Contact Topology

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M392C - Fall 2017

These are lecture notes from Robert Gompf's Contact Topology course M392C given Fall 2017 at UT Austin. The reader should be comfortable with essential notions of differential and algebraic topology as well as the basics of knot theory. A prior knowledge of symplectic topology is recommended but not required. I found parts of Laura Starkston's thesis [3] useful to read alongside these lectures. Bob's proofs tend to be very picture-heavy, which can be hard to translate onto a page, so some proofs herein are not in full rigor. Special thanks to Riccardo Pedrotti for various contributions and corrections. Please send any corrections to `gdavtor@math.utexas.edu`.

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1. Plane Fields and Contact Structures

This is a course on contact manifolds, which are odd dimensional manifolds with an extra structure called a contact structure. Most of our study will focus on three dimensional manifolds, though many of these notions hold for any odd dimension. In this section, we will start with some preliminary definitions to motivate the definition of a contact structure. Then we will embark on a classification of vector bundles in order to classify plane fields, after which we can give a proper definition of a contact structure.

Definition 1.1. A *hyperplane field* on a manifold M is a smoothly varying choice of plane $\xi_x \subset T_xM$.

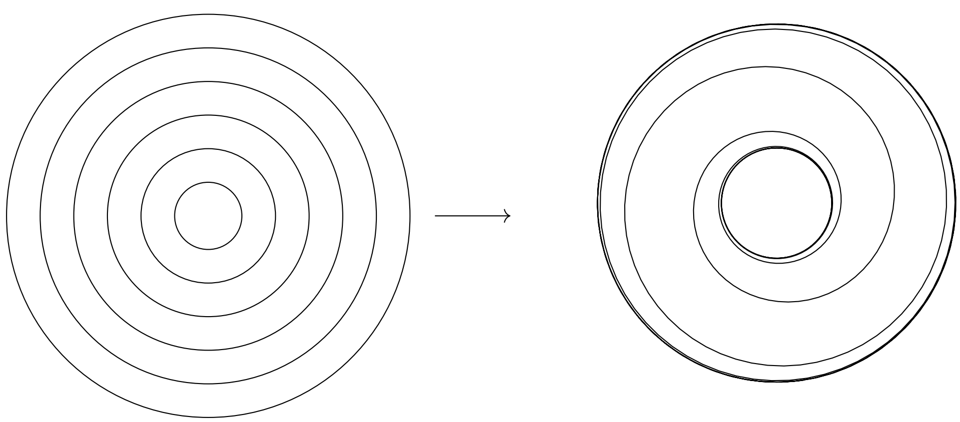
Example 1.2. The simplest plane field on $M = \mathbb{R}^n$ is the horizontal hyperplane field ξ_0; at each point, we assign the horizontal plane described by $dx_n = 0$. These are uniquely tangent to the collection of horizontal planes in $\mathbb{R}^n$.

Hyperplane fields are closely related to *foliations* on manifolds. There are two equivalent definitions of a foliation on a n-manifold M. It is a hyperplane field ξ such that:

1. ξ is locally modeled by the one in the above example. That is, there is a local chart around every point that is diffeomorphic to ξ_0.
2. A collection of surfaces S_α everywhere tangent to ξ such that $M = \sqcup_\alpha S_\alpha$ uniquely.

Sometimes we refer to a foliation by the collection of surfaces S_α. A natural question to ask about a hyperplane field is if it arises from a foliation. When a hyperplane field does (at least locally), it is called integrable. Given a foliation, the collection of tangent spaces is an integrable hyperplane field.

Example 1.3. We can foliate $\mathbb{R}^2 - \{0\}$ by concentric circles, shown below on the left. However, a small perturbation to the hyperplanes can produce a foliation with spirals attracting to circles. In this case, the surfaces are no longer compact as they were in the unperturbed case.



This example demonstrated one of two main stability problems with foliations:

1. Foliations are not stable under perturbation in dimension greater than 1.
2. Even if a perturbation results in another foliation, the resulting surfaces might have very different topologies.

Therefore, to study things that are stable, it's better to look at structures that are *not* foliations. This is the idea behind a contact structure. A contact structure on M is a hyperplane field that does not restrict to an integrable hyperplane field on any submanifold of M. Such a field is called *completely non-integrable*. We will make all of these definitions more precise later.

1.1 Euler Classes ❖

Definition 1.4. A C^k *distribution* ξ on an n-manifold M is an m-plane ξ_x at every point $x \in M$ such that locally ξ is spanned by m linearly independent C^k vector fields. It is called a hyperplane field if $m = n-1$ and a plane field if $m = 2, n = 3$. A smooth distribution is a C^∞ distribution.

Exercise 1.5. Show that a hyperplane field is C^k if and only if it is locally the kernel of a nonzero C^k 1-form.

Proposition 1.6. *$\xi = \ker(\alpha)$ for some nonvanishing 1-form α globally if and only if ξ is co-orientable (i.e. there exists an orientable line field transverse to ξ).*

Another way to define distributions is through the language of vector bundles. Recall that a vector bundle $\pi : E \to B$ is a smooth surjection such that $\pi^{-1}(x)$ is a vector space for every $x \in B$. It must also satisfy the local trivialization condition: around every point $x \in B$ there is a neighborhood U such that $U \times \mathbb{R}^k \cong \pi^{-1}(U)$ and the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\cong"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^k \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

A sub-bundle of $\pi : E \to B$ is a subset $F \subset E$ such that $\pi : F \to B$ is a vector bundle.

Definition 1.7. A *distribution* is a sub-bundle of TM. In particular, a hyperplane field is a codimension 1 sub-bundle of TM.

Proposition 1.8. *If $(F, B, \pi) \subset (E, B, \pi)$ is a sub-bundle, there exists a complementary sub-bundle (F', B, π') such that $F \oplus F' \cong E$.*

An important invariant that can be associated to vector bundles is the Euler class, which is a characteristic class that measures how "twisted" the bundle is. To define this, let $\pi : E \to B$ be an oriented rank k vector bundle over an oriented manifold B. Further, let $\sigma : B \to E$ be a section transverse to the 0 section of π. Since it is transverse, $\sigma^{-1}(0)$ is a codimension k submanifold of B and hence represents a homology class $[\sigma^{-1}(0)] \in H_{n-k}(B, \mathbb{Z})$. By Poincaré duality, there is a corresponding homology class $e \in H^k(B, \mathbb{Z})$.

Definition 1.9. The *Euler class* of $\pi : E \to B$ is $e \in H^k(B, \mathbb{Z})$ as above.

Proposition 1.10. *The Euler class is well defined.*

Proof. (sketch) The idea is to use a cobordism. Let σ_1 and σ_2 be two sections transverse to the zero section. Then these are homotopic (say, by linear homotopy) and so the preimages $\sigma_1^{-1}(0)$ and $\sigma_2^{-1}(0)$ are cobordant. □

Example 1.11. Consider the tangent bundle $E = TB$, so that $k = n$. Then, $e(E) \in H^n(B) \cong \mathbb{Z}$ (so long as B is connected). In this case, $e(E)$ is the Euler characteristic $\chi(B)$ (technically, it is $\langle e(E), [B] \rangle$, where $[B]$ is the fundamental class of B). This can be seen by using the self intersection definition of the Euler characteristic:

$$\chi(M) = I(\Delta, \Delta) = e(N\Delta) = e(T\Delta) = e(TM)$$

Where Δ is the diagonal in $B \times B$, $N\Delta$ is the normal bundle to Δ and $T\Delta$ is the tangent bundle to Δ.

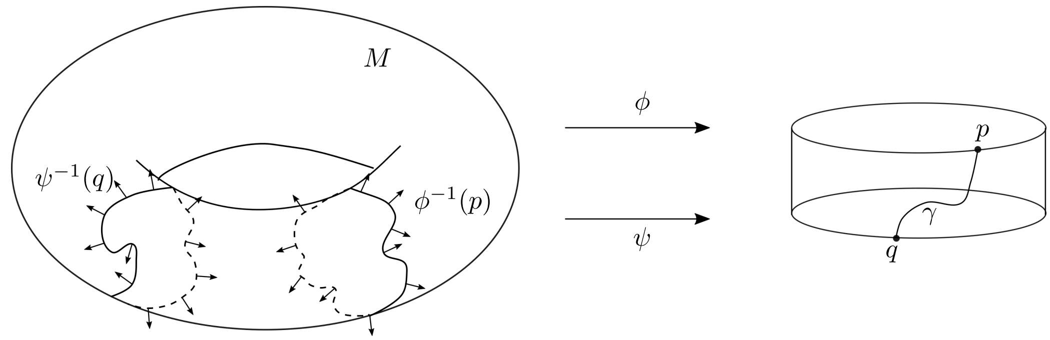


Figure 1.1: The framed submanifolds $\phi^{-1}(p)$ and $\psi^{-1}(q)$ and the induced path γ.

1.2 Classifying Rank 2 Vector Bundles

In general, characteristic classes are not enough to characterize all vector bundles. However, for small ranks they are enough.

Theorem 1.12. *Oriented rank 2 vector bundles over oriented, closed manifolds are classified by the Euler class $e(-)$. That is, for each $x \in H^2(M) \cong H_{n-2}(M)$, there is a unique bundle $E \to M$ such that $e(E) = x$.*

Similarly, rank 1 bundles are classified by $\omega_1 \in H^1(M; \mathbb{Z}_2)$, the Stiefel-Whitney class.

Corollary 1.13. *Oriented plane fields on a closed, oriented 3 manifold are determined up to isomorphism as 2-plane bundles E by $e(E) \in H_1(M)$.*

Remark 1.14. Just because two plane fields have the same Euler class (and are hence isomorphic as sub-bundles of TM) doesn't mean they are homotopic as sub-bundles. As a result, we'll have to use a different technique to answer the question of homotopy.

An important fact is the every oriented 3 manifold has a trivial tangent bundle[1]. Thus we have a (non-unique) isomorphism $\tau : TM \to M \times \mathbb{R}^3$. For now, we fix the trivialization τ. Let ξ be an oriented plane field. Since every plane in $\mathbb{R}^3$ has a unique positive unit normal, such a plane field corresponds to a unit vector field on M, which corresponds to a map $M \to S^2$. Therefore:

$$\{\text{Oriented plane fields up to homotopy}\} \iff [M, S^2]$$

An important understanding of $[M, S^2]$ can be obtained using the Thom-Pontryagin construction.

Thom-Pontryagin Construction

Let M be a closed manifold, and consider the homotopy classes of maps $[M, S^k]$. Given $\phi : M \to S^k$, let $p \in S^k$ be a regular value, so that $\phi^{-1}(p)$ is a closed manifold of codimension k. Given a positive basis for T_pS^k, we get a normal framing on $\phi^{-1}(p)$. For a different $\psi : M \to S^k$ and regular value $q \in S^k$, assume we have a homotopy $\Phi : I \times M \to I \times S^k$ of ϕ and ψ. We then get a path γ connecting p and q in S^k, and hence $\Phi^{-1}(\gamma)$ is a cobordism between $\phi^{-1}(p)$ and $\psi^{-1}(q)$. This cobordism is actually framed by pulling back a framing on γ. See Figure 1.1. The conclusion is that if ϕ and ψ are homotopic, then framed submanifolds are framed cobordant.

Proposition 1.15. *The framed cobordism above is an equivalence relation.*

This means we have a well-defined map between $[M, S^k]$ and equivalence classes of framed cobordism classes of framed codimension k submanifolds.

[1]This related to the fact that odd-dimensional manifolds have trivial Euler characteristic

Theorem 1.16. *This correspondence is a bijection:*

$$[M, S^k] \iff \{\textit{framed cobordism classes of framed codimension } k \textit{ submanifolds}\}$$

Proof:

> To check that this map is onto, let M_0 be a framed codimension k submanifold of M. The framing gives us a map of the normal bundle $NM_0 \to \mathbb{R}^k$ such that the image of $M_0 \times \{0\}$ is 0. But the normal bundle is identified with a tubular neighborhood M_0^ϵ of M_0, so we have a map $\phi : M_0^\epsilon \to \mathbb{R}^k$. We can think of this as a map $\phi' : M_0^\epsilon \to S^k$ by sending the complement of Im(ϕ) to ∞, where we are writing $S^k = \mathbb{R}^k \cup \infty$. Now extend this to a smooth map on all of M by sending $M \setminus M_0^\epsilon$ to ∞ as well. Therefore we have the desired map $\phi' : M \to S^k$.
>
> To check injectivity, let $\phi_0, \phi_1 : M \to S^k$ such that the corresponding framed submanifolds are framed cobordant. We would like to show that $\phi_0 \sim \phi_1$. We extend ϕ_1 and ϕ_0 to a map $\Phi : I \times M \to S^k$ using the given cobordism in the following way: we extend ϕ_0 and ϕ_1 to a map on a tubular neighborhood of the cobordism (seen as a submanifold of $M \times I$), and send the rest of $M \times I$ to a point on S^k like before. Then Φ is the desired homotopy.

□

Remark 1.17. If M is oriented, then so is every framed submanifold.

Example 1.18. Suppose $n = k$ and M connected. This correspondence is:

$$[M, S^k] \iff \{\text{framed cobordism classes of zero dimensional submanifolds}\}$$

The framed cobordism classes of zero manifolds is $\mathbb{Z}$ (given by the signed count) and the map $[M, S^k] \to \mathbb{Z}$ is the degree map. What happens when M is non-orientable? Then the framed cobordism classes are determined by their parity (because same-sign points can be cobordant in this case), so we have $[M, S^k] \to \mathbb{Z}/2\mathbb{Z}$.

1.3 Cobordism classes of framed links ❖

Let's return to the case that we were interested in to begin with: $[M, S^2]$ for a closed, oriented 3 manifold M. We now have:

$$\{\text{Oriented plane fields on M (up to homotopy)}\} \iff [M, S^2] \iff \{\text{Framed cobordism classes of framed links}\}$$

Recall that the first part of this bijection is non-canonical (depends on the trivialization τ). So, now we care about framed links in M. Every link is closed, by the preimage theorem, so they represent elements of $H_1(M; \mathbb{Z})$. This is not a bijection, in general. However, it is onto.

Proposition 1.19. *This is onto.*

Proof:

> Every class in $H_1(M)$ is represented by an oriented link. Since the normal bundle is trivial, any trivialization gives us a framing of the link.

□

Denote Γ_τ to be the map that takes an oriented plane field on M to the corresponding element of $H_1(M)$ using the correspondence above. Recall, we also have an Euler class to associate to a plane field ξ, which is an element of $H^2(M) \cong H_1(M)$. This is not, however, the same as $\Gamma_\tau(\xi)$. To see this, consider the diagram:

$$\begin{tikzcd} \xi \arrow[r] \arrow[d] & TS^2 \arrow[d] \\ M \arrow[r, "\phi_\tau"] & S^2 \end{tikzcd}$$

The top map sends a plane at a point $x \in M$ to the tangent space to the corresponding point on the sphere, and ϕ_τ sends a point $x \in M$ to the unit normal vector of ξ_x. It is not hard to see that this commutes. Now consider the standard vector field on S^2, which flows from the north pole N to the south pole S, and call it v. We can pull this back to a vector field $\sigma = \phi_\tau^*(v)$ in M in a way that is compatible with the above diagram:

$$\begin{tikzcd} \xi \arrow[r] \arrow[d] & TS^2 \arrow[d] \\ M \arrow[r, "\phi_\tau"] \arrow[u, bend left, "\sigma"] & S^2 \arrow[u, bend right, "v"'] \end{tikzcd}$$

We can use this compute $e(\xi)$. The zero set of $\phi_\tau^* v$ is $\phi_\tau^{-1}(N) \cup \phi_t^{-1}(S)$. In fact, ϕ_τ^* is transverse to the zero section. Then:

$$\mathrm{PD}(e(\xi)) = \underbrace{[\phi_\tau^{-1}(N)]}_{\Gamma_\tau(\xi)} + \underbrace{[\phi_\tau^{-1}(S)]}_{\Gamma_\tau(\xi)} = 2\Gamma_\tau(\xi)$$

Where $\mathrm{PD}(-)$ is the Poincaré dual. So, the Euler class and $\Gamma_\tau(\xi)$ aren't quite the same thing (depending on the 2-torsion in $H_1(M;\mathbb{Z})$). In general, Γ_τ is a finer invariant than $e(-)$.

Remark 1.20. There is also an algebraic proof of this identity. In the diagrams above, we have realized ξ as the pullback of the bundle $TS^2 \to S^2$ over the map $M \to S^2$. Euler classes (and characteristic classes in general) behave nicely under pullbacks. That is, the induced map on cohomology $\phi_\tau^* : H^2(S^2) \to H^2(M)$ sends Euler classes to Euler classes:

$$\mathrm{PD}(e(\xi)) = \mathrm{PD}(\phi_\tau^*(e(TS^2))) = \mathrm{PD}(\phi_\tau^*(2[S^2]^*)) = 2\,\mathrm{PD}(\phi_\tau^*[S^2]^*)$$

where $[S^2]^*$ is the chosen generator of $H^2(S^2)$.

Corollary 1.21. *An oriented plane bundle ξ on M is realized as a plane field if and only if $e(\xi) = 2x$ for $x \in H_1(M)$.*

Proof:

> One direction is immediate from our calculation above. For ($\Leftarrow$), surjectivity of Γ_τ gives us ξ' such that $\Gamma_\tau(\xi') = x$. Then $e(\xi') = e(\xi)$, and hence $\xi \cong \xi'$.

□

Suppose we are given two plane fields ξ, ξ' such that $e(\xi) = e(\xi')$. Then $TM \cong \xi \oplus \mathbb{R}$ and $TM \cong \xi' \oplus \mathbb{R}$ (where $\mathbb{R}$ is an oriented line bundle). But $\xi \cong \xi'$ as bundles, so we get an automorphism of TM. In other words, there exists an automorphism of TM sending ξ to ξ'. *The conclusion is that, if we forget τ, all that remains is the Euler class.*

Example 1.22. Consider $M = \mathbb{R}P^3$. We know $H_1(\mathbb{R}P^3) = \mathbb{Z}/2\mathbb{Z}$. Note that only $0 \in H_1$ can be represented as $2x$, so every plane field in $\mathbb{R}P^3$ is trivial as a plane bundle. For fixed choice of τ, however, we get two values of Γ_τ and therefore there are at least two different plane fields that are not homotopic.

Exercise 1.23. Analyze the Lens space $L(p,q)$ in the same way. (Hint: $H_1(L(p,q)) = \mathbb{Z}/p\mathbb{Z}$).

1.3.1 Relative Euler classes

Let M be compact and oriented of dimension n, but maybe not closed. We can do the same derivation as before to define the Euler class of a k bundle $\xi \to M$. In this case, the Euler class will be a relative homology class because the intersection of a submanifold with the zero section can have boundary. That is, $e(\xi) \in H_{n-k}(M, \partial M) \cong H^k(M)$.[2]

Definition 1.24. Suppose $e(\xi|_{\partial M}) = 0$; then there exists a nowhere zero section $v : \partial M \to \xi$. Now, we look for extensions w of v to all of M transverse to the zero section on M. The *relative Euler class* $e(\xi, v) \in H_{n-k}(M) \cong H^k(M, \partial M)$ is $[w^{-1}(0)]$.

[2]This isomorphism is by Poincaré-Lefschetz duality

Remark 1.25. This can be interpreted using the long exact sequence of the pair $(M, \partial M)$:

$$\cdots \longrightarrow H_{n-k}(\partial M) \longrightarrow H_{n-k}(M) \longrightarrow H_{n-k}(M, \partial M) \xrightarrow{\delta} H_{n-k-1}(\partial M) \longrightarrow \cdots$$

Assuming $e(\xi|_{\partial M}) = 0$ is equivalent to saying $\delta(e(\xi)) = 0$ in the above sequence. By exactness, there is a class in $H_{n-k}(M)$ whose image is $e(\xi)$. This is the relative Euler class. Poincaré-Lefschetz duality gives us an isomorphic exact sequence, which shows the same sequence of Euler classes as cohomology elements:

$$\cdots \longleftarrow H^{k+1}(X, \partial M) \longleftarrow H^k(\partial M) \longleftarrow H^k(M) \longleftarrow H^k(M, \partial M) \longleftarrow \cdots$$

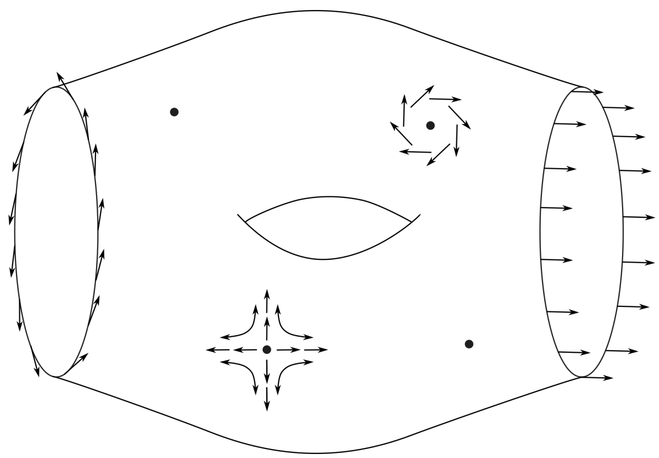


Figure 1.2: A vector field on ∂F extended to F.

Example 1.26. Let $m = k = 2$, so $\xi \to F$ is a 2-plane bundle over a surface F. Suppose also that F is connected and $\partial F \neq \emptyset$. We can choose nonzero vector fields on the boundary as shown in Figure 1.2. The extension will in general have zeros on the interior of F. Then the Euler class is the signed count:

$$e(\xi, v) = \sum_{v(x)=0} \operatorname{sign}_v(x)$$

Changing $e(\xi, v)$ by ± 1 is the same as adding ± 1 twists to $v|_{\partial F}$. So, we can push these zeros to the boundary. For suitable v, therefore, $e(\xi, v) = 0$ and hence $e(\xi) = 0$. Thus every $\xi \to F$ is trivial.

Gompf analogy: F is like a jar full of butterflies whose boundary is the lid; once you release the lid, they all escape and it becomes trivial.

Exercise 1.27. Show that not doing the fix on v above retrieves the Poincaré Hopf theorem as in the boundaryless case. That is, for F compact, connected, show that $\chi(F) = e(TF, v)$ where on each component of ∂F, either v is parallel to ∂F or v is perpendicular to ∂F.

Understanding $[M^3, S^2] \to H_1(M)$

Once again let M be three dimensional manifold, connected and closed. Denote the map $\Gamma : [M^3, S^2] \to H_1(M)$ to be the composition of correspondences shown at the beginning of Section 1.3. We know that this is a surjection, so a natural question is: given $x \in H_1(M)$, what is $\Gamma^{-1}(x)$? In other terms, given two framed links of the same homology class, what is the ambiguity in choosing framed cobordisms between them? Luckily for us, cobordant is the same as homologous in this setting[3], so we just need to focus on the framing of x.

[3]This isn't obvious, but we won't prove it here.

Let η, η' be (nonempty) framed links representing x. Then there is a cobordism $F \subset M \times I$ between η and η'[4]. If we allow for change of framing on one component of ∂F, we can push out any zeros (demonstrated above). This gives us a framing on F. The twists that we had to introduce on one of the components introduces a natural (transitive) $\mathbb{Z}$ action on $\Gamma^{-1}(x)$. It is tempting to say $\Gamma^{-1}(x) \cong \mathbb{Z}$, but this is not quite the case. There is an equivalence relation on $\Gamma^{-1}(x)$ that can be demonstrated by introducing a surface class $\beta \in H_2(M)$. Embed β away from the boundary of $M \times I$, so that it intersects F transversely. As shown in Figure 1.3, we can deform[5] this intersection to get another cobordism F'.

Recall the intersection pairing for M^n closed, oriented:

$$H_k(M^n) \times H_{n-k}(M^n) \to \mathbb{Z}$$

$$[N_1] \cdot [N_2] = I(N_1, N_2)$$

where $I(N_1, N_2)$ denotes the intersection number of $N_1 \cap N_2$. We claim that sewing in β added $e(NF) = ([F]+\beta)\cdot([F]+\beta)$ zeros onto the framing of F (which can be pushed to as many additional twists on a component of η'). This follows from the next two claims.

Claim 1.28. $e(NF')$ detects the number of zeroes of the framing of the normal bundle of F' obtained by extending the framing on the two links

Proof:

Notice that being F' oriented and of codimension 2 in $M \times I$ (which is oriented), a 2-framing is uniquely determined by a non-zero vector field. In fact once we have such vector field v on F' we can define another vector field w on F' by just point-wise rotating counter-clockwise the vector field v. w is well-defined since NF' is orientable (hence we can rotate counter-clockwise in a coherent way) and it's clearly smooth, since v is. Given instead any oriented 2-framing of NF', by applying Gram-Schmidt orthonormalization process (which can be thought as an isotopy) we see that the two vector fields v_1, v_2 are point-wise orthonormal, i.e. one vector field is the other one rotated by $\pi/2$ counter-clockwise (once we choose an orientation, and up to substituting v_1 with $-v_1$).

□

Claim 1.29. Let F' obtained from F and β as described, then $e(NF') = ([F] + \beta) \cdot ([F] + \beta)$

Proof:

Recall that by definition the Euler class (in homology) of a bundle is given by the intersection product of the zero section with a generic section. Take a tubular neighborhood of F' and identify it with the normal bundle. We can use a direction to push a copy of F' transverse to itself and the Euler class will precisely be the homology class defined by the intersection. The latter is $[F'] \cdot [F']$. We conclude by observing that $[F'] = [F] + \beta$.

□

We can now calculate the number of additional zeros to the framing:

$$([F] + \beta) \cdot ([F] + \beta) = [F] \cdot [F] + \beta \cdot \beta + 2[F] \cdot \beta$$

But $[F] \cdot [F] = 0$ because we can just move it off itself disjointly inside $M \times I$. Similarly, $\beta \cdot \beta = 0$. Therefore we have added $2[F] \cdot \beta$ twists to the framing of η'.

Definition 1.30. A class $x \in H_n(X; \mathbb{Z})$/torsion is said to be primitive if $a \neq mb$ for every integer $m > 1$ and $b \in H_n(X; \mathbb{Z})$.

Lemma 1.31. *For any $y \in H_1(M)$ primitive, there exists $\beta \in H_2(M)$ such that $y \cdot \beta = 1$.*

Proof:

[4]The lifting of a cobordism in M to a cobordism in $M \times I$ is justified in Appendix A.1

[5]This deformation can be locally modeled on $\{xy = 0\} \to \{xy = \epsilon\}$ for $x, y \in \mathbb{C}$

Notice that we have:

$$\begin{aligned} y \cdot \beta &= \langle PD(y) \smile PD(\beta), [M] \rangle \\ &= \langle PD(y) \smile \gamma, [M] \rangle \\ &= -\langle \gamma, [M] \frown PD(y) \rangle \\ &= -\langle \gamma, y \rangle \end{aligned}$$

where $\gamma \in H^1(M;\mathbb{Z})$ is the Poincaré dual of β (recall that M is a 3-dimensional manifold). Assume y is not primitive. We will show that that there can't be any surface β such that the intersection product is ± 1. If $y = n \cdot x$ for some $x \in H_1(M)$ and $n \neq \pm 1$, then it's clear by linearity that all the intersection products will be multiple of n, hence not 1. On the other hand, let $y \in H_1(M;\mathbb{Z})$/torsion be primitive. By The Universal Coefficient theorem we know that

$$H_1(M;\mathbb{Z})/\text{torsion} \cong \mathrm{Hom}_{\mathbb{Z}}(H^1(M;\mathbb{Z});\mathbb{Z})$$

Observe now that since M is compact, all the homology groups are finitely generated, in particular $H_1(M;\mathbb{Z})/\text{torsion} \cong \mathbb{Z}^k$, for some $k \in \mathbb{N}$ (the first Betti number). This implies that we can write $y = \sum_{i=1}^{k} \lambda_i x_i$, where x_i's form a basis for $H_1(M;\mathbb{Z})$/torsion. Since y is primitive, it must be that $GCD(\lambda_1, \ldots, \lambda_k) = 1$. This implies that we can find another basis for $\mathbb{Z}$ containing y as one of the generator (use the fact that coprimes coefficients lets you build an unimodular matrix with integer entries). Hence by the Universal Coefficient Theorem we can consider the dual of y, $y^* \in H^1(M)$ and by PD it represents a surface in M intersecting transversely y in a single point.

□

Let $x = dy$ for y primitive and $d \in \mathbb{N}$ (such a d is called the divisibility of x), and take β as in the Lemma above. Then:

$$2x \cdot \beta = 2dy \cdot \beta = 2d$$

Therefore any two framings that differ by $2d$ twists on a boundary component are framed cobordant. Thus $\Gamma^{-1}(x) \cong \mathbb{Z}/2d\mathbb{Z}$. This proves:

Theorem 1.32. *For M a closed, oriented 3-fold, $[M, S^2]$ has a canonical $\mathbb{Z}$ action, and the set of orbits is canonically identified with $H_1(M) \cong H^2(M)$ via Γ. Moreover, for each $x \in H_1(M)$, the orbit $\Gamma^{-1}(x)$ is the necklace $\mathbb{Z}/2d\mathbb{Z}$ where d is the divisibility of x in $H_1(M)/torsion$.*

This is proved with more rigor in Appendix A.1. In particular, there are always torsion elements (perhaps zero) for which $d = 0$, hence:

Corollary 1.33. *Every such M has infinitely many homotopy classes of maps to S^2, and therefore also infinitely many homotopy classes of plane fields that are trivial as bundles.*

Remark 1.34. Unfortunately, we haven't fixed the issue of trivializations. If we want to say anything about plane fields, we'd like to write down information that is independent of τ. See [1] Ch. 11 for more discussion.

1.3.2 4-manifold digression

Definition 1.35. An *almost complex structure* a manifold X is a $\mathbb{C}$-vector space structure on TX. In other words, it is a map $J : TX \to TX$ such that $J \circ J = -\mathrm{id}$ and the following diagram commutes:

$$\begin{tikzcd} TX \arrow[r, "J"] \arrow[d] & TX \arrow[d] \\ X \arrow[r, "\mathrm{id}"] & X \end{tikzcd}$$

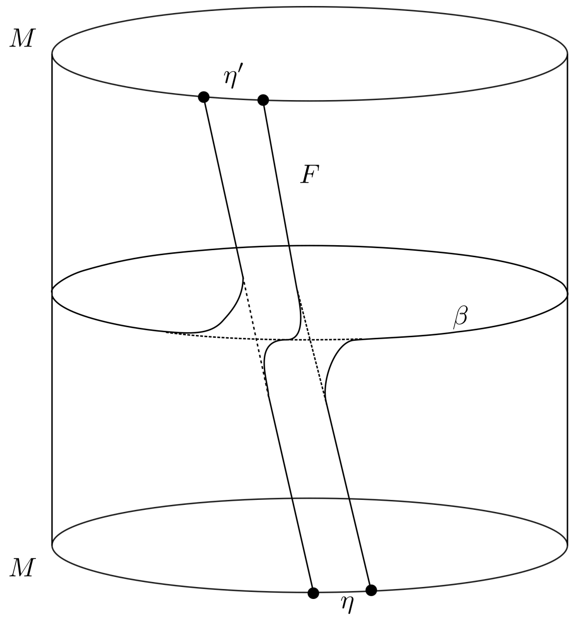


Figure 1.3: Smoothly deforming the transverse intersection of F and β to give a cobordism of η and η'.

Given a codimension 1 submanifold $M \subset X$, then $T_xM \subset T_xX$ is a real codimension 1 subspace for every x. Then $\xi_x := JT_xM \cap T_xM \subset T_xX$ is a complex codimension 1 complex subspace (because it is preserved under applying J). This is the unique maximal complex subspace of T_xM, and it defines a plane field on M.

Example 1.36. $S^{2n-1} \subset \mathbb{C}^n$ inherits a hyperplane field. This is the standard contact structure on S^{2n-1}.

For a closed, almost complex 4 manifold X, there is a relation on three of its invariants:

$$c_1(X)^2 - 2\chi(X) - 3\sigma(X) = 0$$

where c_1 is the first Chern class and $\sigma(X)$ is the signature. Motivated by this, for (X, J) compact with non-empty boundary $\partial X = (M^3, \xi)$, we define

$$\theta(\xi) = c_1(X)^2 - 2\chi(X) - 3\sigma(X)$$

This is a well-defined invariant of the plane field ξ (that is, it is independent of (X, J)). There is a subtlety: what does c_1^2 mean for a manifold with boundary? It turns out it can be defined whenever $e(\xi)$ is a torsion element by using $H_1(M; \mathbb{Q})$.

1.4 Contact Structures ❖

We will now return to the notion of a contact structure and provide a more comprehensive definition. We will start with dimension 3 and define it for arbitrary dimensions after. Recall:

Definition 1.37. A distribution on M is called a *foliation* (or *integrable*) if it is everywhere tangent to submanifolds disjointly filling M.

The Frobeneus theorem characterizes integrable distributions by a formula. Roughly, it says for all vector fields X, Y in ξ, the Lie bracket $[X, Y]$ is in ξ. This is both a necessary and sufficient condition. In the case of hyperplane fields, the Frobeneus theorem is equivalent to:

Theorem 1.38. *ξ is integrable if and only if locally there exists a 1-form α with ker$(\alpha) = \xi$ such that $\alpha \wedge d\alpha = 0$.*

Notice that if α' is another form with ker$(\alpha') = \xi$, then there exists a nowhere zero f such that $\alpha' = f\alpha$ and:

$$\alpha' \wedge d\alpha' = f\alpha \wedge d(f\alpha) = f^2\alpha \wedge d\alpha$$

Therefore a square of f pulls out whenever we transform α by α. Therefore the condition $\alpha \wedge d\alpha = 0$ is independent of α. Further, a co-orientable hyperplane field is integrable if and only if there exists a *global* α such that $\xi = \text{ker}(\alpha), \alpha \wedge d\alpha = 0$.[6]

Example 1.39. Suppose $\alpha = dg$ for some $g : M \to \mathbb{R}$. Then clearly $d\alpha = 0$, so the associated foliation should be integrable. The surfaces are precisely the level sets of g. If $\alpha = fdg$ for some positive function f, then we still get $\alpha \wedge d\alpha = 0$.

Proposition 1.40. *Given $\alpha \in \Omega^1(M)$ nonzero and $\eta \in \Omega^p(M)$, then $\alpha \wedge \eta = 0 \iff \eta|_\xi = 0$, where $\xi = ker(\alpha)$.*

Proof:

> Choose local coordinates at x so that $\xi_x = 0 \times \mathbb{R}^{n-1} \subset \mathbb{R}^n$. Then we write $\eta = \sum \eta_I dx^I$. Notice that $\alpha_x = dx_1$ by construction. Then $\alpha \wedge \eta = 0$ if and only if all nonzero η_I involve dx_1. In other words, $\eta = dx_1 \wedge \zeta$ for some ζ at x.

□

Corollary 1.41. *A hyperplane distribution ξ is integrable if and only if $\xi = ker(\alpha)$ locally for some α such that $d\alpha|_\xi = 0$.*

Definition 1.42. A plane field on a manifold M^3 is a *contact structure* if and only if locally it is ker(α) and $\alpha \wedge d\alpha \neq 0$ everywhere. Equivalently, $d\alpha|_\xi$ is never 0. In the co-orientable case, we can assume that α is global.

The first observation about this definition is that it is an open condition. This is what buys us stability, as mentioned at the start. Additionally, $\alpha \wedge d\alpha$ is a volume form (but it isn't canonical, since we can always re-scale by f). However, such re-scaling only changes by a positive function coefficient (f^2), so at least there is a canonical orientation on M induced by ξ. This is true even when ξ is not co-orientable.

Definition 1.43. A diffeomorphism $\phi : (M, \xi) \to (M', \xi')$ is called a *contactomorphism* if $d\phi_x(\xi_x) = \xi'_{\phi(x)}$ at every point. Equivalently, $\phi^*(\alpha) = \alpha'$ for α, α' local 1-forms cutting out ξ and ξ'.

Definition 1.44. If M is already oriented, we call ξ a *positive* contact structure if the induced contact orientation is the same as the orientation of M. Then we write $\alpha \wedge d\alpha > 0$. Otherwise, it is called a *negative* contact structure and we write $\alpha \wedge d\alpha < 0$.

Definition 1.45. (from Monograph: Eliashberg-Thurston) A plane field ξ on an oriented 3 manifold M is a *positive confoliation* if it is locally given as ker(α) where $\alpha \wedge d\alpha \geq 0$.

1.4.1 Dimension greater than 3

To generalize the definition of contact structure to arbitrary odd dimensions, we replace "never 0" with "non-degenerate." Recall that a bilinear form ω on a vector space V is non-degenerate if, for all $v \in V$ nonzero, there exists $w \in V$ such that $\omega(v, w) \neq 0$. It is skew-symmetric if $\omega(v, w) = -\omega(w, v)$.

Proposition 1.46. *A skew symmetric bilinear form is non-degenerate if and only if $\omega^{\wedge n} \neq 0$.*

Definition 1.47. A hyperplane field on a $2n + 1$ dimensional manifold M is a *contact structure* if $\xi = $ ker(α) locally and $d\alpha|_\xi$ is non-degenerate everywhere (i.e. $\alpha \wedge d\alpha \wedge ... \wedge d\alpha \neq 0$).

In this case, the parity of the nonzero wedge form in the above definition tells us that ξ orients M if n is odd and orients ξ if n is even. What happens when M is even dimensional? We can't get full non-degeneracy, but we can get a 1-dimensional null-space of ξ. There are called even contact structures, and they're not very interesting (for the current discussion, at least). However, even dimensional manifolds can be equipped with a symplectic structure. This is a closed non-degenerate 2-form $\omega \in \Omega^2(M)$.

[6]This can be done by stitching together local forms using partitions of unity and scaling at their intersections.

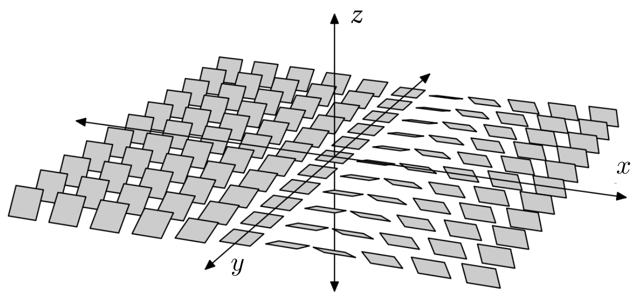


Figure 1.4: The standard contact structure on $\mathbb{R}^3$. (Source: Wikipedia)

Example 1.48. Let $M = \mathbb{R}^{2n+1}$ with coordinates $(x_1, y_1, ..., x_n, y_n, z)$. Take then take:

$$\alpha = dz + \sum_i x_i dy_i$$

This is known as the standard contact structure on $\mathbb{R}^{2n+1}$. We notice $d\alpha = \sum dx_i \wedge dy_i$. Then $\xi = \ker(\alpha)$ is a contact structure. Since $\alpha(\partial_z) = 1$, we find that ∂_z is not in ξ, so this is a never vertical plane field. For $n = 3$, the contact form is $\alpha = dz + xdy$. The plane field is sketched in Figure 1.4.

1.4.2 Contact curves

Definition 1.49. A curve in a contact manifold (M^3, ξ) is *Legendrian* if it is everywhere tangent to ξ. It is *transverse* if it is everywhere transverse to ξ.

In the Legendrian case for $\mathbb{R}^3$, the tangent vectors must satisfy $dz + xdy = 0$. In other words, $\frac{dz}{dy} = -x$. Therefore we can recover the curve from its *front projection* on to the y-z plane. Since these planes are never vertical, closed Legendrian curves must contain cusps (see Figure 1.5). It turns out every link in $\mathbb{R}^3$ is isotopic to a Legendrian knot. Given a closed curve C bounding a region R in the x-y plane, it lifts to a closed Legendrian curve precisely when:

$$0 = -\int_C xdy = \iint_R dx \wedge dy = \text{signed area of } R$$

Remark 1.50. Legendrian curves can be seen as integral submanifolds of maximal dimension. Since we can't integrate the plane field entirely, the best we can do is integrate it along a curve. This serves as a good definition of Legendrian submanifolds in higher dimensions: in the presence of a contact structure, they are integral submanifolds of maximal dimension.

Definition 1.51. We say two Legendrian links are Legendrian (resp. transverse) *isotopic* if they are isotopic through Legendrian (resp. transverse) links.

Legendrian links are completely determined by their front directions modulo Legendrian Reidemeister moves in the plane. These are exactly analogous to the Reidemeister moves in classical knot theory (see Figure 1.6).

Example 1.52. Let $M = \mathbb{R}^3$ and $\xi' = \ker(\alpha')$, where $\alpha' = dz + \frac{1}{2}r^2 d\theta$. In rectangular coordinates, this is $dz + \frac{xdy - ydx}{2}$. This is a positive contact structure because $d\alpha' = dx \wedge dy$. This produces a cylindrically symmetric contact structure.

Exercise 1.53. Let $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ be $\varphi(x, y, z) = (x, y, z + xy/2)$. Show that $\varphi^* \alpha' = \alpha$, where α' is from the example above and α is the standard contact structure. This is an example of a contactomorphism.

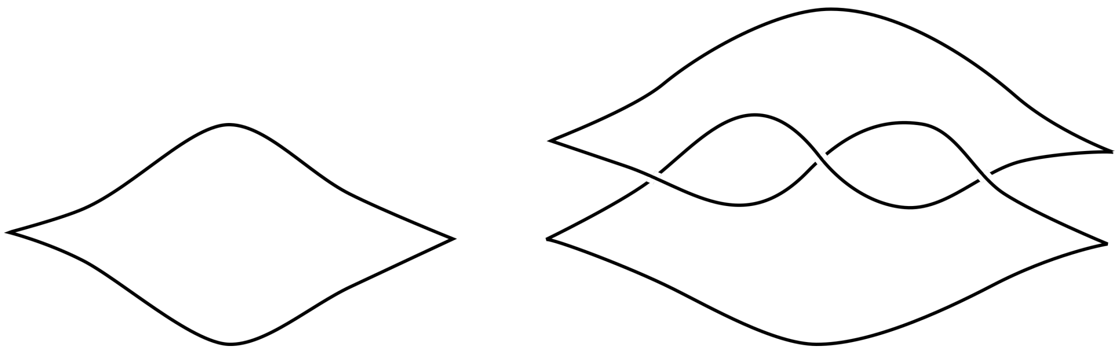


Figure 1.5: Legendrian knots in the y-z plane.

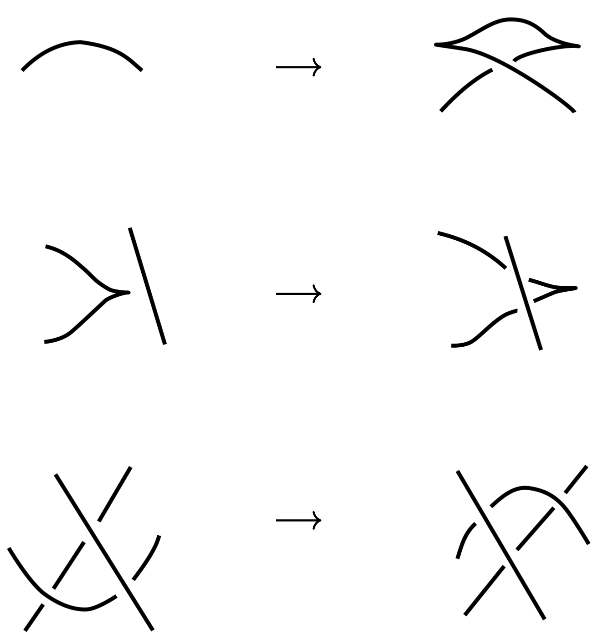


Figure 1.6: Legendrian Reidemeister moves.

2. Gray's Theorem

❖

Our next step in understanding contact structures is Gray's Theorem, which characterizes isotopies of contact structures. It is a critical tool for understanding submanifolds of contact manifolds. In this section, we will show that Legendrian and transverse knots each respectively have local models that are all the same. While this is not true for surfaces, we can use similar techniques to glean properties about the characteristic foliation of the surface, which is a canonical foliation induced by the ambient contact structure.

❖

We'll start with a differential geometry refresher on Lie derivatives. Given a vector field v on M^n, locally we get a flow ϕ_t characterized by $\frac{d\phi_t(p)}{dt} = v(\phi_t(p))$. Given $\alpha \in \Omega^p(M)$, recall that the Lie derivative of α with respect to v by:

$$\mathcal{L}_v(\alpha) := \frac{d}{dt}(\phi_t^*\alpha)\Big|_{t=0}$$

The Lie derivative generalizes the notion of directional derivative. The derivative at any other time is related to the Lie derivative by:

$$\frac{d}{dt}(\phi_t^*\alpha) = \phi_t^*\mathcal{L}_v(\alpha)$$

Definition 2.1. Given a vector field v, then the *contraction* with v is a map $\iota_v : \Omega^p(M) \to \Omega^{p-1}(M)$ by $\alpha(v, w_1, ..., w_{p-1}) = \iota_v\alpha(w_1, ..., w_{p-1})$.

Proposition 2.2. $\iota_v(\alpha \wedge \beta) = \iota_v\alpha \wedge \beta + (-1)^{\dim\alpha}\alpha \wedge \iota_v\beta$

Theorem 2.3. *For a vector field v on M, the Lie derivative is given by:*

$$\mathcal{L}_v = d \circ \iota_v + \iota_v \circ d$$

This is a version of Cartan's formula. We can prove it using the following lemma:

Lemma 2.4. *Suppose that L_1, L_2 are local linear operators on p-forms for all p such that:*

1. $L_i \circ d = d \circ L_i$.
2. $L_i(\alpha \wedge \beta) = L_i\alpha \wedge \beta + \alpha \wedge L_i\beta$.
3. $L_1 = L_2$ *on* $\Omega^0(M)$.

Then $L_1 = L_2$.

Proof:

Let $\alpha \in \Omega^p(M)$ and write $\alpha = \sum \alpha_I dx_I$ locally. We note that $L_1(dx_i) = d(L_1(x_i)) = d(L_2(x_i)) = L_2(dx_i)$, and therefore $L_1(dx_I) = L_2(dx_I)$ for any multi-index I. Then by properties 2 and 3:

$$\begin{aligned} L_1(\alpha) &= \sum L_1(\alpha_I)dx_I + \sum \alpha_I L_1(dx_I) \\ &= \sum L_2(\alpha_I)dx_I + \sum \alpha_I L_2(dx_I) \\ &= L_2(\alpha) \end{aligned}$$

□

Proof (of Theorem 2.3):

Apply Lemma 2.4 with $L_1 = \mathcal{L}_v$ and $L_2 = d \circ \iota_v + \iota_v \circ d$. We have to check each condition:

1. Since $d\phi^*\alpha = \phi^* d\alpha$, we have $\mathcal{L}_v(d\alpha) = d\mathcal{L}_v(\alpha)$. Additionally:
$$\begin{aligned} dL_2 &= d^2\iota_v + d\iota_v d = d\iota_v d \\ &= d\iota_v d + \iota_v d^2 \\ &= (d\iota_v + \iota_v d)d = L_2 d \end{aligned}$$
2. Exercise.
3. Let f be a 0 form. Then $L_2(f) = d(\iota_v f) + \iota_v df = df(v) = \mathcal{L}_v f = L_1(f)$.

□

Given a one parameter family of forms, $\alpha_t \in \Omega^p(M)$, $(0 \leq t \leq 1)$ we have:

$$\frac{d}{dt}(\phi_t^*\alpha_t) = \phi_t^* \frac{d\alpha_t}{dt} + \phi_t^*\mathcal{L}_v\alpha \tag{2.0.1}$$
$$= \phi_t^*\left(\frac{d\alpha_t}{dt} + d(\iota_v\alpha_t) + \iota_v d\alpha_t\right) \tag{2.0.2}$$

We claim that this also works is ϕ_t is an isotopy, where v is the velocity field of the isotopy. This is the case when ϕ_t comes from a time dependent vector field. This can been seen by defining a vector field V on $M \times \mathbb{R}$ by $V(x,t) = (v_t(x), 1)$. This is then a time independent vector field on $M \times \mathbb{R}$, which we can integrate to get a flow. Restricting to the interval, we get an isotopy $M \times I \to M$.

Proposition 2.5. *Let V be a finite dimensional vector space and ω a skew symmetric, bilinear form. Then $T : V \to V^*$ defined by $\iota_v\omega$ is an isomorphism if and only if ω is non-degenerate.*

Proof:

For all $v \neq 0$ in V, there exists w such that $T(v)(w) = \omega(v,w) \neq 0$. Therefore T is injective, and is therefore an isomorphism since $\dim(V) = \dim(V^*)$.

□

Theorem 2.6 (Gray's)**.** *Let ξ_t, $0 \leq t \leq 1$ be a (smoothly varying) one parameter family of contact structures on M that is constant outside a subset U with compact closure in the interior of M. Then there exists an isotopy $\phi_t : M \to M$ such that $\phi_0 = id$, $\phi_t = id$ outside U, and $\phi_t^*\xi_t = \xi_0$ for all t.*

Proof:

Assume for now that ξ is co-oriented, so that ξ_t is globally $\ker(\alpha_t)$ for some 1-form family α_t. For each t, $d\alpha_t|_{\xi_t}$ is non-degenerate. Then by the Proposition above, there is a unique vector field v_t on M such that:

1. v_t lies in ξ_t for all t, i.e. $\iota_{v_t}(\alpha_t) = 0$.
2. $\iota_{v_t}(d\alpha_t|_{\xi_t}) = -\frac{d\alpha_t}{dt}|_{\xi_t}$.

Note that v_t doesn't change if we re-scale α_t. Note that, outside U, $\xi_t = \xi_0$ for all t, so $\alpha_t|_{\xi_0} = \alpha_t|_{\xi_t} = 0$. Then:
$$\left.\frac{d\alpha_t}{dt}\right|_{\xi_t} = \left.\frac{d\alpha_t}{dt}\right|_{\xi_0} = \frac{d}{d}(\alpha_t|_{\xi_0}) = 0$$
Therefore, by property 2 of v_t, $v_t = 0$ for all t outside U. This means we can integrate v_t to a global ϕ_t.

Now all we must show is that $\phi_t^*\xi_t = \xi_0$. To show this, we differentiate $\phi_t^*\alpha_t$ using (2.0.2):

$$\begin{aligned}\frac{d}{dt}(\phi_t^*\alpha_t) &= \phi_t^*\left(\frac{d\alpha_t}{dt} + d(\iota_v\alpha_t) + \iota_v d\alpha_t\right)\\ &= 0 \text{ on } \xi_t.\end{aligned}$$

where we used properties 1 and 2 above. Thus we have:

$$\frac{d}{dt}\left(\phi_t^*\alpha_t\right)\Big|_{\phi_t^*\xi_t} = 0$$

Note that $\phi_t^*\xi_t = \ker(\phi_t^*\alpha_t)$, so there must exist some $f_t : M \to \mathbb{R}$ such that $\frac{d}{dt}\phi_t^*\alpha_t = f_t\phi_t^*\alpha_t$. For each $x \in M$, this corresponds to a curve in T_x^*M whose velocity is radial (i.e. projecting to the sphere is gives a constant curve). Therefore $\phi_t^*\alpha_t$ is constant up to scaling. This implies then that $\phi_t^*\xi_t$ is constant.

□

Scholium 2.7 (Darboux's Theorem). Every contact manifold is locally standard; that is, around every point there are coordinates such that the the contact structure is contactomorphic to the standard contact structure ξ_{std}.

Proof:

Fix a point $p \in M$. There exists a a linear isomorphism $\mathbb{R}^{2n+1} \to T_pM$ that sends ξ_{std} to $\mathbb{R}^{2n} \times \{0\}$ to ξ_p (and all orientations agree). Extend this to a local chart around p. Let $\alpha_t = (1-t)\alpha_{std} + t\alpha$, where $\ker(\alpha) = \xi$ and $0 \leq t \leq 1$. We notice $\alpha_t \neq 0$ at 0, hence near 0. Similarly, $d\alpha_t = (1-t)d\alpha_{std} + td\alpha$ is a symplectic form in a local chart, which implies that α_t is a contact form for all t on a small neighborhood. Since ξ_t is constant at 0, $v_t(0) = 0$ for all t (where v_t comes from the proof of Gray's Theorem). Therefore ϕ_t is defined on some neighborhood of 0 for all t. Then we compose this chart with ϕ_1 to get a contactomorphism of ξ and ξ_{std}.

□

This is analogous to Darboux's theorem for symplectic topology. As an aside, a theorem of Cartan gives a complete list of open conditions on distributions guaranteeing a fixed local model:

1. Contact structures on M^{2n+1}.
2. Even contact structures on M^{2n} (these are classified up to homotopy through such structures by homotopy theory).
3. Line fields (foliations). These are important in dynamical systems.
4. Engel structures: maximally non integrable 2-plane fields on 4-manifolds (not much known about these).

The following is a generalization of Darboux's theorem in the case of a dimension 3 manifold.

Theorem 2.8. *Suppose M^3 is oriented and N contained in the interior of M is compact connected subset. Let ξ_0 and ξ_1 be positive contact structures on M such that $\xi_0|_N = \xi_1|_N$. Then there exists a neighborhood U of N such that id_U is isotopic rel N to a diffeomorphism that is a contactomorphism near N.*

Proof:

Assume ξ_0 and ξ_1 are co-oriented (the non co-orientable case is similar). Choose contact forms α_0 and α_1 inducing the same co-orientation. Let $\alpha_t = (1-t)\alpha_0 + t\alpha_1$. Near N, each α_t is contact (since $d\alpha_0, d\alpha_1$ are positive area forms on $\xi_0|_N = \xi_1|_N$). Now we apply Gray's method as in the proof of Scholium 2.7.

□

2.1 Local models for Legendrian and transverse knots ❖

Suppose that $K_0 \subset M_0$ and $K_1 \subset M_1$ are transverse knots. Then $\xi_i|_{K_i}$ is co-orientable, since an orientation on each knot induces a co-orientation on ξ_i restricted to the knot. There exists a diffeomorphism between tubular neighborhoods of K_i preserving $\xi_i|_{K_i}$, by a version of the Tubular Neighborhood Theorem. Now we can apply Theorem 2.8: every diffeomorphism of tubular neighborhoods is isotopic to a contactomorphism. In other words, all transverse knots have the same local model. The following is one such model that we can easily construct: if we equip $\mathbb{R}^3$ with the contact structure ker$(dz + r^2/2d\theta)$ and mod out by unit z-translations, we get a transverse knot $S^1 \times \{0\} \subset S^1 \times \mathbb{R}^2$ (given by the z-axis).

What happens for Legendrian knots? Let K_1, K_2 be Legendrian, and assume $\xi|_{K_i}$ is co-orientable. Then we get a *contact framing* on K_i (given by taking a normal vector to K_i contained in ξ_i at every point). Then once again, we get contactomorphic tubular neighborhoods as before, but the contact framings must agree up to isotopy. A standard model for this can be constructed in $\mathbb{R}^3$ with $\xi = $ ker$(dz + xy)$. The y-axis is Legendrian, so we mod out by unit y-translation to get a knot in $S^1 \times \mathbb{R}^2$. In the non co-orientable case, instead of a framing we get a contact line field (perhaps not orientable). The standard local model is the same except the $\mathbb{Z}$ action now also flips the other two coordinates: $(x, y, z) \mapsto (-x, y + 1, -z)$.

Theorem 2.9. *Let (M^3, ξ) be a three manifold with a contact structure. Then:*

a) Every transverse knot in (M^3, ξ) has a neighborhood contactomorphic to a neighborhood of the z axis in $(\mathbb{R}^3, dz + r^2/2d\theta)$ mod unit z translations.

b) Every Legendrian knot K with $\xi|_K$ orientable has a neighborhood contactomorphic to a neighborhood of the y axis in $(\mathbb{R}^3, dz + xdy)$ mod unit y translations.

Recall that from classical knot theory, there are two equivalence relations: isotopy (homotopy through embeddings) and ambient isotopy (homotopy through diffeomorphisms of the ambient space). For a compact space, these are the same equivalence relation. This is is a version of the Isotopy Extension Theorem:

Theorem 2.10. *Given $f_t : N \to M$ an isotopy (of embeddings, say) where N is compact, there exists an ambient isotopy $\Phi_t : M \to M$ with compact support in the interior of M such that $\Phi_0 = id_M$ and $\Phi_t \circ f_0 = f_t$.*

Remark 2.11. This only works for the smooth category; it is not true in general topological categories.

Proof:

Define $F : I \times N \to I \times M$ by $(t, n) \mapsto (t, f_t(n))$. The tangent vectors give us a vector field v on $F(I \times N)$ such that $\pi_1(v) = 1$, where π_1 is projection onto the time coordinate. Then we extend this to a compactly supported vector field on $I \times M$ such that $\pi_1(v) = 1$ everywhere. Now there is an induced flow $\Phi : I \times M \to I \times M$. Post composing with the projection on to M gives us the desired ambient isotopy.

□

We would like a similar theorem for the contact category. It is in fact a corollary of Gray's theorem:

Corollary 2.12. *Given $\psi_t : M^3 \to M^3$ an ambient isotopy with $\psi_0 = id_M$ and a fixed contact structure ξ on M. Suppose also that ψ_t preserves ξ outside a subset U with compact closure in the interior of M. Then there exists an ambient isotopy $\psi'_t : M \to M$ through contactomorphisms with $\psi'_0 = id_M$ agreeing with ψ_t outside U.*

Proof:

Let $\xi_t = \psi_t^* \xi$. Applying Gray's theorem to ξ_t, we get maps ϕ_t such that $\phi_t^* \xi_t = \xi_0 = \xi$. By definition of ξ_t, we have:

$$\xi = \phi_t^* \psi_t^* \xi = (\psi_t \circ \phi_t)^* \xi$$

Now we let $\psi'_t = \psi_t \circ \phi_t$.

□

Definition 2.13. A *contact isotopy* is an ambient isotopy through contactomorphisms.

Corollary 2.14. *Every transverse or Legendrian isotopy of links is realized by an ambient contact isotopy.*

2.2 Surfaces and Characteristic Foliations ❖

The case with surfaces is not as easy as with knots. Whereas we were able to show that every Legendrian or transverse knot had the same local model in a contact 3-manifold, such is not generally the case with surfaces in a contact 3-manifold. However, there is a naturally induced foliation on such surfaces, which is known as a characteristic foliation.

Let $F \subset (M^3, \xi)$ be a surface, where ξ is a plane field and everything is oriented. For all $x \in F$, the intersection $\xi_x \cap T_x F$ has dimension 1 or 2. The points where this intersection is transverse ("regular points") gives a singular line field $\mathcal{F}$ on F. In the case where they are the same ("singular points"), we get a singular point field. The singular and regular points can be naturally oriented in the following way. A singular point is positive if the orientations of ξ_x and $T_x F$ agree, and negative if they are opposite. At a regular point x, orient $\mathcal{F}$ so that (v, w) is a positive basis for $T_x F$ if v positive for $\mathcal{F}_x$ and w is positively co-oriented for ξ_x.

Example 2.15. If ξ is a foliation and F is one of the surfaces of the foliation, then the line field $\mathcal{F}$ is singular everywhere, and its orientation is either positive or negative everywhere depending on ξ.

We would like a vector field that cuts out $\mathcal{F}$. Choose $\alpha \in \Omega^1(M)$ such that $\ker(\alpha) = \xi$ and $\alpha(v) > 0$ for positive v. Additionally, let $\omega \in \Omega^2(F)$ be a positive area form. Define v on F such that $\iota_v(\omega) = \alpha|_F$. We note that $v_x = 0$ if and only if $\ker(\alpha)_x = \xi_x$, which means x is singular. Elsewhere, $\alpha(v_x) = \omega(v_x, v_x) = 0$ for any v_x, which means v_x spans $\mathcal{F}_x$. Moreover, by construction of $\alpha(v) > 0$, the direction of v agrees with the orientation of $\mathcal{F}$.

Corollary 2.16. *The field $\mathcal{F}$ determines a singular foliation by Legendrian curves on F given by integrating v.*

This foliation is known as a *characteristic foliation*.

Example 2.17. Let $M = \mathbb{R}^3$ and F be the x-y plane each with the usual orientation. Consider the contact structure induced by $\alpha = dz + r^2/2d\theta$. The singular point is at the origin, and it is positive. The non-singular points are radial lines pointing outward (see Figure 2.1).

Example 2.18. Consider again $M = \mathbb{R}^3$ and F the x-y plane. Now let $\alpha = dz + xdy$. The singular points are the y axis and they are positive. The non-singular curves are lines parallel to the x axis intersecting the y axis at a perpendicular angle. The lines again point away from the singular points (see Figure 2.1).

Theorem 2.19. *Let ξ be a positive contact structure on M^3 and let $F \subset M$ be a surface with induced characteristic foliation $\mathcal{F}$. Then:*

1. *The positive singular points of $\mathcal{F}$ are sources, and the negative singular points of $\mathcal{F}$ are sinks.*
2. *The singular points are nowhere dense.*

To prove this, we will have to develop well-defined notion of divergence near singular points of $\mathcal{F}$.

Definition 2.20. Given a vector field v on M^n with a volume form ω (that is positive), define the divergence $\mathrm{div}_\omega(v) : M \to \mathbb{R}$ by $\mathcal{L}_v \omega = (\mathrm{div}_\omega\, v)\omega$. This is well defined because the space of volume forms is one dimensional.

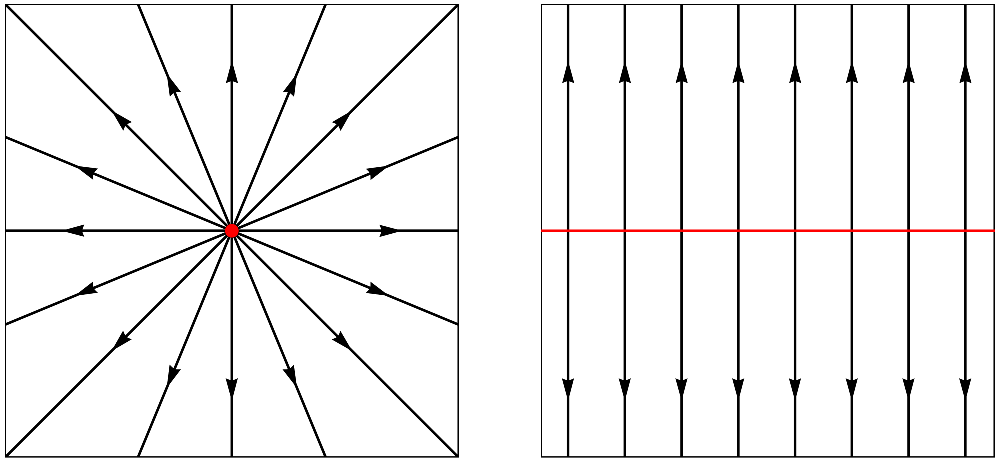


Figure 2.1: Characteristic foliations of the x-y plane under two different contact structures of $\mathbb{R}^3$. Singular points shown in red.

Given positively oriented local coordinates, $\omega = f dx_1 \wedge ... \wedge dx_n$ with $f > 0$. Then by Cartan's formula:

$$\begin{aligned}
\mathcal{L}_v\omega &= d\iota_v\omega + \iota_v d\omega \\
&= d\left(\sum_{i=1}^{n}(-1)^{i-1} f v_i dx_1 \wedge ... \wedge \widehat{dx_i} \wedge ... \wedge dx_n\right) \\
&= \sum_{i=1}^{n}\frac{\partial(f v_i)}{\partial x_i} dx_1 \wedge ... \wedge dx_n \\
&= \underbrace{\frac{1}{f}\sum_{i=1}^{n}\frac{\partial(f v_i)}{\partial x_i}}_{\operatorname{div}_\omega v}\omega
\end{aligned}$$

Then we can compute the divergence using the product rule:

$$\begin{aligned}
\operatorname{div}_\omega v &= \sum_{i=1}^{n}\frac{\partial v_i}{\partial x_i} + \frac{1}{f}\sum_{i=1}^{n}\frac{\partial f}{\partial x_i} v_i \\
&= \nabla\cdot v + d(\ln(f)) \cdot v
\end{aligned}$$

Note that for the standard area in $\mathbb{R}^n$, the function f is constant, so $\operatorname{div}_\omega v = \nabla\cdot v$. When $v(x) = 0$, we again get $\operatorname{div}_\omega v = \nabla\cdot v$. The left hand side of this inequality didn't depend on local coordinates, and the right hand side doesn't depend on the choice of volume form. Therefore neither should depend on either. Moreover, at zeros of v, the sign of $\operatorname{div}_\omega v$ doesn't change under positive re-scaling of v because:

$$\nabla\cdot(gv) = \nabla g \cdot v + g(\nabla\cdot v) = g(\nabla\cdot v)$$

Lemma 2.21. *Let $\mathcal{F}$ be an oriented singular line field determined by v on M (the zeros are of v are the singular points of $\mathcal{F}$). Suppose $v(x) = 0$ and $\operatorname{div}_\omega v \neq 0$ at x. If w also determines $\mathcal{F}$ near x, then $w = gv$ for some smooth function $g : M \to \mathbb{R}$.*

Proof:

Away from singular points, we get a unique $g > 0$ which is the scale factor between w and v. The issue at a zero is that v may vanish faster than w, allowing g to become infinite. Fix local coordinates with $x \leftrightarrow 0$. There exists i such that $\frac{\partial v_i}{\partial x_i} \neq 0$, which means there exists a neighborhood U of x such that $v_i : U \to \mathbb{R}$ is a local submersion at 0. By the Local Submersion Theorem, there exists new coordinates y_j such that $v_i = y_1$. Note that the locus where $v_i = 0$ is the same as where $w_i = 0$ near x. Therefore $w_i(y_1, ..., y_n) = y_1 h(y_1, ..., y_n)$ for some smooth h. Then let $g = \frac{w_i}{v_i} = h$.

□

Definition 2.22. Under the previous hypothesis, define $\operatorname{sign}_x(\operatorname{div} \mathcal{F}) := \operatorname{sign}(\operatorname{div}_\omega v)$. If no such v exists, we say $\operatorname{sign}_x(\operatorname{div} \mathcal{F}) = 0$.

We claim this is well-defined (independent of ω and coordinates) because it equals the sign of $\operatorname{div}_\omega w$ for any $w = gv$ and not vanishing to second order at x.

Theorem 2.23. *Given $F \subset M^3$ be a surface with a plane field ξ on F, everything oriented. Let $\operatorname{sign}_x \xi = \operatorname{sign}_x(\alpha \wedge d\alpha) = \operatorname{sign}(d\alpha|_{\xi_x})$, where ker$(\alpha) = \xi$. Suppose x is a singular point of $\mathcal{F}$ and $\alpha|_F$ doesn't vanish to second order. Then:*

$$\operatorname{sign}_x(\operatorname{div} \mathcal{F}) = (\operatorname{sign}_x \mathcal{F})(\operatorname{sign}_x \xi)$$

Proof:

Let $\iota_v \omega = \alpha|_F$ for any positive choice of α, ω such that $\ker(\alpha) = \xi$. Since v doesn't vanish to second order, we can use it to compute the sign of the divergence of $\mathcal{F}$:

$$(\operatorname{div}_\omega v)\omega = \mathcal{L}_v \omega = d\iota_v \omega + 0 = d\alpha|_F$$

Then:

$$\begin{aligned}\operatorname{sign}_x(\operatorname{div} \mathcal{F}) = \operatorname{sign}(\operatorname{div}_\omega v) &= \operatorname{sign}(d\alpha|_F) \\ &= (\operatorname{sign}(d\alpha|_{\xi_x}))(\operatorname{sign}_x \mathcal{F})\end{aligned}$$

□

As a corollary, we can prove Theorem 2.19. For a positive contact structure, $\operatorname{sign}(\operatorname{div}_x \mathcal{F}) = \operatorname{sign}_x \mathcal{F}$, which means positive singular points are sources are negative singular points are sinks. Since $\operatorname{sign}_x \mathcal{F}$ is never zero by definition, the sign of the divergence of $\mathcal{F}$ at a singular point is always nonzero. Therefore the singular locus of $\mathcal{F}$ must locally lie in 1 manifolds, and so the second part of Theorem 2.19 also follows.

Exercise 2.24. Let $M = \mathbb{R}^3$ and let $\alpha = dz + adx + bdy$ for $(a, b) \in \mathbb{R}^2 - \{0\}$. Determine $\operatorname{sign}(\xi)$ and $\operatorname{div}_\omega v$ for $\mathcal{F}$ on the x-y plane. Check that the characteristic foliation is given by $x^b y^a = c$. Draw a picture and observe how these vary with a and b. How is $\operatorname{sign}(\operatorname{div}_\omega \mathcal{F})$ visible?

Theorem 2.25 (Giroux et. al.)**.** *Let $F \subset M^3$ be a compact surface with boundary (possibly empty) such that $F \cap \partial M$ is a collection of components of ∂F. Let ξ_0, ξ_1 be positive contact structures inducing the the same characteristic foliation $\mathcal{F}$ on F with ∂F Legendrian, $\xi_0 = \xi_1$ on ∂M near F. Then there exists an isotopy ϕ_t of a neighborhood of F with $\phi_0 = id$ and $\forall t \in [0, 1]$, $\phi_t(F) = F$ preserving each leaf of $\mathcal{F}$. Moreover, $\phi_1^* \xi_1 = \xi_0$ and ϕ_t fixes each $p \in M$ on which $\xi_0 = \xi_1$.*

Proof:

Assume ξ_0, ξ_1 are co-orientable (general case follows by double cover). Choose α_i such that $\ker(\alpha_i) = \xi_i$. Then we claim that there exists $g \neq 0$ such that $\alpha_1|_F = g\alpha_0|_F$. This follows since ξ_0 and ξ_1 intersect F in the same way at regular points of $\mathcal{F}$. At singular points, fix ω on F such that $\iota_{v_i}\omega = \alpha_i|_F$. By last time, $\operatorname{div}_\omega v_i \neq 0$ at x, and so we get $v_i = gv_0$ for $g \neq 0$ (see Lemma 2.21). Therefore we can re-scale by g and wlog $\alpha_0|_F = \alpha_1|_F$.

Claim: Let $\alpha_t = (1 - t)\alpha_0 + t\alpha_1$ for $0 \leq t \leq 1$. Then there exists a neighborhood of F such that α_t is contact for each t.

Proof: Identify a neighborhood of F with $F \times \mathbb{R}$, with z being the $\mathbb{R}$ coordinate. Then α_t becomes a family of forms $\beta_{t,z} + u_{t,z}dz$, where $\beta_{t,z} \in \Omega^1(F)$ and $u_{t,z} \in \Omega^0(F)$. Then:

$$d\alpha_t = d\beta_{t,z} + \left(du_{t,z} - \frac{\partial \beta_{t,z}}{\partial z}\right) dz$$

and:

$$\alpha_t \wedge d\alpha_t = \left(\beta_{t,z} \wedge (du_{t,z} - \frac{\partial \beta_{t,z}}{\partial z}) + u_{t,z} d\beta_{t,z}\right) dz$$

Notice that, on F, $\beta_{t,0}$ and $d\beta_{t,0}$ are independent of t since $\alpha_t|_F$ is the same for all t. Therefore at $z = 0$, the above expression of $\alpha_t \wedge d\alpha_t$ is affine in t (i.e. not quadratic), which implies:

$$\alpha_t \wedge d\alpha_t = (1-t)\alpha_0 \wedge d\alpha_0 + t d\alpha_1 \wedge d\alpha_1$$

Therefore $\alpha_t \wedge d\alpha_t$ is also non-degenerate for all t. We can also extend this to a neighborhood of F by continuity. QED.

Now we apply Gray's method. We construct the desired ϕ_t by integrating v_t such that:

- $\iota_{v_t}\alpha_t = 0$ ($\iff$ $v_t \in \xi_t$).
- $\iota_{v_t} d\alpha_t|_{\xi_t} = -\frac{d\alpha_t}{dt}|_{\xi_t}$.

On F, we also have $\alpha_t|_F$ is the same for all F because $\alpha_0|_F = \alpha_1|_F$. This implies $\frac{d\alpha_t}{dt}|_F = 0$. Moreover, we claim v_t lies in $\mathcal{F}$. To show this we must demonstrate $v_t \in T_x F$. This is trivial for x singular, so assume x is regular. Let u span $\mathcal{F}_x$; using the second condition from above:

$$\iota_{v_t} d\alpha_t(u) = -\frac{d\alpha_t}{dt}(u) = 0$$

$$\Rightarrow d\alpha_t(v_t, u) = 0$$

Non-degeneracy of $d\alpha_t$ says $v_t || u$, so indeed $v_t \in T_x F$. Therefore $\phi_t(F) = F$. The flow restricted to F is well defined for all t since ∂F is Legendrian, so the flow doesn't escape on the boundary. Therefore we can extend this flow to a neighborhood of F and it satisfies the required conditions.

$\square$

3. Convex Surfaces

❖

Up to this point, we have shown that surfaces F inside a contact 3 manifold have a canonical directed foliation whose singular points have a well-defined, nonzero, divergence. We have also shown that given two contact structures on the ambient manifold inducing the same foliation, there is a contact isotopy sending one to the other that moves along the leaves. The next step is to allow a perturbation of F, possibly changing the contact structure, and see what invariants there are. To do this, we will restrict our focus to convex surfaces, which we will see carry extra structure in their foliations.

❖

Definition 3.1. Given a plane field ξ on M^3 and a vector field v on M tangent to ∂M, we say v preserves ξ if the corresponding flow does (locally). In other words, if ϕ_t is the flow of v, then $\phi^*\xi = \xi$. If ξ is contact, then we say v is a *contact vector field* if it preserves ξ through contactomorphisms ϕ_t.

Proposition 3.2. *The set of vector fields on M preserving ξ is a vector subspace of all vector fields on M.*

Proof:

Choose α such that $\xi = \ker(\alpha)$. Then v preserves ξ if and only if $\mathcal{L}_v\alpha = f_v\alpha$ for some $f_v : M \to \mathbb{R}$. Using Cartan's formula:

$$d\iota_v\alpha + \iota_v d\alpha = f_v\alpha$$

This is a linear condition in v. That is, if we replace v by $aw + bv$, the same still holds for the function $af_v + bf_w$.

□

Definition 3.3. Given a three manifold with a plane field (M, ξ) and $F \subset M$ is a co-oriented surface, suppose v is a vector field defined near F tangent to ∂M. We say F is *convex* with respect to v if v preserves ξ and v is positively transverse to F everywhere. If such a v exists, we say F is convex.

A few observations:

a) If F is convex with respect to v and with respect to w, then F is convex with respect to $(1-t)v + tw$ for $t \in [0, 1]$.

b) If F is compact and convex with respect to v, then we can identify a tubular neighborhood of F with $F \times (-\epsilon, \epsilon)$ so that $v \leftrightarrow (0, 1)$ and ξ becomes independent of the normal coordinate. If $\xi = \ker(\alpha)$, after scaling α, we can assume α is independent of the vertical coordinate as well: $\mathcal{L}_v\alpha = 0$.

Example 3.4. A standard picture of the second observation is $M = \mathbb{R}^3$, $\alpha = dz + xdy$, and F the x-y plane. Then F is convex with respect to the vertical vector field $\frac{\partial}{\partial z}$. Another example is the z-x plane, which is convex with respect to $v = \frac{\partial}{\partial y}$.

Proposition 3.5. *For ξ a contact structure on M and F a surface, any $p \in F \cap int(M)$ has a convex neighborhood in F.*

Proof:

Identify a neighborhood of p in M with $(\mathbb{R}^3, dz + xdy)$ sending $p \mapsto 0$. Let $w = (1, 0, -y)$; the corresponding flow is $\phi_t(x, y, z) = (x + t, y, z - ty)$. We leave it as an exercise to show that this is a contact flow (i.e. w was contact vector field).[a] Given any vector $v = (v_1, v_2, v_3)$ at 0, we can extend it to a contact vector field $v_1 w + v_2\frac{\partial}{\partial y} + v_3\frac{\partial}{\partial z}$ (since contact vector fields are a vector space). Then choose v transverse

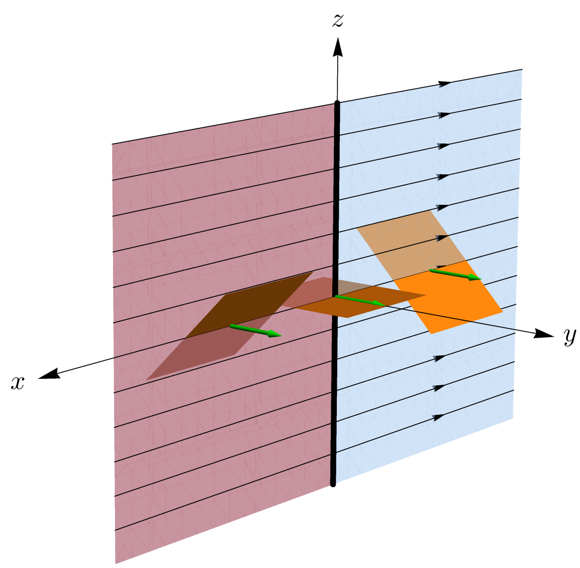


Figure 3.1: The local model for Γ (z axis), R_+ (blue), and R_- (red). The contact structure and vector field $\frac{\partial}{\partial y}$ are shown at three points, one in each region.

to F at 0 and construct such a vector field.

[a]This can be verified either by computing $\phi_t^* \alpha$ or by computing $\mathcal{L}_w \alpha$

□

Let p be a regular point of the characteristic foliation on F induced by a plane field ξ in the interior of M. Then p has local coordinates in F such that $\mathcal{F}$ is horizontal in the z-x plane (this is done by using the t coordinate of the flow as the x axis). Using Proposition 3.5 to choose any v such that F is locally convex near p with respect to v. In our local picture, we can identify v with $\frac{\partial}{\partial y}$ in $\mathbb{R}^3$. Since $\alpha(\partial/\partial z) > 0$, we can re-scale so that $\alpha(\partial/\partial z) = 1$. Since α is invariant under y translations, we can write:

$$\alpha = dz + f(x,z)dy$$

$$\Rightarrow \alpha \wedge d\alpha = df \wedge dy \wedge dz = \frac{\partial f}{\partial z} dx \wedge dy \wedge dz$$

Therefore $\mathrm{sign}(\alpha \wedge d\alpha) = \mathrm{sign}(\partial f/\partial x)$. Therefore ξ is a positive contact structure if and only if $\frac{\partial f}{\partial z} > 0$ ("left twisting"). In contrast, ξ is a foliation if and only if $\frac{\partial f}{\partial x} = 0$.

Now assume ξ is a positive contact structure. Let $\phi(x,y,z) = (f(x,z), y, z)$. Notice that this is an orientation preserving local diffeomorphism of a neighborhood of p to the z-x plane because:

$$\det d\phi = \frac{\partial f}{\partial x} > 0$$

Additionally, $\phi^*(dz + xdy) = dz + f(x,y)dy$. Therefore we can change coordinates so that $\alpha = dz + xdy$. Under all of this, F is still the z-x plane and $\mathcal{F}$ is still parallel to the x axis. This change of coordinates sends $p \mapsto (x_0, 0, 0)$. Without loss of generality, if $x_0 \neq 0$, we can assume $x_0 = \pm 1$ via the contactomorphism $\psi(x,y,z) = (x_0/|x_0|, y, z/|x_0|)$. These correspond to the three different cases of $\alpha(v) > 0, \alpha(v) = 0$, and $\alpha(v) < 0$.

Definition 3.6. Let $F \subset (M, \xi)$ be a surface convex with respect to v, where ξ is a positive contact structure. Moreover, assume that everything is oriented and ∂F is Legendrian. The *dividing set* Γ is $\{p \in F \mid v(p) \in \xi_p\}$. Similarly, let $R_\pm = \{p \in F \mid \alpha(v) >, < 0\}$.

Remark 3.7. Notice that these three sets are disjoint and $F = R_+ \cup \Gamma \cup R_-$. Moreover, $R_\pm$ are open, all of the positive singularities lie in R_+, and all the negative singularities lie in R_-. Also notice that each leaf of $\mathcal{F}$ intersects Γ at most once, and the singular points lie in $R_\pm$ depending on their sign.

The construction above gave us a canonical local model for each of these three regions (shown in Figure 3.1). This model proves:

Proposition 3.8. *Given $F \subset M$ convex with respect to v and ∂F is Legendrian, then Γ is an embedded 1 manifold with $\partial\Gamma = \Gamma \cap \partial F$ and Γ is everywhere transverse to $\mathcal{F}$. Moreover, the leaves of $\mathcal{F}$ are directed from R_+ to R_-.*

Our next job is to show that the topology of the dividing set Γ doesn't depend on the choice of vector field v (i.e. they are all the same up to isotopy).

Theorem 3.9. *Γ is independent of the choice of v up to isotopy preserving each leaf of $\mathcal{F}$.*

Proof:

Given v, w making F convex, we know that F is convex with respect to $v_t = tv + (1-t)w$. This gives us a one parameter family of dividing sets Γ_t. Define:

$$\widehat{\Gamma} = \bigcup_{t\in[0,1]} \{t\} \times \Gamma_t \subset I \times F$$

Observe that $\widehat{\Gamma}$ is the zero set of $\alpha(v_t)$, where α is a 1-form cutting out the contact structure. Locally, this is $(dz + xdy)(\partial/\partial y) = x$. In particular, $\frac{\partial}{\partial x}(\alpha(v_t)) \neq 0$, which implies that 0 is a regular value of $\alpha(v_t)$. Therefore $\widehat{\Gamma}$ is a surface in $I \times F$. Moreover, $\widehat{\Gamma}$ is everywhere transverse to $\{t\} \times F$ and $I \times \{\text{leaf}\}$. This means $F \cap (I \times \{\text{leaf}\})$ is a curve with unique tangent vectors projecting to 1 in I. These tangent vectors can be extended to a vector field on M that projects to 1 on I everywhere. Integrating this vector field gives the desired isotopy.

$\square$

Proposition 3.10. *For F compact, convex, and with Legendrian boundary, the dividing set $\Gamma \neq \emptyset$ for any v.*

Proof:

For any α with $\ker(\alpha) = \xi$, then by Stokes:

$$\int_F d\alpha = \int_{\partial F} \alpha = 0$$

because ∂F is Legendrian. Suppose $\Gamma = \emptyset$, which means either $F = R_+$ or $F = R_-$. Orient F so that $F = R_+$. By convexity, identify a neighborhood of F with $F \times (-\epsilon, \epsilon)$ so that ξ is vertically invariant. By our orientation of F, we have $\alpha(\partial/\partial z) > 0$ (for an orientation preserving choice of local coordinates, of course). re-scale α so that $\alpha = dz + \beta$ for $\beta \in \Omega^1(F)$. Then $d\alpha = d\beta$ and $0 < \alpha \wedge d\alpha = d\beta \wedge dz$, which implies $d\beta$ is a positive area form on F. However:

$$0 = \int_F d\alpha = \int_F d\beta > 0$$

Which is a contradiction.

$\square$

Remark 3.11. A similar argument can show the following facts as well:

a) For each compact surface $\Sigma \subset R_+$, the foliation $\mathcal{F}$ must flow out of $\partial\Sigma$ at some point.

b) For any positive α with $\ker(\alpha) = \xi$, we have $\int_{R_+} d\alpha > 0$ and $\int_{R_-} d\alpha < 0$.

3.1 Giroux's Theorem ❖

Now we consider the converse: given any surface F and any singular foliation $\mathcal{F}$ of that surface, does there exist a contact structure that cuts out $\mathcal{F}$? This is answered (with some assumptions) by Giroux's theorem.

Definition 3.12. Let F be any compact, oriented surface and $\mathcal{F}$ be an oriented singular foliation of F. Let Γ be a compact, co-oriented 1-manifold with $\partial\Gamma = \Gamma \cap \partial F$ transversely. We say Γ *divides* $\mathcal{F}$ if there exists a positive contact structure on $F \times \mathbb{R}$ with characteristic foliation $\mathcal{F}$ and dividing set Γ (with respect to $\partial/\partial z$, where z is the $\mathbb{R}$ coordinate), co-oriented by $\mathcal{F}$.

Before we state the theorem, we will establish some terminology. For any closed leaf of the foliation $\mathcal{F}$, we can define a monodromy associated to the local behavior of the foliation in a neighborhood of the leaf. Take a normal direction to the leaf and identify it with the interval I. Define $\phi : I \to I$ by following the foliation until it intersects the normal again. We say that a closed leaf is *attractive* if $\phi'(0) < 1$ and *repellant* if $\phi'(0) > 1$. Moreover, we define:

$$K_+ = \{+\text{ singular points}\} \cup \{\text{repellant closed leaves}\}$$
$$K_- = \{-\text{ singular points}\} \cup \{\text{attractive closed leaves}\}$$

Theorem 3.13 (Giroux et. al.)**.** *Let F be a compact oriented surface with an oriented singular foliation $\mathcal{F}$ such that ∂F is a union of leaves and critical points. Suppose also that:*

1. *Each leaf limits as $t \to \pm\infty$ to a singular point or closed leaf (t is a parameter on the leaf).*
2. *Each singular point has nonzero divergence and each closed leaf is either repellant or attractive (i.e. $\phi'(0) \neq 1$).*
3. *No leaf runs from K_- to K_+.*
4. *The collection of singular points of $\mathcal{F}$ has finitely many connected components which can be ordered so that each leaf between points of the same sign preserves that order.*[7]

Then there exists a compact, connected one-manifold Γ dividing $\mathcal{F}$. Moreover, it is unique to isotopy preserving the leaves of $\mathcal{F}$ and is characterized as intersecting precisely with those leaves running going from K_+ to K_-.

Remark 3.14. The second requirement that each singular point has nonzero divergence is using the fact that any foliation is *locally* cut out by a vector field vanishing at the singular point. So, for any such v, the second condition is saying $\operatorname{div}_\omega v \neq 0$, where ω is a positively oriented volume form on F. This is well-defined by our discussion of divergence in the previous section (c.f Definition 2.22).

Proof Idea:

> The idea of this proof is to construct globally a volume form ω and a vector field v to get a uniquely defined 1-form $\alpha := \iota_v\omega$ on F. This will hand us a contact structure on F and hence a dividing set Γ.

□

Exercise 3.15. Show that the hypotheses of Theorem 3.13 hold for generic a generic foliation of F.

Corollary 3.16. *A generic closed oriented surface F in a contact 3-manifold (M, ξ) is convex.*

Proof:

> By generic surface, we really mean surface which gives rise to a generic characteristic foliation. Then Theorem 3.13 applies, assuming Exercise 3.15. Therefore there exists an $\mathbb{R}$-invariant contact structure ξ' on $F \times \mathbb{R}$ such that $\mathcal{F}$ is a characteristic foliation induced by ξ' as well. We apply Theorem 2.25 to get a contactomorphism of $F \times (-\epsilon, \epsilon)$ for some $\epsilon > 0$ and a tubular neighborhood of F. This means that F is convex with respect to the image of the vector field $\frac{\partial}{\partial z}$ on $F \times \mathbb{R}$ under this contactomorphism.

[7]This rules out leaves that begin and end at the same singular point, among other cases.

□

3.2 Generic convexity ❖

With some extra work, we can generalize Corollary 3.16 to generic surfaces with boundary. To do this, we will need to understand some local models for ∂F. From now on in this section, (M^3, ξ) is a positive contact manifold, ξ is oriented, and $F \subset M$ is a compact oriented surface with Legendrian boundary. Assume $F \cap \partial M$ is a collection of components γ of F. On any boundary component of F, there are two framings: the normal framing and the contact framing.

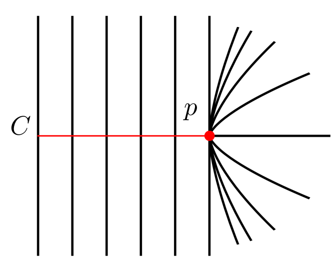
Definition 3.17. For γ a component of ∂F, the *twisting number* of γ relative to F is the number of right hand twists of the contact framing relative to the normal framing on γ. It is denoted $t(\gamma)$.

We can give canonical local models for ∂F depending on the twisting number of a component γ:

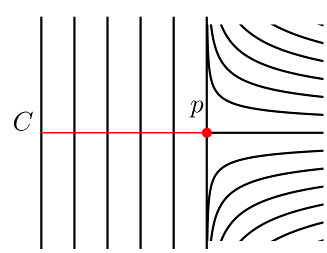
- $(\mathbf{t}(\gamma) < \mathbf{0})$ This model is on $\mathbb{R}^3$ modulo unit x translation with $\xi = \ker(\sin(2\pi n x)dy + \cos(2\pi n x)dz)$. The surface F is the upper z-x half-plane. The number of boundary components of Γ is $2n$. When $\gamma \in \partial M$, we can extend this model by locally modeling ∂M as the x-y plane with transverse vector field $\frac{\partial}{\partial z}$.
- $(\mathbf{t}(\gamma) = \mathbf{0})$ We model this by the y axis in $\mathbb{R}^3$ modulo unit y translation with the contact form $\xi = \ker(dz + xdy)$, where F is the x-y half plane. When $\gamma \subset \partial M$, we can't get ∂M convex if F is convex (because $\frac{\partial}{\partial x}$ is not a contact vector field).
- $(\mathbf{t}(\gamma) > \mathbf{0})$ Finally, when $t(\gamma) > 0$, we claim that there is no convex local model. The conclusion is that, given F as before with standard $\mathcal{F}$ on ∂M near $F \cap \partial M$, we can perturb F relative to its boundary to be convex near ∂F if and only if $t(\gamma) \leq 0$.

Now we perturb F away from ∂F to be generic (Corollary 3.16) so that it is convex and the only singular points are isolated elliptic and hyperbolic points. How do we glue together these perturbations on the interior of F and near ∂F to get convexity? The problem happens when singular curves near ∂F reach into the interior, where there are only isolated singular points.

Definition 3.18. If p is a point on the boundary of a singular curve C, it is *half-elliptic* if the foliation is locally modeled by the diagram on the left and it is *half-hyperbolic* if the foliation is locally modeled by the diagram on the right.



Half-elliptic



Half-hyperbolic

An explicit model for half-elliptic and half-hyperbolic points is given in the exercise below.

Exercise 3.19. Let F be the graph of $f : \mathbb{R}^2 \to \mathbb{R}$ defined by:

$$f(x,y) = \begin{cases} 0 & y \leq 0 \\ bxy & y \geq 0 \end{cases}$$

Where we take the contact structure $dz + xdy$. Show that the singularity at 0 is half-elliptic if $-1 < b < 0$ and half hyperbolic if $b < -1$ or $b > 0$. Show also that ξ twists right relative to F if $b < -1$ and left relative to F if $b > -1$.

In the above exercise, the surface wasn't smooth. However, it can be smoothed out on the x axis to give the same picture of half-elliptic and half-hyperbolic singular points. These are the local models at the edge of the transition between the two regions. We can choose half-elliptic or half-hyperbolic transitions when $t(\gamma) < 0$. For $t(\gamma) > 0$, we can only choose half-hyperbolic transitions which won't be convex. If we assume the twisting is strictly negative, we can glue together the perturbations in a convex manner.

Theorem 3.20. *Given F as above, if ∂M has a standard characteristic foliation near ∂F and all twisting numbers are ≤ 0, we can perturb F relative to its boundary to be convex.*

3.3 The Flexibility Theorem and LeRP ❖

What follows is a series of technical theorems about dividing sets on convex surfaces. They are the tools that allow us to perform multiple procedures on convex surfaces in contact 3-manifolds that will be useful. The first is known as Giroux's flexibility theorem:

Theorem 3.21 (Flexibility Theorem)**.** *Given $F \subset (M, \xi)$ a compact surface, convex with respect to v and ∂F is Legendrian. Let $\mathcal{F}_1$ be the characteristic foliation on F. Suppose that $\mathcal{F}_0$ is another oriented singular foliation with the same co-oriented dividing set Γ. Then there is an isotopy $\phi_t : F \to M$ with ϕ_0 the inclusion and $\phi_1(\mathcal{F}_0)$ induced by ξ. Moreover, ϕ_t fixes Γ, preserves each curve (point) that is a leaf (singular point) of both $\mathcal{F}_i$ (outside a preassigned neighborhood of ∂F), and each $\phi_t(F)$ is convex with respect to v and is contained in a pre-assigned neighborhood of F.*

Proof:

Extend ξ to ξ_1 on $F \times \mathbb{R}$ so that it is $\mathbb{R}$-invariant. By assumption, we also have Γ dividing $\mathcal{F}_0$, so there is ξ_0 on $\mathcal{F} \times \mathbb{R}$ generating $\mathcal{F}_0$. We claim without loss of generality that $\xi_0 = \xi_1$ on $U \times \mathbb{R}$ for some neighborhood U of $\partial F \cup \Gamma$. We do this by isotoping both $\mathcal{F}_1, \mathcal{F}_0$ near Γ to be perpendicularly transverse to Γ. For the boundary, we use a similar argument when $t(\gamma) = 0$ for each component γ. When $t(\gamma) < 0$, since both foliations have the same dividing set Γ, their twisting numbers (with respect to ξ_1 and ξ_1) are the same: $t(\gamma) = -\frac{1}{2}\#\Gamma \cap \gamma$. Then we can arrange the foliations to agree by composing with an isotopy of F, and so by Theorem 2.25 we can make ξ_1 and ξ_0 agree on U via an isotopy.

Now we consider $F \setminus U$, which we can assume to be a compact surface. Let Σ be a component. Since $\Sigma \cap \Gamma = \emptyset$, without loss of generality $\Sigma \subset \mathbb{R}_+$. On $\Sigma \times \mathbb{R}$, we can decompose the one forms α_i determining ξ_i as $\alpha_i = dz + \beta_i$, for $\beta_i \in \Omega^1(F)$. Let $\alpha_t = (1-t)\alpha_0 + t\alpha_1 = dz + \beta_t$, where:

$$\beta_t = (1-t)\beta_0 + t\beta_1$$

Then:

$$\alpha_t \wedge d\alpha_t = (dz + \beta_t) \wedge d\beta_t = (1-t)d\beta_0 \wedge dz + td\beta_1 \wedge dz > 0$$

Now we use Gray's method. We get an isotopy ϕ_t by integrating v_t determined by:

1. $v_t \in \xi_t$.
2. $\iota_{v_t} d\alpha_t|_{\xi_t} = -\frac{d\beta_t}{dt}|_{\xi_t}$.

We see that v_t is $\mathbb{R}$-invariant because all equations above are. On $\partial\Sigma \times \mathbb{R}$, we have $\alpha_0 = \alpha_1$ (since the boundary is contained in U), and therefore $\frac{\partial \beta_t}{dt} = 0 \Rightarrow v_t = 0$. Projecting v_t away from the $\mathbb{R}$ component, we get a well-defined vector field on Σ and so we also get a global isotopy $\overline{\phi}_t$ on Σ. For all $t \in [0,1]$, we have that $\phi_t(F)$ projects diffeomorphically on to F and each $\phi_t(F)$ is convex with respect to $\partial/\partial z$.

All of these isotopies together give us an isotopy ϕ_t such that $\phi_1(\mathcal{F}_0)$ is induced by ξ and each ϕ_t fixes Γ. For any leaf L of both $\mathcal{F}_1, \mathcal{F}_2$, since $\alpha_0|_L = \alpha_1|_L = 0$, we have $\alpha_t|_L = 0 \Rightarrow \frac{d\beta_t}{dt}|_L = 0$. This implies v_t is parallel to L and so it preserves the leaves. All of this works on $F \times \mathbb{R}$; but we would like it to be true for $F \subset M$. To get this, we note that $\exists a > 0$ such that $\phi_t(F) \subset F \times [-a, a]$. It suffices to find a contactomorphism $F \times [-a, a] \to F \times [-\epsilon, \epsilon]$ for $\epsilon > 0$ such that $F \times [-\epsilon, \epsilon]$ is contactomorphic to a neighborhood of $F \subset M$. We start with an isotopy $\psi_t(x, z) = (x, tz)$ for $t \in [\epsilon/a, 1]$; this isn't a contact isotopy *a priori*, so we must modify it. On $F \times [-a, a]$ write $\alpha_1 = udz + \beta$ for $u \in \Omega^0(F)$ and $\beta \in \Omega^1(F)$.

Then $\psi_t^*(\alpha_1) = tudz + \beta := \alpha_t'$. Now we apply Gray's method to get v_t such that $\iota_{v_t} d\alpha_t'|_{\xi_t'} = -udz|_{\xi_t'}$, which is $\mathbb{R}$ invariant. It is also horizontal by plugging in v_t:

$$\iota_{v_t} d\alpha_t'(v_t) = 0 = -udz(v_t) \Rightarrow dz(v_t) = 0$$

This gives us the desired isotopy.

□

The next theorem tells us when a given two curves C and Γ can be realized as a leaf and dividing set of some foliation on a surface. The types of curves C that we can do this for are what is called non-isolated:

Definition 3.22. Given a compact 1-manifold $\Gamma \subset F$, where F is a surface, a curve $C \subset F$ is called *non-isolating* if every component of $F \setminus (\Gamma \cup C)$ has closure intersecting Γ. If C is instead a collection of curves, we say it is non-isolating if each of its components is.

Theorem 3.23. *Let $F \subset (M, \xi)$ be a compact surface and $\Gamma \subset F$ be a compact 1-manifold with $\partial\Gamma = \Gamma \cap \partial F$. Assume that Γ is co-oriented and separates F into regions R_+, R_-, where the co-orientation sends $R_+ \to R_-$. Further, let $C \subset \mathrm{int}(F)$ be a compact 1-manifold whose boundary is contained in Γ. Then there exists a singular foliation $\mathcal{F}$ divided by Γ with the property that $\partial F \cup C$ is a union of leaves and singular points if and only if C is non-isolated.*

Corollary 3.24 (Legendrian Realization Principle)**.** *Let $F \subset (M, \xi)$ be a compact, convex surface with ∂F Legendrian and dividing set Γ. Let $(C, \partial C) \subset (\mathrm{int}(F), \Gamma)$ be a non-isolating 1-manifold. Then there is an isotopy ϕ_t of F such that $\phi_t(F)$ is convex, ϕ_0 is the inclusion, $\phi_1(\Gamma)$ is the dividing set of $\phi_1(F)$ and $\phi_1(C)$ is Legendrian.*

Note that if $C_0 \subset C$ is a compact Legendrian submanifold with $\partial C_0 \subset \Gamma$, we can assume ϕ_t preserves C_0.

3.3.1 Modifying Dividing Sets

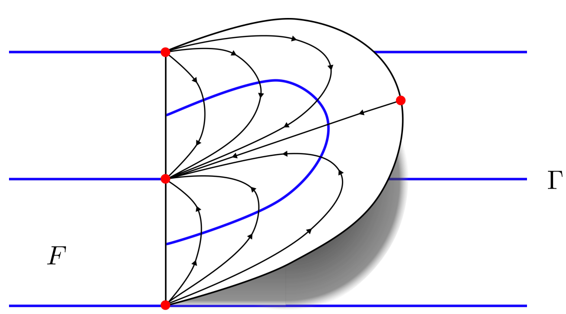
Theorem 3.25. *Let $F \subset (M, \xi)$ be convex with Legendrian boundary and $F \cap \partial M = \partial F$. Let $C \subset R_+$ be a non-isolating circle in the interior of F or let C be a boundary component of F disjoint from Γ. Then $\exists$ an isotopy of F relative to its boundary changing Γ by adding a tubular neighborhood of C to R_-.*

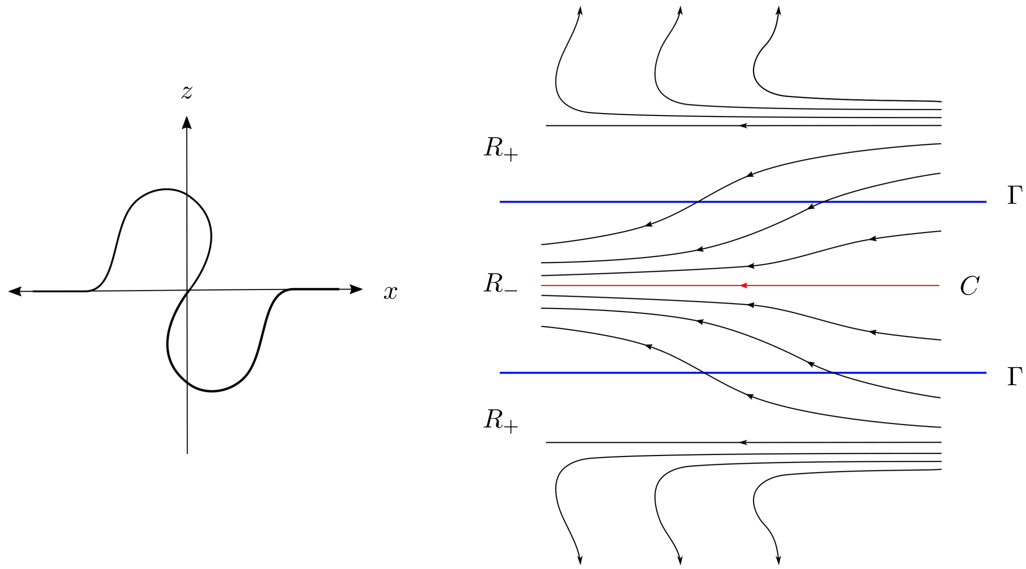
Proof:

By the Legendrian Realization Principle (LeRP), we can assume that C is Legendrian in both cases. Since $C \cap \Gamma = \emptyset$, we have $t(C) = 0$. Therefore the local model for C is the y-axis in the x-y plane in $\mathbb{R}^3$ with contact structure $dz + xdy$, modulo unit y translations. We can then locally perturb F by adding a wrinkle near the y axis, as shown in Figure 3.2a. Doing so introduces two new dividing curves that bound a new R_- region containing C (Figure 3.2b).

□

Definition 3.26 (Honda)**.** A *bypass* for a convex $F \subset (M, \xi)$ is a convex disk D with Legendrian boundary transverse to F with part of the boundary contained in F as shown below. Dividing sets are shown in blue and singular points are shown in red.



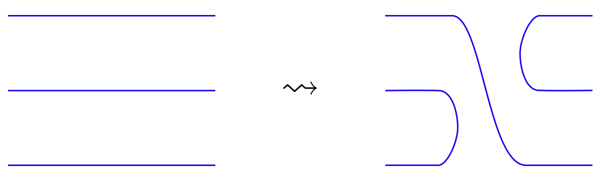


(a) Adding a wrinkle to F along the y axis. (b) The resulting foliation of F year the y axis.

Figure 3.2: Perturbation of the surface F near C, as viewed (a) from a transverse slice (x-z plane) and (b) from above (stretched out x-y plane).

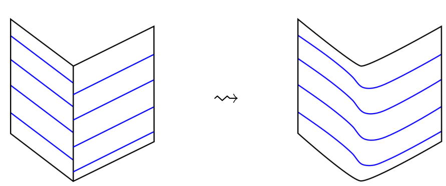
Remark 3.27. We can assume there are only positive singularities on the boundary component of D not touching F by pairwise canceling positive and negative singular points (using the Flexibility Theorem).

Theorem 3.28 (Honda)**.** *Given a bypass D, we can isotope F near D to be convex with dividing set Γ as shown below.*



Proof (*sketch*)*:*

If γ is the component of D contained in F, we can complete it to a Legendrian loop. By LeRP, we can assume this is a Legendrian circle C of F (i.e. a leaf of F). Then we build a wall above C underneath D as shown in Figure 3.3. Note how we have thickened the wall slightly. Our strategy from here will be to round the corners of this wall, both at its base where it meets F and on top. The rounding of the intersection of two surfaces joins their dividing sets as shown below.



The rule is that at an upward bend Γ shifts left and at a downward bend Γ shifts right. When we round the corners of the wall above C, the dividing sets then connect as shown in Figure 3.4. The resulting dividing set structure is isotopic to what we set out for.

□

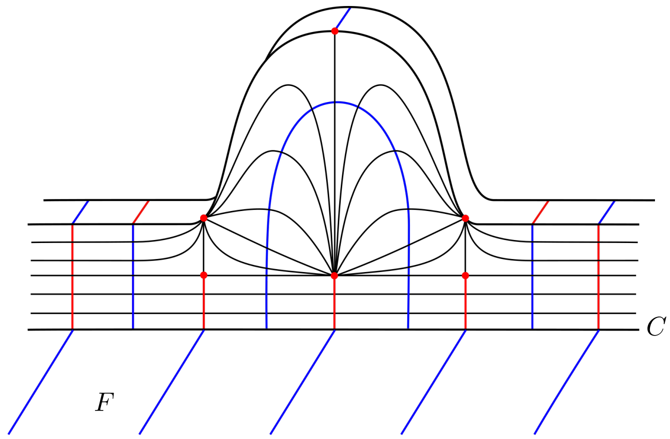


Figure 3.3: The bypass disk D raised up along C and thickened. Singular points and curves shown in red, dividing sets show in in blue.

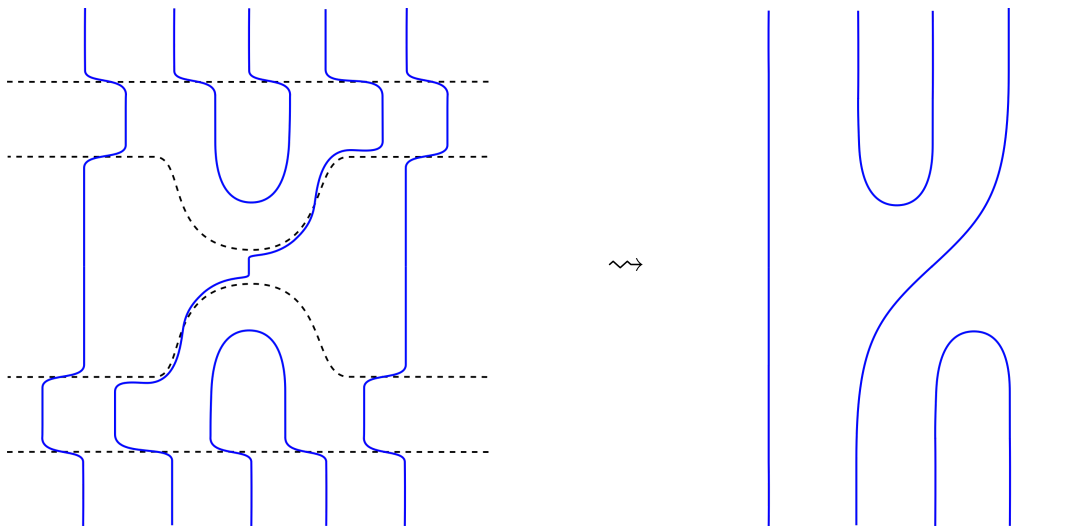
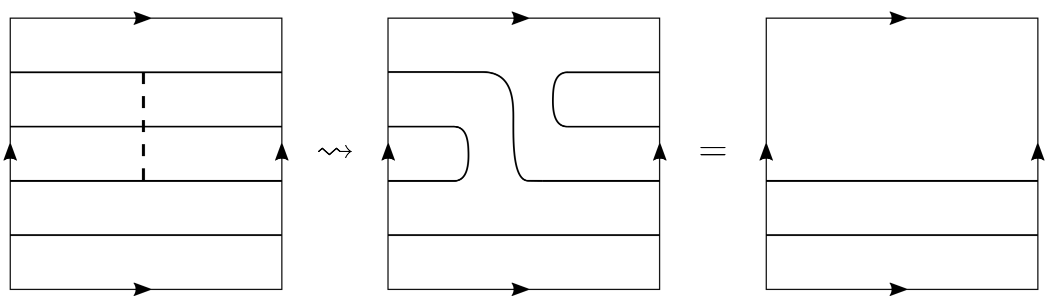
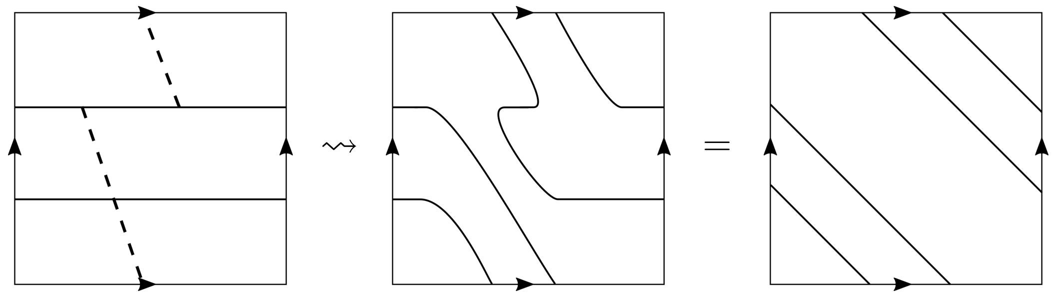


Figure 3.4: Dividing sets on F after smoothing the edges of the bypass disk and wall.

Example 3.29. Consider the torus T^2. Assume for now that no curve of Γ bounds a disk. Then all dividing curves must be parallel and there must be an even number of them (so that they divide T^2 into two well-defined regions R_+, R_-). Then if there exists a bypass D, we can perform a bypass operation on Γ:



By repeating this operation, we can reduce Γ to two dividing curves. A further bypass operation the remaining two dividing curves produces:



This is known as a Dehn twist. At the level of homotopy, if we write $\pi_1(T^2) = \langle a, b\rangle$ where a represents both components of Γ, we have multiplied each component of Γ with by a representative of b.

This construction is related to the classification of contact structures on Lens spaces $L(p,q)$ and on T^2-bundles over S^1 done by Giroux and Honda independently.

4. Contact structures in 3-Manifolds

❖

We now have enough convex surface machinery to begin analyzing and classifying contact structures in 3-manifolds. We'll begin by proving an important theorem of Martinet, which says that all 3-manifolds have a contact structure. From there, we will define tight and overtwisted contact structures, with some brief discussion on current knowledge of their classification in 3 and higher dimensions. From there, the remainder of the section will specialize to tight contact structures in dimension 3, which are in some sense the more interesting of the two in dimension 3.

❖

Theorem 4.1 (Martinet, '71)**.** *Every closed, oriented 3-manifold admits a contact structure.*

Proposition 4.2. *Let M be an oriented 3-manifold.*

(a) *Every link L in M is C^0 small isotopic to a Legendrian link.*

(b) *Every oriented Legendrian link L in M is C^∞ small isotopic to a transverse link that is positive with respect to the co-orientation of ξ.*

Proof:

We can use the standard local model discussed in Section 1.4.2 for each link component. This was the y axis in $(\mathbb{R}^3, dz + xdy)$ modulo unit y translations. An arbitrarily small horizontal translation in the x direction will make L transverse to ξ and the orientation induced can be either positive or negative depending on which direction we translate. This proves part (b).

For part (a), we can assume that L is generic with respect to ξ, which means there are only finitely many points of tangency to ξ. In between these points, we "take the spiral staircase" near L, which is an arbitrarily close (in C^0) Legendrian curve near L. See Figure 4.1.

□

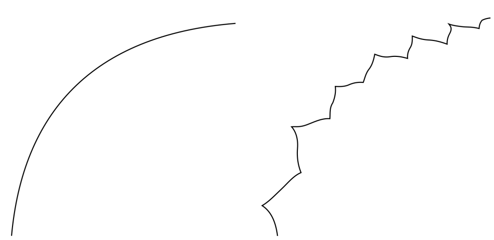


Figure 4.1: Front projection of a C^0 Legendrian approximation to a curve in $\mathbb{R}^3$. (Source: [3].)

Recall that $S^3 = \partial B^4 \subset \mathbb{C}^2$ inherits a plane field which we will call the standard structure on S^3. To do construct it, write S^3 as the preimage of 1 of the function r^2. Therefore $\ker(d(r^2))$ along S^3 is TS^3. Moreover, there is a complex structure J on TS^3 inherited from $\mathbb{C}^2$. Write $\alpha = d(r^2) \circ J$. Then we claim that $\xi = \ker(\alpha|_{S^3})$ is a positive contact form.

Exercise 4.3. Verify that this is a positive contact form by showing that $d(r^2) \wedge \alpha \wedge d\alpha$ is a positive volume form on $\mathbb{C}^2$, and therefore $\alpha \wedge d\alpha$ is a positive volume form on S^3. Do this by showing that:

$$\alpha = x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2$$

and computing $\alpha \wedge d\alpha$. Can you show that this form restricted to $S^3 - \{*\}$ is contactomorphic to $(\mathbb{R}^3, dz + xdy)$?

Theorem 4.4 (Rohlin '51, Wallace)**.** *Every closed, oriented 3-manifold is obtained from S^3 by surgery on a link L. (i.e. by cutting a tubular neighborhood of L out of S^3 and gluing in its place $S^1 \times D^2$.)*

Proof (of Theorem 4.1):

> Write M^3 as a surgery on $L \subset S^3$. Since S^3 has the standard contact structure ξ constructed above, we can make L transverse to ξ by Proposition 4.2. The standard model of a transverse circle is the z axis in $(\mathbb{R}^3, dz + r^2/2d\theta)$ modulo unit z translation. In this model, a tubular neighborhood of the z axis is a torus whose foliation is a collection of lines at constant slope. We call these standard cylinders.
>
> By excising a neighborhood of L and gluing copies of $S^1 \times D^2$ to get M, we get a foliation on $\partial(S^1 \times D^2)$ induced by the contact structure on S^3 consisting of lines of a certain slope. We can assume the slope is nonzero by shrinking the neighborhood. Note that we can change $S^1 \times D^2$ by a Dehn twist, so without loss of generality this spiral is left-handed as in the local model we just constructed. Therefore we can identify $\partial(S^1 \times D^2)$ with a standard cylinder. Now we apply Giroux's theorem (Theorem 2.25) to glue by a contactomorphism in a neighborhood of $\partial(S^1 \times D^2)$.

□

Remark 4.5. While we used Rohlin and Wallace's result to prove Martinet's theorem, this is not how it was originally proved.

4.1 Tight and Overtwisted Contact Structures ❖

Definition 4.6. Suppose $K \subset (M^3, \xi)$ is Legendrian and null-homologous. Then for any oriented surface F with $\partial F = K$ (known as a *Seifert surface*), we get a framing on K. Define $\mathrm{tb}(K) := t(K)$, the twisting number of K relative to this framing on F. This is known as the *Thurston-Bennequin invariant*.

Remark 4.7. A fact we won't prove here is that Seifert surfaces for a null homologous knots K always exist and always determine the same framing on K (up to homotopy).

Example 4.8. In $\mathbb{R}^3$ with the standard contact structure and K with the blackboard framing (from the front projection of K), we call the number of twists in this framing relative to F is called the *writhe* of K, denoted $w(K)$. The Thurston-Bennequin invariant is then $\mathrm{tb}(K) = w(K) - \#$of left cusps. The writhes of the knots pictured in Figure 1.5 are 0 and $+3$, respectively. The Thurston-Bennequin invariants are then -1 and $+1$, respectively.

Remark 4.9. It is important to note that any one can add as many left (and right) cusps to a knot as one wishes, so the Thurston-Bennequin invariant can be made as negative as we wish for any knot.

Definition 4.10. A contact manifold (M, ξ) is *overtwisted* if it has a Legendrian unknot K with $\mathrm{tb}(K) = 0$. Equivalently, it is overtwisted if there exists an embedded disk $D \subset M$ such that $T_pD = \xi_p$ for all $p \in \partial D$ (called an *overtwisted disk*). It is called *tight* if it is not overtwisted.

A result of Bennequin in the 70's is that $(\mathbb{R}^3, \xi_{\text{std}})$ is tight. The key difference between tight and overtwisted is in the regime of maximizing $\mathrm{tb}(-)$. If (M, ξ) is overtwisted, we can realize any knot type as Legendrian with any value of $\mathrm{tb}(-)$; if it is tight, then every null-homologous knot has a maximal finite value of $\mathrm{tb}(-)$.

Example 4.11. Consider $\mathbb{R}^3$ with the contact structure $dz + r^2/2d\theta$. Modify the contact structure by adding a 2π twist in a cylindrical region. In this case, moving out radially from $r = 0$ to $r = \infty$ is $2\pi + \pi/2$ twists (whereas before it was just $\pi/2$). The x-y plane is a convex surface, with transverse vector field $\partial/\partial z$, and with the contact structure we have just constructed, there is a singular circle centered at the origin. This is a null-homologous Legendrian unknot K with $\mathrm{tb}(K) = 0$. Thus we have constructed overtwisted contact structure. The twist we introduced to do this is known as a *Lutz twist*.

For any (M^3, ξ) and any unknot, isotope it to be transverse to ξ. Identifying it with the z axis in standard model in the above example, we can perform a Lutz twist to produce an overtwisted contact structure ξ' on M. Thus, by Martinet's theorem, every closed, oriented 3-manifold has an overtwisted positive contact structure.

Example 4.12. Now consider $\mathbb{R}^3$ with $\alpha_0 = \sin(2\pi nx)dy + \cos(2\pi nx)dz$. Let $\alpha_t = (1-t)\alpha_0 + tdx$. One can check that for any $t \in [0,1)$, α_t is contact. This is a homotopy through confoliations to dx.

The conclusion from the above example is that ξ' and ξ are homotopic (through confoliations). We have just unrigorously proved:

Proposition 4.13. *Every contact structure ξ is homotopic to an overtwisted contact structure.*

There are recent stronger statements of this result for both regular contact structures in higher dimensions, but also for even contact structures. The current thinking is that there is also a similar tight/overtwisted dichotomy for Engel structures on four manifolds.

Theorem 4.14 (Eliashberg, 1980's)**.** *Every homotopy class of plane fields on a closed, oriented 3-manifold contains a unique isotopy class of overtwisted contact structures.*

Theorem 4.15 (Borman, Eliashberg, Murphy '14)**.** *In any odd dimension greater than 1, there is a notion of an overtwisted contact structure. Moreover, they satisfy the h-principle. In other words, the inclusion of overtwisted hyperplane fields supporting a non-degenerate 2-form into the space of all contact structures is a homotopy equivalence.*

Theorem 4.16 (McDuff)**.** *Even contact structures on M^{2n} satisfy the h-principle.*

Conjecture 4.17. *There exists a tight/overtwisted dichotomy for Engel structures on M^4 and the overtwisted ones satisfy the h-principle.*

4.2 Euler Classes of Tight Contact Structures ❖

The focus for the remainder of this class will be dimension three manifolds with tight contact structures. Recall from Exercise 1.27, for any compact oriented surface F, the Euler characteristic $\chi(F)$ is $e(TF, v)$, where v is a nonvanishing vector field on ∂F that is either normal to ∂F or tangent to ∂F. The idea for proving this is to cap off the boundaries with disks and extend the vector field over the resulting surface. Then use Poincaré-Hopf. If $F \subset (M, \xi)$ is a closed, oriented surface in a contact three-manifold, then we have a natural pairing:

$$\langle e(\xi), [F] \rangle = \mathrm{PD}(e(\xi)) \cdot [F] = e(\xi|_F) \in \mathbb{Z}$$

If F instead had Legendrian boundary, we'd like to do something similar. A good choice of nonvanishing vector field v on ∂F is the tangent vector field to ∂F, since it is contained in ξ.

Definition 4.18. The *rotation number* of $F \subset (M, \xi)$ is $r(F) = e(\xi|_F, v)$, where v is as above. If $\partial F = \emptyset$, the rotation number is just $e(\xi_F)$.

Remark 4.19. Just as above, this can also be seen as the pairing $\langle e(\xi, v), [F] \rangle$ in relative homology and cohomology. It can also be seen as $\mathrm{PD}(e(\xi, v)) \cdot F$ where $\mathrm{PD}(e(\xi, v)) \in H_1(M \setminus \partial F)$.

Suppose $t(\gamma) = 0$ for each component γ of F. Perturb F (rel boundary) to be convex with respect to some v, which gives us $\Gamma \cap \partial F = \emptyset$. Observe that on ∂F, $\xi = \pm TF$, so we can change v to be normal to ∂F in the direction of the characteristic foliation $\mathcal{F}$. Then we extend v generically over F so that it generates $\mathcal{F}$ on the interior of $\mathcal{F}$. At the zeros of v, $\xi = \pm TF$ so we can perturb ξ to be equal to $\pm TF$ near these points. This ensures $\mathrm{ind}_\xi v_x = \pm \mathrm{ind}_{TF} v_x$, where $\mathrm{ind}_\xi v_x$ is the index of v when thought of as section of ξ and $\mathrm{ind}_{TF} v_x$ is the index of v when thought of a section of TF. Note that the resulting plane field will not be a contact structure. Then:

$$\begin{aligned} r(F) &= \sum_{v(x)=0} \mathrm{ind}_\xi v_x \\ &= \sum_{v(x)=0, x \in R_+} \mathrm{ind}_\xi v_x + \sum_{v(x)=0, R_-} \mathrm{ind}_\xi v_x \\ &= \sum_{v(x)=0, x \in R_+} \mathrm{ind}_{TF} v_x - \sum_{v(x)=0, R_-} \mathrm{ind}_{TF} v_x \\ &= \chi(R_+) - \chi(R_-) \end{aligned}$$

We have thus proven a special case of:

Given a compact, convex, oriented surface $F \subset (M, \xi)$ with Legendrian boundary, then $r(F) = \chi(R_+) - \chi(R_-)$. In particular if F is closed, then $\langle e(\xi), [F] \rangle = \chi(R_+) - \chi(R_-)$.

The proof for $t(\gamma) < 0$ uses a similar idea by excising a neighborhood of ∂F.

Theorem 4.21 (Giroux criterion)**.** *Let $F \subset (M, \xi)$ be a compact, connected, convex surface with Legendrian boundary (everything oriented). Then F has a tight neighborhood if and only if:*

- *(If $F \ncong S^2$) Γ has no circle bounding a disk in F.*
- *(If $F \cong S^2$) Γ has only one component.*

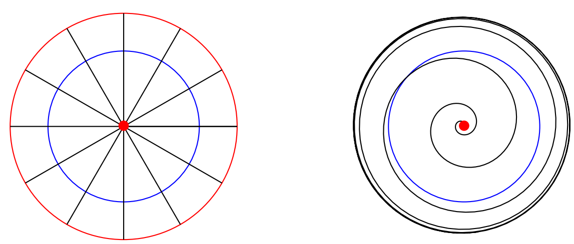
Proof:

We will prove the forward implication by contrapositive and not prove the other direction. That is, assume that both conclusions above are false. Without loss of generality, we claim we can assume that Γ is not connected. To see this, suppose otherwise, i.e. Γ is connected. Then $F \ncong S^2$ and Γ is a circle bounding a disk. We then add a wrinkle by Theorem 3.25 and introduce more components of Γ, so we have reduced to the case where Γ is not connected.

Now we assume that there exists a circle component of Γ bounding a disk. Take Γ_0 a circle component that contains no other components, and C surrounding Γ_0. We note that C is non-isolating. By LeRP, we can make C Legendrian. Then C bounds an overtwisted disk, so every neighborhood of F must be overtwisted (i.e. not tight).

□

Corollary 4.22. *Every overtwisted contact structure contains disks as shown below.*



(a) (b)

Where red denotes singular points, black denotes foliation leaves, and blue denotes Γ.

Proof:

Use the overtwisted disk constructed in the proof of Theorem 4.21 and the Flexibility Theorem to realize either foliation.

□

Theorem 4.23. *For $F \subset (M, \xi)$ a closed, oriented surface, with ξ a tight contact structure, let:*

$$\widehat{F} = \coprod_{F_i \ncong S^2} F_i$$

Where each $F_i \subset F$ is a connected component of F. Assume that ξ is tight. Then $|\langle e(\xi), F \rangle| \leq -\chi(\widehat{F})$.

Proof:

First assume that F is connected. Perturb F so that it is convex, and apply Theorem 4.21. If $F \cong S^2$ (and therefore $\widehat{F} = \emptyset$) then $R_\pm \cong D^2$ so $\chi(R_+) = \chi(R_-)$ and $\langle e(\xi), F\rangle = 0 = -\chi(\emptyset)$ by Theorem 4.20. If $F \not\cong S^2$, then $\chi(R_\pm) \leq 0$. Again by Theorem 4.20, we have:

$$\langle e(\xi), F\rangle = \chi(R_+) - \chi(R_-) \leq -\chi(F)$$

because Γ is a disjoint union of circles, meaning $\chi(\Gamma) = 0$, and $F = R_+ \sqcup R_- \sqcup \Gamma$. Doing the same but with F reversely oriented, gives $|\langle e(\xi), F\rangle| \leq -\chi(\widehat{F})$.

Now let $F = \coprod F_i$. Then by linear properties of $\langle \cdot, \cdot \rangle$ and $\chi(-)$, we have:

$$\begin{aligned} |\langle e(\xi), F\rangle| &= \left|\sum \langle e(\xi), F_i\rangle\right| \\ &\leq \sum |\langle e(\xi), F_i\rangle| \\ &\leq \sum (-\chi(\widehat{F_i})) = -\chi(\widehat{F}) \end{aligned}$$

□

Corollary 4.24. *For a closed, oriented 3-manifold, there are only finitely many Euler classes of tight contact structures.*

In order to prove this, we will use two facts:

1. For any closed, oriented 3-manifold M, a class in $H^2(M)$/torsion $\cong H_1(M)$/torsion is determined by its pairing with $H_2(M)$.
2. Every element $H_2(M)$ is realized by a closed, oriented surface (not necessarily connected).

The Corollary then follows by considering $H_2(M) \cong \mathbb{Z}^n$ (there is no torsion in penultimate dimension). Choose a basis of this module and represent it by surfaces $F_1, ..., F_n$. Given a tight contact structure ξ, we can consider $|\langle e(\xi), F_i\rangle| \leq -\chi(\widehat{F_i})$. There are only finitely many possibilities for $e(\xi)$ for every i, by Fact 1. Therefore there are only finitely many $e(\xi) \in H^2(M)$/torsion. Since torsion subgroups are finite, the Corollary follows.

Remark 4.25. Recall for a fixed trivialization τ of TM, we get the invariant Γ_τ defined in Section 1.3. The above Corollary the also says that there are only finitely many values of Γ_τ realized by a tight ξ.

Theorem 4.26 (Colin, Giroux, Honda)**.** *Only finitely many homotopy classes of plane fields are realized by tight contact structures.*

Example 4.27. $M = S^1 \times S^2$. Then $H_2(M) \cong \mathbb{Z}$ generated by $\{*\} \times S^2$. By Theorem 4.23, every tight contact structure has $e(\xi) = 0$. In fact, one can show that there exists a unique tight contact structure on M.

Example 4.28. Let $M = T^3$ be the 3-torus. Then $H_2(T^3) \cong \mathbb{Z}^3$. A basis is given by tori each with a single point in one of the three components. Once again, since $\chi(T^2) = 0$, we have $e(\xi) = 0$ for any tight contact structure ξ. In this case, all tight contact structures are homotopic through confoliations, however there is more than one such tight structure.

We can strengthen Theorem 4.23 by considering F with possibly non-empty boundary and using Theorem 4.20. Note that for a disconnected surface $F = \sqcup F_i$, the rotation number adds: $r(F) = \sum r(F_i)$.

Theorem 4.29. *If $F \subset (M\ \xi)$ is a compact, oriented surface with Legendrian boundary and ξ is tight, let $t_1, ..., t_n$ be the twisting numbers of the boundary components. Then $|r(F)| + \sum_{i=1}^n t_i \leq -\chi(\widehat{F})$.*

Proof:

As in our proof of Theorem 4.23, it suffices to show $\sum^n t_i - r(F) \leq -\chi(\widehat{F})$ because we get the other inequality by reversing the orientation of F. Without loss of generality, we assume that there are no components of F diffeomorphic to S^2. We also assume that all $t_i \leq 0$, so that we can perturb F to be convex by Theorem 3.20. Write $F = R_+ \sqcup \Gamma \sqcup R_-$. The number of arcs in Γ (i.e. components of Γ with boundary in ∂F) is $-\sum^n t_i$, so that:

$$\chi(F) = \chi(R_+) + \chi(R_-) + \sum^n t_i$$

We can place bounds on $\chi(R_\pm)$ by considering the number of disks in F bounded by Γ and ∂F. The Giroux criterion rules out interior disks, since ξ is tight, so that:

$$\#\text{disks} \leq -\sum^n t_i$$

In particular:

$$\chi(R_-) + \sum^n t_i \leq 0$$

Finally:

$$\begin{aligned}
\chi(F) &= \chi(R_+) + \chi(R_-) + \sum^n t_i \\
&\leq \chi(R_+) - \chi(R_-) - \sum^n t_i \\
&= r(F) - \sum^n t_i
\end{aligned}$$

The last equality came from Theorem 4.20.

□

We will return to justify the assumption that $t_i \leq 0$ in our proof of Theorem 5.3.

4.3 Classification Theory ❖

Given a closed, oriented 3-manifold M, let $\mathcal{T}(M)$ denote the set of all tight, positive, oriented contact structures modulo isotopy. If M is compact with boundary, there is an induced foliation $\mathcal{F}$ on ∂M for any contact structure. Assume ∂M is convex, and let $\mathcal{T}_{\mathcal{F}}(M, \Gamma)$ be the set of all tight, positive, oriented contact structures for a fixed foliation $\mathcal{F}$ on ∂M divided by Γ, modulo isotopy. Similarly, let $C(M, \Gamma)$ be the same thing but without assuming tightness.

Theorem 4.30. *Given $\mathcal{F}_0, \mathcal{F}_1$ divided by Γ, there is a canonical bijection between $\mathcal{T}_{\mathcal{F}_0}(M, \Gamma)$ and $\mathcal{T}_{\mathcal{F}_1}(M, \Gamma)$.*

Proof (sketch):

Flow ∂M with foliation $\mathcal{F}_0$ along a convex vector field inside M to get another copy of ∂M with foliation $\mathcal{F}_1$ (using the Flexibility Theorem). This extends to a contact isotopy on all of M, and hence gives us a map $\mathcal{T}_{\mathcal{F}_0}(M, \Gamma) \to \mathcal{T}_{\mathcal{F}_1}(M, \Gamma)$. One can check that it is a bijection.

□

In light of this, we drop the foliation subscript and just write $\mathcal{T}(M, \Gamma)$. Given two three manifolds (M_0, ξ_0) and (M_1, ξ_1) with boundary, the above theorem gives us a procedure for gluing them along the boundary components with the same dividing sets Γ_i. The resulting contact structure may not be tight, however. This gives a canonical map:

$$\mathcal{T}(M_0, \Gamma_0) \times \mathcal{T}(M_1, \Gamma_1) \to C(M_0 \cup_{\Gamma_i} M_1, \Gamma_j)$$

Where Γ_j are the remaining dividing sets that don't match up anywhere.

Example 4.31. Consider $M = S^2 \times I$. There is exactly one choice of Γ on ∂M that gives tight contact structures, which is a great circle along each component. A Theorem of Eliashberg says $\mathcal{T}(S^2 \times I, \Gamma)$ has a unique element.

There are some corollaries of this example:

Corollary 4.32. *$\mathcal{T}(B^3, \Gamma)$ has a unique element and $S^3, \mathbb{R}^3$ have unique positive tight contact structures.*

Proof:

First we assume $\#\mathcal{T}(B^3, \Gamma) = 1$. There are two copies of B^3 on S^3, the north and south pole caps. The complement of these caps is $S^3 \times I$. Since $S^2 \times I$ has a unique tight contact structure, removing the pole caps leaves us with a unique tight contact structure, and each cap only has one tight contact structure. So the result of gluing the caps back on can only produce one tight contact structure on S^3, which is the standard one.

To see that $\#T(B^3, \Gamma) = 1$ and $\#T(\mathbb{R}^3, \Gamma) = 1$, we foliate each space by copies of S^2. The details are left to the reader.

□

Corollary 4.33 (Colin,Makar-Limanov)**.** *Given a tight three manifold (M, ξ). Let $\phi_t : S^2 \to M^3$ be an isotopy. Perturb it so that $\phi_0(S^2)$ and $\phi_1(S^2)$ are convex with the same characteristic foliation (using the Flexibility Theorem). Then there is a contact isotopy sending $\phi_0(S^2)$ to $\phi_1(S^2)$.*

Proof:

For any t_1, t_2 sufficiently close, there exists a $S^2 \times I$ neighborhood containing $\phi_{t_1}(S^2)$ and $\phi_{t_2}(S^2)$. We can modify the isotopy to pass through finitely many smoothly embedded $S^2 \times I$'s. Without loss of generality, all $S^2 \times I$ boundaries are convex, so that the contact structure on the $S^2 \times I$ they bound is standard, and therefore I invariant. Thus we can modify ϕ on $[t_1, t_2]$ to be a contact isotopy. Concatenating all of these perturbed isotopies gives the desired contact isotopy.

□

A basic fact from three manifold theory is that every compact oriented 3-manifold has a unique prime decomposition under connected sum $\#$. Then the question arises: are there any contact structures on the connect sum that don't come from gluing two contact structures? The answer is no:

Theorem 4.34 (Colin, Makar-Limanov)**.** *Let M_0, M_1 be connected 3 manifolds with respective dividing sets Γ_0, Γ_1 on their boundaries. There is a canonical bijection $\mathcal{T}(M_0, \Gamma_0) \times \mathcal{T}(M_1, \Gamma_1) \to \mathcal{T}(M_0 \# M_1, \Gamma_0 \cup \Gamma_1)$.*

Proof:

Suppose we are given (M_i, ξ_i) tight for $i = 0, 1$. We want to construct a tight structure on $M_0 \# M_1$. Let $B_i \subset M_i$ be a three ball with convex boundary. This is unique up to isotopy, which is just a topological fact. In fact, it is unique up to contact isotopy and perturbation of the foliation on its boundary, by the Corollary above. We use these to perform the connect sum $M_0 \# M_1$, which induces a contact structure $\xi_0 \# \xi_1 \in C(M_0 \# M_1, \Gamma_0 \cup \Gamma_1)$. This is well-defined because B_i were unique up to contact isotopy. To show that $\xi_0 \# \xi_1$ is tight, suppose it isn't; then there is an overtwisted disk D in $M_0 \# M_1$ that must intersect $S = \partial B_1 \equiv \partial B_2$. There is an isotopy shrinking D to a disk in M_1. Reversing this isotopy pushes the sphere S off of D. By the previous Corollary, there is a contact isotopy that does the same thing (up to perturbing the foliation on S). This gives us a contactomorphism of $(M_0 \# M_1, \xi_0 \# \xi_1)$ pushing S off of D, which realizes an overtwisted disk $D \subset M_1$, a contradiction. Therefore $\xi_1 \# \xi_1 \in \mathcal{T}(M_0 \#, M_1, \Gamma_0 \cup \Gamma_1)$, hence the map is well-defined.

To show surjectivity, let ξ be a contact structure on $M_0 \# M_1$. Take a convex sphere S and cut along S. Then glue in tight balls B_i to M_i. If the resulting structures on M_0 and M_1 were overtwisted, then we could shrink B_i away from the overtwisted disk and find an overtwisted disk in $M_0 \# M_1$, a contradiction. So the resulting structure is tight on each M_0, M_1.

To show injective, suppose $\xi_0 \# \xi_1$ and $\xi_0' \# \xi_1'$ are isotopic contact structures. Let $S = \partial(B^3)$ be the sphere along which M_1 and M_2 were glued, and let ϕ_t be the isotopy of the two structures. There is then a contact isotopy ψ sending $S \to \phi_1(S)$ by the Corollary above. Then the composition $\psi \circ \phi_t$. This fixes

S and sends $\xi_0 \to \xi_0'$ and $\xi_1 \to \xi_1'$ on the respective components $M_i - \text{int}(B^3)$. This isotopy extends to all of M_i, so that $\xi_i \sim \xi_i'$.

□

Corollary 4.35. $\#\mathcal{T}(S^2 \times S^1) = 1$.

Proof:

Write $S^2 \times S^1$ as a self sum of S^3 (i.e. S^3 glued to itself along two holes) and use Theorem 4.34.

□

4.4 Classification using Tori and Lens Spaces ❖

Consider the Lens space $L(p, q)$, for p and q coprime. This is a quotient of S^3 and is a prime 3 manifold (i.e. it cannot be written as a nontrivial connect sum of two three manifolds).

Theorem 4.36. *If $[a_1, a_2, a_3, ..., a_m]$ is the continued fraction expansion of $-\frac{p}{q}$ for p and q coprime, then:*

$$\#\mathcal{T}(L(p,q)) = \prod_{i=1}^{m}(|a_i| - 1)$$

Definition 4.37. A contact structure on M is called *universally tight* if it is tight and the pullback to an oriented cover of M is also tight.

Definition 4.38. An embedded torus $T^2 \subset M$ is called *incompressible* if the induced map $\pi_1(T^2) \to \pi_1(M)$ is an injection. M is called *atoroidal* if no such torus exists.

An example of a universally tight contact structure is $\xi_m = \ker(\sin(2\pi mx)dy + \cos(2\pi nx)dz)$ on the three torus T^3. It can be shown that the pullback of ξ_m to the universal cover $\mathbb{R}^3$ is the standard contact structure on $\mathbb{R}^3$, which is tight.

Definition 4.39. The *Giroux torsion* of a contact three manifold (M, ξ) is the maximal m such $(T^2 \times I, \xi_m)$ can be embedded contactomorphically into M.

The following are a few recent results in classifying contact structures in prime three manifolds.

Theorem 4.40 (Colin et. al. 2000)**.** *Every closed, oriented, prime three-manifold with an incompressible torus has infinitely many tight structures, each distinguished by Giroux torsion.*

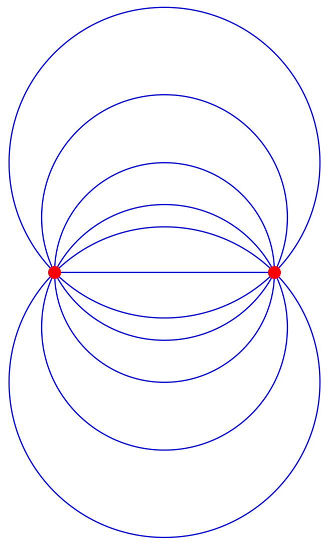
Theorem 4.41 (Colin, Giroux, Honda)**.** *Every closed, oriented, atoroidal three manifold has only finitely many tight structures.*

Theorem 4.42. *A closed, oriented, prime 3-manifold M with $H_2(M) \neq 0$ admits a universally tight positive contact structure.*

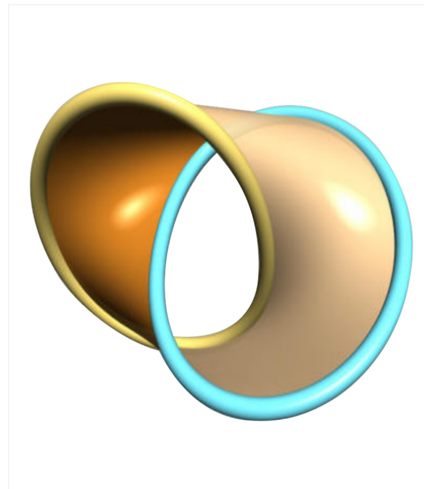
Proof Idea:

The original proof uses a result of Gabai from 1983 which used cut and paste methods to show that M admits a taut foliation. Eliashberg-Thurston in the mid 90's showed how to perturb a tight foliation into a tight contact structure. A newer proof, due to Honda, Kazez, and Matić in 2001, used Gabai's cut and paste method directly in a contact setting. The $H_2(M) \neq 0$ requirement comes from the fact that you need a homologically essential foliation to get a taut foliation.

□



(a) Cross-section of the bubble open book decomposition of S^3. Knot shown in red.



(b) Seifert surface of a negative Hopf link, which is a fiber of the negative Hopf decomposition of S^3.

Figure 4.2: Examples of open book decompositions of S^3. Second image source: Jack van Wijk, Eindhoven University of Technology

Example 4.43. Consider the Poincaré Homology Sphere. One way to construct it is $\Sigma = SO(3)/\text{icosahedral group}$. Since $SO(3) \cong SU(2)/\pm I$ and $SU(2) \cong S^3$, the Poincaré Homology Sphere is a quotient of S^3 by a discrete subgroup of $SU(2)$. Then the standard structure ξ on S^3 descends to a tight contact structure on Σ.

Theorem 4.44 (Honda)**.** *The reversely oriented Poincaré homology sphere $\overline{\Sigma}$ admits no positive tight contact structure.*

Corollary 4.45. *$\Sigma \# \overline{\Sigma}$ admits no tight contact structure.*

4.5 Using Open Book Decompositions ❖

Definition 4.46. An *open book decomposition* of a three manifold M is a link $L \subset M$ (called the *binding*) and a bundle structure $\pi : M \setminus L \to S^1$ by Seifert surfaces. In other words, $\pi^{-1}(\theta)$ is the interior of a compact surface in M whose boundary is L. The fibers of π are called *pages*.

Example 4.47. The simplest example of an open book decomposition of S^3 is given by considering the collection of "bubbles" bounded by an unknot in S^3 (see Figure 4.2a). Another such decomposition of S^3 is called the Hopf decomposition H_+, which is along positive Hopf links. The negative Hopf decomposition H_- is the same but along negative Hopf links.

Definition 4.48. The *monodromy* of an open book decomposition with fiber F is $\phi : F \to F$ such that:

$$M \setminus L \cong F \times I/(x, 1) \sim (\phi(x), 0)$$

as bundles over S^1, where $\phi = \text{id}$ near ∂F. The monodromy determines M and the open book decomposition.

Theorem 4.49. *Given open books B_0, B_1 for M_0, M_1 respectively, there is an open book $B_0 \# B_1$ on $M_0 \# M_1$. This depends on the choice of an embedding $(I, \partial I) \hookrightarrow (F, \partial F)$ for each manifold, where F is the fiber.*

Proof:

It suffices to construct this in C^0 and then perturb it to C^∞. Embed an interval I into a fiber F_0 of M_0 so that $\partial I \subset \partial F_0$. Take a tubular neighborhood of I in M_0, which we can take to be a rectangular box as shown in Figure 4.3a. Since we are working in C_0, we can bend the fibers so that the top and bottom of the box are leaves. Doing this for both M_0, M_1, we then glue M_0 and M_1 along this ball by matching the foliations rotated by 90 degrees (see figure 4.3b). This operation is called a plumbing. The resulting manifold has an open book decomposition once we smooth the edges of the box.

□

Definition 4.50. Given an open book B_0 for M_0, its positive or negative *Hopf stabilization* is $B_0 \# H_\pm$ on $M_0 \# S^3 = M_0$.

Returning to the contact setting, we can give an alternative proof of Martinet's theorem (due to Thurston-Winkelkemper '75) that uses open book decompositions. An old theorem of Alexander says that any M^3 admits an open book decomposition B. Choose a 1-form $\beta \in \Omega^1(F)$, where F is a fiber of B, such that $d\beta$ is a positive area form and β is "standard" near ∂F. Let $\beta_t = (1-t)\beta + t\phi^*\beta$, where $0 \leq t \leq 1$ and ϕ is the monodromy of B. These are all "standard" near the boundary because ϕ is the identity near ∂F, and $d\beta_t$ is a positive area form. Since we are gluing $F \times I$ along ϕ, this extends to a form η on $M \setminus L$ by extending to the I coordinate z. Let $\alpha = dz + \epsilon\eta$ for $\epsilon > 0$. Then:

$$\begin{aligned}\alpha \wedge d\alpha &= (dz + \epsilon\eta) \wedge \epsilon d\eta \\ &= \epsilon(\underbrace{dz \wedge d\eta}_{(+)\text{ vol. form}} + \underbrace{\epsilon\eta \wedge d\eta}_{\text{small}})\end{aligned}$$

We can then choose ϵ small enough so that $\alpha \wedge d\alpha$ is positive. Moreover, we can extend α to L by pulling back the standard contact structure of a transverse knot in $(\mathbb{R}^3, dz + r^2/2d\theta)$. Therefore α defines a contact structure on all of M.

4.5.1 The Open Book Correspondence

Definition 4.51. An open book decomposition B *supports* a contact structure ξ if there exists a one-parameter family of plane fields $\xi_t, 0 \leq t \leq 1$, such that $\xi_0 = \xi$, for all $t < 1$, ξ_t is a contact structure transverse to thie binding, and ξ_1 defines the foliation away from the link of B.

Theorem 4.52. *Every open book supports a unique positive contact structure up to isotopy.*

Exercise 4.53. Show that H_+ supports the standard contact structure ξ on S^3 using the Hopf fibration. Show also that H_- supports a different homotopy class of contact structure, which must therefore be overtwisted because $\#\mathcal{T}(S^3) = 1$.

A fact we won't prove is that, given B_i supporting ξ_i on M_i for $i = 0, 1$, then $B_0 \# B_1$ supports $\xi_1 \# \xi_1$ on $M_0 \# M_1$. An immediate corollary, using Exercise 4.53, is:

Corollary 4.54. *Any positive Hopf stabilization preserves the contact structure. Any negative Hopf stabilization gives an overtwisted contact structure.*

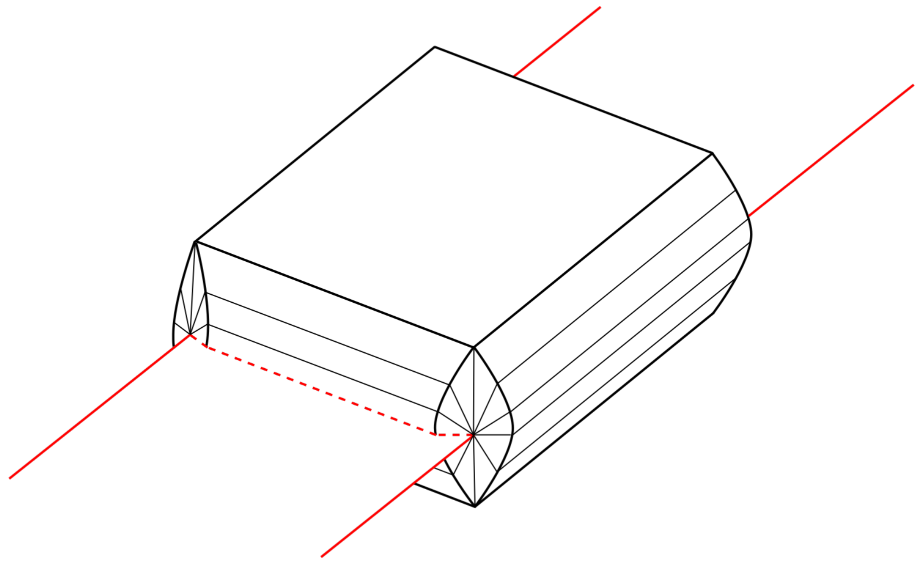
Therefore we have a well-defined map:

$$\{\text{open books on } M\}/(+)\text{ Hopf stab. \& isotopy} \to \{(+)\text{ contact structures}\}/\text{isotopy}$$

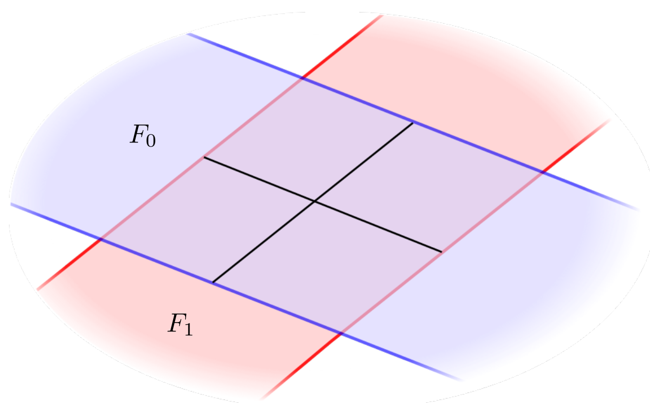
A theorem of Giroux '03 says that this is map actually a bijection. We'll call this the Open Book Correspondence.

Theorem 4.55 (Harer Conjecture)**.** *Every fibered link in S^3 is made by plumbing and deplumbing Hopf bands, i.e. any two open books in S^3 are related by Hopf stabilizations.*

Proof Idea (*Goodman '0?*):



(a) A tubular neighborhood of I deformed to a box whose top and bottom are leaves of B_0. F shown in red and all other leaves intersecting the box shown in black.



(b) Gluing the neighborhoods of I by a rotation by 90 degrees. Embeddings of I shown in black.

Figure 4.3: The plumbing operation to construct an open book decomposition $B_0 \# B_1$

Given two open books, look at the corresponding contact structures via the correspondence above. After negative Hopf stabilization, these are overtwisted. By adding more negative Hopf stabilizations, we can arrange these to be homotopic (and overtwisted), and therefore they are isotopic (by Theorem 4.14). Therefore they represent the same equivalence class of open book, and hence the books we started with were related by Hopf stabilizations.

□

Open questions about the Open Book Correspondence:

1. How do we characterize open books supporting *tight* contact structures? It is known that if the monodromy of an open book can be written as a composite of right handed Dehn twists, then the corresponding structure is tight. The converse, however, is not true.
2. What is the minimum genus g of pages of open book supporting a contact structure? Etnyre proved that ξ overtwisted $\Rightarrow g = 0$. The converse isn't true, since H_+ is has $g = 0$ but supports the tight structure ξ_{std} on S^3.
3. What is the smallest number of binding components supporting a given contact structure?

5. Legendrian Knot and Link Theory

Suppose we are given an oriented Legendrian knot $K \subset (M^3, \xi)$ that is nullhomologous. Recall the Thurston-Bennequin invariant $\mathrm{tb}(K) = t(F, K)$, where F is a Seifert surface. Another invariant we had was $r(F) = \langle e(\xi, v), [F] \rangle$, where v is tangent to K. A natural question is: under what circumstances is $r(F)$ an invariant of K and not of F? Notice that given Seifert surfaces F_1, F_2 for K, we have:

$$r(F_2) - r(F_1) = \langle e(\xi, v), [F_2] - [F_1] \rangle = \langle e(\xi), [F_1 - F_2] \rangle$$

Where $F_1 - F_2$ denotes the disjoint union of F_1 and $\overline{F}_2$. To justify the last equality, let $j : (M, *) \to (M, K)$ be the inclusion. Then $j^* : H^2(M, K) \to H^2(M)$ is an isomorphism and sends $e(\xi, v)$ to $e(\xi)$ and moreover $j_*([F_1 - F_2]) = [F_1] - [F_2]$. Then use the fact that $\langle j^* \cdot, \cdot \rangle = \langle \cdot, j_* \cdot \rangle$.

If we assume $e(\xi) = 0$, the above equality shows that $r(F)$ is independent of F and only depends on K. In this case, we write it as $r(K)$. Thus we have two knot invariants, $\mathrm{tb}(K)$ and $r(K)$ which are preserved under contactomorphism.

Example 5.1. For $M = \mathbb{R}^3$ with ξ_{std}, every $K \subset \mathbb{R}^3$ is nullhomologous and $e(\xi) = 0$ because $H^2(\mathbb{R}^3) = 0$. Then $\mathrm{tb}(K)$ and $r(K)$ are defined for all oriented Legendrian knots in $\mathbb{R}^3$.

Recall from Example 4.8 that we showed $\mathrm{tb}(K) = w(K) - \#$left cusps, where $w(K)$ is the writhe of K. To see what $r(K)$ is, we trivialize ξ by $\partial/\partial x$ and measure twisting relative to TK. Using front projections, a downward left cusp produces a positive twist and an upward right cusp produces a negative twist, so:

$$r(F) = \#\text{downward left cusps} - \#\text{upward right cusps}$$

We could have also done this relative to $-\partial/\partial x$, to get:

$$r(F) = \#\text{downward right cusps} - \#\text{upward left cusps}$$

Yet a third way to describe $r(F)$ comes from averaging the two above:

$$r(F) = \frac{1}{2}\left(\#\text{downward cusps} - \#\text{upward cusps}\right)$$

This is what we will use, since it is least confusing.

Now suppose that $K \subset (M, \xi)$ is a nullhomologous, positively oriented transverse knot. Given a Seifert surface F for K, we can choose the outward normal v to F to be in ξ. Then we define the self-linking number $\ell(K) = -\langle e(\xi, v), [F] \rangle$. Again this is independent of F when $e(\xi) = 0$.

Proposition 5.2. *For K Legendrian as above, it has a canonical positive transverse pushoff τK (Proposition 4.2b). Then:*

$$\ell(\tau K) = \mathrm{tb}(K) - r(K)$$

Proof:

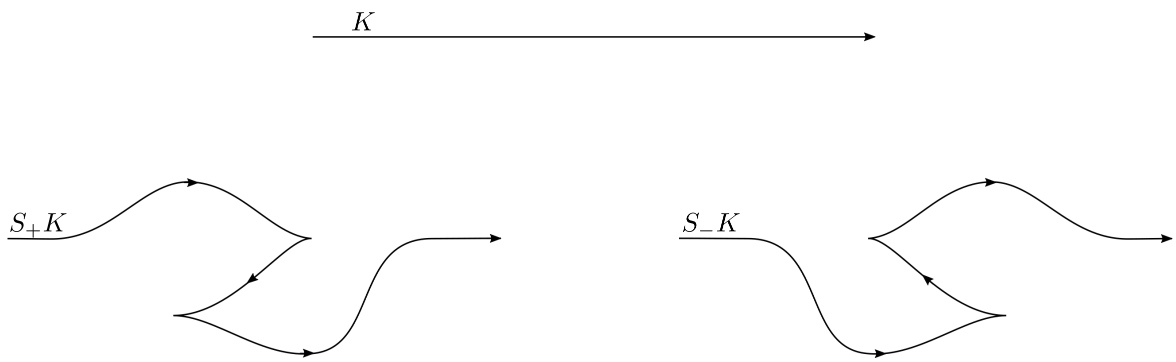
Let v_K be tangent to K, v_F be the outward normal to F, and v_ξ be the contact framing in ξ. Then $r(K) = \langle e(\xi, v_K), [F] \rangle = \langle e(\xi, v_\xi), [F] \rangle$ because v_ξ is perpendicular to K. This is also equal to $\langle e(\xi, v_\xi, [\tau F] \rangle$, since τ is a local perturbation of the boundary. Then by definition of $\mathrm{tb}(K)$:

$$\langle e(\xi, v_\xi), [\tau F] \rangle = \underbrace{\langle e(\xi, v_F), [\tau F] \rangle}_{-\ell(\tau K)} + \mathrm{tb}(K)$$

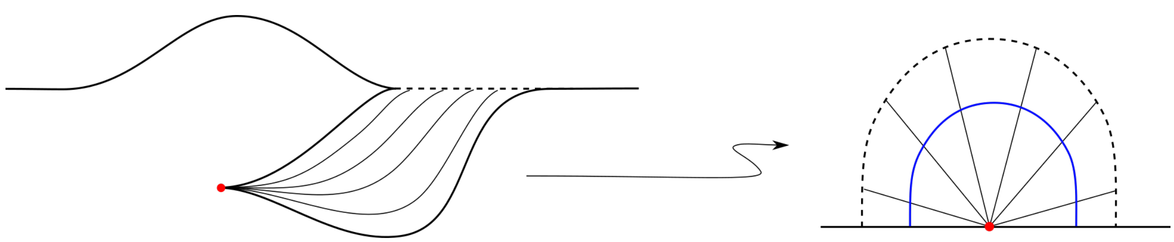
$\square$

5.1 Knot Operations ❖

Let $\mathcal{K}(M)$ be the set of all oriented knots in a three manifold M modulo isotopy. Then let $\mathcal{L}(M,\xi)$ be the set of all oriented Legendrian knots modulo contact isotopy. Also let $T(M,\xi)$ be the set of all positive transverse knots modulo contact isotopy. Then the pushoff τ is a map $\mathcal{L}(M,\xi) \to T(M,\xi)$. Now we will define stabilizations $S_\pm : \mathcal{L}(M,\xi) \to \mathcal{L}(M,\xi)$. To define these, consider the local model for a Legendrian knot K ($\mathbb{R}^3$ modulo unit y translation). Then the stabilization operations are:



For K as before, we can see that $\operatorname{tb}(S_\pm K) = \operatorname{tb}(K) - 1$ and $r(S_\pm K) = r(K) \pm 1$. Notice that applying a stabilization gives us a bypass disk attached to K by filling in part of $S_\pm K$ with a family of Legendrian curves:



Where the cusp marked in red became a singular point and the family of curves comprise the foliation on the bypass.

Theorem 5.3. *For ξ tight with $e(\xi) = 0$ and $K \in \mathcal{L}(M,\xi)$ nullhomologous, oriented, then:*

$$\operatorname{tb}(K) + |r(K)| \leq -\chi(F)$$

where F is any Seifert surface of K.

Proof:

It suffices to show that $\operatorname{tb}(K)+r(K) \leq -\chi(F)$ (the other equality comes from reversing the orientation of K). From Theorem 4.29, we already know that for any compact, connected, oriented F with nonempty boundary:

$$|r(F)| + \sum_i t_i \leq -\chi(F)$$

where t_i are the twisting numbers of the boundary components, when each $t_i \leq 0$. Performing S_- enough times, we can ensure that $\operatorname{tb}(\partial F) \leq 0$, so that each $t_i \leq 0$.[a] This proves the result, since $\operatorname{tb}(K) =$

$\sum_i t_i$.

[a]This also justifies our assumption of $t_i \leq 0$ in our proof of Theorem 4.29

□

As a consequence of this inequality, given any knot in S^3, we can define an invariant $\mathrm{TB}(K) = \max(\mathrm{tb}(\overline{K}))$, where the maximum is taken over Legendrian representatives $\overline{K}$ of K.

Theorem 5.4. *Let $\kappa : \mathcal{L}(M,\xi) \to \mathcal{K}(M)$ be the forgetful map. Then κ induces a bijection $\overline{\kappa} : \mathcal{L}(M,\xi)/S_\pm \to \mathcal{K}(M)$.*

Proof Idea:

Surjectivity is a consequence of Proposition 4.2. To show injectivity, we note that an isotopy between knots is a 1-parameter family of knots. Any two knots representing the same isotopy class can be approximated by Legendrian curves and moreover the family of knots that are the isotopy can also be taken to be Legendrian (we need to allow $S_\pm$ operations on the family to get this work).

□

Theorem 5.5. *The transverse pushoff $\tau : \mathcal{L}(M,\xi) \to \mathcal{T}(M,\xi)$ induces a bijection $\overline{\tau} : \mathcal{L}(M,\xi)/S_- \to \mathcal{T}(M,\xi)$. Similarly, it induces a bijection $\mathcal{L}(M,\xi)/S_+ \to \mathcal{T}(M,-\xi)$.*

Proof:

Proposition 4.2b shows that τ is well defined up to transverse isotopy. First we note that $\tau S_- K = \tau K$. To show this, look at the standard model of K in $\mathbb{R}^3$ mod y translation. The transverse pushoff of $S_- K$ is positively transverse to ξ everywhere and is thus transverse isotopic to τK. This shows that $\overline{\tau}$ is well-defined. To show surjectivity, take a tubular neighborhood of K in $\mathcal{T}(M,\xi)$, which is modeled by the z axis in $(\mathbb{R}^3, dz + r^2/2d\theta)$ mod z translation. The Legendrian helix around this neighborhood has transverse pushoff that is transverse isotopic to K.

To show injectivity, suppose $K, K' \in \mathcal{L}(M,\xi)$ with τK and $\tau K'$ contact isotopic. Without loss of generality, $\tau K = \tau K'$. Take a tubular neighborhood N of τK such that $K \subset \partial N$ and similarly take N' a neighborhood of $\tau K = \tau K'$ with $K' \subset \partial N'$. Let $N'' \subset N \cap N'$ be a standard tubular neighborhood of τK with $K'' \subset \partial N''$ Legendrian with $\tau K'' = \tau K$. We will show that K, K'' are isotopic up to S_-, then the same argument applies to K', K'' and hence K and K' must also be isotopic same up to S_-. We see that $N - \mathrm{int}\, N'' \cong T^2 \times I$ with tight ξ. Moreover we can assume that it has a convex boundary and each component has 2 dividing curves. These are classified. Extend K to a $K \times I \subset T^2 \times I$. The dividing set Γ on $K \times I$ gives bypasses coming from stabilizations, i.e. K'' is made from K by stabilizing. Moreover, these stabilizations must be negative because $\ell(\tau K) = \mathrm{tb}(K) - r(K)$ being the same for K and K'' implies that there can be no S_+ stabilizations.

□

Corollary 5.6. *For $K \in \mathcal{T}(M,\xi)$ with ξ tight and K nullhomologous, then $\ell(K) \leq -\chi(F)$.*

Proof:

Write $K = \tau K'$. Then $\ell(K) = \ell(\tau K') = \mathrm{tb}(K') - r(K') \leq -\chi(F)$ by Proposition 5.2.

□

5.2 Knot Simplicity ❖

Theorem 5.7 (Eliashberg-Fraser '98, Etnyre-Honda '01)**.** *Suppose an oriented knot K is an unknot in (M^3,ξ), ξ tight, or a torus knot or a figure eight knot in (S^3, ξ_{std}), then Legendrian representatives of K are classified by $\mathrm{tb}(K), r(K)$ and* *transverse representatives are classified by $\ell(K)$. Furthermore, they all come from representatives with* $\mathrm{tb}(K) = \mathrm{TB}(K)$ *by $S_\pm$ and τ operations. Moreover, the maximal* TB *representative is unique, except for left-handed torus knots.*[8]

This theorem doesn't generalize to other types of knots, as shown in the next example which uses the following theorem:

Theorem 5.8 (Etnyre-Honda '02)**.** *Suppose $K = K_1\#...\#K_n$ in S^3, where each K_i is prime. Then $\kappa^{-1}(K)$ is given by $\kappa^{-1}(K_1) \times ... \times \kappa^{-1}(K_n)$ up to commuting stabilizations between summands and permuting topologically equivalent summands.*

The proof idea of this is similar to the proof we wrote for characterizing tight contact structures on $M_1\#M_2$. In fact, connected sums of knots K_i can be thought of special case of relative connected sums of relative three manifolds (M_i, K_i).

Example 5.9. Consider the connect sum of reversely oriented torus knots $K = \overline{T}_{p,q}\#\overline{T}_{p',q'}$ (for different p', p and q', q). What are the Legendrian representatives of K with $\mathrm{tb}(-)$ maximal? The above theorem says that, for each summand $\mathrm{tb}(-)$ and $r(-)$ are invariants of maximal tb represenatives of K. However, $r(K) = r(K_1) + r(K_2)$ for any $K = K_1\#K_2$, so there are are non-equivalent Legendrian representatives with the same value of $r(-)$.

Definition 5.10. A knot K is *Legendrian simple* (resp. *transverse simple*) if its Legendrian (resp. transverse) representatives are classified by $\mathrm{tb}(-), r(-)$ (resp. $\ell(-)$).

It is now known that there exist families of knots that are neither Legendrian simple nor transverse simple. The first examples of Legendrian non-simple knots came from contact homology.

5.3 Stein Manifolds ❖

Definition 5.11. A *Stein manifold* is a complex submanifold of $\mathbb{C}^n$ that is a closed subset.

Remark 5.12. By the Maximum Modulus principle, any Stein manifold cannot be compact.

A simple Stein manifold is $\mathbb{C}^n$ itself. Letting $\phi(z) = ||z||^2$, the level sets of this map are contact manifolds, (for example, $S^3 \subset \mathbb{C}^2$). The contact form α is $d(r^2) \circ i$ restricted to the level sets, where i is the multiplication by i map. This follows from the fact that $d\alpha$ is the standard symplectic form on $\mathbb{C}^n$. More generally, for any Stein manifold V, $\phi|_V$ has contact level sets that are tight. After a generic translation of V, ϕ is a Morse function and all of its critical points have index $\leq \dim_{\mathbb{C}} V = \frac{1}{2}\dim_{\mathbb{R}} V$.

Theorem 5.13. *A complex manifold V is Stein if and only if there exists a proper and bounded below $\phi : V \to \mathbb{R}$ such that all regular level sets are pseudo convex.*

We haven't defined pseudo convex, but know that it is equivalent to being tight contact away from critical points for $\dim_{\mathbb{C}} V = 2$, and in dimension greater than 2 it implies (but is not equivalent to) tight contact away from critical points.

Theorem 5.14 (Eliashberg '90)**.** *A smooth $2n$ manifold for $n > 2$ admits a Stein structure if and only if there exists an almost complex structure and exhausting Morse function with all indices $\leq n$.*

Definition 5.15. A *Stein domain* is $\phi^{-1}((-\infty, a])$ for a regular value a.

Definition 5.16. (M, ξ) is called *Stein fillable* if it is the boundary of a Stein domain.

Since a Stein domain is a Khäler manifold, it follows that Stein fillable $\Rightarrow$ symplectically fillable, which implies tightness.

Theorem 5.17 (Lisca-Matić, Kronheimer-Mrowka)**.** *Let F be a compact, connected, oriented surface embedded smoothly in a Stein surface X ($\dim_{\mathbb{C}} V = 2$). If F is not a nullhomotopic sphere, then $F\cdot F + |\langle c_1(TX), [F]\rangle| \leq -\chi(F) = 2g(F) - 2$.*

[8] In the left-hand torus knot case, it is still known which values of $\mathrm{tb}(-)$ and $r(-)$ are realized.

Remark 5.18. For a complex line bundle L, $c_1(L) = e(L)$.

Corollary 5.19. *Suppose K is a Legendrian knot in the boundary (M, ξ) of a Stein domain X and $F \subset X$ is a compact, connected oriented surface with $\partial F = K$. Then $\mathrm{tb}(K) + |r(F)| \leq -\chi(F)$.*

Here, $r(F)$ denotes $\langle c_1(TX, v), [F] \rangle$ (since F isn't a subset of M, we can't use our previous definition). This is a reasonable definition because $c_1(TX)|_M = e(\xi)$. This generalizes Theorem 5.3.

Proof Idea:

> Add a 2-handle to X along K with framing $\mathrm{tb}(K) - 1$. Then we get a new stein surface $\widehat{F} \subset \widehat{X}$ with $\widehat{F} \cdot \widehat{F} = \mathrm{tb}(K) - 1$ and $\langle c_1(TX), \widehat{F} \rangle = r(F)$. Since $\chi(\widehat{F}) = \chi(F) - 1$, the result follows by the previous theorem.

□

Remark 5.20. An overtwisted disk D violates this formula because $\mathrm{tb}(K) = 0$ and $\chi(D) = 0$. This proves that the standard contact structure on $\mathbb{R}^3$ is tight.

Corollary 5.21. *For $K \subset S^3$, $\mathrm{TB}(K) \leq 2$(4-ball genus of K) $- 2$.*

Appendix

A.1 More details on Framed Cobordisms of Framed Links ❖

In Section 1.3, we claimed that framed knots in M in the same homology class were cobordant in $M \times I$, and that moreover this cobordism F could be embedded in $M \times I$. To justify this, consider the edge homomorphism:

$$\mathfrak{ed}\colon \Omega_*^{SO}(X) \to H_*(M;\mathbb{Z})$$

For $* \leq 3$, this is an isomorphism, which sends $[X, f]$ to $f_*[X] \in H_*(M;\mathbb{Z})$. For $* = 4$ it is a surjection (since the edge map doesn't see the signature). It's important to notice that the map f is not required to be an embedding (and in general the problem of representing an homology class via an embedded submanifold is way harder than representing it just via a continuous map). In any case, for 1-dimensional homology classes (i.e. 1 dimensional submanifolds) we can homotope the map f to be a an embedding, hence we can work with embedded links representing homology classes.

Notice now that so far, this machinery tells us that whenever two links (which have to be think as pairs $(\coprod S^1, \imath)$) represent the same homology class in M, there is an abstract cobordism between them. Can we embed such cobordism into M? (i.e. as a cobordism of links into M) the answer is yes, and such cobordism takes the name of Seifert surface. A proof of its existence can be found in [2], page XXI of the introduction.

For the existence of a bordism connecting the links η and η' we can argue as follows. Consider the long exact sequence of the pair $(M \times I, \partial(M \times I))$, where $\partial(M \times I) = (M \times 0) \sqcup (M \times 1)$.

$$\longrightarrow H_2(M \times I, \partial(M \times I);\mathbb{Z}) \xrightarrow{\delta} H_1(\partial(M \times I);\mathbb{Z}) \xrightarrow{i_*} H_1(M \times I;\mathbb{Z}) \longrightarrow$$

and consider the element $[\eta] - [\eta'] \in H_1(\partial(M \times I);\mathbb{Z})$. Since they both represent the same class in $H_1(M;\mathbb{Z})$, using the homotopy equivalence between M and $M \times I$ and the fact that inclusion at level 0 is the same map in homology as inclusion at level 1, we have that $i_*([\eta] - [\eta']) = 0 \in H_1(M \times I;\mathbb{Z})$. Hence there is a class $\alpha \in H_2(M \times I, \partial(M \times I);\mathbb{Z})$ s.t. $\delta(\alpha) = [\eta] - [\eta']$. Since we can represents relative 2-homology classes of a four manifolds with surfaces with boundaries (with given boundary) we conclude that there exists an orientable surface F whose boundary is $\eta \sqcup -\eta'$ (where the minus means reverse orientation). F realizes an oriented cobordism between η and η' in $M \times I$.

Lemma A.1. *The $\mathbb{Z}$ action on $\Gamma^{-1}(x)$ for any $x \in H_1(M)$ is transitive.*

Proof:

> Recall that the action is the defined in the following way: given a link α_0 with a framing, then $n \cdot (\alpha_0)$ is defined to be α_0 with the framing obtained by adding n-twists to the original framing on it. Now let us consider another link α_1 in $\Gamma^{-1}(x)$ with its framing. We know that there is an oriented bordism connecting α_0 to α_1. We extend, via a partition of unity argument, the framings on these links to the interior of the bordism. Clearly there might be some zeroes here but we can push them off in the direction of α_0 in order to get a framing. Doing this, we are changing the framing on α_0 by a certain integer n. This means that $n \cdot \alpha_0$ is framed cobordant to α_1, proving transitivity.

□

This shows that $\Gamma^{-1}(x) \cong \mathbb{Z}/m\mathbb{Z}$ for some m (possibly zero). The content of Theorem 1.32 is that the integer m is twice the divisibility of x. The proof is as follows:

Proof (of Theorem 1.32):

As before, we start working in $M \times I$. Let α be a link in $\Gamma^{-1}(x)$, then it's enough to prove that any framed cobordism between $i_0(\alpha)$ and $i_1(\alpha)$ induced a framing on $i_1(\alpha)$ which differs by $2d$ twists form the framing induced by the trivial framed cobordism. Let F be such framed cobordism. If we introduce some zeroes on he framing on F we can assume that the framing on $i_0(\alpha)$ and $i_1(\alpha)$ coincide. Clearly F is no more a framed cobordism but it keeps being an oriented cobordism, i.e. an oriented compact surface with boundary α. This suggests that if we glue $M \times 0$ to $M \times 1$ we can consider $\tilde{F}$, F with the boundary components identified, to be a surface without boundary. Before doing that notice that by our previous claims it's enough to compute $e(N_{F\subset M\times I}F) = F \cdot F$. Now thanks to the fact that the framing on the boundary coincide, the normal bundle $N_{F\subset M\times I}(F)$ factors as a well-defined normal bundle $N_{\tilde{F}\subset M\times S^1}(\tilde{F})$ and any framing on the first induces a framing on the second and vice-versa. Hence we can compute $e(N\tilde{F}) = \tilde{F} \cdot \tilde{F}$. Now by Künneth theorem applied to $H_2(M \times S^1; \mathbb{Z})$ we have

$$H_2(M \times S^1; \mathbb{Z}) \cong H_1(M; \mathbb{Z}) \otimes H_1(S^1; \mathbb{Z}) \oplus H_2(M; \mathbb{Z})$$

which geometrically means that any 2-homology class (for example $\tilde{F}$) is uniquely determined by a 2 homology-class β in M plus a 2-homology class represented by a cylinder over a link in $H_1(M)$ with glued boundary components. After moving the surface representing β to $M \times \frac{1}{2}$ (parametrized as a rotation in this "torus"), we see that

$$\begin{aligned} \tilde{F} \cdot \tilde{F} &= ([\alpha \times S^1] + \beta) \cdot ([\alpha \times S^1] + \beta) \\ &= 2[\alpha \times S^1] \cdot \beta \end{aligned}$$

Since the intersection product only cares about a neighbor of where the transverse intersection takes place, it should be clear that $[\alpha \times S^1] \cdot \beta = [\alpha \times I] \cdot \beta = kd$, where d is the divisibility of $x = [\alpha]$ and k is any integer number. This concludes the proof.

$\square$

References

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[3] Starkston, Laura. *Contact Structures and Classifications of Legendrian and Transverse Knots*, Master's thesis, Harvard University. `http://www.math.harvard.edu/theses/senior/starkston/starkston.pdf`.