac{v}{1+Ev}P\right\rangle_+}\left(\left\langle \frac{v}{1+Ev}\left(k - \frac{\langle vk\rangle_+}{\langle vP\rangle_+}P\right)\right\rangle_+\right). \tag{4.6}$$

Proof. The proof of this theorem follows the same strategy as the proof of Theorems 1 and 2. We shall not redo this proof since it relies on exactly the same strategy, and we will give only some hints.

The first brick of the proof is the study of the homogeneous-in-x problem. It is clear that $\tilde{P}$ is a solution of

$$E\tilde{P}\partial_v(g\tilde{P}^{-1}) = Q(g\tilde{P}^{-1}M)M^{-1}\tilde{P}. \tag{4.7}$$

Actually, any solution g such that $g/\tilde{P} \in L^\infty$ is a multiple of $\tilde{P}$. Indeed, it is enough to prove that a solution g of (4.7) such that $\int g\,dv = 0$ is nothing but the identically vanishing function: to this end, we consider a function h which solves $E\partial_v h + Q(h) = g$. Such a solution exists since $\int g\,dv = 0$ (see [30]). Multiplying (4.7) by $h/\tilde{P}$ and integrating leads to $\int g^2\tilde{P}^{-1}\,dv = 0$.

The second property to be noticed is that $\int v\tilde{P}\,dv$ and E have opposite signs. This is why the sign of E is inverted in Theorem 7. With these remarks, one can reproduce the proofs of subsections 3.1 and 3.3 as well as the proof of Theorem 4 (subsection 3.2). We conjecture that the results of subsection 3.2 can be translated to the adjoint problem (4.4). This would imply that, in the case $E < 0$, the unique solution converges as x tends to $+\infty$ towards a multiple of $\tilde{P}$.

Let us now solve (4.3) approximately by proceeding similarly to the Chapman–Enskog expansion method. We recall that $E > 0$ in this section, so that $\langle vg\rangle$ has to be prescribed. Since the problem is linear, the solution is given up to a multiplication factor, and we choose g such that

$$\langle vg\rangle = -1.$$

The first step of the approximate resolution of (4.3) is to introduce a diffusion approximation: introducing an artificial small parameter δ, we consider

$$v\partial_x g + E\tilde{P}\partial_v(g\tilde{P}^{-1}) = \frac{1}{\delta}Q(g\tilde{P}^{-1}M)M^{-1}\tilde{P}. \tag{4.8}$$

Using the series expansion

$$g = g^0 + \delta g^1 + \cdots$$

in (4.8) and collecting terms of equal order in δ gives to $O(1)$

$$Q(g^0\tilde{P}^{-1}M) = 0$$

or

$$g^0 = \rho\tilde{P}. \tag{4.9}$$

To $O(\delta)$ we have

$$g^1 = M^{-1}\tilde{P}Q^{-1}\left[M\tilde{P}^{-1}(v\partial_x g^0 + E\tilde{P}\partial_v(g^0\tilde{P}^{-1}))\right].$$

Using (4.9) and $Q(vM) = -vM$, one obtains

$$g^1 = -v\tilde{P}\partial_x\rho.$$

The solvability conditions for the $O(\delta^2)$-equations give

$$\partial_x \langle v g^1 M \tilde{P}^{-1} \rangle + E \langle M \partial_v (g^1 \tilde{P}^{-1}) \rangle = 0.$$

Using the special form of g^1, this yields

$$\partial_x^2 \langle v^2 M \rangle \rho + E \partial_x \langle M \rangle \rho = 0$$

or the following equation for ρ:

$$\partial_x^2 \rho + E \partial_x \rho = 0. \tag{4.10}$$

Equation (4.10) is the drift-diffusion equation associated to the kinetic half space equation (4.2). We note that this is in contrast to (4.1). In this case the associated drift-diffusion equation is $\partial_x^2 \rho - E \partial_x \rho = 0$. The solution of (4.10) can be determined exactly up to two parameters:

$$\rho(x) = A e^{-Ex} + B, \quad \text{where} \quad A, B \in \mathcal{R}.$$

Next we compute an approximation $\hat{g}$ of g solving the following equation:

$$\begin{aligned} v \partial_x \hat{g} + E \tilde{P} \partial_v (\hat{g} \tilde{P}^{-1}) &= \rho \tilde{P} - \hat{g}, \\ \hat{g}(0, v) &= 0, \quad v > 0. \end{aligned} \tag{4.11}$$

This equation has been obtained from (4.3) by substituting the first order approximation $\rho \tilde{P}$ for g in $\langle g \tilde{P}^{-1} M \rangle$ into (4.3), where ρ is determined from the drift-diffusion equation (4.10). Notice that $\langle v \hat{g} \rangle$ is no longer independent of x. Now (4.11) can be further simplified by using the above approximation also in those terms in (4.11) involving E. One obtains

$$\begin{aligned} v \partial_x \hat{g} &= \rho(x) \tilde{P} - \hat{g}, \\ \hat{g}(0, v) &= 0, \quad v > 0. \end{aligned} \tag{4.12}$$

The solution of (4.12) can be given explicitly. Assuming boundedness at infinity of the solution, we get

$$\hat{g}(x, v) \tilde{P}^{-1} = \begin{cases} \dfrac{A}{1 - Ev} (e^{-Ex} - e^{-\frac{x}{v}}) + B(1 - e^{-\frac{x}{v}}), & v > 0, \\ \dfrac{A}{1 - Ev} e^{-Ex} + B \, , & v < 0. \end{cases}$$

In particular, $g(\infty, v) = B \tilde{P}$ and

$$\hat{g}(0, v) \tilde{P}^{-1} = \begin{cases} 0, & v > 0, \\ \dfrac{A}{1 - Ev} + B \, , & v < 0. \end{cases}$$

We determine A and B by

$$\left\langle v \left\{ \begin{array}{c} \hat{g}(\infty, v) \\ \hat{g}(0, v) \end{array} \right\} \right\rangle = -1,$$

the closest analogue to $\langle v g \rangle = -1$. This yields

$$B = -\frac{1}{\langle v \tilde{P} \rangle} = \frac{1}{\langle v P \rangle}$$

and

$$A = \frac{1}{\langle vP \rangle} \frac{\langle vP \rangle_-}{\left\langle \frac{v}{1+Ev} P \right\rangle_+}.$$

Here and in the following we use the notation

$$\langle f \rangle_+ = \int_{v>0} f(v)dv, \qquad \langle f \rangle_- = \int_{v<0} f(v)dv.$$

We mention that the g- approximation can be iterated considering the equation for the remaining term $g - \hat{g}$ instead of (4.3) and proceeding as before. Now, one transforms backwards, $v \to -v$, to get the desired approximation of the solution g of (4.2).

The following observation is crucial for the whole scheme: if f is a solution of (4.1) and g one of (4.2),

$$\begin{aligned}
&\partial_x(\langle vf(x,v)g(x,v)P^{-1}\rangle) \\
&\quad = \langle v(\partial_x f)(x,v)g(x,v)P^{-1}\rangle + \langle v(\partial_x g)(x,v)f(x,v)P^{-1}\rangle \\
&\quad = \langle [Q(f) - E\partial_v f]gP^{-1}\rangle \\
&\qquad + \langle f[EgP^{-1}\partial_v P - E\partial_v g - Q(gP^{-1}M)M^{-1}P]P^{-1}\rangle.
\end{aligned}$$

Since

$$\begin{aligned}
\langle Q(f)gP^{-1}\rangle &= \langle Q(f)(gP^{-1}M)M^{-1}\rangle \\
&= \langle fQ(gP^{-1}M)M^{-1}PP^{-1}\rangle
\end{aligned}$$

and

$$\begin{aligned}
\langle E\partial_v fgP^{-1}\rangle &= -\langle Ef\partial_v(gP^{-1})\rangle \\
&= -E\langle f(\partial_v g)P^{-1}\rangle + E\langle fg\partial_v PP^{-2}\rangle,
\end{aligned}$$

we get

$$\partial_x(\langle vf(x,v)g(x,v)P^{-1}\rangle) = 0.$$

In other words, $\langle v\varphi gP^{-1}\rangle$ is an invariant in x. Using this invariant, we get

$$\langle vf(\infty,v)g(\infty,v)P^{-1}(v)\rangle = \langle vf(0,v)g(0,v)P^{-1}(v)\rangle,$$

and substituting gives

$$\langle vn_\infty g(\infty,v)\rangle = \int_{v>0} vk(v)g(0,v)P^{-1}(v)dv.$$

Or, with $\langle vg(x,v)\rangle = 1$,

$$n_\infty = \int_{v>0} vk(v)g(0,v)P^{-1}(v)dv,$$

and thus (4.5) holds. In addition, since

$$g(0,v) \sim \hat{g}(0,v),$$

$\hat{g}$ is given by

$$\hat{g}(0,v)P^{-1} = \begin{cases} \dfrac{A}{1+Ev} + B, & v > 0, \\ 0, & v < 0, \end{cases}$$

with A and B determined above.

Altogether, we obtain

$$\begin{aligned} n_\infty = & \frac{\langle vk \rangle_+}{\langle vP \rangle_+} \\ & + \frac{\langle vP \rangle_-}{\langle vP \rangle \left\langle \frac{v}{1+Ev} P \right\rangle_+} \left(\left\langle \frac{v}{1+Ev} \left(k - \frac{\langle vk \rangle_+}{\langle vP \rangle_+} P \right) \right\rangle_+ \right) , \end{aligned} \tag{4.13}$$

and so (4.6) holds and the proof of the theorem is completed. □

Remark. If $k(v) = \lambda P(v)$, we obtain from the above formula the correct value $n_\infty = \lambda$.

Remark. For E tending to 0 we have

$$\frac{1}{1+Ev} \sim 1 - Ev \sim 1 - Ev + O(E^2).$$

Moreover, $P \to M$ as $E \to 0$. Thus, one obtains in the limit the same result as, for example, in [22], namely,

$$n_\infty = \frac{\langle vk \rangle_+}{\langle vM \rangle_+} + \left\langle v^2 \left(k - \frac{\langle vk \rangle_+}{\langle vM \rangle_+} M \right) \right\rangle_+ .$$

4.2. Computation of the Albedo operator. The outgoing density $f(0,v)$, $v < 0$, of (4.1) can be computed as follows. We proceed in a similar same way as before; however, now $f(\infty, v) = n_\infty$ is known. We start directly with (4.1).

The drift-diffusion equation for this equation is determined by the same procedure as above. One obtains $\partial_x^2 \rho - E \partial_x \rho = 0$. Looking for solutions bounded at infinity, one obtains

$$\rho = B,$$

B a constant. Substituting as before $\rho P(v)$ for f in $E\partial_v f$ and $\langle f \rangle$ in (4.1) gives

$$\begin{aligned} v\partial_x \hat{f} &= \rho P - \hat{f}, \\ \hat{f}(0,v) &= k(v),\ v > 0. \end{aligned}$$

The solution is

$$\hat{f}(x,v) = \begin{cases} k(v)e^{-\frac{x}{v}} + B(1 - e^{-\frac{x}{v}})P(v), & v > 0, \\ BP(v), & v < 0. \end{cases}$$

In particular, one obtains

$$\hat{f}(\infty, v) = BP(v)$$

and therefore $B = n_\infty$. Moreover, $\hat{f}(0,v) = n_\infty P(v)$, $v < 0$. Again, considering the equation for the remainder term $f - \hat{f}$, one obtains a better approximation of the outgoing function.

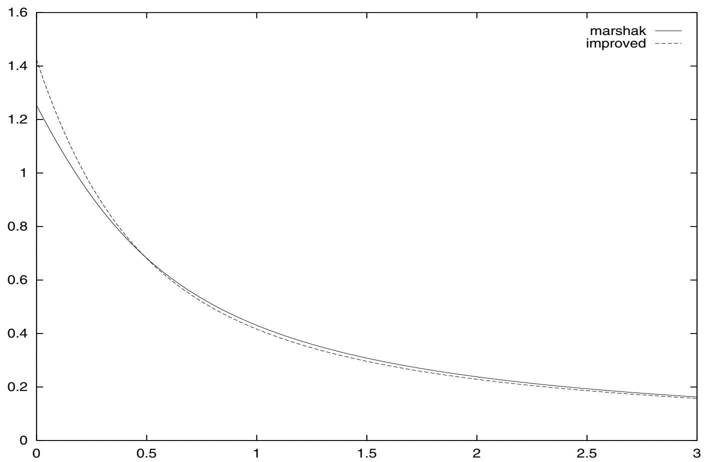

FIG. 1. *Asymptotic states for* $E \in [0, 3]$.

4.3. The Maxwell conditions. The following method was developed by Maxwell [27] and Marshak [26] (see also [6]) to derive approximate boundary conditions. In order to determine n_∞, one equalizes the half-fluxes at the boundary and at infinity, i.e.,

$$\int_{v>0} v\varphi(0,v)dv = \int_{v>0} v\varphi(\infty,v)dv,$$

which means in our context

$$n_\infty \int_{v>0} vP(v)dv = \int_{v>0} vk(v)dv$$

or

$$n_\infty = \frac{\langle vk\rangle_+}{\langle vP\rangle_+}. \tag{4.14}$$

This equality provides correct orders of magnitude in many situations. We observe that the value obtained by the procedure in section 4.1 obviously contains the term one obtains from the Marshak approximation (4.14). However, additionally, a correction term appears in (4.13). The Maxwell approximation of the outgoing distribution is simply

$$\varphi(0,v) = \varphi(\infty,v) = n_\infty P(v), \quad v < 0,$$

with n_∞ given by (4.14).

4.4. Numerical results. We used $k(v) = vM(v)$ to get in Figure 1 the asymptotic values for different values of the electric field $E > 0$. We computed these values by the approximations (4.14), labeled "marshak," and (4.13), labeled "variational." As E tends to 0 one obtains the same results as, for example, in [11, 22]: $n_\infty = 1.2533$ for the Maxwell–Marshak method and $n_\infty = 1.4245$ for the above approximation procedure, which is in case $E = 0$ equivalent to the so-called variational method; see, e.g., [25]. The true solution in this case is known: its numerical value is $n_\infty = 1.4371$.

REFERENCES

[1] M. A. Anile, J. A. Carrillo, I. M. Gamba, and C. W. Shu, *Approximation of the BTE by a relaxation-time operator: Simulations for a 50nm-channel Si diode*, VSLI Design J., 13 (2001), pp. 349–354.

[2] K. Aoki and Y. Sone, *Gas flows around the condensed phase with strong evaporation or condensation*, in Advances in Kinetic Theory and Continuum Mechanics, Proceedings of a Symposium in Honor of H. Cabannes, S. Gatignol, ed., Springer-Verlag, Berlin, 1991, pp. 43–54.

[3] M. D. Arthur and C. Cercignani, *Nonexistence of a steady rarefied supersonic flow in a half space*, Z. Angew. Math. Phys., 31 (1980), pp. 634–645.

[4] G. Baccarani, A. Gnudi, D. Ventura, and F. Odeh, *Two-dimensional MOSFET simulation by means of a multidimensional harmonic expansion of the Boltzmann transport equation*, Solid State Electron., 36 (1993), pp. 575–582.

[5] H. U. Baranger and J. W. Wilkins, *Ballistic structure in the electron distribution function of small semiconducting devices*, Phys. Rev. B, 36 (1987), pp. 1487–1502.

[6] C. Bardos, R. Santos, and R. Sentis, *Diffusion approximation and computation of the critical size*, in Numerical Solutions of Nonlinear Problems (Proceedings of the meeting in Rocquencourt, 1983), INRIA, Rocquencourt, France, 1984, pp. 1–39.

[7] N. Ben Abdallah and H. Chaker, *The high field asymptotics for degenerate semiconductors*, Math. Models Methods Appl. Sci., 11 (2001), pp. 1253–1272.

[8] N. Ben Abdallah and P. Degond, *On a hierarchy of macroscopic models for semiconductors*, J. Math. Phys., 37 (1996), pp. 3306–3333.

[9] N. Ben Abdallah and J. Dolbeault, *Relative entropies for the Vlasov–Poisson system in bounded domains*, C. R. Acad. Sci. Paris Sér. I Math., 330 (2000), pp. 867–872.

[10] J. A. Carrillo, I. M. Gamba, O. Muscato, and C. W. Shu, *Comparison of Monte Carlo and deterministic simulations of a silicon diode*, in Transport in Transition Regimes, IMA Vol. Math. Appl. 135, Springer-Verlag, New York, 2004, pp. 75–84.

[11] C. Cercignani, *The Boltzmann Equation and Its Applications*, Springer, New York, 1988.

[12] C. Cercignani, I. M. Gamba, and C. D. Levermore, *High field approximation to a Boltzmann–Poisson system and boundary conditions in a semiconductor*, Appl. Math. Lett., 10 (1997), pp. 111–118.

[13] C. Cercignani, I. M. Gamba, and C. D. Levermore, *A drift-collision balance for a Boltzmann–Poisson system in bounded domains*, SIAM J. Appl. Math., 61 (2001), pp. 1932–1958.

[14] F. Coron, *Computation of the asymptotic states for linear halfspace problems*, Transport. Theory Statist. Phys., 19 (1990), pp. 89–114; errata in 19 (1990) p. 581.

[15] F. Coron, F. Golse, and C. Sulem, *A classification of well-posed kinetic layer problems*, Comm. Pure Appl. Math., 41 (1988), pp. 409–439.

[16] P. Degond, *A model of near-wall conductivity and its application to plasma thrusters*, SIAM J. Appl. Math., 58 (1998), pp. 1138–1162.

[17] P. Degond, *An infinite system of diffusion equations arising in transport theory: The coupled spherical harmonic expansion model*, Math. Models Methods Appl. Sci., 11 (2001), pp. 903–932.

[18] P. Degond, N. Ben Abdallah, and S. Génieys, *An energy transport model for semiconductors derived from the Boltzamnn equation*, J. Statist. Phys., 84 (1996), pp. 205–231.

[19] G. Frosali, C. V. M. van der Mee, and S. L. Paveri-Fontana, *Conditions for runaway phenomena in the kinetic theory of particle swarms*, J. Math. Phys., 30 (1989), pp. 1177–1186.

[20] I. M. Gamba, J. A. Carrillo, and C. W. Shu, *Computational macroscopic approximations to the 1-D relaxation-time kinetic system for semiconductors*, Phys. D, 146 (2000), pp. 289–306.

[21] L. Garrigues, P. Degond, V. Latocha, and J. P. Boeuf, *Electron transport in stationary plasma thrusters*, Transport Theory Statist. Phys., 27 (1998), pp. 203–221.

[22] F. Golse and A. Klar, *A numerical method for computing asymptotic states and outgoing distributions for kinetic linear half-space problems*, J. Statist. Phys., 80 (1995), pp. 1033–1061.

[23] W. Greenberg, C. van der Mee, and V. Protopopescu, *Boundary Value Problems in Abstract Kinetic Theory*, Birkhäuser Boston, Cambridge, MA, 1987.

[24] A. Klar, *A numerical method for kinetic semiconductor equations in the drift-diffusion limit*, SIAM J. Sci. Comput., 20 (1999), pp. 1696–1712.

[25] S. K. Loyalka, *Approximate method in the kinetic theory*, Phys. Fluids, 14 (1971), pp. 2291–2294.

[26] R. E. Marshak, *The Milne problem for a large plane slab with constant source and anisotropic scattering*, Phys. Rev., 72 (1947), pp. 47–50.
[27] J. C. Maxwell, *The Scientific Papers of J. C. Maxwell*, Dover, New York, 1965.
[28] F. Poupaud, *Diffusion approximation of the linear semiconductor equation*, J. Asympt. Anal., 4 (1991), pp. 293–317.
[29] F. Poupaud, *Boundary value problems for the stationary Vlasov-Maxwell system*, Forum Math., 4 (1992), pp. 499–527.
[30] F. Poupaud, *Runaway phenomena and fluid approximation under high fields in semiconductor kinetic theory*, ZAMM Z. Angew. Math. Mech., 72 (1992), pp. 359–372.
[31] A. Saul, P. Dmitruk, and L. Reyna, *High electric field approximation in semiconductor devices*, Appl. Math. Lett., 5 (1992), pp. 99–102.
[32] C. Schmeiser and A. Zwirchmayr, *Elastic and drift-diffusion limits of electron-phonon interaction in semiconductors*, Math. Models Methods Appl. Sci., 8 (1998), pp. 37–54.
[33] C. E. Siewert and J. R. Thomas, *Strong evaporation into a half space* II, Z. Angew. Math. Phys., 33 (1982), pp. 202–218.
[34] S. A. Trugman and A. J. Taylor, *Analytic solution of the Boltzmann equation with applications to electron transport in inhomogeneous semiconductors*, Phys. Rev. B, 33 (1986), pp. 5575–5583.
[35] A. Yamnahakki, *Second order boundary conditions for the drift-diffusion equations of semiconductors*, Math. Models Methods Appl. Sci., 5 (1995), pp. 429–455.