A REVISION ON CLASSICAL SOLUTIONS TO THE CAUCHY BOLTZMANN PROBLEM FOR SOFT POTENTIALS

RICARDO J. ALONSO AND IRENE M. GAMBA

Abstract. This short note complements the recent paper of the authors [2]. We revisit the results on propagation of regularity and stability using L^p estimates for the gain and loss collision operators which had the exponent range misstated for the loss operator. We show here the correct range of exponents. We require a Lebesgue's exponent $\alpha > 1$ in the angular part of the collision kernel in order to obtain finiteness in some constants involved in the regularity and stability estimates. As a consequence the L^p regularity associated to the Cauchy problem of the space inhomogeneous Boltzmann equation holds for a finite range of $p \geq 1$ explicitly determined.

1. Introduction and previous results

In [2] we have proven existence of distributional and classical (regular) solutions for the Cauchy problem to the Boltzmann equation with soft potentials with both initial data sufficiently small and data close to a local Maxwellian. The corresponding collision kernel consisted in a product of two functions, one depending on the relative speed of interacting particles and the other part on the scattering angle for the corresponding interaction. The soft potential law was taken to be a power law on the relative speed on the the interaction for a parameter $-\lambda$ with $-1 \leq \lambda < n-1$, where n is the space dimension. The angular part of the collision kernel was taken to be integrable.

The purpose of the present note is to revise the statement of the regularity and stability results as well as to provide a condition for a finiteness of constants controlling L^p bilinear estimates of the collisional integral. In this way, the present manuscript completes the analysis undertaken in [2].

The existence and uniqueness result only requires L^1 integrability of the angular part of the collision kernel. However, we will see here that the propagation of regularity will require more that just integrability because is based in bilinear estimates of the collision operator. We will prove such propagation without imposing conditions on the size of ∇f_0 in any particular norm.

As recently studied in [2], we considered the standard model in the kinetic theory of gases given by the Boltzmann Transport Equation in the particular case of soft potentials (i.e. collision kernels with singular forms of the relative speed), under

2000 *Mathematics Subject Classification.* 76P05, 35D05.

Key words and phrases. Boltzmann equation for soft potentials; generalized and classical solutions, stability in L^p spaces.

The authors acknowledge partial support from NSF grant DMS- and DMS 0807712 respectively. Support from the Institute from Computational Engineering and Sciences at the University of Texas at Austin is also gratefully acknowledged.

S^{n-1} integrability assumption in the angular part of the collision kernel to be described below. More specifically,

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f,f) \ \text{ in } \ (0,+\infty) \times \mathbb{R}^{2n} \tag{1.1}$$

with initial condition $f(0,x,v) = f_0(x,v)$. The collision kernel was given by $B(|u|, \hat{u}\cdot\sigma) = |u|^{-\lambda}\ b(\hat{u}\cdot\sigma)$ with $\lambda \in [-1, n-1)$ in the case of small initial data and $\lambda \in [0, n-1)$ in the case of data close to a local Maxwellian.

We have shown in [2] that existence of global in time distributional solutions near vacuum or near a local Maxwellian can be obtain by a modified Kaniel-Shinbrot iteration argument on exponential decaying spaces in both spacial and phase variables and just integrability on the angular part of the collision kernel, i.e. $b(\hat{u}\cdot\sigma) \in L^1(S^{n-1})$ (Grad's cut-off assumption). We recall these two Theorems 1.1 and 1.2, proved in [2], for completeness of this presentation.
In the Section 2 the L^p regularity results that we developed in [2] are revisited. First, the statement of Theorem 2.1 with the range of p exponents for the loss operator as proven in [1] is presented. In particular, we revise the range of the convexity exponents p, q, r of the bilinear form associated with the loss operator with soft potentials, in our recent work [2], Theorem 4.1, Part ii).
Second, Theorems 2.2; 2.3 and 2.4 on section 2 revisit the proofs of Theorems 4.1, 4.2 and 4.3 in [2] respectively. They correctly state the range of exponents and impose an extra requirement in the integrability for the angular part of the cross section, namely $b(\hat{u}\cdot\sigma) \in L^{\alpha}(S^{n-1})$ with $\alpha > 1$.

Recall the following notion of distributional (or *mild*) solutions and the existence and uniqueness result from [2] that provides the ground to our regularity estimates. These result only requires L^1 integrability of the angular part of the collision kernel and does not need to be revised.

- The trajectory operator $f^{\#}(t,x,v) := f(t, x+tv, v)$ gives the evaluation along the trajectories of the transport operator $\partial_t + v\cdot\nabla$ and reduces equation (1.1) to

$$\frac{df^{\#}}{dt}(t) = Q^{\#}(f,f)(t) \ \text{ with } \ f(0) = f_0. \tag{1.2}$$

- A *distributional solution* in $[0,T]$ of problem (1.1) is a function

$$f \in W^{1,1}(0,T; L^{\infty}(\mathbb{R}^{2n}))$$

  that solves (1.2) a.e. in $(0,T] \times \mathbb{R}^{2n}$.

In addition, recall the following Maxwellian weighted spaces. For $M_{\alpha,\beta}(x,v) := \exp\left(-\alpha|x|^2 - \beta|v|^2\right)$, the space of functions bounded by a space-velocity Maxwellian, is denoted by

$$\mathcal{M}_{\alpha,\beta} = L^{\infty}(\mathbb{R}^{2n},\ M^{-1}_{\alpha,\,\beta})\ ,$$

and endowed with the norm

$$\|f\|_{\alpha,\beta} = \left\| f\ M^{-1}_{\alpha,\beta} \right\|_{L^{\infty}(\mathbb{R}^{2n})}.$$

The following existence and uniqueness theorems for small data and for close to local Maxwellian data were proven in [2]. These results only require L^1 integrability of the angular part of the collision kernel.

Theorem 1.1. *(Distributional solutions for small initial data) The Cauchy Boltzmann problem has a unique global distributional solution if* $\|f_0\|_{\alpha,\beta} \leq 4k_{\alpha,\beta}{}^{-1}$, *where the constant* $k_{\alpha,\beta}$ *is given in Lemma 2.2 of* [1]*. Moreover, such distributional solution satisfies*

$$\left\|f^{\#}\right\|_{L^\infty(0,T;\mathcal{M}_{\alpha,\beta})} \leq C := \frac{1-\sqrt{1-4k_{\alpha,\beta}\,\|f_0\|_{\alpha,\beta}}}{2k_{\alpha,\beta}}\,, \tag{1.3}$$

for any $0 \leq T \leq \infty$.

In the case of the existence of bounded distributional solutions with initial data close to local Maxwellian, we first recall a few notions necessary to describe the result clearly. Define the *distance* between two Maxwellian distributions and ϵ*–closeness* to a Maxwellian as follows: For two Maxwellians $M_i = C_i M_{\alpha_i,\beta_i}$ $(i = 1,2)$ the *distance* between them is given by

$$d(M_1, M_2) := |C_2 - C_1| + |\alpha_2 - \alpha_1| + |\beta_2 - \beta_1|.$$

We say that f is ϵ*–close* to the Maxwellian distribution $M = C\,M_{\alpha,\beta}$ if there exist Maxwellian distributions M_i $(i = 1,2)$ such that $d(M_i, M) < \epsilon$ for some small $\epsilon > 0$, and $M_1 \leq f \leq M_2$.

Theorem 1.2. *(Distributional solutions with initial data close to a local Maxwellian) Let* $f_0(x,v)$ *be* ϵ*–close to a local Maxwellian distribution* $C\,M_{\alpha,\beta}(x-v,v)$ $(\alpha > 0,\ \beta > 0,\ C > 0)$*. Then, for sufficiently small* ϵ *depending on the initial parameters* C, α, β*, the model parameters* $\lambda, \|b\|_{L^1(S^{n-1})}$ *and dimension* n*, the Cauchy Boltzmann problem* (1.1) *has a unique solution satisfying*

$$C_1(t)\ M_{\alpha_1,\beta_1}\left(x-(t+1)v, v\right) \leq f(t,x,v) \leq C_2(t)\ M_{\alpha_2,\beta_2}\left(x-(t+1)v, v\right), \tag{1.4}$$

for all $t \geq 0$*, for some continuous functions* $0 < C_1(t) \leq C \leq C_2(t) < \infty$*, and parameters* $0 < \alpha_2 \leq \alpha \leq \alpha_1$ *and* $0 < \beta_2 \leq \beta \leq \beta_1$*. Moreover, the case* $\beta = 0$ *(infinite mass) is permitted as long as* $\beta_1 = \beta_2 = 0$.

The proof of these two theorems can be found in [2].

2. Classical solutions and stability for soft potentials

In the second part of [2] we addressed the *classical solutions* to problem (1.1) as any function in $[0,T]$ satisfying both, $f(t) \in W^{1,1}(0,T;L^\infty(\mathbb{R}^{2n}))$ and $\nabla_x f \in L^1(0,T;L^p(\mathbb{R}^{2n}))$ for some $1 \leq p$, which solves the Cauchy Boltzmann problem (1.1) a.e. in $[0,T) \times \mathbb{R}^{2n}$.

The collision kernel was assumed to be

$$\begin{gathered} B(|u|, \hat{u}\cdot\sigma) = |u|^{-\lambda}\ b(\hat{u}\cdot\sigma) \text{ with } 0 \leq \lambda < n-1, \text{ and} \\ b(\hat{u}\cdot\sigma) \in L^1(S^{n-1}). \end{gathered} \tag{2.1}$$

We revise the quote and statement of Theorem 4.1 in [2] which was misstated for the loss part of the collisional integral from [1]. The corrected quoted theorem reads,

Theorem 2.1. *[New Theorem 4.1] Fix $n \geq 3$ and let the collision kernel satisfy assumptions* (2.1) *above. Then,*

(i) for $1 < p, q, r < \infty$ with $1/p + 1/q + \lambda/n = 1 + 1/r$,

$$\|Q_+(f,g)\|_{L^r_v(\mathbb{R}^n)} \leq C_+ \left\| |u|^{-\lambda} \right\|_{L^{n/\lambda}_w(\mathbb{R}^n)} \|f\|_{L^p_v(\mathbb{R}^n)} \|g\|_{L^q_v(\mathbb{R}^n)} . \tag{2.2}$$

(ii) Moreover, assume that

$$B(|u|, \hat{u}\cdot\sigma) = \Phi(u)\, b(\hat{u}\cdot\sigma)$$

with $\Phi \in L^s(\mathbb{R}^n)$ radially symmetric and non increasing. Then, for $1 \leq p, q, r \leq \infty$ with $1/p + 1/q + 1/s = 1 + 1/r$ one can estimate

$$\|Q_+(f,g)\|_{L^r_v(\mathbb{R}^n)} \leq C_+ \|\Phi\|_{L^s(\mathbb{R}^n)} \|f\|_{L^p_v(\mathbb{R}^n)} \|g\|_{L^q_v(\mathbb{R}^n)} . \tag{2.3}$$

Meanwhile, the negative collision operator satisfies,

(iii) for $1 < p, q, r < \infty$ with $1/p + 1/q + \lambda/n = 1 + 1/r$ and $r < p$,

$$\|Q_-(f,g)\|_{L^r_v(\mathbb{R}^n)} \leq C_- \left\| |u|^{-\lambda} \right\|_{L^{n/\lambda}_w(\mathbb{R}^n)} \|f\|_{L^p_v(\mathbb{R}^n)} \|g\|_{L^q_v(\mathbb{R}^n)} . \tag{2.4}$$

(iv) Assume that

$$B(|u|, \hat{u}\cdot\sigma) = \Phi(u)\, b(\hat{u}\cdot\sigma)$$

with $\Phi \in L^s(\mathbb{R}^n)$ radially symmetric and non increasing. Then, for $1 \leq p, q, r \leq \infty$ with $1/p + 1/q + 1/s = 1 + 1/r$ and $r \leq p$ one can estimate

$$\|Q_-(f,g)\|_{L^r_v(\mathbb{R}^n)} \leq C_- \|\Phi\|_{L^s(\mathbb{R}^n)} \|f\|_{L^p_v(\mathbb{R}^n)} \|g\|_{L^q_v(\mathbb{R}^n)} . \tag{2.5}$$

The constants $C_\pm := C(n,p,q,r,b)$ are specify below.

The proof of this theorem and estimates can be found in sections 4 and 5 of [1]. In elastic interactions one can assume, arguing by symmetry, that $b(\cdot)$ has support on $[0,1]$. Under this assumption, the constants $C_\pm$ can be chosen as

$$C_+ = C(n,p,q,r) \int_{-1}^{1} \frac{1}{(1-s)^{\frac{n-\lambda}{2a}}}\, b(s)(1-s^2)^{\frac{n-3}{2}} ds$$

$$\text{and} \quad C_- = C(n,p,q,r)\ \|b\|_{L^1(S^{n-1})},$$

for some positive constants $C(n,p,q,r)$. The parameter a satisfies,

$$\max\left\{ \frac{1}{r'}\,,\, 1 - \frac{1}{q(1-\frac{\lambda}{n})} \right\} < \frac{1}{a} < \min\left\{ \frac{1}{r'(1-\frac{\lambda}{n})}\,,\, \frac{1}{q'} \right\}. \tag{2.6}$$

The prime in (2.6) denotes the conjugate exponent. In order to obtain regularity estimates, one needs that b is such that the constant C_+ is finite. That means a stronger than Grad's cut–off condition will be needed for the revised versions of Theorem 4.2, 4.3 and 4.4 in [2]. The simplest requirement is $b \in L^\alpha(S^{n-1})$ for some $\alpha > 1$. The careful reader noted that estimates *(iii)* and *(iv)* for the negative part of the collision operator come with the conditions $r < p$ and $r \leq p$ respectively. We will see in the proof below that this condition caps strongly the propagation of smoothness in L^p.

2.1. **Spatial L^p regularity estimates.** As mentioned, we add the condition $b \in L^\alpha(S^{n-1})$ to assure that the constant C_+ is finite. Note however, that there is no constraint on the size of $\nabla_x f_0$ in the space $L^1 \cap L^p$.

Theorem 2.2. *[New Theorem 4.2] Fix $n \geq 3$, $T \in (0,\infty]$ and $\alpha > 1$. Assume that $b \in L^\alpha(S^{n-1})$ and that f_0 satisfies the smallness assumption of Theorem 1.1 or is near to a local Maxwellian as in Theorem 1.2. In addition, assume that $\nabla_x f_0 \in (L^1 \cap L^p)(\mathbb{R}^{2n})$ for some $p > 1$. Then, there is a unique classical solution f to problem (1.1) in the interval $[0,T]$ satisfying the estimates (2.2–2.5) such that for any $p \in [1, \beta(n,\alpha,\lambda))$*

$$\sup_{t\in[0,T]} \|\nabla_x f\|_{L^p(\mathbb{R}^{2n})} \leq C \|\nabla_x f_0\|_{L^p(\mathbb{R}^{2n})}, \tag{2.7}$$

with constant $C = C(n,p,\lambda,\|b\|_{L^\alpha(S^{n-1})})$. The constant $\beta(n,\alpha,\lambda) > 1$ is computed below.

Proof. Denote by $D := D_x$ the *difference* operator in space and τ the *translation* operator. Recall that, for $p > 1$, estimate (4.5) in [2] reads

$$\frac{d\|Df\|^p_{L^p}}{dt} \leq p\int_{\mathbb{R}^n} \|Df\|^{p-1}_{L^p_v(\mathbb{R}^n)} \Big(\|Q_+(Df,f)\|_{L^p_v(\mathbb{R}^n)} + \|Q_+(\tau f, Df)\|_{L^p_v(\mathbb{R}^n)} + \|Q_-(\tau f, Df)\|_{L^p_v(\mathbb{R}^n)} \Big)\, dx. \tag{2.8}$$

All the terms in the right hand side of (2.8) can be estimated using Theorem 2.1. Indeed, using *(i)* with $(p,q,r) = (p, \frac{n}{n-\lambda}, p)$ one has

$$\|Q_+(Df,f)\|_{L^p_v(\mathbb{R}^n)} \leq C_+ \left\||u|^{-\lambda}\right\|_{L^{n/\lambda}_w(\mathbb{R}^n)} \|Df\|_{L^p_v(\mathbb{R}^n)} \|f\|_{L^q_v(\mathbb{R}^n)}.$$

Notice that for this choice of exponents the lower bound condition on the parameter a given in (2.6) reduces to $1/p' < 1/a$. Using Hölder inequality we have

$$C_+ \leq C(n,p,q,r) \left(\int_{-1}^{1} \frac{1}{(1-s)^{\frac{n-\lambda}{2a}\alpha'}} (1-s^2)^{\frac{n-3}{2}}\, ds \right)^{1/\alpha'} \|b\|_{L^\alpha(S^{n-1})}.$$

This constant C_+ is finite provided that the choice of exponent $1 < p < \infty$ satisfies $\frac{n-\lambda}{2p'}\alpha' - \frac{n-3}{2} < 1$. In this way, it is possible to choose a in the valid range (2.6) satisfying $\frac{n-\lambda}{2a}\alpha' - \frac{n-3}{2} < 1$. A simple computation identifies a bound for the p-range:

$$p < \beta_0(n,\alpha,\lambda) := \begin{cases} \infty & \text{when } \frac{n-\lambda}{n-1}\alpha' \leq 1 \\ \left(\frac{n-\lambda}{n-1}\alpha'\right)' & \text{when } \frac{n-\lambda}{n-1}\alpha' > 1. \end{cases} \tag{2.9}$$

Same estimate for the operator $Q_+(\tau f, Df)$ is valid since $b(\cdot)$ was chosen to have support in $[0,1]$. The estimate for $Q_-(\tau f, Df)$ follows using *(iii)* in Theorem 2.1 with $(p,q,r) = (\frac{n}{n-\lambda}, p, p)$. The restriction $r < p$ translates in this context to $p < \frac{n}{n-\lambda}$. Choosing $\beta(n,\alpha,\lambda) = \min\{\beta_0, \frac{n}{n-\lambda}\}$ and using inequality (2.8),

$$\frac{d\|Df\|^p_{L^p}}{dt} \leq p\, C \int_{\mathbb{R}^n} \|Df\|^p_{L^p_v(\mathbb{R}^n)} \Big(\|f\|_{L^q_v(\mathbb{R}^n)} + \|\tau f\|_{L^q_v(\mathbb{R}^n)} \Big)\, dx \quad \text{for any} \quad p \in (1, \beta(n,\alpha,\lambda)).$$

With this estimate one concludes (2.7). The case of $p = 1$ is based on *(ii)* and *(iv)* from Theorem 2.1. First, split the potential in two non-increasing and radially symmetric ones

$$|u|^{-\lambda} = \Phi_1(u) + \Phi_2(u),$$

where $\Phi_1 \in L^s(\mathbb{R}^n)$ for any $1 \leq s < n/\lambda$ and $\Phi_2 \in L^\infty(\mathbb{R}^n)$ (see the proof of Theorem 4.4 [2]). The $L^1_v(\mathbb{R}^n)$ norm of the positive and negative collision operators can be all estimated in the same way. For instance, using *(iv)* in Theorem 2.1,

$$\begin{aligned} \|Q_-(f,\nabla f)\|_{L^1_v(\mathbb{R}^n)} &\leq \|Q_{-,\Phi_1}(f,\nabla f)\|_{L^1_v(\mathbb{R}^n)} + \|Q_{-,\Phi_2}(f,\nabla f)\|_{L^1_v(\mathbb{R}^n)} \\ &\leq C\Big(\|\Phi_1\|_{L^1(\mathbb{R}^n)}\|f\|_{L^{s'}_v(\mathbb{R}^n)}\|\nabla f\|_{L^1_v(\mathbb{R}^n)} \\ &\qquad + \|\Phi_2\|_{L^\infty(\mathbb{R}^n)}\|f\|_{L^1_v(\mathbb{R}^n)}\|\nabla f\|_{L^1_v(\mathbb{R}^n)}\Big). \end{aligned}$$

Using the upper Mawxellian control on f one concludes

$$\|Q_-(f,\nabla f)\|_{L^1(\mathbb{R}^{2n})} \leq \frac{C}{(1+t)^{n/s'}}\|\nabla f\|_{L^1(\mathbb{R}^{2n})}.$$

Note that $n/s' > 1$ if we choose $s \in (\frac{n}{n-1}, \frac{n}{\lambda})$. Using *(ii)* in Theorem 2.1 and the same procedure, one concludes the same estimate for $Q_+(f,\nabla f)$ and $Q_+(\nabla f, f)$. Therefore,

$$\frac{d\|\nabla f\|_{L^1}}{dt} \leq \frac{C}{(1+t)^{n/s'}}\|\nabla f\|_{L^1}.$$

Using Gronwall's lemma,

$$\|\nabla f\|_{L^1}(t) \leq C\|\nabla f_0\|_{L^1} \quad \text{for all} \quad t\in[0,T].$$

□

2.2. **Phase space L^p regularity estimates.**

Theorem 2.3. *[New Theorem 4.3] Fix $n \geq 3$, $T \in (0,\infty]$ and $\alpha > 1$. Assume that $b \in L^\alpha(S^{n-1})$ for some $\alpha > 1$ and that f_0 satisfies the smallness condition of Theorem 1.1 or is near to a local Maxwellian as in Theorem 1.2 . Assume also that the full gradient $\nabla f_0 \in (L^1\cap L^p)(\mathbb{R}^{2n})$ for some $p > 1$. Then, f satisfies the estimate*

$$\|\nabla_v f\|_{L^p(\mathbb{R}^{2n})}(t) \leq C\Big(\|\nabla_v f_0\|_{L^p(\mathbb{R}^{2n})} + t\|\nabla_x f_0\|_{L^p(\mathbb{R}^{2n})}\Big), \qquad (2.10)$$

with $p \in [1, \beta(n,\alpha,\lambda))$. The constant $C = C(n,p,\lambda,\|b\|_{L^\alpha(S^{n-1})})$ is independent of time.

Proof. Follow the proof of Theorem 4.3 in [2] using the estimates given in the proof of Theorem 2.2 in the argument. □

We stress that Theorems 2.2 and 2.3 *do not* assume smallness in the size of ∇f_0 in the space $(L^1\cap L^p)(\mathbb{R}^{2n})$. Moreover, the constants C given in (2.7) and (2.10) depend on the initial datum only by means of its Maxwellian norm.

2.3. L^p **Stability.** Following the method of proof of Theorem 4.4 in [2] and correcting the estimates there using the discussion given in the proof of Theorem 2.2 proves the following stability result.

Theorem 2.4. *[New Theorem 4.4] Fix $n \geq 3$, $T \in (0,\infty]$ and $\alpha > 1$. Assume that $b \in L^{\alpha}(S^{n-1})$. Let f and g distributional solutions of problem (1.1) associated to the initial data f_0 and g_0 respectively. Assume that these data satisfies the condition of Theorem 2.3 (or Theorem 3.1) in [2]. Then, there exists a positive constant C independent of time such that for any $p \in [1, \beta(n,\alpha,\lambda))$,*

$$\sup_{t\in[0,T]} \|f-g\|_{L^p(\mathbb{R}^{2n})} \leq C \ \|f_0 - g_0\|_{L^p(\mathbb{R}^{2n})}\,, \tag{2.11}$$

Moreover, for f_0 and g_0 sufficiently small in $\mathcal{M}_{\alpha,\beta}$ it holds

$$\left\|(f-g)^{\#}\right\|_{L^{\infty}(0,T;\mathcal{M}_{\alpha,\beta})} \leq C \ \|f_0 - g_0\|_{L^{\infty}(\mathcal{M}_{\alpha,\beta})}\,. \tag{2.12}$$

The constant C depends on the initial data only by mean of its Maxwellian norm, that it, $C := C(\|f_0\|_{L^{\infty}(\mathcal{M}_{\alpha,\beta})}, \|g_0\|_{L^{\infty}(\mathcal{M}_{\alpha,\beta})})$.

References

[1] Alonso, R., Carneiro, E. and Gamba, I.: Convolution inequalities for the Boltzmann collision operator, *Commun. Math. Phys.*, **298**, 293 - 322 (2010).

[2] Alonso, R. and Gamba, I.: Distributional and classical solutions to the Cauchy Boltzmann problem for soft potentials with integrable angular cross section, *Journal of Statistical Physics*, **137**, no. 5-6, 1147 - 1165 (2009).

Dept. of Computational & Applied Mathematics, Rice University, Houston, TX 77005-1892

E-mail address: Ricardo.J.Alonso@rice.edu

Department of Mathematics & ICES, University of Texas at Austin, Austin, TX 78712-1082.

E-mail address: gamba@math.utexas.edu