Introduction to Affine Kac-Moody Algebras and Quantum Groups

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Course abstract: Affine Lie algebras are a certain generalization of a finite dimensional Lie algebra $\mathfrak{g}$ which is used, roughly speaking, to capture the representation theory of $\mathfrak{g}((t))$. These algebras have importance in representation theory and string theory. We'll start by quickly reviewing the vocabulary of representation theory of compact Lie groups, including a brief discussion on why semisimple Lie algebras are classified by their Cartan matrix. (There are no prerequisites to this course, but experience with finite dimensional semisimple Lie algebras certainly will help.) We will then define the notion of a Kac-Moody algebra using the notion of a generalized Cartan matrix. We will then specialize to the case of an affine Kac-Moody algebra, and explain some of the basics of its representation theory. The remainder of the course will be about the relation of representations of affine Kac-Moody algebras to quantum groups, which will end with a review of the celebrated Kazhdan-Lusztig equivalence $(\hat{\mathfrak{g}}_\kappa\text{-Mod})^{G(\mathcal{O})} \cong \mathrm{Rep}_q(G)$.

Please send me typos if you find them! Thank you to Thiago Landim, Claire Mirocha, Alberto San Miguel Malaney, and Raul Sanchez for doing this.

Contents

1 Kac-Moody Algebras as Generalizations of Semisimple Lie Algebras

We'll quickly recall the representation theory of a finite dimensional semisimple Lie algebra $\mathfrak{g}$ over[1] $\mathbb{C}$; for more information, see [FH91] or [Hum78] for the original sources or the Introduction to Representation Theory mini course last year [Gan20]. This first day is definitely a little dense and faster than the other days will be, so I will have (virtual) office hours upon request for anyone that wants to learn the language of representation theory a little better.

1.1 The Finite Kac-Moody Algebra A_1

We'll start with the representation theory of the smallest dimensional semisimple Lie algebra, $\mathfrak{sl}_2$, the Lie algebra of traceless matrices with Lie bracket given by commutator of matrices. We will see later that this is the finite Kac-Moody algebra of type A_1.

Recall there are three distinguished elements (up to scalar multiple) $h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ such that representations of $\mathfrak{sl}_2$ are equivalently modules for the *universal enveloping algebra*

$$U(\mathfrak{sl}_2) := \mathbb{C}\langle e, f, h\rangle/(ef - fe = h, he - eh = 2e, hf - fh = -2f).$$

Finite dimensional representations of $\mathfrak{sl}_2$ (or any semisimple Lie algebra) are *completely reducible*; this implies that any finite dimensional representation of $\mathfrak{sl}_2$ splits as a direct sum of its irreducible representations. Therefore it remains to classify the irreducible representations. Furthermore, it turns out that any $\mathfrak{sl}_2$ representation is a direct sum of its various h eigenspaces, and one can check that e (respectively, f) applied to an element in the $\lambda \in \mathbb{C}$ eigenspace maps to the $\lambda + 2$ (repsectively, $\lambda - 2$) eigenspace. We can check by hand the *PBW theorem* for $U(\mathfrak{sl}_2)$, which says that any element of $U(\mathfrak{sl}_2)$ can be written as a polynomial of terms where all powers of f appear to the left of the powers of h, and all powers of h can be written to the left of all powers of e. This more or less implies (after some algebra, see 11.1 of [FH91]):

Theorem 1.1. (Classification of Finite Dimensional Representations of $\mathfrak{sl}_2$) For any finite dimensional $U(\mathfrak{sl}_2)$ module V, there exists a unique up to nonzero scalar vector $v \in V$, called a *highest weight vector*, such that $ev = 0$ and $hv = nv$ for some nonnegative integer $n \in \mathbb{N}^{\geq 0}$.

Note this discussion is symmetric in e and f in the sense that representations can also be classified by the lowest weight, a nonpositive integer. We will see this as the first manifestation of the *Weyl group* of a semisimple Lie algebra, which for $\mathfrak{sl}_2$ is the finite group generated by a single reflection, $\mathbb{Z}/2\mathbb{Z}$.

1.2 The Finite Kac-Moody Algebra A_2

Next, we'll start with the classical next example in the story of representation theory of semisimple Lie algebras, $\mathfrak{sl}_3$, which again we will see as a finite Kac-Moody algebra of type A_2. There are a lot of terms here, so see Appendix A for a summary of the notation. The natural replacement for the notion of h is the two dimensional abelian[2] Lie algebra $\mathfrak{h}$ of diagonal matrices[3]. Instead of looking for eigenvalues of a single h as above, we will look for eigenvalues of the entire Lie algebra $\mathfrak{h}$:

Definition 1.2. Let V be an $\mathfrak{h}$ representation. We say that a nonzero $v \in V$ is an *eigenvector* for the *eigenvalue* $\lambda \in \mathfrak{h}^*$ if for any $h \in \mathfrak{h}$, $hv = \lambda(h)v$.

[1] In fact, much of what we're going to be talking about works for an *arbitrary* algebraically closed field of charactersitic zero. Often times in geometric representation theory, people work over an arbitrary closed field of characteristic zero to signal that they are not using the topology of $\mathbb{C}$ in their results. In particular, most of these results apply for fields such as $\overline{\mathbb{Q}_\ell}$, which are abstractly isomorphic to $\mathbb{C}$ but not as topological fields. However, the specification of the field $\mathbb{C}$ will be important in the setting of quantum groups, since the relationship to affine Kac-Moody algebras to quantum groups is given by exponentiation, see Section 4.

[2] A Lie algebra is *abelian* if its Lie bracket is the zero bracket.

[3] I may also write this as $\mathfrak{t}$ in lecture, but will try to be consistent.

Remark 1.3. We could have applied a similar procedure for $\mathfrak{sl}_2$ by setting $\mathfrak{h} := \mathbb{C}h$. One can check that if v above is an eigenvalue for h, it is an eigenvalue for the one dimensional abelian Lie algebra $\mathbb{C}h$ as in Definition 1.2.

Just as for $\mathfrak{sl}_2$, it turns out that for any semisimple Lie algebra $\mathfrak{g}$, any finite dimensional representation V can be written as a direct sum of its $\mathfrak{h}$ eigenvalues. Let's apply this to the one canonical representation of $\mathfrak{g}$ we get just by nature of $\mathfrak{g}$ being a Lie algebra, the *adjoint representation* $\mathfrak{g} \to \mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$ given by $X \mapsto [X, -]$.

Remark 1.4. At first, the notion of the adjoint representation may look abstract. The perspective of Lie groups may help here–if $G = \mathrm{SL}_n$ (or is any Lie group or algebraic group), then G naturally acts on the tangent space at the identity $\mathfrak{g} := T_1(G)$. It can't act by left or right multiplication, though, because that would move the basepoint. Therefore the only way we can act is by conjugation. Taking the derivative of this action yields the adjoint action.

Example 1.5. The adjoint representation of $\mathfrak{sl}_2$ is an irreducible representation of highest weight 2. This implies that there is a one dimensional eigenspace[4] of h with eigenvalue 2, 0, and -2. One nonzero eigenvector in each of these eigenspaces is e, h, and f respectively.

We now consider the eigenspaces of the adjoint representation of $\mathfrak{sl}_3$. To do this, following 12.1 of [FH91], it helps to choose indicator functions of the diagonal entries L_1, L_2, and L_3, for example, $L_2(\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -a-b \end{pmatrix}) := b$. Then $\mathfrak{h}^* \cong \{aL_1 + bL_2 + cL_3 : a + b + c = 0\}$.

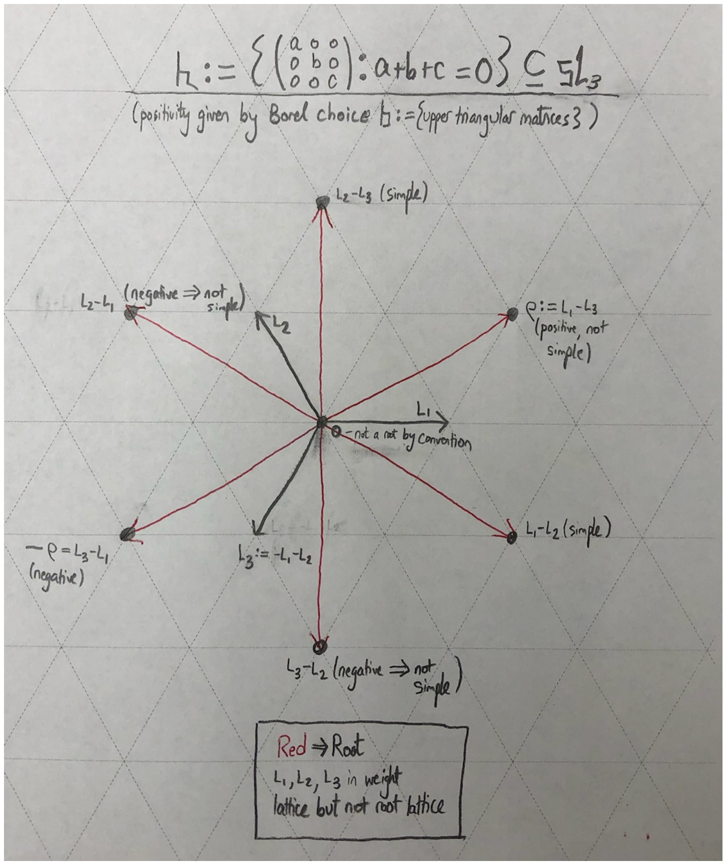
Example 1.6. One can check that the adjoint representation of $\mathfrak{sl}_3$ is an 8 dimensional irreducible representation. Except for the $0 : \mathfrak{h} \to \mathbb{C}$ eigenspace, which is two dimensional, there are six one dimensional eigenspaces. Two of them are given by $e_{L_1-L_2} := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $e_{L_2-L_3} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Specifically, these two elements have eigenvalues $L_1 - L_2$ and $L_2 - L_3$ respectively.

A useful fact is that *these two eigenvalues determine all others.* Specifically, given these two roots, one can check that $e_{L_1-L_3} := [e_{1,2}, e_{2,3}]$ is another–it will have eigenvalue $L_1 - L_3 = (L_1 - L_2) + (L_2 - L_3)$. Furthermore, given one of the $e_{i,j}$ we've constructed we can construct the *associated* $\mathfrak{sl}_2$ *triple* to the root (:= eigenvalue in $\mathfrak{h}^*$ of the adjoint representation). For example, given the eigenvalue/root $L_1 - L_3$, the associated $\mathfrak{sl}_2$ triple is given by $(e_{L_1-L_3}, h_{L_1-L_3}, f_{L_1-L_3})$ where $h_{L_1-L_3} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and $f_{L_1-L_3} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Note that $f_{L_1-L_3}$ is also an eigenvalue of the adjoint representation, with eigenvalue $L_3 - L_1$, and similarly $f_{L_1-L_2}$ and $f_{L_2-L_1}$.

We are soon in a position to generalize the notion of *positivity*[5] for general $\mathfrak{g}$. With our choice of $\mathfrak{h}$, we can plot our eigenvalues of the adjoint representation in the plane (with a clearer picture available by clicking here):

[4]If you want to be fancy, you can call these eigenspaces of the one dimensional Lie algebra $\mathbb{C}h$ with eigenvalues given by the linear functions sending h to 2, 0, and -2 respectively. This will allow one to match the representation theory of $\mathfrak{sl}_2$ more closely with the general representation theory of semisimple Lie algebras.

[5]In fact, this problem actually comes up for $\mathfrak{sl}_2$. Informally, this follows because we could have replaced e with $-f$ and vice versa and h with $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. In representation theory, it is often useful to remember that this is a choice up to action of the Weyl group $\mathbb{Z}/2\mathbb{Z}$.



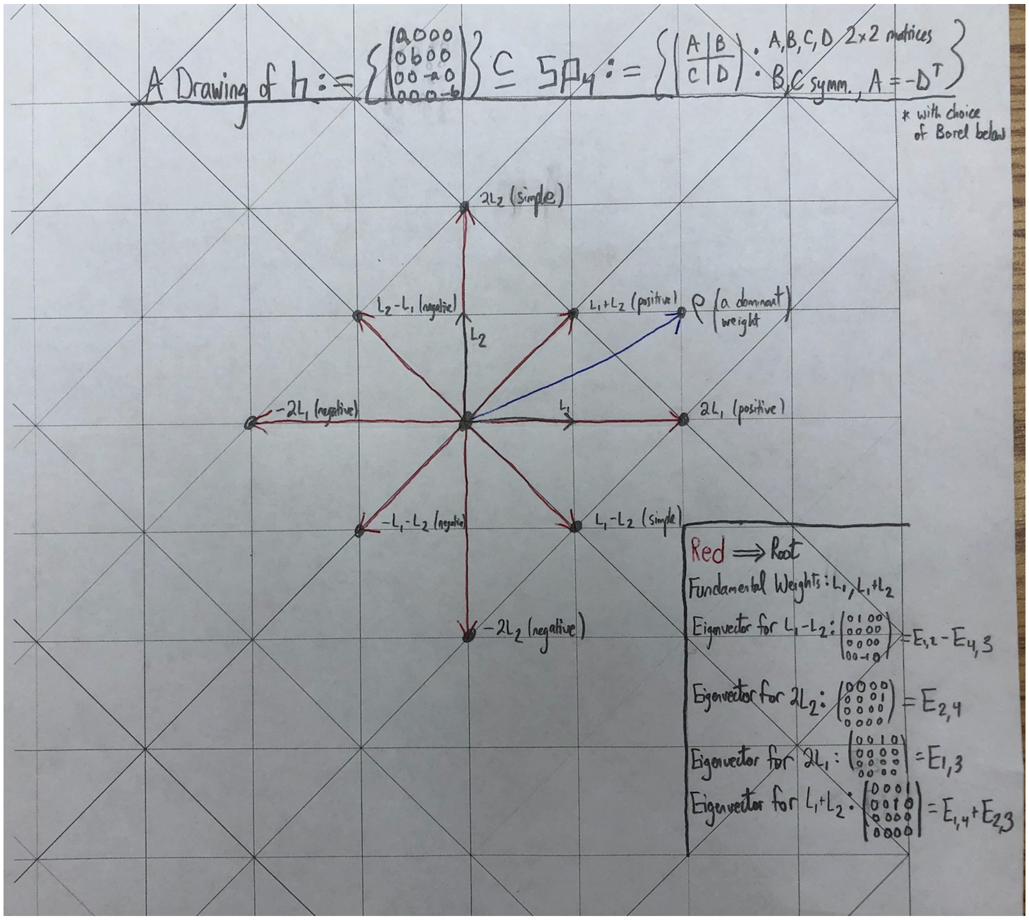
Therefore, we see that a choice of positivity is equivalent to the choice of two roots which are not the negative of each other. Implicitly above, we chose our two simple roots as $L_1 - L_2$ and $L_2 - L_3$. Note, however, this was a choice, and there are in fact 6 different choices we could have made! These choices are permuted simply transitively by the Weyl group, the group of reflections generated by the *root hyperplanes*, drawn above in the picture. Note that the Weyl group S_3 is generated by reflections in the root hyperplanes corresponding to the simple roots (and correspond to the permutations swapping 1 and 2 and swapping 2 and 3). Just as before, this data determines the finite dimensional representations of $\mathfrak{sl}_3$. In fact, tomorrow in Section 2, we will see that this data turns out to determine the Lie algebra itself.

1.3 The Finite Kac-Moody Algebra B_2

We will now discuss a bit about the Lie algebra/finite Kac-Moody algebra of type $B_2 = C_2$, given by the Lie algebra $\mathfrak{so}_5 \cong \mathfrak{sp}_4$. We have seen that to classify representations of this Lie algebra, it is of tremendous interest to diagonalize the adjoint representation. One can check (see the first two pages of chapter 16 of [FH91]) that, with choice of $\mathfrak{h} \subseteq \mathfrak{sp}_4$ given by

$$\mathfrak{h} := \{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & -b \end{pmatrix} : a, b \in \mathbb{C}\}.$$

With this choice we can perform a similar analysis of the adjoint representation of $\mathfrak{sp}_4$ and obtain (with a clearer picture available by clicking here):



Furthermore, just as above we can choose two simple roots. Just as before, the simple roots must be such that exactly half of the roots can be obtained as sums of the simple roots. A standard choice is to choose the roots $2L_1$ and $L_1 - L_2$. The main takeaway is that the roots here are of *different* length, unlike what happens in type A_n (i.e. the semisimple Lie algebra $\mathfrak{sl}_n$). However, the structure is still rigid (eg, the angles between all roots are an angle you know from trig class) and we will see tomorrow in Section 2 that one can recover $\mathfrak{sp}_4$ from this geometric data and choice of simple roots.

2 Affine Kac-Moody Algebras as Specific Kac-Moody Algebras

2.1 Recovering A Lie Algebra from Its Cartan Matrix

Yesterday, we saw that much of the information can be determined about a semisimple Lie algebra from the action of the Weyl group on $\mathfrak{h}^*$ with choice of simple roots. Now we will discuss the converse–specifically, if $\mathfrak{g}$ is a semisimple Lie algebra with a choice of Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ and a choice of Borel subalgebra (i.e. choice of positive direction) $\mathfrak{b} \subseteq \mathfrak{g}$, what precise data do we need to recover our Lie algebra? Let us fix these choices of $\mathfrak{h} \subseteq \mathfrak{b} \subseteq \mathfrak{g}$ for today. We saw yesterday that it was important to determine the simple roots in the plane, which we will now define:

Definition 2.1. A *simple root* is a positive root which cannot be written as the sum of two positive roots.

It turns out that given all simple roots, we can recover all the roots by repeatedly taking Lie brackets. Therefore, if we wish to be minimalist about the data we are keeping around, we should only focus on the simple roots. Now we will discuss a very cool fact about how *different* simple roots interact, which will give you a taste of Lie theory:

Proposition 2.2. ($[e_\alpha, f_\beta] = 0$ if $\alpha \neq \beta$)) Assume that $\alpha, \beta \in \mathfrak{h}^*$ are two *distinct* simple roots, and let e_α and f_β be the associated image of e (respectively f) in the $\mathfrak{sl}_2$ triple associated to α (respectively β).

We will prove this with a helpful lemma:

Lemma 2.3. If $\alpha, \beta : \mathfrak{h} \to \mathbb{C}$ are any roots and $e_\alpha, e_\beta \in \mathfrak{g}$ are in the α, β eigenspaces[6], then $[e_\alpha, e_\beta] = \mathrm{ad}(e_\alpha)(e_\beta)$ is in the $\alpha + \beta$ eigenspace.

Proof. Let $h \in \mathfrak{h}$. We wish to compute $\mathrm{ad}(h)([e_\alpha, e_\beta]) = [h, [e_\alpha, e_\beta]]$. Since this is a representation of Lie algebras, we have:

$$[h, [e_\alpha, e_\beta]] = -[e_\beta, [h, e_\alpha]] - [e_\alpha, [e_\beta, h]]$$

and so we can use the fact that the Lie bracket is anti-commutative multiple times to obtain

$$[h, [e_\alpha, e_\beta]] = [[h, e_\alpha], e_\beta] + [e_\alpha, [h, e_\beta]].$$

Therefore, since e_α and e_β are (generalized) eigenvalues for the adjoint action, we get that

$$[h, [e_\alpha, e_\beta]] = \alpha(h)[e_\alpha, e_\beta] + \beta(h)[e_\alpha, e_\beta]$$

and so $[e_\alpha, e_\beta]$ has (generalized) eigenvalue $\alpha + \beta$. □

Proof of Proposition 2.2. By Lemma 2.3, we have that $[e_\alpha, f_\beta]$ has eigenvalue $\alpha - \beta$. Swapping α and β if necessary, we may assume that $\alpha - \beta$ is positive. However, this violates the simplicity of α, since $\alpha = \beta + (\alpha - \beta)$. □

Remark 2.4. We can now re-interpret Proposition 2.2 as saying that if α and β are distinct simple roots, then $\mathrm{ad}(f_{\beta)(e_\alpha} = 0$. In particular, *if α and β are distinct roots, then e_α is a lowest weight vector for the $\mathfrak{sl}_2$ triple $(e_\beta, h_\beta, f_\beta)$.*

Now we fix all simple roots of $\mathfrak{g}$. We see that for *any* two roots α, β, that e_α is an eigenvalue for h_β. It is of interest to record this eigenvalue. Let us arbitrarily order our simple roots and write a_{ij} such that $\mathrm{ad}(h_i)(e_j) = a_{ij} e_j$. It turns out these a_{ij} are precisely what is needed to recover our semisimple Lie algebra:

Definition 2.5. The matrix given by the a_{ij} above (again, for a fixed $\mathfrak{h} \subseteq \mathfrak{b} \subseteq \mathfrak{g}$) is called the *Cartan matrix for $\mathfrak{g}$.*

We will now work through four examples:

Example 2.6. We saw that $\mathfrak{sl}_3$ has $\dim(\mathfrak{h}^*) = 2$, which implies the associated Cartan matrix is 2x2. We also saw that $\mathfrak{h}^*$ has two simple roots of the same length, $\alpha := L_1 - L_2$ and $\beta := L_2 - L_3$. Since $s_\alpha(\beta) = \beta - (-1)\alpha$, we obtain one entry of the Cartan matrix associated to this data: $\begin{pmatrix} ? & -1 \\ ? & ? \end{pmatrix}$. Since α and β have the same length, we can do the computation with the roles of α and β switched and obtain that the (2,1) entry is also -1. Finally, $s_\alpha(\alpha) = \alpha - 2\alpha$ and similarly for β, so the full Cartan matrix is:

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Example 2.7. We saw yesterday that $\mathfrak{sl}_2$ has $\dim(\mathfrak{h}^*) = \dim(\mathbb{C}h) = 1$, so the Cartan matrix is 1×1. In particular, $\mathfrak{h}^*$ one simple root, and so by Remark 2.8 we see the Cartan matrix is the 1×1 matrix given by (2).

Remark 2.8. Note that for any $\mathfrak{sl}_2$ triple we have, by defintion, that $ad(h_i)e_i = 2e_i$. Therefore, *the diagonal entries of a Cartan matrix will always be 2.*

[6] It turns out that the eigenspaces for roots are one dimensional, so this notation hopefully shouldn't cause too much confusion.

Example 2.9. Now we compute the Cartan matrix for $\mathfrak{sp}_4$, which as before will be a 2×2 matrix since the dimension of our choice of Cartan is 2. By Remark 2.8, our diagonal entries will always be 2. Let α be the short root and β be the long root. Then we see that $s_\alpha(\beta) = \alpha - (-1)\beta$ and $s_\alpha(\beta) = \beta - (-2)\alpha$, so in particular we see that the Cartan matrix is given by:

$$\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$$

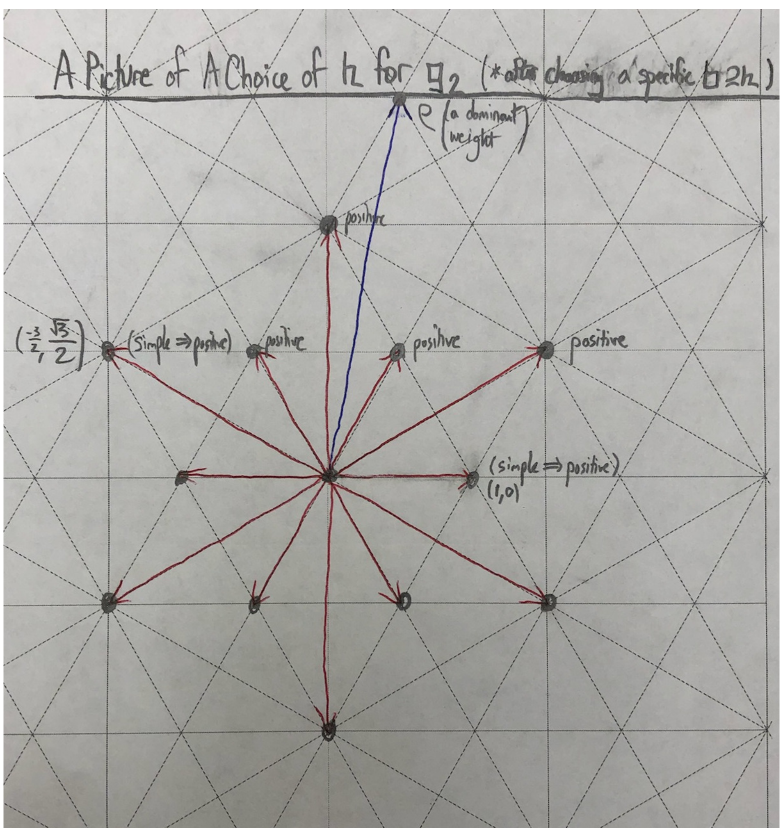
Note that, so far, our off diagonal entries are always nonpositive.

Example 2.10. For type $A_1 \times A_1$ let α_1 be the simple root associated to the first factor and α_2 be the simple root associated to the second. Then we see that $s_{\alpha_1}(\alpha_2) = \alpha_2 - 0\alpha_1$ (since the two roots are perpendicular– these correspond to a semisimple Lie algebra which is not simple). This is symmetric in swapping the two α above, so the generalized Cartan matrix is:

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Exercise 2.11. Fix an $\mathfrak{h} \subseteq \mathfrak{b} \subseteq \mathfrak{g}$ with n simple roots. Prove that for any distinct i, j, if $\mathrm{ad}(h_i)(e_j) = a_{i,j}e_j$, then $-a_{i,j} + 1$ is the dimension of the associated $\mathfrak{sl}_{2,i}$ representation generated by e_j. (Hint: We saw that e_j is a lowest weight vector if $i \neq j$. What happens if $i = j$?)

Exercise 2.12. The root system for the exceptional Lie algebra $\mathfrak{g}_2$ is given as follows (with a clearer picture available by clicking here):



Compute the Cartan matrix of $\mathfrak{g}_2$. Try to avoid computing the adjoint action ($\mathfrak{g}_2$ is 14 dimensional!) and instead try to use Exercise 2.11.

2.2 Generalized Kac-Moody Algebras

We will now start by recording some of the properties that a Cartan matrix of a semisimple Lie algebra and discuss how we can use them to recover the Lie algebra itself.

Definition 2.13. A *generalized Cartan matrix* (GCM) of rank n is an $n \times n$ matrix $A = \left(a_{ij}\right)$ of integers such that

- (See Remark 2.8) $a_{ii} = 2$.
- (See Remark 2.4) If $i \neq j$, $a_{ij} \leq 0$.
- (Orthogonality is a symmetric relation) We have $a_{ij} = 0$ if and only if $a_{ji} = 0$.

Remark 2.14. Sometimes, authors add the condition that a GCM is symmetrizable (see Definition 2.19 below). I am choosing to follow the definitions of chapter 27 of [EMTW20], which is slightly weaker than the notion of a symmetrizable GCM, see [6] for the exact comparison.

Example 2.15. Any Cartan matrix of a semisimple Lie algebra is a generalized Cartan matrix.

Example 2.16. Define the *generalized Cartan matrix of affine type* $\tilde{A}_1$ as $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$.

It turns out that this data is all that is needed to recover the Lie algebra of a semisimple Lie algebra. To see this, we first note that we can recover $\mathfrak{g}$ as a vector subspace of the universal enveloping algebra $U\mathfrak{g}$ as the 'degree one' piece (in fancier language, $U\mathfrak{g}$ is graded and the vector space of grading 1 is canonically $\mathfrak{g}$, as a Lie algebra). Therefore, to recover $\mathfrak{g}$ it suffices to recover $U\mathfrak{g}$ (as a graded algebra).

Theorem 2.17. *Given a semisimple Lie algebra $\mathfrak{g}$, let $\left(a_{ij}\right)$ be its associated (generalized) Cartan matrix of dimension n. Then the universal enveloping algebra $U\mathfrak{g}$ is given as the algebra generated by symbols e_i, h_i, f_i $(1 \leq i \leq n)$, such that, for all i, j:*

1. (The Lie algebra of a torus is abelian) $[h_i, h_j] = 0$.
2. (h_i eigenvalues are given by i^{th} row of GCM) $[h_i, e_j] = a_{ij} e_j$ and $[h_i, f_j] = -a_{ij} f_j$.
3. (The triple e_i, h_i, f_i is an $\mathfrak{sl}_2$ triple) $[e_i, f_i] = h_i$.
4. (e_j is the lowest weight vector of an irrep of $\mathfrak{sl}_{2,i}$ for $i \neq j$) $[e_i, f_j] = 0$ if $i \neq j$.
5. (The *Serre relation*, i.e. the dimension of the above irrep is $1 - a_{ij}$) We have $\mathrm{ad}(e_j)^{1-a_{ij}}(e_i) = 0$ and $\mathrm{ad}(f_j)^{1-a_{ij}}(f_i) = 0$ for $i \neq j$.

Remark 2.18. We can re-interpret the Serre relation above in Theorem 2.17, via identifying $\mathrm{ad}(e_j)e_i = e_j e_i - e_i e_j$ in the universal enveloping algebra, as the relation:

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \binom{1-a_{ij}}{s} e_j^s e_i e_j^{a_{ij}+1-s} = 0.$$

and the obvious analogoue for the f_i and f_j.

2.3 Finite (Generalized) Cartan Matrices

We have seen that any semisimple Lie algebra $\mathfrak{g}$ (with choices of $\mathfrak{h} \subseteq \mathfrak{b}$) can be recovered from its generalized Cartan matrix. We now seek to answer the question: which generalized Cartan matrices can appear?

Given a generalized Cartan matrix, we have seen that we can recover its universal enveloping algebra above in Theorem 2.17. One can informally summarize this procedure as follows. Specifically, starting with a simple root e_i, we may repeatedly apply the various operators $[e_j, -]$ to obtain all of the eigenvalues of the adjoint representation. Now, we can ask: Why does this process ever stop? Note that the Serre relation in Theorem 2.17 yields that repeatedly applying the operator $[e_j, -]$ for the *same* e_j must eventually terminate, but, as we will see below in Section 3.1, nothing in Theorem 2.17 states that, for example, $(\mathrm{ad}(e_i)\mathrm{ad}(e_j))^n$ vanishes for any n.

The answer is given in the fact that semisimple Lie algebras have a canonical positive definite bilinear form, known as the *Killing form*. This means that we have an intuitive notion of distance on $\mathfrak{h}$. We won't emphasize the Killing form too much in this course (see [Gan20]), other than discussing how one can get the Killing form from the generalized Cartan matrix.

Definition 2.19. A GCM A is *symmetrizable* if it can be written as a product $A = DS$ for some diagonal matrix D and symmetric matrix S.

Example 2.20. It turns out the finite Cartan matrices are all symmetrizable. For example, all Cartan matrices of type A_n are themselves symmetric, and for type B_2, we have:

$$\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

Since symmetric matrices over $\mathbb{Z} \subseteq k$ have real eigenvalues, we can make the following distinction of generalized Cartan matrices based on how much access we have to a usual notion of 'distance':

Definition 2.21. We say that a symmetrizable GCM A is *finite type* if the associated symmetric matrix has all positive eigenvalues.

Then it turns out that semisimple Lie algebras are in bijective correspondence with positive definite, symmetrizable GCM! We will say this in a way closer to the classification of simple Lie algebras via connected Dynkin diagrams:

Indecomposable GCM

Assume you are given this generalized Cartan matrix and know it comes from a semisimple Lie algebra $\mathfrak{g}$:

$$A := \begin{pmatrix} 2 & 0 & -2 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

Automatically, we see that this associated semisimple Lie algebra cannot be irreducible. The reason is is because this the second simple root is perpendicular to the others. For example, we see that $[h_1, e_2] = a_{12}e_2 = 0$, $[e_1, e_2] = 0$ (by the Serre relation), and $[e_1, f_2] = 0$ and so in the universal enveloping algebra of $\mathfrak{g}$ we have that the $\mathfrak{sl}_2$ triple indexed by the first root commutes with the $\mathfrak{sl}_2$ triple indexed by the second root, so the representation theory reduces to the representation theory of the product. (In fact, using definitions below, one can show this is a *realization* of the Lie algebra $\mathfrak{sp}_4 \times \mathfrak{sl}_2$). This motivates the following definition:

Definition 2.22. An *decomposable GCM* is an $n \times n$ generalized Cartan matrix for which there exists two nonempty, disjoint subsets $I, J \subseteq \{1, ..., n\}$ such that $a_{ij} = 0$ if $i \in I$, $j \in J$. We say a GCM is *indecomposable* if it is not decomposable.

Compare this to Example 2.10, which is decomposable.

Classification of Simple Lie Algebras

With that being said, we can now state one of the fundamental theorems of representation theory in terms of generalized Cartan matrices!

Theorem 2.23. There is a canonical bijection

$$\{\text{Simple Lie Algebras}\} \xrightarrow{\sim} \{\text{Indecomposible, finite type, symmetrizable GCM}\}.$$

Tomorrow, we'll ask the question: What happens if you apply the same procedure of Theorem 2.17 to *other* GCM?

3 Affine Kac-Moody Algebras as Generalizations of the BGG Category $\mathcal{O}$

Yesterday, we saw that simple Lie algebras were given by indecomposible, finite type, symmetrizable GCM. Furthermore, given such a GCM, we can go backwards via an explicit generators and relations procedure. However, we may apply the universal enveloping algebra procedure to *any* GCM, and obtain the notion of a Kac-Moody algebra (and, whatever an affine Kac-Moody algebra is, it's a particular kind of Kac-Moody algebra)!

3.1 Kac-Moody Algebras

The idea of a Kac-Moody algebra is simple, given the above information. We take the relations given to us by Theorem 2.17, except that we don't require our generalized Cartan matrix actually be a Cartan matrix, i.e. actually come from a semisimple Lie algebra. Then, for a generalized $n \times n$ Cartan matrix $A = \big(a_{ij}\big)$, the *commutator of the Kac-Moody algebra* $\mathfrak{g}'(A)$ is the free Lie algebra generated by symbols h_i, e_i, f_i for $i \in \{1, ..., n\}$ such that:

1. (The Lie algebra of a torus is abelian) $[h_i, h_j] = 0$.
2. (h_i eigenvalues are given by i^{th} row of GCM) For all $i, j \in \mathfrak{h}$, $[h_i, e_j] = a_{ij}e_j$ and $[h_i, f_j] = -a_{ij}f_j$.
3. (The triple e_i, h_i, f_i is an $\mathfrak{sl}_2$ triple) $[e_i, f_i] = h_i$.
4. (e_j is the lowest weight vector of an irrep of $\mathfrak{sl}_{2,i}$ for $i \neq j$) $[e_i, f_j] = 0$ if $i \neq j$.
5. (The *Serre relation*, i.e. the dimension of the above irrep is $1 - a_{ij}$) We have $\text{ad}(e_j)^{1-a_{ij}}(e_i) = 0$ and $\text{ad}(f_j)^{1-a_{ij}}(f_i) = 0$ for $i \neq j$.

If A is symmetrizable, then we say our Kac-Moody algebra is *symmetrizable*.

Remark 3.1. One can generalize this definition slightly further and make this into the actual definition of a Kac-Moody algebra used by Kac in his original book [Kac90]. Specifically, given a generalized GCM of rank r, we may want to specify by hand a choice of roots $\{\alpha_j\}$ in a chosen vector space $\mathfrak{h}^*$ of dimension[7] $2n - r$ and a choice of coroots $\{\alpha_i^\vee\}$ in the dual space $\mathfrak{h}$. (For a semisimple Lie algebra, we can find the coroots via the Killing form. However, for a general Kac-Moody algebra we will also have to specify them by hand. However, this choice is unique up to isomorphism [Kac90, Proposition 1.1]). The full definition of a Kac-Moody algebra can be found, for example, on the Wikipedia page. In the finite dimensional case, the h_i span $\mathfrak{h}$. In the general case, we must spell this condition for all $\mathfrak{h}$ explicitly, since there is no condition that the roots must span $\mathfrak{h}^*$.

From here on out, we will focus on those Kac-Moody algebras whose associated generalized Cartan matrix is indecomposable.

Finite, Affine, and Indefinite Kac-Moody Algebras

In general, Kac-Moody algebras are more difficult to answer questions about (for example, any non-finite Kac-Moody algebra turns out to be necessarily infinite dimensional). However, there is one class of examples of Kac-Moody algebras, known as *affine Kac-Moody algebras*, for which more is understood. One way to motivate this is as follows–if $A = DS$ is a symmetrizable generalized Cartan matrix, we got a lot of mileage if we knew that S was positive definite. We weaken this condition slightly.

Definition 3.2. Assume $\mathfrak{g}$ is a symmetrizable Kac-Moody algebra such that its generalized Cartan matrix $A = DS$ is indecomposable.

- If S is positive definite, we say that $\mathfrak{g}$ is *finite*.

[7]Other than so that this agrees with the semisimple Lie algebra case (i.e. $2n - r = 2n - n = n = \dim(\mathfrak{h})$ for a Cartan matrix of a semisimple Lie algebra), I'm not sure why this dimension condition is needed or what it buys us.

- If S is not positive definite but is positive semidefinite, we say that our irreducible[8] $\mathfrak{g}$ is an affine Kac-Moody algebra.
- If S is not positive semidefinite, we say the GCM is *indefinite*.

Remark 3.3. It turns out that S cannot be negative definite or negative semidefinite.

We can now rephrase yesterday's result that we have already seen regarding classification of semisimple Lie algebras in terms of Kac-Moody algebras, see Theorem 2.23:

Theorem 3.4. The finite Kac-Moody algebras are precisely those Lie algebras given by semisimple Lie algebras.

Example: 'Affine A_1' $= \tilde{A}_1$

It is high time for an example. Consider the generalized Cartan matrix of *affine type* A_1, denoted $\tilde{A}_1$, given by the matrix

$$A := \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

We will explore the commutator of the Kac-Moody algebra associated to A, given by generators and relations above. Let κ denote the element $h_1 + h_2$. Here is an exercise that is a good exercise which is more or less a check to make sure you are following the definitions:

Exercise 3.5. The element κ is *central*, i.e. $[\kappa, X] = 0$ for all X in our commutator of the Kac-Moody algebra.

This fact is important enough that we will keep track of it in our notation and denote our commutator of the Kac-Moody algebra $\hat{\mathfrak{sl}}_{2,\kappa}^{\text{poly}}$. It's a bit strange that this commutator of the Kac-Moody algebra has a central element–note that the Lie algebra $\mathfrak{sl}_2$ has no central elements because it is a semisimple Lie algebra.[9] Since κ is central in $\hat{\mathfrak{sl}}_{2,\kappa}^{\text{poly}}$, it determines a Lie subalgebra, and the Lie algebra quotient $\hat{\mathfrak{sl}}_{2,\kappa}^{\text{poly}}/k\kappa$ is defined. We will first classify this Lie algebra:

Proposition 3.6. There is a canonical isomorphism of Lie algebras $\hat{\mathfrak{sl}}_{2,\kappa}^{\text{poly}}/\mathbb{C}\kappa \cong \mathfrak{sl}_2[t^{\pm 1}]$, where $\mathfrak{sl}_2[t^{\pm 1}]$ denotes the *loop algebra of* $\mathfrak{sl}_2$ $\mathfrak{sl}_2[t^{\pm 1}] := \mathfrak{sl}_2 \otimes k[t^{\pm 1}]$ with Lie bracket given by the rule $[X \otimes p(t), Y \otimes q(t)] = [X, Y] \otimes p(t)q(t)$.

We will sketch the proof of this proposition here, but this is seriously worth doing to get a sense of what's going on! First, we'll construct the map, which is one of the most crucial ideas. The map will be given by sending $e_1 \mapsto e \otimes 1$, $h_1 = -h_2 \mapsto h \otimes 1$, $f_1 \mapsto f \otimes 1$, $e_2 \mapsto f \otimes t$, and the following exercise:

Exercise 3.7. Where must f_2 in the above isomorphism map to to be a Lie algebra map? (Hint: Any element of the loop algebra can be written as a finite sum of terms of the form $X_d \otimes t^d$ where $d \in \mathbb{Z}$).

We now compute some images of the span of the positive roots of $\hat{\mathfrak{sl}}_{2,\kappa}^{\text{poly}}$, at least heuristically verifying surjectivity. This paragraph will be largely informal. What elements in the loop algebra can we get just with the (images) of e_1 and e_2? We know $[e_i, e_i] = 0$ for both i, so our first move must be $[e_1, e_2] \mapsto [e \otimes 1, f \otimes t] = h \otimes t$. Then we can either apply $[e_1, -]$ or $[e_2, -]$. We have, for example,

$$[e_1, [e_1, e_2]] \mapsto [e \otimes 1, h \otimes t] = -2h \otimes t$$
$$[e_2, [e_1, e_2]] \mapsto [f \otimes t, h \otimes t] = 2f \otimes t^2$$

So we have heuristically justified that *the positive (Lie algebra) span of the two positive simple roots of* $\hat{\mathfrak{sl}}_{2,\kappa}^{poly}$ *is the Lie algebra of objects of the form* $e \otimes 1 + \sum_d X_d \otimes t^d$, for $X_d \in$ a finite collection of $d \in \mathbb{N}^{>0}$.

[8]In general, given the usual data, if the generalized Cartan matrix has corank 1, we also say it is an affine Kac-Moody algebra, which is clearly consistent with our definition for indecomposable GCM.

[9]Don't confuse this with the *Casimir element*, an element which is central in the *universal enveloping algebra* $U(\mathfrak{sl}_2)$.

Exercise 3.8. Justify this statement until you are satisfied. (Hint: You can enumerate all possible sequences of e_1, e_2 which don't vanish. For example, as above, the first two elements which must be paired by the Lie bracket yields $[e_1, e_2]$. Then, we have a choice. If we choose e_1, we end up with $[e_1, [e_1, e_2]]$, but $[e_1, [e_1, [e_1, e_2]]] = 0$–why? This pattern will allow you to make a tree of possible sequences.

3.2 Motivation: Why Kac-Moody algebras?

The computation of Section 3.1 might seem to suggest that affine Kac-Moody algebras may be difficult to get a grip on. This turns out to be not quite true, in part due to an alternate description of affine Kac-Moody algebras as an extension of the loop algebra, see Section 3.3. However, they are certainly not as easy as finite Kac-Moody algebras, and are in particular all non-finite Kac-Moody algebras are infinite dimensional Lie algebras. Therefore, we take time to discuss some motivation of (affine) Kac-Moody algebras. This subsection is a bit dense, and nothing will be used from these sections in what follows!

Motivation from String Theory

One of the original reasons people were interested in this subject is its connections to topics in physics, such as *conformal field theory*. There is a certain two dimensional conformal field theory known as the Wess-Zumino-Witten or WZW model whose associated symmetry algebra is an affine Lie algebra. This theory 'lives at the boundary of Chern-Simons theory.' Unfortunately, the connection to physics is outside the scope of this course.

Motivation from (Quantum) Geometric Langlands

Another reason that one might care about Kac-Moody algebras is their manifestation in Geometric Langlands. For example, one of the first steps in the Geometric Langlands program is known as the *geometric Satake* theorem, which says that the abelian category of representations of a reductive algebraic group G can be realized as a certain object known as *perverse sheaves* on an object known as the the affine Grassmannian for $\check{G}$, the *Langlands dual group* for G. For example, if $G = SO_{2n+1}$, which is type B, the Langlands dual group is Sp_{2n}, which is type C. This is an amazing theorem, and one of the key ideas understanding the Grassmannian in the original full-generality proof of Mirkoviĉ Vilonen [MV07] uses the representation theory of affine Kac-Moody algebras in their proof.

Furthermore, the parameter κ plays a role in one form of the *local geometric Langlands program.* The local geometric Langlands program (in broad strokes!) conjectures an equivalence of $(\infty, 2)$-categories

$$\mathcal{D}_\kappa(\mathcal{L}(G))\text{-Mod}(\text{Cat}) \simeq \mathcal{D}_{\frac{1}{\kappa}}(\mathcal{L}(\check{G}))\text{-Mod}(\text{Cat})$$

of (DG-)categories with an action of $\mathcal{D}_\kappa(G)$ and (DG-)categories with an action of $\mathcal{D}_{\frac{1}{\kappa}}(\mathcal{L}(\check{G}))$. We won't discuss this any further, other than to discuss a Whittaker-Satake version of this theorem, which says that certain representations of Kac-Moody algebras can also be realized as twisted sheaves on the affine Grassmannian.

Theorem 3.9. ([CDR] Non-factorizable, Parabolic Fundamental Local Equivalence) There is an equivalence of derived categories

$$\text{Whit}_\kappa(Gr_G) \simeq (\hat{\check{\mathfrak{g}}}_{\check{\kappa}}\text{-Mod})^{G(\mathcal{O})}$$

which induces an equivalence of underlying abelian categories.

Motivation from Representation Theory in Positive Characteristic

It is known that a certain category of representations of the Kac-Moody algebra (specifically, the category of $G(\mathcal{O})$-integrable representations at nontrivial levels which are p-torsion) are 'analogous' to characteristic p representation theory. For example, *Lusztig's conjecture*, a conjecture for character formulas of simple representations, was first proven for Kac-Moody algebras.

Furthermore, characteristic p representation theory also motivates the general abstract notion of generalized Cartan matrices. While we won't go into them here, generalized Cartan matrices have different *realizations*. We won't explicitly define them here, see 27.2 of [EMTW20], however, see:

Example 3.10. Consider the (generalized) Cartan matrix for $\mathfrak{sl}_3$. The associated Weyl group is S_3. It acts both on the three dimensional $\mathbb{Z}$-module(!) $\mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3$ (the GL_3-*realizaiton*) and, as we have seen before, its two dimensional quotient space $\mathfrak{t}_{\mathbb{Z}}$ associated to $\mathfrak{sl}_3$, i.e. $\mathfrak{t}_{\mathbb{Z}} := \{aL_1 + bL_2 + cL_3 : a+b+c = 0\}$.

The advantage of working over $\mathbb{Z}$ is that we can base-change to fields which are not necessarily of different fields of characteristic zero! For example, it is currently believed that the category of representations $\mathrm{Rep}_L(G)$ of a (split) reductive algebraic group G over a field L of characteristic $p > 0$ is controlled by the *affine* Weyl group, i.e. the Weyl group associated to the associated affine GCM; see 27.5 of [EMTW20] where this is spelled out more explicitly in terms of the *diagrammatic Hecke category*, and see [RWa] for some recent progress on this in terms of tilting characters.

3.3 Affine Kac-Moody Algebras as An Extension of the Loop Algebra

Polynomial Algebras vs. Taylor Series

We first take a bit of motivation from algebraic geometry. In algebraic geometry, when we are given a ring like $k[t]$ with choice of point $(t=0)$ (or $\mathbb{Z}$ with a choice of point (p) for a prime integer p), we can consider what's happening at the *formal neighborhood* of the point $k[[t]] := \{\sum_{i=0}^{\infty} a_i t^i : a_i \in k\}$ (respectively, $\mathbb{Z}_p := \{\sum_{i=0}^{\infty} a_i t^i : a_i \in \{0, 1, ..., p-1\}\}$) or the *formal punctured neighborhood* $k[[t]][t^{-1}]$ (respectively, $\mathbb{Q}_p := \mathbb{Z}_p[\frac{1}{p}]$). One can heuristically justify this as follows. If we are working at a line whose functions are $k[t]$, then if we zoom in very very close to the point $(t=0)$, the polynomial $1-t$ should be invertible. Since $\frac{1}{1-t} = 1 + t + t^2 + ...$, we have already admitted some power series into the discussion, and it becomes technically convenient sometimes to allow *all* power series into the discussion.

One might worry that technical complications arise when transitioning from something like the localized algebra $k[t]_{(t)} := \{\frac{p(t)}{q(t)} : q(0) \neq 0\}$ to the Taylor series. This is solved by putting a *topology* on our ring. Specifically, we will put the *inverse limit topology* on our ring, defining open sets $U \subseteq k[[t]]$ to be those U for which $p \in U$ implies $p + t^n k[[t]] \subseteq U$ for some n. While this topology is not easy to visualize (for example, $k[[t]]$ admits a norm which satisfies the inequality $|f(t), g(t)| \leq \max(|f(t)|, |g(t)|)$, a much stronger version of the triangle inequality that's not satisfied by the usual norm on vector spaces), it's the correct topology for a lot of algebraic purposes.

Affine Lie Algebras

Given the discussion of Section 3.3, we would like to replace the loop algebra we saw in Section 3.1 with the algebra of *Laurent series* $K := k[[t]][t^{-1}]$. Therefore, we wish to work with a notion of affine Kac-Moody algebras which quotient to a *loop algebra* $\mathfrak{g}((t)) := \mathfrak{g} \otimes K$. However, it is also useful (for example, in the geometric Langlands program), to keep track of our central element κ that we deserved. Therefore, for our simple Lie algebra $\mathfrak{g}$, we want a Lie algebra $\hat{\mathfrak{g}}_\kappa$ which has a one dimensional central Lie subalgebra, and quotients to the loop algebra. In fancier language, we would like $\hat{\mathfrak{g}}_\kappa$ to fit into a short exact sequence of Lie algebras:

$$0 \to k\kappa \to \hat{\mathfrak{g}}_\kappa \to \mathfrak{g}((t)) \to 0$$

and, in turn, these turn out to be classified by their *second Lie algebra cohomology group* $H^2_{\mathrm{Lie}}(\mathfrak{g}((t)), k)$, see, for example, 7.6 of [Wei94]. Since we don't want to delve too much into homological algebra, we will simply state the result:

Proposition 3.11. If $\mathfrak{g}$ is a simple Lie algebra, then $H^2_{\mathrm{Lie}}(\mathfrak{g}((t)), k)$ is a one dimensional vector space with a canonical basis given by the Killing form.

In particular, there is a one parameter family of central extensions of the loop algebra $\mathfrak{g}((t))$, and, after choosing one, we label this choice by κ and let $\hat{\mathfrak{g}}_\kappa$ denote the *affine Lie algebra.*

Remark 3.12. Note affine Lie algebras are technically not affine Kac-Moody algebras per our conventions, but closer to their 'completion'.

One can explicitly choose some scalar multiple of the Killing form which we denote $\tilde{\kappa}$ and see that the Lie bracket on $\hat{\mathfrak{g}}_\kappa$ is given by the Lie bracket on elements in the loop algebra (since κ is central):

$$[X \otimes p(t), Y \otimes q(t)] = [X, Y] \otimes p(t)q(t) - \tilde{\kappa}(X, Y)\mathrm{Res}_{t=0}(fdg)\kappa \tag{1}$$

Because of this clash of notation, some authors prefer to denote κ by c or $\mathbf{1}$. The second term is related to the *gauge transformations* of local systems on the punctured disk, see [Ras].

4 Quantum Analogues

Today, we'll discuss the *category of representations of the quantum group.* To avoid technical complications, today, we will stick to *simply-laced Lie groups/algebras*, which means that all roots of the associated Lie algebra are of the same length.This means that our semisimple Lie algebra (which is a product of simple Lie algebras by a definition or theorem, depending on how you view the theory) only has simple factors of type A (the type of $\mathfrak{sl}_n$), D (the type of SO_{2n}), or E_6, E_7, or E_8. Everything here is defined for all semisimple Lie algebras and the associated theorems are true.

We will also let $k = \mathbb{C}$, since the relationship between representations of quantum groups and representations of affine Lie algebras is given by an exponential map, it turns out.

The motivation for quantum groups is as follows. One idea that you might have in studying representations of a group is to *deform* a group. The idea is that we might view a group G as a *family* G_t (where t is either a complex number or some formal parameter). Then we might gain information about the group itself by studying the G_t where $t \neq 0$. Unfortunately, *no such deformations of a semisimple group exist*[10].

This means that we will have to think differently if we want to use ideas in deformation theory to solve representation theoretic problems. In particular, quantum group is not defined as a group! In fact, the only thing that we will define in this course is *representations* of a quantum group. Although non-commutative geometry can be used to define the notion of a quantum group (see, eg, [hjf]), we will stick to defining the category of representations of a quantum group. This is analogous to studying categories of *modules* over the universal enveloping algebra instead of the universal enveloping algebra in and of itself.

To achieve this deformation, we must actually define a *quantum analogue* of the nonzero integers. We work in the field $\mathbb{Q}(v)$, where v is a formal parameter.

4.1 Quantum Analogues of Integers

Definition 4.1. If $n \in \mathbb{Z}$, the 'graded quantum analogue of n' is $[n] := \frac{v^n - v^{-n}}{v - v^{-1}}$.

Remark 4.2. This term is nonstandard–we will denote any nonstandard term below in quotes and italicize standard terms.

Remark 4.3. Set $q = v^2$. Note that we have an equality:

$$[n] = \frac{v^n}{v}\frac{1 - v^{2n}}{1 - v^2} = v^{n-1}\frac{1 - q^n}{1 - q} =: v^{n-1}[n]_q$$

where $[n]_q$ is the *q-analogue* of n. Note that $[n]_q = 1 + q + ... + q^{n-1}$. In particular, $\lim_{q \to 1}[n]_q = n$, which jusifies the term for the q-analogue of n. The v^{n-1} term specifies the grading, as we can informally identify v as in cohomological degree 1. Similarly, we can informally identify our q as a cohomological parameter in degree 2. In fact, we can identify this element with an object called the *Bott parameter*, see 3.1 in [Sul] for more information on q analogues and [RWb], 16.2 for how this identification is used in representation theory and, in particular, why one may regard q as in cohomological degree 2, justifying the term 'grading' in Definition 4.1.

[10] Unfortunately, I don't know of a reference for this fact and haven't actually seen an explicit statement written down anywhere. If you know of one, let me know because I would love to include it here!

Definition 4.4. Define the *quantum factorial* of $n \in \mathbb{N}^{>0}$ as $[n]! := [n][n-1]...[2][1]$ and $[0]! := 1$. We also set[11] the 'quantum binomial coefficient' as:

$$\begin{bmatrix} n \\ r \end{bmatrix} := \frac{[n]!}{[r]![n-r]!} = \frac{[n][n-1]...[n-r+1]}{[r]!}.$$

4.2 Quantum Universal Enveloping Algebra

We will now define the *quantum universal enveloping algebra* of a simply-laced $\mathfrak{g}$. The reader may wish to stick to the case of $\mathfrak{g} = \mathfrak{sl}_2$.

Definition 4.5. Let $\mathfrak{g}$ be a simply laced Lie algebra. The *quantum universal enveloping algebra* is the free $\mathbb{Q}(v)$-algebra generated by symbols $K_i^{\pm 1}, e_i$, and f_i for each simple root i, subject to the following relations:

1. $K_i^{\pm 1} K_i^{\mp 1} = 1$ and all K's commute.
2. (h_i eigenvalues are given by graded quantum analogue of i^{th} row of GCM) $K_i e_j K_i^{-1} = v^{a_{ij}} e_j$ and the 'obvious analogue' for the negative part $K_i f_j K_i^{-1} = v^{-a_{ij}} f_j$
3. (Analogue[12] of $[e, f] = h$, i.e. (e_i, h_i, f_i) form a quantum $\mathfrak{sl}_2$ triple) $[e_i, f_i] = \frac{K_i - K_i^{-1}}{v - v^{-1}}$
4. (Analogue of Remark 2.4) For distinct i, j we have $[e_i, f_j] = 0$.
5. (The positive *quantum Serre relation*, see Remark 2.18) If $i \neq j$,
$$\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1 - a_{ij} \\ s \end{bmatrix} e_i^{1-s-a_{ij}} e_j e_i^s = 0.$$
6. (The negative quantum Serre relation)
$$\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1 - a_{ij} \\ s \end{bmatrix} f_i^{1-s-a_{ij}} f_j f_i^s = 0.$$

Exercise 4.6. Check that in the case of $a_{ij} = -1$ (which is always the case for type A if $i \neq j$), the positive quantum Serre relation states
$$e_i^2 e_j + e_j e_i^2 = (v + v^{-1}) e_i e_j e_i.$$

Note that quantum groups are in fact not actually groups at all! One reason that this term has been adopted is that the algebra $U_q(\mathfrak{g})$ shares many properties that a ring A has if $G = \mathrm{Spec}(A)$ is an *algebraic group*, i.e. a group object of varieties. For example, $G = \mathrm{SL}_2$ can be viewed as an algebraic group via $\mathrm{SL}_2 := \mathrm{Spec}(\mathbb{C}[a, b, c, d]/(ad - bc))$. Algebraic geometers will note that the $\mathbb{C}$ points of the variety SL_2 (in the sense of maps from $\mathrm{Spec}(\mathbb{C})$) are the same thing as points of $\mathrm{SL}_2(\mathbb{C})$. This algebraic structure is called a *Hopf algebra*. We won't go into the structure of a Hopf algebra today, except to remark that:

Proposition 4.7. If a ring A is a Hopf algebra, the representations of A (as a Hopf algebra) acquire a canonical braided monoidal structure.

[11] as far as I know, this is a nonstandard term, but the notation is standard

[12] Unfortunately, this notion doesn't seem to be as cleanly mapped to the usual universal enveloping algebra, partly due to the exponentiated torus and partly due to the fact that these ideas first arose from solving the *Yang-Baxeter equations*. The way you can get the usual relation $[e, f] = h$ is to view $K := q^h$ and view $q := e^{\hbar/2}$. Then if you take the power series of $\frac{K - K^{-1}}{v - v^{-1}}$, you will get the first term h, i.e. if you send $\hbar \to 0$ you will get h. I learned this from [Pan], which has extended $\mathfrak{sl}_2$ examples.

4.3 $\mathbf{Rep}_q(\mathrm{SL}_2)$ for Generic q

Today, we will work with the simplest non-abelian case. We'll make the following definition:

Definition 4.8. A *Kac-De Concini representation of quantum* SL_2 *at a parameter* $q_0 \in \mathbb{C} \setminus \{0, \pm 1\}$ is a module over the ring $\mathbb{C} \otimes_{\mathbb{C}[v^{\pm 1}]} \mathbb{C}[v^{\pm 1}]\langle e, f, K^{\pm 1}/(KeK^{-1} = v^2 e, KfK^{-1} = v^{-1}f, ef - fe = \frac{K-K^{-1}}{v-v^{-1}})$ where the left hand module structure is given by $\mathbb{C}[v^{\pm 1}] \to \mathbb{C}$ is given by setting $v = q_0$.

Remark 4.9. Equivalently, this representation can be realized as a module over $\mathbb{C}[v^{\pm 1}]\langle e, f, K^{\pm 1}/(KeK^{-1} = v^2 e, KfK^{-1} = v^{-1}f, ef - fe = \frac{K-K^{-1}}{v-v^{-1}})$ where the parameter v acts by scaling by q_0. The condition that $q_0 \neq \pm 1$ is only included for simplicity, and one can define this in a more uniform way by incorporating the symbol $\begin{bmatrix} K;0 \\ 1 \end{bmatrix}$, which we think of as formally adding in $\frac{K-K^{-1}}{v-v^{-1}}$, i.e. adding an extra generator $\begin{bmatrix} K;0 \\ 1 \end{bmatrix}$ and modding out by the relations $ef - fe = \begin{bmatrix} K;0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} K;0 \\ 1 \end{bmatrix}(v - v^{-1}) = K - K^{-1}$.

Remark 4.10. Similarly, working over $\mathbb{C}$ is not needed to make this definition. However, we will see below that the theory is very, very different if we allow v to act by zero, and so this is more essential.

Today, we will see which Kac-De Concini representations of the quantum group we can obtain by mimicking the usual proof of the usual classification of irreducible $\mathfrak{sl}_2$ representations. We will do this by mimicking the usual proof of the classifications of irreps of $\mathfrak{sl}_2(\mathbb{C})$ and throw out any q_0 where problems arise. (We will see that drastically different behavior happens when q_0 is a root of unity, and these will be the q_0 we throw out and discuss tomorrow).

Finding a Highest Weight Vector Classically

Recall in the classical story that a *highest weight vector* is a vector killed by e and scaled by h. Let us fix a finite dimensional $\mathfrak{sl}_2$ representation V_{cl}. How do we know it is diagonalizable at all? One way is to note that an $\mathfrak{sl}_2$ module is in particular a $\mathbb{C}[h] \hookrightarrow U(\mathfrak{sl}_2)$ module. Since our module is finite dimensional, by the Jordan normal form of a matrix, there must exist some vector v_{cl} with some complex h-eigenvalue λ. One can check that the operator e sends object in the h-eigenspace λ to an object in the h-eigenspace $\lambda + 2$. Therefore, we can keep applying e to v_{cl} until we can't anymore, and we obtain a highest weight vector.

Finding a Highest Weight Vector for Quantum Groups

We can proceed similarly for modules over U_q, because any U_q-module is in particular and $\mathbb{C}[K^{\pm 1}]$-algebra, and the same analysis above shows that there is *some* (nonzero!) K-eigenvalue λ (with nonzero eigenvalue).

Exercise 4.11. Assume we are given a finite dimensional U_q-module V, and $v_\lambda \in V$ has K-eigenvalue $\lambda \in \mathbb{C} \setminus 0$. Compute the K-eigenvalue of ev. (Do not look down if you do not want the answer.)

We can similarly repeatedly apply e to our nonzero eigenvector u until $e^{m+1}u = 0$ for some (minimal) m. Then we obtain that the vector $e^m u$ is nonzero and has K-eigenvalue $q_0^{2m}\lambda$.

Computing the Full Representation Given a Highest Weight Vector Classically

Now assume we are given a highest weight vector u of highest (real part, a priori) weight λ in a finite dimensional irreducible $\mathfrak{sl}_2$-representation. We can similarly compute that $f^n u$ is in the $\lambda - 2n$ eigenspace. One can also repeatedly show the following proposition by induction:

Proposition 4.12. If u is a highest h-weight λ vector in an $\mathfrak{sl}_2$ representation, then $efu = \lambda u$ and more generally

$$ef^n u = (\lambda + (\lambda - 2) + ... + (\lambda - 2n + 2))f^{n-1}u$$

This in particular shows that e fixes the vector space spanned by the various $f^n u$. Furthermore, by finite dimensionality, we may choose some n mimimal such that $f^n u = 0$. Therefore we in particular see that

$$(n\lambda - n(n-1))f^{n-1}u = (\lambda + (\lambda - 2) + ... + (\lambda - 2n + 2))f^{n-1}u = ef^n u = 0$$

so therefore by assumption on n we have that $n\lambda - n(n-1) = 0$ and so $\lambda = n - 1$. Therefore our highest weight must be an integer!

Computing the Full Representation Given a Highest Weight Vector for Kac-De Concini Quantum Group Representations

Similarly to Exercise 4.11, we can compute that if u is some vector for a naive representation of a quantum group with K-eigenvalue λ, then fu has K-eigenvalue $q_0^{-2}\lambda$. However, note that if q_0 is a root of unity, we may apply f many times and get back to the eigenspace we started! Therefore, from here on out, we make our only assumption: *We assume q_0 is not a root of unity.* Therefore, we will now call these the *representations of the quantum group of* SL_2, when q_0 is not a root of unity.

Here on after, the proof is nearly identical to the $\mathfrak{sl}_2$ case. For example, the analogue of Proposition 4.12 is straightforward but tedious induction argument:

Proposition 4.13. In U_v, if $m \in \mathbb{N}^{\geq 0}$, we have

$$[e, f^m] = [m]f^{m-1}\frac{v^{1-m}K - v^{m-1}K^{-1}}{v - v^{-1}} = [m]\frac{v^{m-1}K - v^{1-m}K^{-1}}{v - v^{-1}}f^{m-1}$$

We can use this proposition to apply the same trick we applied in the classical case. Namely, if n is the minimal nonnegative integer such that $f^n u$ is zero, then we can use this expression to show that $[m]\frac{q^{n-1}K - q^{1-n}K^{-1}}{q-q^{-1}} = 0$. Now (again, critically!) the quantum integer $[m]$ is nonzero because q_0 is not a root of unity. Therefore, we see $(q^{1-n}\lambda)^2 = 1$ and so we in particular obtain that $\lambda^2 = q^{2n-2}$, so in particular, $\lambda = \pm q^{n-1}$.

We have therefore shown uniqueness of the following theorem:

Theorem 4.14. Fix an irreducible representation of the quantum group of unity and some $q_0 \in \mathbb{C} \setminus 0$ which is not a root of unity. Then for each integer n, there are precisely 2 representations of dimension $n+1$–specifically, one with highest weight which has K-eigenvalue q_0^n, and one with highest weight with K-eigenvalue $-q^n$.

As in the classical case, the existence can be shown by hand for generic q_0 using these generators and relations. Unfortunately, we obtain drastically different behavior when q_0 is a root of unity.

5 Representations of $\hat{\mathfrak{g}}_\kappa$ and $\mathbf{Rep}_q(G)$

Today, we'll finally discuss the representation theory of quantum groups and affine Lie algebras. We'll first pick up where we left off yesterday.

5.1 Introduction to Representation Theory of Quantum Groups

Problem: Characteristic p Lie Algebras Don't See the Whole Group

In representation theory over $\mathbb{C}$, one can use the exponential map to show that representations of a simply connected Lie group are equivalently given by representations of the associated Lie algebra. The exponential map for a given tangent vector $X \in \mathfrak{g}$ can be informally described as $X \mapsto \sum_{m=0}^{\infty} \frac{X^m}{m!}$. Since we need multiplicative inverses to all integers, this suggests that such a simple trick will not work in characteristic p representation theory.

Example 5.1. Consider the representation of SL_2 over an algebraically closed (for safety) field of k of positive characteristic p, ∇_p, given by degree p homogeneous polynomials in two variables, say x, y. This has a simple subrepresentation given by the two dimensional vector space spanned by x^p and y^p, which we write as the subrepresentation L_p. This representation has associated map to GL_2 given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^p & b^p \\ c^p & d^p \end{pmatrix}$$

and is equivalently the *first Frobenius twist* of the standard two dimensional SL_2 representation. One can check in a few ways that the restriction to the Lie algebra is trivial. Here is one: we identify the tangent space at the identity with the set $\{\begin{pmatrix} 1+a\epsilon & b\epsilon \\ c\epsilon & 1+d\epsilon \end{pmatrix} : a,b,c,d \in k\}$, i.e. the $\mathrm{Spec} k(\epsilon)/(\epsilon^2)$ points of SL_2 where the restriction at $\epsilon = 0$ (i.e. the pullback by the map $\mathrm{Spec}(k) \to \mathrm{Spec} k(\epsilon)/(\epsilon^2)$) is the identity point. We then compute:

$$\begin{pmatrix} 1+a\epsilon & b\epsilon \\ c\epsilon & 1+d\epsilon \end{pmatrix} \mapsto \begin{pmatrix} 1+pa\epsilon & 0 \\ 0 & 1+pd\epsilon \end{pmatrix}$$

(using the fact $\epsilon^2 = 0$) and therefore since $p = 0$ in k we see that this is also the trivial representation. The fix here is *divided powers*.

Solution: Divided Powers in Characteristic p Representation Theory

Here is one trick which is used in many places in number theory, algebraic geometry, and beyond. We can view any object in $\mathfrak{g}$ (over a field of any characteristic) as giving rise to a global vector field on G (since all vector fields on a Lie/algebraic group are trivializable, we can just 'G-around' the vector), we can view the tangent vectors as differential operators. However, in characteristic p, not all differential operators come from powers of degree 1 differential operators.

Example 5.2. Again let k be a field of characteristic p and consider the ring of polynomial differential operators on $\mathbb{A}^1$. (These are defined recursively–the functions are the degree 0 differential operators and then inductively any endomorphism D on functions for which $[D, -]$ maps the degree n differential operators into degree $n-1$ differential operators is a differential operator). Then one can check that the differential operator $(\frac{d}{dx})^{(p)} := \frac{(d/dx)^p}{p!}$ is defined since if we take p-many derivatives of any polynomial, each term will have a p we can inforamlly factor out (and $(p-1)!$ is invertible in k).

There is more to be said here (for example, search 'restricted Lie algebra') but for now we will take this as motivation.

Representations of Quantum Groups

We have a similar tradeoff to make in the theory of quantum groups[13] $Rep_q(G)$. Given a semisimple Lie type, we can take the associated quantized universal enveloping algebra as in Section 4.2. However, as we saw in the SL_2 case, it is inconvenient to expect ourselves to be able to work with the algebra $\mathbb{C}(v)$, this in effect prevents us from setting v to be any number! Therefore we work with the associated *integral forms* of the quantized universal enveloping algebra. Let $\mathcal{A} := \mathbb{Z}[v^{\pm 1}]$.

Definition 5.3. The *Kac-De Concini integral form* for the quantum group is the $\mathcal{A}$-subalgebra of the quantized universal enveloping algebra generated by the various e_α, f_α, and $K_\alpha^{\pm 1}$. The *Lusztig integral form* for the quantum group is the $\mathcal{A}$-subalgebra generated by the Kac-De Concini integral form and all *v-divided powers* $e_\alpha^{(n)} := \frac{e_\alpha^n}{[n]!}$ and $f_\alpha^{(n)} := \frac{f_\alpha^n}{[n]!}$, where again $[n]$ denotes the quantum factorial.

Definition 5.4. A *De Concini-Kac representation* (respectively a *Lusztig representation*) *of the quantum group* over $\mathbb{C}$ with chosen parameter $q \in \mathbb{C}$ is a module over the Kac-De Concini (respectively, Lusztig) integral form for the quantum group specialized so that q acts by c.

[13] So as to remain consistent with most of the literature, contrary to what we did in class I will denote the category of representations at a fixed parameter $q \in \mathbb{C} \setminus 0$.

Remark 5.5. The above definitions work over a ring C with choice of nonzero $q \in C$. However, it is often convenient to impose a finiteness condition in the Lusztig setting (for example, requiring that only finitely many divided powers act nontrivially).

We have thus obtained one piece of the Kazhdan-Lusztig equivalence–the category $\mathrm{Rep}_q(G)$ of finite dimensional representations of the (Lusztig form of the) quantum group!

5.2 Introduction to Representation Theory of Affine Lie Algebra

Recall that the affine Lie algebra for a fixed simple Lie algebra $\mathfrak{g}$ is a central extension of $\mathfrak{g}((t))$ by a central element $k\kappa$, with distinguished basis element κ. We impose the following 'local constancy' condition on our representation.

Definition 5.6. A module V over the affine Lie algebra $\hat{\mathfrak{g}}_\kappa$ is said to be *smooth* if for all $v \in V$, there exists some $n > 0$ for which $t^n\mathfrak{g}[[t]]$ acts by zero.

Remark 5.7. It turns out that the cocycle which determines the extension of an affine Lie algebra canonically splits on the Lie algebra $\mathfrak{g}[[t]]$, so in particular this condition is well defined (a priori, $t^n\mathfrak{g}[[t]]$ is only a quotient).

We can now define our category of representations $\hat{\mathfrak{g}}_\kappa$-Mod:

Definition 5.8. Let $\hat{\mathfrak{g}}_\kappa$-Mod denote the category of smooth $\hat{\mathfrak{g}}_\kappa$ representations for which the canonical basis element κ acts by the identity.

Remark 5.9. Authors define this category in two equivalent ways–one is to *always* work with the canonical central extension given by the Killing form and require that the canonical basis element κ act by some choice of scalar, and one is to allow the scalar itself to vary. These are equivalent.

Highest Weight Categories

Now, we'll discuss the affine Lie algebra analogue of the Kazhdan-Lusztig equivalence. This can seem a bit out of nowhere, so we'll start off with the analogue in the finite dimensional story.

Definition 5.10. The *BGG category* $\mathcal{O}$ for a given semisimple Lie algebra $\mathfrak{g}$ with choice of $\mathfrak{h} \subseteq \mathfrak{b}$ is the abelian category of finitely generated $\mathfrak{g}$-modules V for which each $n \in \mathfrak{n}$ acts *locally nilpotently*, that is, for each $v \in V$ the space $U(\mathfrak{n})v$ is finite dimensional and $\mathfrak{t}$ acts semisimply.

The BGG category is one of the first examples of a *highest weight category.* The idea is this: there are 'easy to define' objects, called the *standard objects*, which admit maps to other objects of interest. In this case, the easy to define objects will be the *Verma modules* for each $\lambda \in \mathfrak{h}^*$.

Definition 5.11. For each $\lambda \in \mathfrak{h}^*$, the *Verma module* is $\Delta_\lambda := U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{C}$, where the $U\mathfrak{b}$-module structure on $\mathbb{C}$ is given by the ring map $U(\mathfrak{b}) \to U(\mathfrak{b}/\mathfrak{n}) \simeq U(\mathfrak{h}) = \mathrm{Sym}(\mathfrak{h}) \to \mathbb{C}$.

It's a fact that every Verma module has a unique simple quotient. The interplay between these two objects was one of the original motivating studies of the BGG category $\mathcal{O}$.

Example 5.12. Fix some $\lambda \in \mathbb{C}$. The Verma module Δ_λ for $\mathfrak{sl}_2$ (where we regard λ in $\mathfrak{h}^*$ as a functional which sends $h \in \mathfrak{sl}_2$ to the number λ) has a highest weight vector v of highest weight λ, a one dimensional $\lambda - 2$ eigenspace (spanned by fv) and so on and so forth, to infinity!

In particular, if λ is a positive integer, Δ_λ is a nontrivial extension of the trivial representation and the representation $\Delta_{-\lambda-1}$. This suggests more complicated behavior for category $\mathcal{O}$ than the category of finite dimensional representations.

Example 5.13. It turns out that representations of a reductive algebraic group over a field of characteristic p also form a highest weight category, although it's easier to see the *dual* picture with the *costandard objects.* Specifically, for SL_2, the *co*standard objects ∇_m are labeled by the nonnegative integers m (identified as the dominant weights of SL_2) and have a unique simple *sub*object. (The standard objects here are called *Weyl modules* and are given in general by identifying Δ_m (and more generally Δ_λ for any dominant integral weight) with the top cohomology of a certain line bundle on the flag variety.

Category $\mathcal{O}_\kappa$

We now have some more motivation behind us, and can ask ourselves–what plays the role of the Borel here? The answer turns out to be the group of positive loops $G(\mathcal{O}) := G(\mathbb{C}[[t]])$! (This is because the group $G(\mathcal{O})$ is generated by the usual simple roots in a Kac-Moody algebra.)

Definition 5.14. We define the *affine category* $\mathcal{O}_\kappa$ for a given $\hat{\mathfrak{g}}_\kappa$ as those smooth $\hat{\mathfrak{g}}_\kappa$ representations V such that for each n, the space $V(n) := \{v \in V : t^n\mathfrak{g}[[t]]v = 0\}$ is finite (this is an analogue of finite generation).

The analogue of the Verma modules, the 'easy to define' objects of $\mathcal{O}_\kappa$, are called the *Weyl modules*. We will do the example of $\mathfrak{sl}_2$, but for a general semisimple Lie algebra, the same ideas apply, see section 8 of [KL91]. Let us fix a κ.

Proposition 5.15. Choose some finite dimensional irreducible representation V of SL_2, with highest weight vector v. Set Δ_κ^V, as a vector space, as the vector space $V \oplus t^{-1}V \oplus t^{-2}V \oplus \ldots$. There is a canonical affine $\mathfrak{sl}_2$ module structure on Δ_κ^V (so that, in particular, our central element scales by κ) such that v killed by the positive roots (i.e. $e \otimes 1 + t\mathfrak{sl}_2[[t]]$) and scaled by h. We have $\Delta_\kappa^a \in \mathcal{O}_\kappa$.

The idea of this proposition is not too hard–we take the (completed) universal enveloping algebra and first use the tensor product to kill anything of the form $e \otimes 1 + t\mathfrak{sl}_2[[t]]$. We can also arrange h and κ to act by the appropriate scalars (in fact, the collection of h and $e \otimes 1 + t\mathfrak{sl}_2[[t]]$ span an Lie subalgebra known as the *Iwahori subalgebra*). Therefore, this representation of the Kac-Moody algebra has a canonical highest weight vector, *and* has a canonical simple quotient (since it is a direct sum of its h-eigenspaces, where we regard the eigenvalues distinct if they have a distinct power of t on them, and therefore any submodule is h-diagonalizable and since the eigenvalues are well ordered, the sum of two h-submodules which both individually don't have our highest weight vector also do not have a highest weight vector in the submodule spanned by the two). Thus, we have a canonical simple quotient, as desired.

5.3 The Kazhdan-Lusztig Equivalence

We can finally state the Kazhdan-Lusztig equivalence!

Theorem 5.16. Fix some $\kappa \in \mathbb{C}$ which is not in the real ray $[-\frac{1}{2}, \infty)$. There is an equivalence of highest weight, braided monoidal categories

$$Rep_q(G) \simeq \mathcal{O}_\kappa$$

where $q := e^{\pi i \kappa}$.

The proof of this theorem is pretty involved. For example, we haven't really touched on the braided monoidal structure on either of these categories! Just to give you a taste of one of the interesting new ideas of [KL91]. How can we get that the structure of $\mathcal{O}_\kappa$ is braided monoidal? The answer is this–given two representations, smoothness and the analogue of finite generation allow us to view these representations as actually living over certain points of $\mathbb{P}^1$ (explicitly, we are viewing the t as the uniformizer of different points). Then, the braidedness of the monoidal structure comes from the fact that we can move the point around in locally constant paths and not change the representation. This also heuristically explains why this monoidal structure need not be symmetric–if we fix one point and move the other around, and then move the other point around the first in the same direction, that's the same as fixing one point and moving the second all the way around. This is a nontrivial loop in the fundamental group!

Remark 5.17. The requirement that κ is not in the real ray is actually required! In fact, the level $\kappa = \frac{1}{2}$ is called the *critical level*, and is where a lot of Geometric Langlands occurs. In fact, often times in Langlands, all levels are shifted so that $\kappa = -\frac{1}{2}$ is regarded as the origin (we used this convention above in surveying the non factorizable fundamental local equivalence above).

A Cheat Sheet for $\mathfrak{sl}_3$

This cheat sheet is largely copied from [Gan20]. All of this information can be found in a less condensed form in [FH91] or [Hum78].

Term	**(Usual) Value for $\mathfrak{sl}_3$**	**Determined By**
$\mathfrak{sl}_3$	{traceless 3×3 matrices}	No Choices
Borel Subalgebra $\mathfrak{b}$	{upper triangular traceless matrices}	Choice
Unipotent Radical $\mathfrak{n} := [\mathfrak{b}, \mathfrak{b}]$	{strictly upper triangular matrices}	$\mathfrak{b}$
$\mathfrak{h}$ (or $\mathfrak{t}$)	{diagonal traceless matrices}	*
Opposite Borel $\mathfrak{b}^-$	{lower triangular traceless matrices}	$\mathfrak{b}$
Opposite Unipotent Radical $\mathfrak{n}^-$	{strictly lower triangular matrices}	$\mathfrak{b}$
$\mathfrak{h}^* = Hom_{Vect}(\mathfrak{h}, \mathcal{C})$	$\{(L_1, L_2, L_3) : L_1 + L_2 + L_3 = 0\}$	$\mathfrak{h}$
Weight Lattice Λ	$\mathbb{Z}\text{-}Span\{L_1, L_2, L_3\}$	$\mathfrak{h}$
Roots R	$\{L_i - L_j\}\ i, j \in \{1, 2, 3\}, i \neq j$	$\mathfrak{h}$
$\mathfrak{sl}_{2,\alpha}$	$\mathfrak{sl}_2$ Lie-subalgebra spanned by $f_\alpha, h_\alpha, e_\alpha$	Root α
Root Lattice Λ_R	$\mathbb{Z}$-span of roots	$\mathfrak{h}$
Positive Roots R^+	$\{L_i - L_j\}\ i, j \in \{1, 2, 3\}, i < j$	$\mathfrak{b}, \mathfrak{h}$
Simple Roots	$\{L_1 - L_2, L_2 - L_3\}$	$\mathfrak{b}, \mathfrak{h}$
Weyl Group	S_3 via $\sigma(L_i) = L_{\sigma(i)}$	$\mathfrak{b}, \mathfrak{h}$
Simple Reflections	$\{(1, 2), (2, 3)\}$	$\mathfrak{b}, \mathfrak{h}$
Dominant Weyl Chamber	$\{aL_1 + bL_2 + cL_3 : a \geq b \geq c\}$	$\mathfrak{b}, \mathfrak{h}$
Parabolic Subalgebra** $\mathfrak{p}_s$	$\mathcal{C}$-Span$(\mathfrak{b}, f_s)$	$\mathfrak{b}$, simple root s
Killing Form κ	$\kappa(M, N) = 6\text{Tr}(MN)$	No Choices
Coroot $\alpha^\vee := 2\alpha/\kappa(\alpha, \alpha)$	$\alpha^\vee = h_\alpha \in \mathfrak{h}$	Root α
Fundamental Weights	$L_1, -L_3$	$\mathfrak{b}, \mathfrak{h}$
$\rho := \sum_{r \in R^+} r/2$	$L_1 - L_3$	$\mathfrak{b}, \mathfrak{h}$

*Canonically $\mathfrak{b}/\mathfrak{n}$ for any choice of Borel $\mathfrak{b}$ but a choice makes it a subset of $\mathfrak{g}$

**Caution: These are the nontrivial parabolic subalgebras ($\mathfrak{b}$ and $\mathfrak{g}$ are parabolic subalgebras) and in general parabolic subalgebras correspond in an order preserving bijection with subsets of the simple roots.

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