Four Manifold Topology

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1. Introduction

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The main question in the theory of manifolds is classification. Manifolds of dimension 1 and 2 have been classified since the 19th century. Though 3 dimensional manifolds are not yet classified, they have been well-understood throughout the 20th century to today. A seminal result was the Poincaré Conjecture, which was solved in the early 2000's. For dimensions $n \geq 4$, manifolds can't be classified due to an obstruction of decidability on the fundamental group. One way around this is to consider simply connected n-manifolds; in this case there are some limited classification theorems. For $n \geq 5$, tools like surgery theory and the S-cobordism theorem developed in the 1960's allowed mathematicians to approach classification theorems as well. The case of $n = 4$ does not have the same machinery, but work by Freedman in 1981 resulted in a classification of simply connected closed topological 4-manifolds. A year later, Donaldson showed that smooth 4-manifolds are very different from higher dimensional smooth manifolds. Modern incarnations of Donaldson's theory include Seiberg-Witten theory and Heegaard-Floer theory.

1.1 Examples of 4-Manifolds ❖

The primary objects of study for this class are closed simply connected 4-manifolds. Some examples of such objects are $S^4, S^2 \times S^2$ and $\# n S^2 \times S^2$ (the n-fold connected sum of $S^2 \times S^2$, see Definition below). Note that while the first homology $H_1(X)$ clearly doesn't distinguish these, the second homology group does. The second homology group plays an important role in 4-manifold theory. By Poincaré Duality, $H_3(X) = 0$ as well. The second homology also has no torsion, since $H_1(X)$ has no torsion. Thus the homology picture of a simply connected closed 4-manifold is just the second Betti number b_2.

Definition 1.1. If X and Y are oriented n manifolds, then the connected sum $X \# Y$ is the space obtained by gluing X and Y along the boundary of embedded disks in X and Y.

One can prove with Mayer-Vietoris that:

Proposition 1.2. *There is a canonical isomorphism $H_i(X \# Y) \cong H_i(X) \oplus H_i(Y)$ for $1 \leq i \leq n-1$.*

Another example of a simply connected 4-manifold is $\mathbb{CP}^2$, which is also a complex manifold. It can also be seen as $\mathbb{CP}^1 \sqcup_f B^4$, where f is the Hopf fibration. The second Betti number is $b_2 = 1$, which is odd. It then follows that $\# n\mathbb{CP}^2$ is a simply connected 4 manifold with $b_2 = n$.

While $S^2 \times S^2$ and $\mathbb{CP}^2 \# \mathbb{CP}^2$ have the same Betti numbers, and hence homology, there is yet a homological invariant that distinguishes them. This is the intersection pairing. Recall that if X is a compact, connected, oriented n manifold, the intersection pairing:

$$Q_X : H_k(X) \times H_{n-k}(X) \to \mathbb{Z}$$

In low dimensions ($n \leq 4$), we will prove that every homology class is represented by a compact oriented submanifold. Hence the intersection pairing can be defined by $[Y] \cdot [Z] = Y \cdot Z = I(Y, Z)$. In dimension 4, the only interesting intersection pairing is on the middle homology. Since $n = 2k$ for k even, Q_X is symmetric. To compute $Q_{S^2 \times S^2}$, choose a basis of H_2 as $\alpha = S^2 \times \{p\}$ and $\beta = \{q\} \times S^2$. It is an exercise to compute that $\alpha \cdot \alpha = \beta \cdot \beta = 0$ and $\alpha \cdot \beta = 1$, hence:

$$Q_{S^2 \times S^2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

While the matrix itself is not an invariant, the associated bilinear form is an invariant. One can check as a corollary of the above proposition that $Q_{X \# Y} = Q_X \oplus Q_Y$. Therefore:

$$Q_{\# n S^2 \times S^2} = \bigoplus^n Q_{S^2 \times S^2}$$

To compute $Q_{\mathbb{CP}^2}$, we choose a basis of $H_2(\mathbb{CP}^2) \cong \mathbb{Z}$. One such generator is $e = [\mathbb{CP}^1]$, where we are seeing $\mathbb{CP}^1$ as a complex line in $\mathbb{CP}^2$. Any two representatives of e are complex lines in $\mathbb{CP}^2$, which always meet at exactly one point. Therefore $e \cdot e = \pm 1$. To determine the sign of this intersection, we make a short digression to complex geometry.

Proposition 1.3. *Complex manifolds have canonical orientations.*

Proof:

Given a $\mathbb{C}$-linear isomorphism $L : \mathbb{C}^n \to \mathbb{C}^n$ in $GL(n, \mathbb{C})$, it can be seen as an element of $GL(2n, \mathbb{R})$ that commutes with the map $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ representing multiplication by i. Moreover, $\det_{\mathbb{R}} L = |\det_{\mathbb{C}} L|^2 > 0$.[a] Therefore L preserves orientation. A smooth $f : \mathbb{C}^n \to \mathbb{C}^n$ is holomorphic iff df is $\mathbb{C}$-linear, since $df \circ J = J \circ df$ is equivalent to the Cauchy-Riemann equations. Therefore the transition functions of the atlas are complex linear and thus preserve orientation. Once $\mathbb{C}^n$ has been oriented in the natural way (as a direct sum of n copies of $\mathbb{C}$), every local coordinate chart is oriented and every transition preserves orientation. Therefore any complex manifold is oriented.

[a] This can be seen by diagonalizing L.

□

Theorem 1.4. *Two transverse complex submanifolds of a complex manifold intersect positively.*

Returning to $\mathbb{CP}^2$, we see that e is canonically oriented and $e \cdot e = 1$. Therefore $Q_{\mathbb{CP}^2} = \langle [1] \rangle$ (where the brackets denote the bilinear form generated by the matrix $[1]$). Thus the intersection form on $\mathbb{CP}^2$ is multiplication of integers. Note that $(me) \cdot (me) = m^2 > 0$, so it is also positive definite. Note that $Q_{\overline{\mathbb{CP}^2}} = \langle [-1] \rangle$, where $\overline{\mathbb{CP}^2}$ is $\mathbb{CP}^2$ with the opposite orientation. This is a negative definite intersection form, and therefore $\mathbb{CP}^2$ and $\overline{\mathbb{CP}^2}$ have different intersection forms and are therefore different as oriented manifolds. In fact they are not homeomorphic (preserving orientation) not homotopy equivalent (preserving orientation).

Now we can conclude that:

$$Q_{\#k\mathbb{CP}^2 \# \ell \overline{\mathbb{CP}^2}} = \begin{bmatrix} I_k & 0 \\ 0 & -I_\ell \end{bmatrix}$$

Theorem 1.5. *Every symmetric bilinear form on $\mathbb{R}^m$ is uniquely diagonalizable; i.e. there exists a basis in which it is given by:*

$$\begin{bmatrix} I_{b_+} & 0 & 0 \\ 0 & I_{b_-} & 0 \\ 0 & 0 & 0_n \end{bmatrix}$$

and therefore b_+, b_- and n are invariants of the form.

We note that for $M = \#k\mathbb{CP}^2 \# \ell \overline{\mathbb{CP}^2}$, the invariants are $b_+ = k, b_- = \ell, n = 0$. More generally, for a $4m$ dimensional manifold (closed and oriented), Q_X is unimodular (i.e. $\det Q_X = \pm 1$) and hence $n = 0$ and $b_+ + b_- = b_{2m}(X)$. The *signature* of a compact oriented $4k$-manifold is $\sigma(X) := b_+ - b_-$. The signature of a manifold of any other dimension is defined to be zero.

Exercise 1.6. Show that $\sigma(S^2 \times S^2) = 0$, thus distinguishing $S^2 \times S^2$ from $\mathbb{CP}^2 \# \mathbb{CP}^2$. Show further that, even though $\sigma(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}) = \sigma(S^2 \times S^2)$, the manifolds $S^2 \times S^2$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ are not diffeomorphic.

Proposition 1.7. *The set of 4-manifolds $\{\#nS^2 \times S^2, \#k\mathbb{CP}^2 \# \ell \overline{\mathbb{CP}^2}\}$ are all different oriented homotopy types for $n > 0$ and $k, \ell \geq 0$.*

As an example, we solve the second part of Exercise 1.6. Recall the basis $\{\alpha, \beta\}$ the basis for $H_2(S^2 \times S^2)$. Note that $(a\alpha + b\beta)^2 = 2ab$. Thus the square of any homology class must be even (this means $Q_{S^2 \times S^2}$ is even as a symmetric bilinear form). However, in $H_2(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2})$the generators $\{e_1, e_2\}$ have odd square and hence the bilinear form $Q_{\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}}$ is not even.

Definition 1.8. A symmetric bilinear form on $\mathbb{Z}^n$ is *even* if every element has even square. Otherwise it is *odd*. Equivalently, it is even if every basis consists of elements of even square.

2. Signatures and 4-Manifolds

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2.1 Signatures of Smooth Manifolds ❖

Recall our definition of the signature of a manifold:

Definition 2.1. The *signature* of a compact oriented $4k$-manifold is $\sigma(X) := b_+ - b_-$, where b_+ is the dimension of a maximal positive definite subspace and b_- is the dimension of a maximal negative definite subspace. The signature of a manifold of any other dimension is defined to be zero.

Some properties of the signature that can be taken as exercises to prove are:

- If $\partial X = \emptyset$ and $X = Y \cup_\partial Z$ then $\sigma(X) = \sigma(Y) + \sigma(Z)$.
- $\sigma(\overline{X}) = -\sigma(X)$.
- If $X = \partial W$ with W compact and oriented, then $\sigma(X) = 0$.
- $\sigma(X \times Y) = \sigma(X)\sigma(Y)$.

Definition 2.2. Let X, Y be closed, oriented k-manifolds. Then X and Y are *cobordant* if there exists a compact oriented W such that $\partial W = \overline{X} \sqcup Y$.

Cobordism is an equivalence relation on closed oriented manifolds of a given dimension, denoted by Ω_k. We can add cobordism classes by disjoint union. We can also multiply by Cartesian product. One can show that Ω_k is an abelian group (with additive inverses given by reversing orientation). Hence $\Omega_* = \bigoplus_k \Omega_k$ is a ring (called the *oriented cobordism ring*). Moreover, σ is well-defined on cobordism classes and is compatible with disjoint union and Cartesian product. Therefore $\sigma : \Omega_* \to \mathbb{Z}$ is a ring homomorphism (in fact surjective).

Exercise 2.3. Show that $\Omega_k = \mathbb{Z}$ for $k = 0$ and $\Omega_k = 0$ for $k = 1, 2$.

By tensoring with $\mathbb{Q}$, the ring $\Omega_* \otimes \mathbb{Q}$ is a polynomial ring generated by $\mathbb{CP}^{2k}$ for $k \in \mathbb{N}$.

2.2 Complex Projective Varieties ❖

Recall that a collection of homogeneous polynomials in $n+1$ complex variables cut out a well-defined zero locus in $\mathbb{CP}^n$, called a projective variety. As usual, singularities will exist for general polynomials, but we are most interested in those which are smooth. In this case, the resulting manifold will be a complex manifold.

Example 2.4. Let p be a homogenous polynomial of degree $d > 0$. Then the associated variety $V(p) \subset \mathbb{CP}^n$ is called a hypersurface. "Most of the time" this surface is non-singular. In fact:

Theorem 2.5. *For generic $p \in \mathbb{C}[x_0, ..., x_n]$ homogenous, $V(p) \subset \mathbb{CP}^n$ is a manifold and the diffeomorphism type depends on n.*

Proof idea:

> We can prove non-singularity by using the usual regular value theory. Writing down the condition for the variety to be singular, it is easy to see that the parameter set of homogenous polynomials in $\mathbb{C}[x_0, ..., x_n]^{(n)}$ for which $V(p)$ is singular is itself a projective variety Z in $\mathbb{CP}^N$ for some N. In particular, $\operatorname{codim}_{\mathbb{C}}(Z) \geq 1 \Rightarrow \operatorname{codim}_{\mathbb{R}}(Z) \geq 2$. Therefore any two non-singular varieties can be connected by a 1-parameter family of non-singular varities (since the real codimension is at least 2).

□

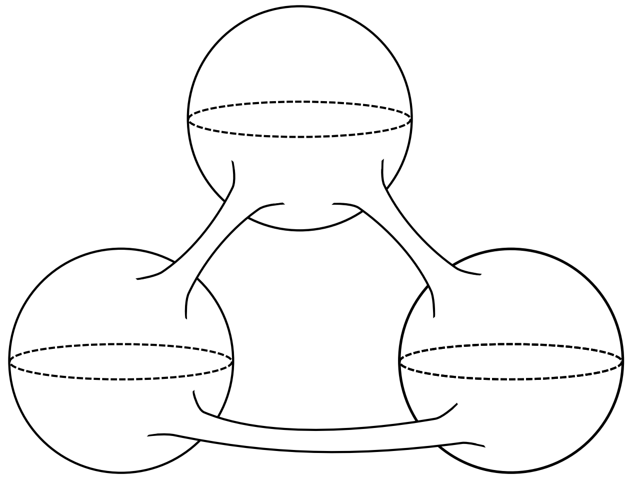

Figure 2.1: Perturbing a product of three linear polynomials and looking at the corresponding variety results in a surface with three spheres, every pair of which is joined by a connect sum. The result is diffeomorphic to a torus.

Example 2.6. Let $p_d(z) = \sum_{i=1}^{n} z_i^d$. It is easy to check that $V(p_d(z))$ is nonsingular. For $n \geq 2$, this surface is connected and for $n \geq 3$ it is simply connected. For $n = 2$, it is a surface of real dimension 2, so we can ask about its genus. For $d = 1$, clearly $V(p_d) = \mathbb{CP}^1 \subset \mathbb{CP}^2$ and so its genus is zero. For general d, first consider a polynomial q_d which is a product of d linear polynomials. The variety $V(q_d)$ is a union of d copies of $\mathbb{CP}^1$, with singularities at the intersection points. We then we perturb q_d to make $V(q_d)$ smooth. Around each intersection point in a coordinate neighborhood, the perturbation replaces the singular point with the annulus $\mathbb{C} - \{0\}$. Topologically, this operation is a connected sum. Therefore a nonsingular degree d curve is obtained by d spheres by connected summing each pair (see Figure 2.1). Then one can check that the resulting genus is $g = \frac{(d-1)(d-2)}{2}$.

Example 2.7. Now consider the same polynomial but in $\mathbb{CP}^3$. As usual, $V(p_1) = \mathbb{CP}^2$; moreover, $V(p_2) = \mathbb{CP}^1 \times \mathbb{CP}^1 = S^2 \times S^2$ and $V(p_3) = \mathbb{CP}^2 \# 6\overline{\mathbb{CP}}^2$. For degree 4, we find a new surface type: the K3 surface. The invariants $b_2(V(p_d))$ and $\sigma(V(p_d))$ increase cubically with d and the parity of the intersection form $Q_{V(p_d)}$ is the same as the parity of d. For example, for $d = 4$ we have $b_2 = 22$ and $\sigma = -16$ with and the parity is even.

Definition 2.8. A closed smooth 4-manifold X is called *irreducible* if in any smooth splitting $X = X_1 \# X_2$ one X_i is a homotopy sphere.

For $d \geq 4$, the surface $V(p_d) \subset \mathbb{CP}^3$ is irreducible.

Definition 2.9. A *complete intersection* is a transverse intersection of smooth hypersurfaces.

By our discussion above, there exists a unique complete intersection for each choice of degrees. However, most algebraic manifolds are not complete intersections.

2.3 Unimodular Forms ❖

The following is an important example of an even symmetric bilinear form of rank 8

$$E_8 = \left\langle \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix} \right\rangle$$

As an exercise, diagonalize this matrix over $\mathbb{Q}$ and show that its determinant is 1. Therefore it is unimodular. The diagonalization will also show that it is positive definite (i.e. $b_+ = 8$ and $b_- = 0$). Therefore its signature is 8 as well. We would like to understand if there is a closed manifold X whose intersection form is E_8.

2.3.1 Indefinite Forms

Example 2.10. Some examples of indefinite unimodular forms that we can now make are:

$$\oplus k\langle 1\rangle \oplus \ell\langle -1\rangle$$

$$\pm(\oplus k E_8 \oplus \ell H), \quad (H = Q_{S^2\times S^2})$$

The first class of signatures can be any rank, any signature, and are odd. The second class can realize any rank, have signature divisible by 8 and are even.

Theorem 2.11. *Indefinite forms are classified by their rank, signature, and parity.*

As a result, since the second Betti number, signature and parity of a K3 surface are known, we can conclude from the above theorem that $Q_{K3} \cong -(\oplus 2E_8 \oplus 3H)$. The following theorem completes our classification of indefinite forms:

Theorem 2.12. *If Q is an even unimodular symmetric bilinear form, then $\sigma(Q)$ is divisible by 8.*

Consider any unimodular symmetric bilinear form Q on $A \cong \mathbb{Z}^n$, we can reduce mod 2 to get $Q|_2$ over $A|_2 = A \otimes \mathbb{Z}/2 = \mathbb{Z}/2^n$. Given $x, y \in A|_2$, then $(x+y)^2 = x^2 + y^2$ in $A|_2$. Therefore $\phi : A|_2 \to \mathbb{Z}/2$ sending $y \mapsto y^2$ is a homomorphism. Since $\phi \in (A|_2)^*$, we can represent it as $(-)\cdot x$ for some unique $x \in A|_2$. A lift of x to A is known as a characteristic element:

Definition 2.13. A *characteristic element* of A is any $x \in A$ such that for all $y \in A$, $x \cdot y \equiv y \cdot y$ mod 2.

We have just shown that characteristic elements always exist and form a coset of $2A$ in A.

Lemma 2.14. *If x is characteristic for Q, a unimodular symmetric bilinear form, then $x^2 \equiv \sigma(Q)$ mod 8.*

Proof:

We first make the observation that if x and y are characteristic elements for Q_1 and Q_2, respectively, then $x+y$ is characteristic for $Q_1 \oplus Q_2$. Now consider the form $Q' = Q \oplus \langle 1\rangle \oplus \langle -1\rangle$. This is odd and indefinite and thus by Theorem 2.11 and Example 2.10, it is isomorphic to $\oplus k\langle 1\rangle \oplus \ell\langle -1\rangle$. Clearly the Lemma is true for $\langle 1\rangle$ and $\langle -1\rangle$. Therefore the square of the characteristic element x' of Q' is the sum of the squares of the characteristic elements of the $\langle \pm 1$ components, which shows $x'^2 = k - \ell$. By the way we constructed Q', the characteristic element x of Q has the same square as x', therefore $(x')^2 = x^2 = k - \ell = \sigma(Q')$. Moreover, $\sigma(Q) = \sigma(Q')$.

Finally, we claim that if x and y are characteristic, then $x^2 \equiv y^2$ mod 8. This follows by writing $y = x + 2z$ and expanding.

□

Proof of Theorem 2.12:

If Q is even, then 0 is characteristic for Q, hence by the above lemma $0 = 0^2 \equiv \sigma(Q) \bmod 8$.

□

2.3.2 Definite Forms

In the definite case, $|\sigma(Q)| = \text{rank}(Q)$. Since we know that $\sigma(Q)$ must be divisible by 8, the number of positive definite even forms is given in Figure 2.2. For even or odd definite forms, the number of forms of a particular rank is finite; however, the number grows very fast with the rank, unlike what we found with the indefinite case.

Rank	Number of even, positive definite forms
8	1
16	2
24	24
⋮	⋮
40	> 10^{50}

Figure 2.2: Number of positive definite even unimodular forms of a given rank. The number grows super exponentially.

2.4 Realizing Unimodular Forms ❖

The connection between unimodular forms and classifying topological 4-manifolds was solved by Freedman in the 1980's. His classification is stated as follows:

1. Every even form is realized by exacly one such manifold.
2. Every odd form is realized by exactly two such manifolds, distinguished by the Kirby-Siebermann invariant $ks \in H^4(X; \mathbb{Z}/2) \cong \mathbb{Z}/2$.

We will return to this in more detail later in the course. Work on topological manifolds and smooth manifolds in the mid 20th century led to the discovery that $ks(X) \neq 0 \Rightarrow X$ not smoothable. Therefore, as a corollary of Freedman's classification, there exist closed simply connected topological 4-manifolds with no smooth structure.

Theorem 2.15 (Rokhlin)**.** *Every closed, smooth, simply connected even* 4*-manifold has signature divisible by* 16*.*

If X is even and simply connected (not necessarily smoothable), then $ks(X) = \sigma(X)/8 \bmod 2$, so Rokhlin's theorem can be seen as a precursor to the Kirby-Siebermann invariant.

Theorem 2.16 (Donaldson)**.** *If a closed, oriented, smooth* 4*-manifold has a definite form, then it is diagonalizable over* $\mathbb{Z}$*.*

Corollary 2.17. *Every closed, simply connected, odd, smooth 4-manifold is homeomorphic to* $\# \pm \mathbb{CP}^2$*.*

This follows from the fact that the odd indefinite forms from Example 2.10 are diagonalizable and the even indefinite forms from that Example are not diagonalizable.

3. Handlebodies and Kirby Diagrams

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Definition 3.1. Given manifolds X, Y and an embedding $\phi_0 : Y \to \mathrm{int}(X)$, an *isotopy* of ϕ_0 is a homotopy $\phi_t : Y \to X$ through embeddings defined at $t = 0$.

Theorem 3.2 (Smooth Isotopy Extension). *Given an isotopy $\phi_t : Y \to \mathrm{int}(X)$ with Y compact, then this extends to an ambient isotopy; i.e. there exists an isotopy $\Phi_t : X \to X$ through diffeomorphisms such that $\Phi_0 = \mathrm{id}_X$ and $\Phi_t \circ \phi_0 = \phi_t$ for all t. Moreover, Φ_t has compact support in the interior of X.*

For a proof, see [1]

Definition 3.3. An *n-dimensional k-handle* is $H = D^k \times D^{n-k}$ attached to a manifold X^n by an embedding $\phi : \partial D^k \times D^{n-k} \to \partial X$ (the resulting space is denoted $X \cup_\phi H$). The *core* of the handle is $D^k \times \{0\}$, the *attaching sphere* is $\partial D^k \times \{0\}$, the *attaching region* is $\partial D^k \times D^{n-k}$, the *cocore* is $\{0\} \times D^{n-k}$, and the *belt sphere* is $\{0\} \times \partial D^{n-k}$ (see Figure 3.1).

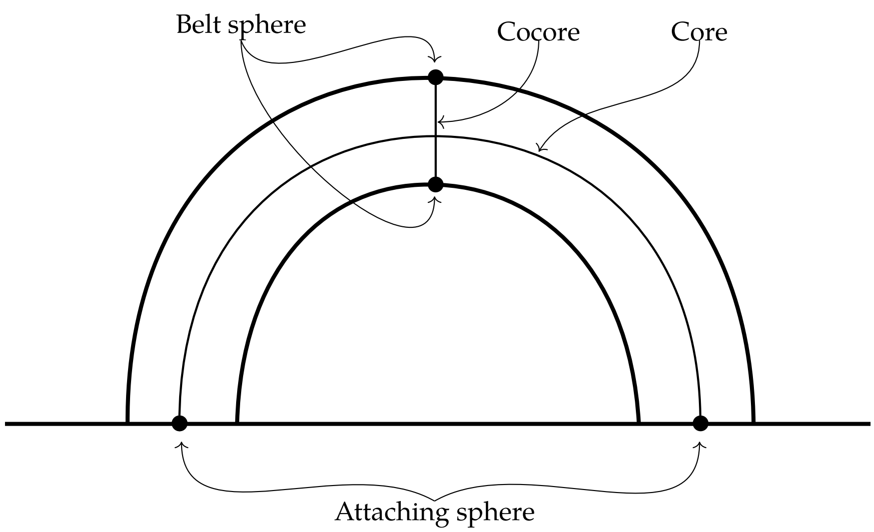


Figure 3.1: Anatomy of a handle [1].

Remark 3.4. As described in the above definition, the resulting manifold with an n handle has corners. However, there is a canonical way to smooth such a manifold, which we relegate to [1]. Therefore we take any manifold with a handle attached to be smooth in this context.

Remark 3.5. In the homotopy category, the attaching of a k-handle is the same as attaching a k-cell (i.e. they are homotopic). However, we want the thickness of a handle so that we can smooth the resulting manifold and work in the smooth category.

Proposition 3.6. *If (H, ϕ) is a k-handle, the diffeomorphism type of $X \cup_\phi H$ is determined by the isotopy class of ϕ.*

Proof:

We work in a collar $I \times \partial X$ of ∂X in X. Gluing a handle (H, ϕ_0) on $\{0\} \times \partial X$ and (H, ϕ_1) on $\{1\} \times \partial X$, we can take an isotopy ϕ_t of ϕ_0 and ϕ_1 and extend it to the collared X via the Isotopy Extension Theorem. This produces the desired diffeomorphism. For details see [1].

□

We note that, given $\phi : S^{k-1} \to \partial X$ such that $\phi(S^{k-1})$ has trivial normal bundle, then we can extend this to a map $S^{k-1} \times D^{n-l} \to \partial X$. This extension is uniquely determined up to isotopy by the trivialization (i.e. the framing of the normal bundle). As an example where this fails, consider the central circle of the Möbius band. This has a nontrivial normal bundle. In fact, there are lots of ways to embed spheres in a manifold such that the image does not have trivial normal bundle.

Proposition 3.7. *The data of a map $\phi : S^{k-1} \to \partial X$ with trivial normal bundle and a framing of its normal bundle uniquely determines a k handle attached to X up to diffeomorphism only depending on the isotopy class of ϕ and the trivialization.*

Example 3.8. If $n = 4$ and $k = 2$ with $X = B^4$, then $\partial X = S^3$. The attaching map will specify a knot in S^3, the theory of which is very rich and known.

Given an embedding $S^{k-1} \to Y^{n-1}$ with a trivial normal bundle, fix a framing of this bundle f_0. Every framing now corresponds to a map $S^{k-1} \to GL(n-k)$. There is a strong deformation retract $GL(n-k) \to O(n-k)$ given by the Grahamn-Schmidt process, so we can take this to be a map $S^{k-1} \to O(n-k)$. Fixing $p \in S^{k-1}$, after composing ϕ by a fixed element of $O(n-k)$, we can assume that the two framings agree at p. Thus homotopy (or isotopy) classes of based framings corresponds bijectively to $\pi_{k-1}(O(n-k), p)$. This bijection becomes canonical once we choose the framing f_0; thus it is more accurate to say the homotopy classes of based framings is a $\pi_{k-1}(O(n-k), p)$ torsor.

Special cases:

1. An n dimensional 0 handle is just a copy of D^n attached along $\emptyset$.

2. For 1 handles, the framings are in bijection with $\pi_0(O(n-1))$, which is the set of path components of $O(n-1)$. There are exactly two such components, and so once you have attached the 0 sphere, there are two corresponding framings that can happen. One is the annulus and one is the Möbius strip. Since the latter makes the resulting manifold non-orientable, we see there is a unique way to attach a 1-handle preserving orientation (when ∂X has one component).

3. When $k = n$, we have $\pi_{n-1}(O(0)) = 0$. So there is only one framing.

4. When $k = n-1$, we have:
$$\pi_{n-2}O(1) = \begin{cases} \mathbb{Z}/2 & n = 2 \\ 0 & n \neq 2 \end{cases}$$
The case of $n = 2$ is for 1 handles on surfaces (which was discussed in 2.). Otherwise, there are no framing issues. Therefore attaching the handle is determined by embedding a S^{n-1} into ∂X. For $n \geq 7$, that-taching a n handle attached to D^n creates an exotic sphere based on which embedding is chosen. In low dimensions, however, it is enough to specify the image of the attaching map.

Proposition 3.9. *For $n \geq 3$, there is a unique oriented diffeomorphism type of $D^n \cup \{\ell \text{ 1-handles}\}$.*

3.1 Attaching a handle to the disk ❖

Proposition 3.10. *Given X^n connected, there exists a unique isotopy class of embedding $\phi : D^k \to X$ for $k < n$. If $k = n$, there are at most 2, differing by a reflection.*

Proof:

We can assume that $\phi(0)$ and $d\phi_0$ are the same for any ϕ by an appropriate local isotopy. Let $\phi_t = \frac{1}{t}\phi(tx)$. As $t \to 0$, the maps ϕ_t approach id_{D^k}, where we are thinking of $D^k \subset \mathbb{R}^n$ as a standard k disk embedded into a local coordinate chart.

□

Corollary 3.11. *There is a unique unknotted embedding $S^{k-1} \to S^{n-1}$ when $k < n$.*

If we attach a k-handle H to $D^n = D^k \times D^{n-k}$ along an unknotted S^{k-1}, without loss of generality we can assume the attaching map sends $S^{k-1} \times D^{n-k} \subset H$ to $S^{k-1} \to D^{n-k} \subset D^n$. Collapsing the D^{n-k} dimension of the resulting space produces a D^{n-k} bundle over S^k with structure group $O(n-k)$[1]. In fact, every D^{n-k} bundle over S^k arises this way. Therefore manifolds obtained by attaching a handle to a disk are classified by disk bundles over a sphere.

Example 3.12. Recall attaching a 1 handle to D^1 gave either an annulus or a Möbius band; these are the only two disk bundles over S^1 up to diffeomorphism. This classification holds for attaching a 1 handle to any D^n.

Example 3.13. Now consier a 2 handle attached to D^n. The structure group is $\pi_1(O(n-2))$, which is:

$$\pi_1(O(n-2)) = \begin{cases} 0 & n \leq 3 \\ \mathbb{Z} & n = 4 \\ \mathbb{Z}/2 & n \geq 5 \end{cases}$$

For $n = 4$, the $\mathbb{Z}$ structure group describes the twisting of the framing around S^2. Each element $n \in \mathbb{Z}$ determines the resulting 4 manifold X_n, where we are writing $X_0 = S^2 \times D^2$ (the trivial D^2 bundle). The next exercise shows that the intersection forms are:

$$Q_{X_n} \cong \langle [n] \rangle$$

Additionally, the Euler number of X_n is n.

Exercise 3.14. Let X_n be a manifold constructed in the previous example. Show that $H_2(X_n) \cong \mathbb{Z}$ and that $Q_{X_n} \cong \langle [n] \rangle$. A solution can be found in Appendix A.2.

More generally, consider D^m bundles over S^2 for $m \geq 3$. The structure group is now $\pi_1(O(m)) = \pi_2(SO(m)) = \mathbb{Z}/2$. Therefore we end up with exactly two classes of S^2 bundles: $S^2 \times D^m$ and $S^2 \widetilde{\times} D^m$. Each of these has boundary $S^2 \times S^{m-1}$ and $S^2 \widetilde{\times} S^{m-1}$. In particular, we care about the case with $m = 3$, which is the four manifold $S^2 \widetilde{\times} S^2$.

3.2 Handlebodies ❖

Definition 3.15. A *handlebody* is a compact manifold X^n exhibited as built by handles from $\emptyset$. If X has boundary $\partial X = \partial_- X \sqcup \partial_+ X$, then a *relative handlebody* is an exhibition of X built by attaching handles to $I \times \partial_- X$.

Theorem 3.16. *Every smooth compact manifold X has a handle decomposition (or relative handle decomposition).*

The proof of this idea uses Morse theory. Start with a smooth $f : X \to I$ with $f^{-1}(0) = \partial_- X$ and $f^{-1}(1) = \partial_+ X$. Then f can be perturbed generically so that the critical points are nondegenerate and quadratic having a local model $f(x_1, ..., x_n) = -\sum_{i=1}^k x_i^2 + \sum_{i=k+1}^n x_i^2$. The integer k is called the index of the critical point. One can show that fo each critical point of index k corresponds to attaching a k-handle. There is a similar theorem and proof in the PL (piecewise linear) category as well.

Theorem 3.17. *Every TOP n-manifold admits a handle decomposition except for unsmoothable such 4-manifolds.*

Example 3.18. There is a handle decomposition of $\mathbb{CP}^n$ given by considering the coordinate patches $\psi_i : \mathbb{C}^n \to \mathbb{CP}^n$, $(z_1, ..., z_n) \mapsto [z_1 : ... : 1 : ... : z_n]$. Let $D \subset \mathbb{C}$ be the unit disk and consider the "poly disk" $D \times .... \times D \subset \mathbb{C}^n$. For each i let $B_i = \psi_i(D \times ... \times D) \subset \mathbb{CP}^n$. Normalize homogeneous coordinates on $\mathbb{CP}^n$ so that each $|z_i| \leq 1$ and there exists at least one coordinate with $|z_i| = 1$. This shows that $\mathbb{CP}^n = \bigcup_{i=0}^n B_i$. The B_i intersect only on their boundaries. We can think of B_0 as a 0-handle, B_1 as a 2-handle attached to it along $\psi_0(S^1 \times D \times ... \times D)$. Repeating this, we see that each B_k is a $2k$ handle attached to $\bigcup_{i=0}^{k-1} B_i$. This exhibits the handle decomposition of $\mathbb{CP}^n$ and the reader will note that it mimics the cell decomposition of $\mathbb{CP}^n$.

[1] Equivalently, a rank $n-k$ vector bundle over S^k.

Example 3.19. In the case of $\mathbb{CP}^2$, the decomposition is 0-handle $\cup$ 2-handle $\cup$ 4-handle. By our analysis in the previous section, this is some surface X_n with a 4-handle attached. It turns out that $n = 1$. Since $\partial X_1 = S^3$, restricting the S^2 bundle X_1 to its boundary gives the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$.

Exercise 3.20. Exhibit a handlebody decomposition of the torus T^2 and $\mathbb{RP}^2$. See [1] for hints and more examples.

Theorem 3.21. *Every handlebody can be perturbed so that the handles are attached in order of increasing index.*

Proof:

> Suppose H is a k handle and G is an ℓ handle with $l \leq k$ and consider $(X \cup H) \cup G$. We would like to show that this is isotopic to $(X \cup G) \cup H$. The belt sphere of H has dimension $n - k + 1$ and the dimension of the attaching sphere of G is $\ell - 1$. The sum of these dimensions is $n + (\ell - k) - 2$. Since $l \leq k$, this sum is less than $n - 1 = \dim \partial X$. Therefore the belt sphere of H and the attaching sphere of G can be taken to be disjoint by transversality. Now we can construct an isotopy that pushes G off of H by flowing along a vector field that pushes off of the cocore of H. Thus in gluing G to $X \cup H$ we can assume that H and G are disjoint, proving the claim.

□

We assume from now on that handlebodies are constructed by attaching handles by increasing index.

Definition 3.22. Given a handle decomposition of $(X, \partial_- X)$, there is a *dual decomposition* of $(X, \partial_+ X)$ given by "turning X upside down." Begin with a collar $I \times \overline{\partial_+ X}$ glued to X alon $\partial_+ X$ and remove the collar $I \times \partial_- X$ on which the handlebody is built. Then notice that every k handle of the decomposition corresponds can be thought of as an $(n - k)$ handle glued to the part of X above it.

3.3 Handle operations ❖

In this subsection we describe two important handle operations: cancellation and sliding. Consider a manifold X with a copy of $D^n = D^k \times D^{n-k}$ boundary connected summed to ∂X along the lower hemisphere D_-^{k-1} of $\partial D^k = S^{k-1}$. This does not change the diffeomorphism type of X, since it is gluing along a contractible region. Now take a neighborhood of the upper hemisphere D_+^{k-1} of ∂D^k, which gives us the decomposition $D^k = D_0^k \cup_{D_+^{k-1}} D_+^{k-1} \times I$. Then $D_+^{k-1} \times I \times D^{n-k}$ is a $k - 1$ handle attached to ∂X and $D_0^k \times D^{n-k}$ is a k handle attached to that. This is an example of a *cancelling pair* of handles (see [1] Figure 4.7).

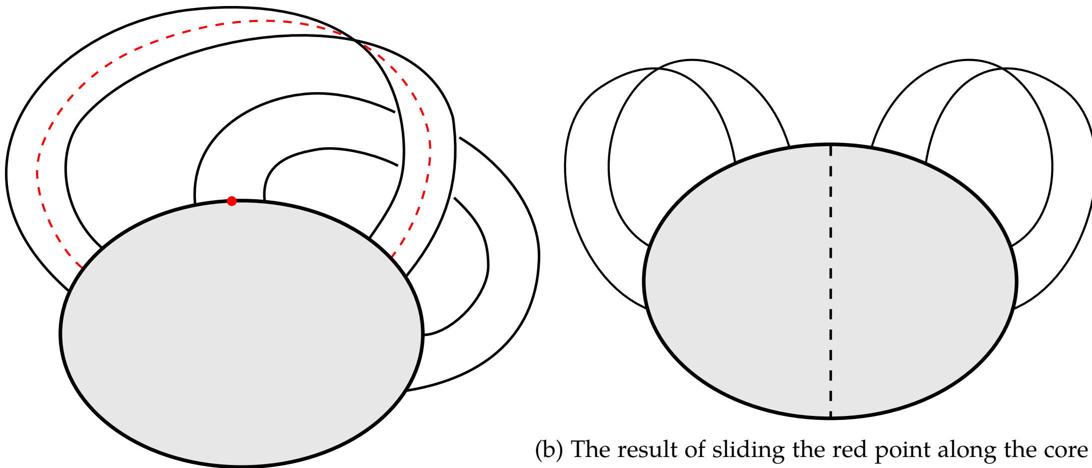
Proposition 3.23. *A $k - 1$ handle h_{k-1} and a k handle h_k can be cancelled providing that the attaching sphere of h_k interescts the belt sphere of h_{k-1} transversely at a single point.*

A proof of this Proposition can be found in [1]. Recall from our proof of Theorem 3.21 that we were able to construct a slide of one handle off of another. This is an example of a more general operation called a handle slide. Given two handles h_1 and h_2 of the same index attached to ∂X, a handle slide of h_1 over h_2 is the operation of moving the attaching sphere A of h_1 across the belt sphere B of h_2 through isotopy. Since h_1 and h_2 have the same index, there is a point p at which h_1 and h_2 intersect transversely and at which their tangent spaces span a codimension 1 subspace. Therefore there are exactly two directions to push A off of B to make them disjoint (as in the proof of Proposition 3.21) in a handle slide procedure.

Example 3.24. The Klein bottle K has a handle decomposition given by attaching two 1 handles, one with a twist, to D^2 (shown in the left of Figure 3.2) and 2 handle to cap it off. Using a handle slide, we can slide the untwisted handle along the core of the twisted handle to untangle the handles (shown on the right). What results is two twisted handles that are attached to a disk. The resulting space can then be interpreted as the boundary sum of two copies puctured $\mathbb{RP}^2$'s (the dotted line indicating where the sum is taking place). Attaching the final 2 handle, we have shown that $K \cong \mathbb{RP}^2 \# \mathbb{RP}^2$.

Exercise 3.25. Classify compact dimension 2 manifolds using handle slide theory.

Proposition 3.26. *Every compact connected $(X, \partial_- X)$ has a relative handle decomposition with no 0-handles if $\partial_- X \neq \emptyset$ and exactly one 0-handle if $\partial_- X = \emptyset$. Moreover, there is exactly 1 n-handle if $\partial_+ X = \emptyset$ and no n-handle if $\partial_+ X \neq \emptyset$.*



(a) A handle decomposition of the punctured Klein bottle.

(b) The result of sliding the red point along the core of the twisted handle.

Figure 3.2: Using handle slides to show that $K \cong \mathbb{RP}^2 \# \mathbb{RP}^2$.

Proof:

Suppose there is more than one 0-handle in any handle decomposition of X. Since X is connected, these 0-handles must be connected by 1-handles. In such a situation, the belt sphere of the 0 handle and the attaching sphere of the 1-handle meet transversely at one point. Applying Proposition 3.23, the 1 handle and the 0 handle cancel out, diminishing the number of 0 handles by 1. This shows that we can always reduce the number of 0 handles to one (or zero if $\partial_- X \neq \emptyset$). Applying the same argument to the dual decomposition of $(X, \partial_+ X)$ shows the n-handle claim.

□

Theorem 3.27 (Cerf)**.** *For a compact manifold $(X^n, \partial_- X)$, any two handle decompositions are related by handle sides, pair creation/cancellations, and isotopy on the levels.*

Proof idea:

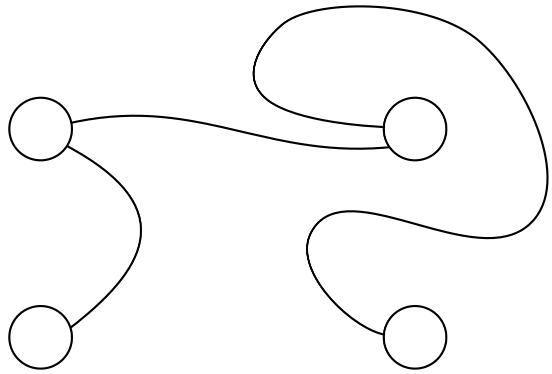
Let f_0 and f_1 be the corresponding Morse functions to the 2 handlebodies, which can be taken to be self-indexing (this is a generic property[a]). Because $[0, 1]$ is contractible, the maps f_0, f_1 are homotopic preserving the preimages of their boundaries (namely $\partial_- X$ and $\partial_+ X$). Let f_t be a generic homotopy; then for finitely many t, f_t is not a self-indexed Morse function. At these points, f_t is either not Morse or not self-indexing. The non-Morse situation corresponds to a handle cancellation/creation, and the non-self-indexing corresponds to handle slides.

[a] Technically, self-indexing isn't generic, but rather a generic Morse function can be re-scaled to be self-indexing by a scalar monotonic function

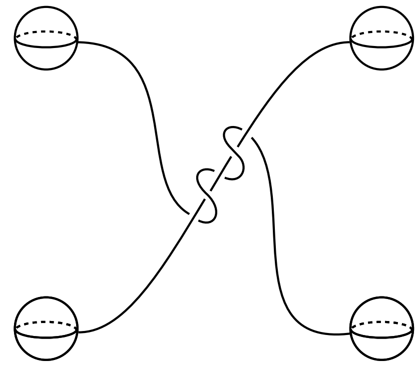
□

3.4 Dimension 3 - Heegaard Diagrams ❖

Let X be a connected 3 dimensional manifold with $\partial_- X = \emptyset$. Then any handle decomposition can be assumed to have exactly 1 0-handle and 1 3-handle. Suppose that there are g 1-handles and n 2-handles. Let X_k denote the union of handles of index up to k. If X is oriented, then $X_1 = \natural_g S^1 \times D^2$, a solid multi-torus. The 2-handles attach to disjoint circles in which the framings are trivial. Now assume $\partial X = \emptyset$ and turn X upside down. Then $X - \text{int}(X_1)$ is a 0-handle $\cup_n$ 1-handles. Since the boundary of $X - \text{int}(X_1)$ is ∂X_1, we must have $n = g$. This



(a) Example of a Heegaard diagram.



(b) Example of a Kirby Diagram, sans framings.

Figure 3.3

expresses X as $X = H \cup_\phi H$, where H is a 1-handlebody and $\phi : H \to H$ is a diffeomorphism. This is called a *Heegaard Splitting*.

A good way to visualize the 1-handlebodies H is to identify the boundary of their zero handle (a copy of S^2) with the paper together with a point at infinity. Then 1-handles consist of pairs of disks that are identified with eachother (reversing orientation). A 2-handle is then glued along a circle embeded in the paper. See Figure 3.3. This is an example of a Heegaard diagram.

3.5 Dimension 4 - Kirby Diagrams ❖

A Kirby diagram for a smooth 4 manifold is the same idea as a Heegaard diagram, but in one dimension higher (see Figure 3.3). We identify the boundary of the 0-handle with $\mathbb{R}^3 \cup \{\infty\}$. Then 1-handles are pairs of spheres in $\mathbb{R}^3$ that are identified with each other (reversing orientation). The 2-handles are glued along embedded circles attached to these spheres. Since these have dimension and codimension greater than 1, we have to also consider the framings of these 2-handles

If X is closed ($\partial X = \emptyset$), then $\partial(X_2) \cong \partial(\natural_\ell S^1 \times D^3) \cong \#_\ell S^1 \times S^2$. Because of the theorem below, we can conclude that $X_2 \cup$ 3,4 handles is a unique closed manifold.

Theorem 3.28. *Every self-diffeomorphism of $\#_\ell S^1 \times S^2$ extends to $\natural_\ell S^1 \times D^3$.*

Thus we don't need to write the 3 and 4 handles in the diagram. This analysis shows that all of the complexity of a closed 4-manifold lies in the 2-handles (namely how they are attached to the spheres and their framings).

Example 3.29. Consider D^2 bundles over S^2 again. These are $D^4 \cup_{K_0}$ 2-handle, where K_0 is an unknot. The associated Kerby diagram has no 1-handles and a single 2-handle attached with a certain number of twists. These twists can be represented by a parallel copy of K_0 pushed off in the direction of the framing, as shown in the left of Figure 3.4. This is the Kirby diagram for X_n, where n is the number of right handed twists in the framing. The right hand side is how we represent these twists of the framing in a more compact way.

3.6 Framings and Links ❖

Every oriented link L in S^3 has a Seifert surface, i.e. a compact, oriented, connected, surface F with $\partial F = L$.

Theorem 3.30. *For every oriented X^4, each $\alpha \in H_2(X)$ is represented by a closed oriented surface (and we can assume it is connected if X is connected).*

Proof sketch:

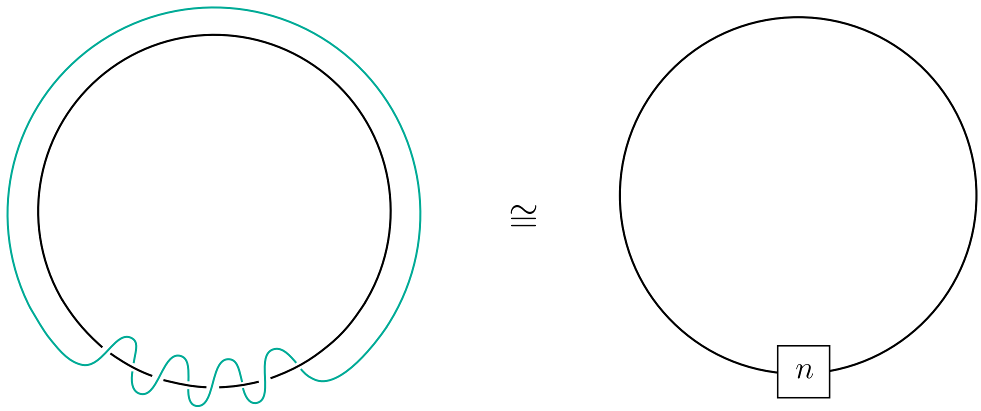


Figure 3.4: The Kirby diagram for X_n.

> WLOG we can assume that X is compact and connected, since α has compact support. Write X as a handlebody so that α is contained in $X_2 = 0$-handles $\cup$ 1-handles $\cup$ 2-handles. Represent α by a union of core disks. Then band these together preserving orientation so that the new boundary is disjiont from the 1-handles (lies in S^3). Then attaching a Siefert surface to this gives us a closed surface representing α.

□

Definition 3.31. Given two knots $K_1, K_2 \in S^3$, the *linking number* $\text{lk}(K_1, K_2)$ is the number of crossings of K_1 under K_2 counted with sign when projected to a plane.

Definition 3.32. For a framed knot K in S^3, the *framing coefficient* is $\text{lk}(K, K')$ where K' is K pushed off K by the framing.

Definition 3.33. The *blackboard framing* of a knot diagram is given by the normal vector in the plane. This is not invariant under isotopy in $\mathbb{R}^3$.

Theorem 3.34. *The framing coefficient of the blackboard framing is given by the writhe $w(K)$, which is the signed number of self -crossings.*

Theorem 3.35. *The outward or inward normal to any Seifert surface of a knot has framing coefficient* 0.

Proof:

> This follows from the fact that $\text{lk}(K, K')$ can be computed by taking a Siefert S surface for K and taking the intersection number of K' and S. Since the outward or inward normal framing pushes K' off of S normally, there will be no intersection.

□

Now we return to the case of D^2 bundles over S^2; these are handlebodies with a 0-handle and some number of 2-handles. Recall these are specified by a framed link in S^3.

Definition 3.36. The *linking matrix* of an oriented, ordered, framed link S^3 is $A = (a_{ij})$ where:

$$a_{ij} = \begin{cases} \text{lk}(K_i, K_j) & i \neq j \\ \text{framing coeff.} & i = j \end{cases}$$

where K_i is the i'th component of the link.

Example 3.37. Consider the link diagram showin in Figure 3.5. There are two knots K_1 and K_2 with framings 2 and -1, respectively. The linking number $\text{lk}(K_1, K_2)$ is -2. Therefore the linking matrix is $a = \begin{pmatrix} 2 & -2 \\ -2 & -1 \end{pmatrix}$.

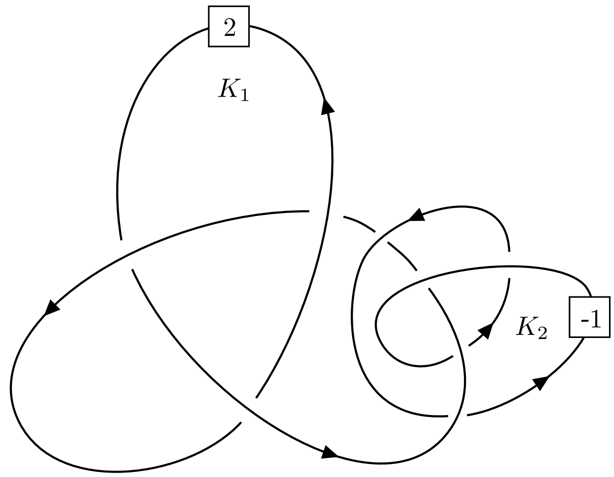


Figure 3.5: An example of a framed link diagram.

Theorem 3.38. *For X a handlebody with one 0-handle and some number of 2-handles, we have $H_2(X) \cong \mathbb{Z}^m$ where $m = \#$link components and Q_X is given in the obvious ordered basis by the linking matrix.*

This is seen by representing each homology class as the core of each handle capped off by a Seifert surface to each knot. Then the intersection numbers of these classes can be computed by the intersection of the Seifert surface of one of the classes with the knot of the other class. This is the linking number $\mathrm{lk}(K, K')$.

Example 3.39. What is the link diagram for $S^2 \times S^2$? We write $S^2 = D^2_+ \cup D^2_-$ so that $S^2 \times S^2 = D_- \times D_- \cup D_+ \times D_- \cup D_- \times D_+ \cup D_+ \times D_+$. We are thinking of $D_- \times D_-$ as the 0 handle and $D_+ \times D_+$ as the 4 handle. Then attaching $D_+ \times D_-$ to the 0 handle gives the trivial disk bundle over S^2, and similarly so does attaching $D_- \times D_+$ to the 0 handle. These correspond to two unknots with zero framing in the knot diagram (shown below HOPF LINK).

Exercise 3.40. Draw the link diagram for $S^2 \widetilde{\times} S^2$.

Remark 3.41. We can get a boundary sum of two Kirby diagrams by joining the two diagrams disjointly. This allows us to draw, for example, $\#k \pm \mathbb{CP}^2$ and $\#kS^2 \times S^2$.

3.6.1 Doubling

Let X be a compact manifold with boundary. We define its *double* to be $DX = \partial(I \times X)$ after rounding the corners of $I \times X$. This is equivalently $DX = X \cup_{\mathrm{id}_\partial} \overline{X}$. For example, if X is the twice punctured disk, then DX is the 2-torus. If X is a handlebody with a 0 handle and some number of 1-handles and 2-handles, then DX is given by taking the dual decomposition of X and attaching its 2 handles to X (and then the 3 and 4 handles thereafter in a unique way). The attaching of the extra 2 handles is done by adding a 0 framed meridian to each existing 2 handle of X.

4. Kirby Calculus and Surgery Theory

❖

4.1 Blowups and Kirby Calculus ❖

Exercise 4.1. If P is the E_8 plumbing, then $P\#\mathbb{CP}^2$ is the diagram with lefthanded trefoil knot (framing -1) connected sum with $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$. Hint: blow up at the fart left link of P.

Exercise 4.2. Given a framed link $L \subset S^3$, prove any handle slide can be realized by blowing up and down ± 1 framed unknots.

The previous exercise shows that handle slides can be realized as an algebraic operation via the Kirby diagram. In a similar manner, handle cancellation can also visulized using the Kirby diagram (CANCELLATION-FIG).

Theorem 4.3. *A* 2-3 *handle pair can be cancelled in a Kirby diagram of a closed oriented* 4 *manifold after handle slides if and only if there exist slides producing the zero framed unknot separate from the rest of the diagram.*

Exercise 4.4. a) If $X =$ 0-handles $\cup_m$ 2-handles, then show $DX \cong \#mS^2 \times S^2$ or $\#m\mathbb{CP}^2\#m\overline{\mathbb{CP}}^2$.

b) If $X =$ 01-handle $\cup_\ell$ 11-handles $\cup$ 2handles is simply connected, then $DX\#\ell S^2 \times S^2 \cong \#mS^2 \times S^2$ or $\#m\mathbb{CP}^2\#m\overline{\mathbb{CP}}^2$.

4.2 Surgery ❖

What happens to the boundary of a manifold when we attach a k-handle? Attaching a handle of one dimension higher than the boundary is an example of surgery on the boundary:

Definition 4.5. Given $\phi_0 : S^\ell \to M^m$ a framed embedding (up to isotopy), this extends uniquely up to isotopy to $\phi : S^\ell \times D^{m-\ell} \to M^m$. Then an ℓ *surgery* on ϕ_0 is the process of cutting out $S^\ell \times \mathrm{int}(D^{m-1})$ and gluing in $D^{\ell+1} \times S^{m-\ell-1}$ by $\mathrm{id}_{S^\ell \times S^{m-\ell-1}}$.

Conversely, we can think of any surgery operation as attaching an $\ell+1$ handle to $I \times M$ and looking at $\partial_+ M$. Specifically, surgery on (M^m, ϕ_0) is obtained from $M - \mathrm{int}(S^\ell \times D^{m-\ell})$ by attaching an $\ell + 1$ handle and an m handle.

Theorem 4.6. *Two closed, oriented dimension m manifolds are oriented (cobordant) if and only if they are related by a sequence of surgeries (where* 0 *surgeries preserve orientation).*

Proof Sketch:

> Given a cobordism W, take a morse function f such that $f^{-1}(0) = \partial_- W$ and $f^{-1}(1) = \partial_+ W$ and build the handle structure in the usual way via critical points. The other direction works in the opposite way; take a morse function that realizes the handle decomposition.

□

Corollary 4.7. *Two oriented* 4*-manifolds are related by surgery if and only if they have the same signature.*

Theorem 4.8. *Suppose $N \subset M$ is a properly embedded submanifold of codimension r. Then the map $i_*\pi_1(M - N) \to \pi_1(M)$ is an isomorphism if $r \geq 3$ and an epimorphism if $r = 2$. Moreover, when $r = 2$, then $\ker(i_*)$ is generated by meridians of N connected to the basepoint.*

Proof:

Given $\alpha \in \pi_1(M)$, if $r \geq 2$, then α is represented by a loop disjoint from N by dimension considerations. Therefore i_* is onto. If $r \geq 3$, then every map of a disk can be assumed disjoint from N. Therefore anything in the kernel of i_*, which has a bounding disk in M that can be perturbed to be disjoint from N, is zero. The remainder of the proof is left as an exercise.

□

Given an orientable manifold M^m with $m \geq 4$ and a map $\bigsqcup_1^k S^1 \to M$, let M' be obtained by surgery on these circles. Then we claim that $\pi_1(M') \cong \pi_1(M)/\langle \gamma_1, ..., \gamma_k \rangle$, where γ_1 is the homotopy class of the ith circle in $\pi_1(M)$. This follows from the fact that removing $\text{int}(S^1 \times D^{m-1})$ does not affect change $\pi_1(M)$ by the previous theorem (the circle have codimension 3). But then gluing in the handles annhilates the circles in homotopy.

Theorem 4.9. *For $m \geq 4$ every finitely presented group G is $\pi_1(M)$ for some closed, smooth, oriented M^m.*

Proof:

First create $\#_n S^1 \times S^{m-1}$ where n is the number of generators of G. The fundamental group is now the free group on the number of generators. Then we can represent the relators by embedded circles and surger them out. The resulting manifold X_G is the boundary $X_G = \partial W_G$ where W_G has 0,1 and 2 handles. In fact, $W_G = I \times Y_G = DY_G$ where Y_G is a 4-manifold with a 0 handle, k 1-handles and ℓ 2-handles (where k is the number of generators and ℓ is number of relations). The surgery to kill the relators can be represened in a Kirby diagram of Y_G by attaching various 2 handles. Then doubling gives a Kirby diagram for X_G.

□

If we choose to do this construction on the balanced trivial presentation (with $k = \ell$), then W is contractible because it is simply connected and has no homology past dimension two[2]. Therefore ∂W is homotopy equivalent to S^4. This implies it is homeomorphic to S^4 by Freedman's Theorem.

Theorem 4.10. *Suppose $C \subset X^4$ is a nullhomotopic circle. Then surgery on C is $X \# S$ where $S = S^2 \times S^2$ or $S = S^2 \widetilde{\times} S^2$ depending on framing.*

Proof:

First note that generically homotopy ⇒ isotopy for circles when $n \geq 4$ for dimension reasons (by pushing off the homotopy annuli off each other). Without loss of generality, write $X \cong X \# S^4$ and $C = S^1 \times \{p\} \subset \partial(D^2 \times D^3)$. Now surgery on C in S^4 is $\partial(D^2 \times D^3 \cup_{D^2 \times \partial\{0\}} D^2 \times D^3)$, which is the boundary of a D^3 bundle over S^2. There are only two such manifolds: $S^2 \times S^2$ and $S^2 \widetilde{\times} S^2$.

□

Theorem 4.11. *If X is odd and simply connected, then both surgeries on a nullhomotopic circle yield diffeomorphic manifolds.*

Proof:

Simple connectivity and the Hurewicz theorem tell us that $H_2(X) \cong \pi_2(X)$. Observe that since Q_X is odd, there exists $\alpha \in H_2(X)$ such that α^2 is odd. By our first observation, α is represented by a sphere S with only transverse double point singularities. It is not hard to show that:

$$\alpha^2 = e(\nu S) + 2(\#\text{double points})$$

where $e(-)$ is the Euler number. Therefore we conclude that $e(\nu S)$ is odd. Going back to the nullhomotopic circle C, we can assume that C is the Arctic circle in S. When we push C down to the South polar cap, the framing of C changes by a sign since the normal bundle of S has an odd twist. There-

[2] $k = \ell$ implies that $\chi(M) = 1$

fore performing on either the south or north cap of S gives the two possibilities of $X\#S^2\times S^2$ and $X\#S^2\widetilde{\times}S^2$.

□

An application of these theorems is:

Theorem 4.12 (Markov). *There is no algorithm for classifying smooth, closed, oriented 4-manifolds.*

Proof:

Let $P=\langle g_1,...,g_k \mid r_1,...,r_\ell\rangle$ be a presentation. A fact from combinatorial group theory is that there is no algorithm for determining whether P presents the trivial group. Construct $X_P=\partial W_P$ such that $\pi_1(X_P)$ is presented by P, by Theorem 4.9. Let Z_P be obtained by surgering out the 1 handles in X_P. In other words, we are sending $X_1=\#_k S^1\times D^3\to\#_k D^2\times S^2$. Thus $Z_P=$ 0-handle $\cup_{k+\ell}$ 2-handles. By theorem 4.10, we have $DZ_P\#\mathbb{CP}^2=\#_{k+\ell}S^2\widetilde{\times}S^2\#\mathbb{CP}^2=(k+\ell+1)\mathbb{CP}^2\#(k+\ell)\overline{\mathbb{CP}}^2$. If $\pi_1(X)$ is trivial, then by theorem 4.11, then $DZ_P\#\mathbb{CP}^2\cong X\#(k+1)\mathbb{CP}^2\#k\overline{\mathbb{CP}}^2$. If $\pi_1(X)$ is not trivial, then $\pi_1(DZ_P)=1$ and hence $DZ_P\#\mathbb{CP}^2\ncong X\#(k+1)\mathbb{CP}^2\#k\overline{\mathbb{CP}}^2$. Then if we have an algorithm that determines if $(k+\ell+1)\mathbb{CP}^2\#(k+\ell)\overline{\mathbb{CP}}^2\cong X\#(k+1)\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$, then that will in turn determine if $\pi_1(X)$ is trivial or not. Therefore such an algorithm can't exist.

□

Theorem 4.13 (Wall). *Suppose that X,Y are smooth, closed, simply connected and oriented 4 manifolds with $Q_X\cong Q_Y$. Then:*

(a) There exists $k\in\mathbb{Z}_+$ such that $X\#_kS^2\times S^2\cong Y\#_kS^2\times S^2$ (X and Y are stably diffeomorphic).

(b) X and Y are h-cobordant, i.e. there exists a cobordism W from X to Y with inclusions $X,Y\hookrightarrow W$ that are homotopy equivalences.

Given any two closed simply connected 4 manifolds, we can connect sum with appropriate numbers of $\pm\mathbb{CP}^2$ to make the rank, signature, and pairity of thier intersection forms coincide. Since we can ensure that these forms are indefinite, these three characteristics determine their forms and hence $Q_X\cong Q_Y$. Then applying the theorem above proves:

Corollary 4.14. *Any two closed, smooth, simply connected 4 manifolds are diffeomorphic after connect-summing with enough $\pm\mathbb{CP}^2$.*

Theorem 4.15 (Freedman). *A simply connected h-cobordism between topological 4-manifolds X,Y is homeomorphic to $I\times X$.*

Theorem 4.16 (Freedman). *Let Σ^3 be a homology 3-sphere. Then $\Sigma=\partial\Delta$ for some contractible topological 4-manifold Δ.*

Proof:

Look at $\Sigma\times S^1$ and $S^3\times S^1$, which have the same homology. Note that $\pi_1(\Sigma\times S^1)=\pi_1(\Sigma)\times\mathbb{Z}$. Now we surger out $\pi_1(\Sigma)$ to get $X\cong S^3\times S^1\#kS^2\times S^2$, which is a homology equivalence. Now $\pi_1(X)\cong\mathbb{Z}$. We use Freedman to surger out new $S^2\times S^2$'s giving us a topological manifold Y with the same homology as $S^3\times S^1$ and $\pi_1(Y)\cong\mathbb{Z}$. Therefore Y is a homotopy equivalence to $S^3\times S^1$. Now consider the universal cover $\tilde{Y}\cong S^3\times\mathbb{R}$ which contains a copy of Σ. One-point compactifying at each end gives $S^4\supset\Sigma$. Cutting this space along Σ and taking one component gives the required Δ.

□

4.3 Surgery in 3-manifolds ❖

Let T^n be the n-torus, which can be thought of as $\mathbb{R}^n/\mathbb{Z}^n$. Notice that every element of $SL(n,\mathbb{Z})$ gives an orientation preserving diffeomorphism of T^n. This gives an injection $SL(n,\mathbb{Z}) \hookrightarrow \pi_0(\text{Diff}_+(T^n))$ (the "diffeotopy group of T^n", which is the diffeomorphisms of T^n up to isotopy). This is in fact an isomorphism when $n \leq 3$. This shows that there are lots of self-diffeomorphisms of $T^2 = S^1 \times S^1 = \partial(S^1 \times D^2)$.

Definition 4.17. A *Dehn surgery* on a 3-manifold M cuts out $S^1 \times D^2$ and glues it back in by any diffeomorphism of its boundary.

A solid torus $S^1 \times D^2$ is determined by a framed knot. Note that a Dehn surgery the same as gluing a 2-handle $\cup$ 3-handle to a new boundary $S^1 \times S^1$, and therefore this gluing is determined by the image of a primitive homology class $\alpha \in H_1(S^1 \times S^1)$. We orient the knot K, which gives us a meridian μ and longitude λ. Now $\alpha = p\mu + q\lambda$ for some relatively prime p, q. Changing the orientation of K flips the sign of λ and μ, but the resulting 3-manifold doesn't change. Therefore only the pair $\pm(p,q)$ matters, which is to say that the rational number $\frac{p}{q} \in \mathbb{Q} \cup \{\infty\}$ uniquely determines the Dehn surgery. This is called the *surgery coefficient* or *slope*.

Remark 4.18. A link in S^3 with elements of $\mathbb{Q} \cup \{\infty\}$ attached to the components uniquely determines a 3-manifold M by applying Dehn surgery. This is called Dehn surgery diagram. If all coefficients are in $\mathbb{Z}$ (called integral Dehn surgery), then M is exhibited as ∂(0-handle$\cup$2-handles), where the framings of the 2 handles are the surgery coefficients of the links.

Example 4.19. Surgery on any link in M with surgery coefficient ∞ (i.e. $q = 0$) is a trivial operation.

Example 4.20. If K is an unknot with the ratio $-\frac{p}{q}$ attached, the Dehn surgery produces the Lens space $L(p,q)$. When $q = 1$, the space $L(p,1)$ is an S^1 bundle over S^2. As an exercise, show that $L(2,1) \cong \mathbb{RP}^3$.

Theorem 4.21. $\Omega^3 = 0$.

Proof sketch:

Given a closed oriented 3-manifold M, we can find an immersion $M \to \mathbb{R}^5$. The singular set is a union of circles. We surger on these circles to get $M' \subset \mathbb{R}^5$ with M' cobordant to M. Since M' is a manifold of codimension 2, it admits a Seifert "hypersurface." This shows that M' (and hence M) is null-cobordant.

□

Theorem 4.22. *Integral Dehn surgery on S^3 gives all closed oriented 3-manifolds.*

Proof:

Given M closed and oriented, write $M = \partial X^4$ by above. Then we can surger out the 1 and 3 handles to ensure that $X =$ 0-handle $\cup$ 2-handles. Therefore M can be obtained by integral Dehn surgery, since X has only 2 and 0 handles.

□

Theorem 4.23. *Any two integral Dehn surgery diagrams of a fixed M^3 are related by blowing up/down.*

4.3.1 Rolfsen Moves and Slam Dunks

Definition 4.24. A Rolfsen move on an unknotted link K with coefficient $\frac{p}{q}$ is achieved by performing an n-fold Dehn twist on the complement of a neighborhood of K. This can be seen as cutting out the disk spanning K, rotating it around n times, and pasting it back in (see Figure 4.1).

The result of performing a Rolfsen move is the unknotted coefficient becomes $\frac{p}{q+np}$, where n was the number of Dehn twists performed. Any links K_i passing through K also become twisted n times and their surgery coefficients change from r_i to $r_i + n(\text{lk}(K_i, K))^2$. By using Rolfsen moves on the unknot with $-\frac{p}{q}$ coefficient (which is $L(p,q)$), we have shown:

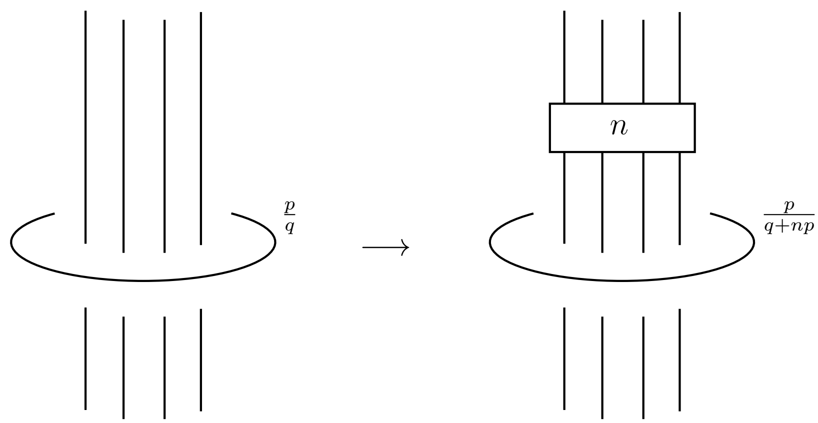


Figure 4.1: A Rolfsen twist, which is performed by applying an n-fold Dehn twist to the complement of the unknotted component (which is a torus).

Corollary 4.25. $L(p,q) \cong L(p,q+np)$.

A related operation is called the *slam-dunk*. Let K_1, K_2 be components of a link such that K_1 is a meridian of K_2 and let $\frac{p}{q}$ be the surgery coefficient of K_1. Also suppose that the surgery coefficient of K_2 is integral. If M is the manifold obtained by surgery on just K_2, then K_1 is a knot in M. We can pull this knot into the gluing torus T of K_2 so that T is also a tubular neighborhood of K_1. But then performing the surgery on K_1 is cutting out T again and gluing by $\frac{p}{q}$. We can instead combine the two gluings into one, and one can work out that the correct surgery coefficient is $n - \frac{q}{p}$.

Exercise 4.26. Let K be the Hopf link where one link has coefficient 0 and the other has coefficient n. Show using Slam Dunks that the resulting manifold by applying surgery is S^3.

4.4 Spin Structures ❖

4.4.1 Constructing a Characteristic Class

Let X be a manifold and let $\xi \to X$ be a vector bundle (usually TX). Choose a triangulation of X. An trivialization on the zero cells that extends to a trivialization on the once cells actually extends to a trivialization on the entire manifold because all higher simplices have connected boundary. Therefore one can define an orientation of ξ to be a trivialization of $\xi|_{X_0}$ that can be extended over X_1 (up to homotopy).

Fix a trivialization τ over $\xi|_{X_0}$. Let $c(\tau) \in C^1(X;\mathbb{Z}/2)$ be zero on a 1-cell σ if and only if we can extend τ over σ. Then $c(\tau) = 0$ if and only if τ extends to a trivialization on X_1. This is called the *obstruction cochain*. Given another trivialization τ' and a zero cell σ_0, let $d(\tau,\tau')(\sigma_0)$ be zero if $\tau|_{\sigma_0} = \tau'|_{\sigma_0}$ and otherwise let it be 1. The map $d(\tau,\tau') \in C^0(X;\mathbb{Z}/2)$ is called a *difference cochain*. Note that $\langle d(\tau,\tau'), \partial\sigma_1\rangle = \langle c(\tau) - c(\tau'), \sigma_1\rangle$. But the left hand side is the same as $\langle \delta d(\tau,\tau'), \sigma_1\rangle$, where δ is the coboundary map. Therefore $\delta d(\tau,\tau') = c(\tau) - c(\tau')$. Finally, we claim that:

$$\langle \delta c(\tau), \sigma_2\rangle = \langle c(\tau), \partial\sigma_2\rangle = 0$$

This follows from choosing a trivialization τ' on σ_2 such that $c(\tau') = 0$, exhibiting $c(\tau)$ as a coboundary. Since this is true for all σ_2, we have $\delta c(\tau) = 0$, and hence $c(\tau)$ is a cocycle up to arbitrary coboundaries. Therefore we get a well-defined class $w_1(\xi) = [c(\tau)] \in H^1(X;\mathbb{Z}/2)$. This is the first *Stiefel-Whitney class*, which is an example of a characteristic class. It has the property that $w_1 = 0$ if and only if ξ admits an orientation. A nice fact about w_1 is that $X - \mathrm{PD}w_1$ is orientable, where PD is the Poincaré dual map.

Suppose $w_1 \neq 0$ and let τ, τ' be orientations of ξ. Then $\delta d(\tau,\tau') = c(\tau) - c(\tau') = 0$ and therefore $d(\tau,\tau')$ is a cocyle in $C^0(X;\mathbb{Z}/2)$. It represents a 0 cohomology class uniquely, since $H^0(X;\mathbb{Z}/2) \cong Z^0(X;\mathbb{Z}/2)$. Therefore differences in orientations correspond bijectively to $H^0(X;\mathbb{Z}/2) \cong H^n(X;\mathbb{Z}/2)$ and hence the number of orientations is $2^{b_0(X)}$.

4.4.2 Constructing Spin Structures

Let $\xi \to X$ be an orientable vector bundle of rank at least 3. To construct a spin structure on ξ, we can follow a similar construction to above but shifted up by one:

Definition 4.27. A *spin structure* on $\xi \to X$ (where X is given a cell structure) is an orientation preserving trivialization of $\xi|_{X_1}$ that extends over $\xi|_{X_2}$ (up to homotopy).

Note that the fact that ξ is orientable means there always exists a trivialization of $\xi|_{X_1}$. Given such a trivialization τ, we get an obstruction cochain $c(\tau) \in C^2(X; \mathbb{Z}/2)$ defined as we did with w_1. Given another trivialization τ' and assume without loss of generality (after a homotopy) that $\tau|_{X_0} = \tau'|_{X_0}$ (i.e. they determine the same orientation), we get a difference cochain which compares τ and τ' on 1-simplices. Once again we get:

$$\langle \delta d(\tau, \tau'), \sigma_2 \rangle = \langle d(\tau, \tau'), \partial \sigma_2 \rangle = \langle c(\tau) - c(\tau'), \sigma_2 \rangle$$

Therefore $\delta d(\tau, \tau') = c(\tau) - c(\tau')$. Moreover $c(\tau)$ is a cocycle independent of τ up to coboundaries (which can be shown in the same way as before). Now we get the second Stiefel-Whitney class $w_2(\xi) := [c(\tau)] \in H^2(X; \mathbb{Z}/2)$, which is independent of τ. It has the property that $w_2(\xi) = 0$ if and only if ξ admits a spin structure.

Remark 4.28. We can alternatively define a spin structure to be a trivialization of $\xi|_{X_2}$. Since $\pi_2(\mathrm{SO}(n)) = 0$, given a trivialization over X_1 there is only one way to extend it to a trivialization on X_2, so this definition is indeed equivalent to our first one. The same argument shows that it extends over X_3.

Proposition 4.29. *For an oriented 3-manifold X, we have $w_2(X) := w_2(TX) = 0$.*

Corollary 4.30. *If X is an oriented 3-manifold, then TX is trivial.*

If X^4 is closed and connected, then the obstructions to trivializing TX are the first two Stiefel-Whitney classes w_1, w_2, the first Pontryagin class p_1 and the Euler class e. If X is spin (i.e. TX admits a spin stucture), then this reduces to just p_1 and e.

If τ, τ' are spin structures on X, we homotope them so that $\tau = \tau'$ on X_0. Therefore the difference cochain $d(\tau, \tau')$ is well defined and is a cocycle (because $c(\tau) = c(\tau') = 0$). Changing the homotopy changes $d(\tau, \tau')$ by δb for some b. Therefore we get a difference class $\Delta(\tau, \tau') = [d(\tau, \tau')] \in H^1(X; \mathbb{Z}/2)$. Thus we see that $H^1(X; \mathbb{Z}/2)$ is a torsor for the group of spin structures on X. In other words, spin structures are classified by $H^1(X; \mathbb{Z}/2)$.

Proposition 4.31. *Given $\xi \to X$ and $\xi' \to X'$ (where X and X' have cell structures) and suppose there is a map $f : X \to X'$ lifting to a bundle map $F : \xi \to \xi'$. Then $w_2(\xi) = f^* w_2(\xi')$, every spin structure s on ξ' pulls back to a spin structure $f^* s$ on ξ and $\Delta(f^* s_1, f^* s_2) = f^* \Delta(s_1, s_2)$.*

Proof:

After homotopy, we can assume that for each k, $f(X_k) \subset X'_k$ (i.e. we can assume f is a cell map). Then every trivialization τ over X'_1 pulls back to $f^*\tau$ on X_1. Then it is not hard to see that $f^\sharp c(\tau) = c(f^*\tau)$, where $f^\sharp$ is the induced map on cochains. Passig to cohomology this formula becomes $w_2(\xi) = f^* w_2(\xi')$. The other two statements follow similarly.

To see that this equality holds independent of homotopy, suppose we have a homotopy F of f. Without loss of generality, $F(I \times X_k) \subset X'_{k+1}$. Since spin structures extend over X_2, $f^*\tau$ is independent of homotopy.

□

Corollary 4.32. *Spin structures are independent of the choice of cell structure.*

Proof:

Apply the previous proposition to $f = \mathrm{id}_X$ between cell structures.

□

Appendix

A.1 Bonus Lecture: Exotic $\mathbb{R}^4$ ❖

This lecture follows §9.4 of [1]. Let $X = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$. Then $b_+ = 1$ and $b_- = 9$. Let the generators of $H_2(X)$ be $e_0, e_1, \ldots e_9$ with $e_0 \cdot e_0 = 1$ and $e_i \cdot e_i = -1$ for $i > 1$. Let $\alpha = 3e_0 + \sum_{i=1}^{8} e_i \in H_2(X)$. Then $\alpha^2 = 9 - 8 = 1$ and α is characteristic in $\langle 1 \rangle \oplus 8\langle -1 \rangle$. Then $\langle \alpha \rangle^{\perp}$ is negative definite (because $\alpha^2 = 1$) and odd, and therefore $\langle \alpha \rangle^{\perp} \cong -E_8 \oplus \langle -1 \rangle$.

Proposition A.1. *α as above is not represented by a smoothly embedded sphere.*

Proof:

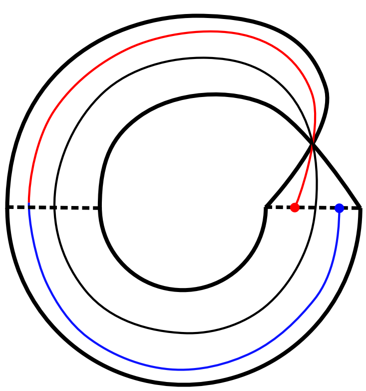
> If it were, then there would be a sphere $S \subset X$ with normal Euler number 1. Then a tubular neighborhood of S is diffeomorphic to the D^2 bundle X_1, which is $\mathbb{CP}^2 - B^4$. Excising this submanifold from X and gluing a disk in its place yields a smooth manifold with intersection form $-E_8 \oplus \langle -1 \rangle$. This contradicts Donaldson's theorem.

□

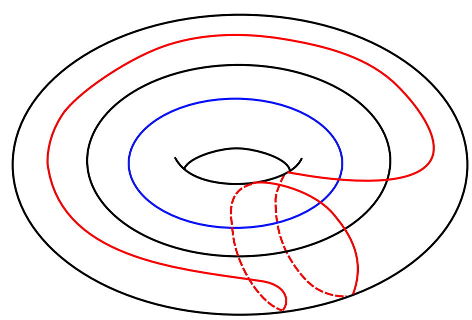
As a consequence of the Hurewicz theorem, α can be represented by an *immersed* sphere, however. By attaching a Casson handle, we can obtain the same situation as in the proof above, but instead the tubular neighborhood U of S is *homeomorphic* to $\mathbb{CP}^2 - \{pt\} \supset \mathbb{CP}^2 - \mathrm{int}(D^4)$. Let R denote the complement of $\mathbb{CP}^2 - \mathrm{int}(D^4)$ inside $\mathbb{CP}^2 - \{pt\}$. Then R is homeomorphic to $\mathbb{R}^4$, but it is not diffeomorphic to $\mathbb{R}^4$ because otherwise we could smoothly embed it into S^4 (which is negative definite). Then replacing this S^4 with U would contradict Donaldson's theorem as before. This is an example of an exotic $\mathbb{R}^4$, a space that is homeomorphic to $\mathbb{R}^4$ but not diffeomorphic. Note that $R \not\cong \overline{R}$, and so $\overline{R}$ and $R \natural \overline{R}$ are also exotic $\mathbb{R}^4$'s.

A.2 Solutions to selected Exercises ❖

Exercise 3.14. Since X_n is a disk bundle over S^2, it deformation retracts to S^2. Therefore its homology $H_2(X_n)$ must be $\mathbb{Z}$. A generator for this class is the cores of the two disks which are joined along S^1 (see left figure below). To find the self intersection of this class, we perturb it off of itself (red and blue curves in left figure).



(a) Schematic of a 2 handle attached to another 2 handle with a twist. The generator of $H_2(X_n)$ is shown in black. A pushed off version of itself is shown in red and blue, which are joined at the red and blue spheres inside the attaching region.



(b) The attaching region, which is a solid torus. The blue curve represents the blue dot from the figure on the left, and the red curve the red dot. The number of twists of the red curve represents the twisting of the handle framing.

Figure A.1: The manifold X_n (left) and the attaching region (right).

We see that, in order to make the red and blue curves coincide on the right figure, the red curve will have to intersect the black curve n times. Each of these intersections is positive. This shows that the self intersection of this generator is n, and so $Q_{X_n} \cong \langle [n] \rangle$.

References

[1] Gompf, Robert. Stipsicz, András. *4-Manifolds and Kirby Calculus*. Graduate Studies in Mathematics, American Mathematical Society.