Complex Geometry

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These are lecture notes from the Fall 2020 course M392C Complex Geometry at the University of Texas at Austin taught by Prof. Bernd Seibert. The prerequisites for following these notes is single variable complex analysis, manifold theory, (e.g. from Guillemin and Pollack), and some familiarity with commutative algebra, algebraic geometry, and sheaf theory. Please email me at `gdavtor@math.utexas.edu` with any typos and suggestions.

1. Introduction

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(*from the syllabus*)

Complex geometry is the study of spaces endowed with holomorphic functions. The field connects the theory of complex functions in one variable with differential geometry (differentiable functions) and algebraic geometry (polynomial functions). Complex geometry indeed provides an ample source of examples for differential geometry, while providing the geometric picture for algebraic geometry.

The aim of the class is to provide a working knowledge of the subject to enable further study in any of the many particular areas of research. The focus is thus on introducing techniques, illustrated by typical examples, but without any particular application in mind.

Topics

The topics covered will roughly be:

I *Local theory:* Holomorphic functions, complex vector spaces, differential forms, analytic sets.

II *Compelx Manifolds:* Topology, examples, divisors, line bundles, coverings, blowing up.

III *Kähler Manifolds:* Kähler identities, Hodge theory, Lefschetz theorems.

IV *Special topics:* According to interests of participants, e.g. K3 surfaces, deformation theory, vector bundles.

Texts

The main reference is Huybrechts' textbook "Complex Geometry" ([Huy05]). An additional great source is Griffiths/Harris' classic "Principles of Algebraic Geometry" ([GH78]). Both books are available online from the UT library. Another text is "Holomorphic Functions of Several Variables" by Kaup and Kaup [KBK83].

2. Local Theory

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Lecture 8/27

2.1 Holomorphic functions of several variables ❖

There are three equivalent ways of formulating complex differentiability (holomorphicity) that one learns in an introductory complex analysis class.

1. A function $f : \mathbb{C} \to \mathbb{C}$ can be seen as a pair of functions $u, v : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x+iy) = g(x,y)+ih(x,y)$. For f to be complex differentiable, it must first be differentiable in the real sense; that is there exists a Jacobian at every point:
$$Df_p = \begin{pmatrix} \partial_x g & \partial_y g \\ \partial_x h & \partial_y h \end{pmatrix}_p$$
For f to be complex differentiable one also requires that this matrix, seen as a linear map of $\mathbb{R}^2 \to \mathbb{R}^2$, is complex linear under the standard identification of $\mathbb{R}^2$ and $\mathbb{C}$. In other words, it must be of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. This means that:
$$\partial_x g = \partial_y h$$
$$\partial_x h = -\partial_y g$$
These are also known as the **Cauchy-Riemann equations**. Changing coordinates to $z = x + iy$ and $\bar{z} = x - iy$ and defining the operators:
$$\partial_z := \frac{1}{2}(\partial_x - i\partial_y), \qquad \partial_{\bar{z}} := \frac{1}{2}(\partial_x + i\partial_y)$$
the Cauchy-Riemann equations are equivalent to $\partial_{\bar{z}} f = 0$.

2. The second equivalent formulation is that $f : U \to \mathbb{C}$ is holomorphic if for any point $z_0 \in U$ there exists a ball $B_\epsilon(z_0) \subset U$ such that it is a convergent **power series** in this ball:
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad \forall z \in B_\epsilon(z_0)$$

3. The third formulation is the **Cauchy integral formula**. A function $f : U \to \mathbb{C}$ is holomorphic if and only if it is continuously differentiable and for any $B_\epsilon(z_0) \subset U$ the folloiwng holds:
$$f(z_0) = \frac{1}{2\pi i} \int_{\partial B_\epsilon(z_0)} \frac{f(z)}{z - z_0}\, dz$$

When we look at functions of several complex variables, the definition of complex differentiability stays roughly the same. The standard complex coordinates on $\mathbb{C}^n$ are $z_1, ..., z_n$ where $z_i = x_i + iy_i$. Some take the convention to never use "i" as both an index and as a complex number; we will not use this convention and hope that it is clear which is which depending on the context.

Definition 2.1. Let $U \subset \mathbb{C}^n$ be open and let $f : U \to \mathbb{C}$ be a continuously differentiable function. Then f is holomorphic or complex differentiable if $w \mapsto f(z_1, ..., z_{i-1}, w, z_{i+1}, ..., z_n)$ is holomorphic for all $z_1, ..., z_{i-1}, z_{i_1}, ..., z_n \in U$ and for all $i = 1, ..., n$.

A more explicit way to say this is that all coordinate functions $z_i = x_i + iy_i$ satisfy the Cauchy-Riemann equations. Equivalently, $\partial_{\bar{z}_i} f = 0$ for all $i = 1, ..., n$. The Cauchy integral formula can be obtained by applying the one variable version to each component:

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|z_i - w_i| = \epsilon_i} \frac{f(\zeta_1, ..., \zeta_n)}{(\zeta_1 - z_n) \cdots (\zeta_n - z_n)} d\zeta_1 \cdots d\zeta_n$$

The natural domain on which this applies is different, however. It is the *polydisk* $D_\epsilon(w) = \{z_1, ..., z_n \ : \ z_i - w_i < \epsilon_i \ \forall \ i\}$, which is a product of disks of size ϵ_i and centers w_i. The standard polydisk Δ^n has unit sizes in each dimension and is centered at 0. Note that the boundary $\partial(\Delta^n)$ is not the same as $\{z \in \mathbb{C}^n \ : \ |z_i| = 1\}$ (known as the Shilov boundary).

Just as in the one variable case, the Cauchy integral formula can be used to derive the power series representation of f in a polydisk $D_\epsilon(w) \subset U$.

$$f(z) = \sum_{i_1, \ldots, i_n = 0}^{\infty} a_{i_1, \ldots, i_n} (z_1 - w_1)^{i_1} \cdots (z_n - w_n)^{i_n}$$

Definition 2.2. For any open subset $U \subset \mathbb{C}^n$, we denote $\mathcal{O}(U)$ to be the collection of holomorphic functions $f : U \to C$. Given an inclusion $U \subset V$, there is a natural restriction map $\mathcal{O}(V) \to \mathcal{O}(U)$.

2.1.1 Domains of Holomorphy

Definition 2.3. Given $U \subset \mathbb{C}^n$ open, the *holomorphic hull* is a domain $V \supset U$ with the following property. If $W \supset V$ is any set whose induced map $\mathcal{O}(W) \to \mathcal{O}(V)$ is bijective, then $W = V$.

For example, fuppose $f \in \mathcal{O}(\Delta^2 \setminus \{0\})$. Then f extends over 0; i.e. the restriction map $\mathcal{O}(\Delta^2) \to \mathcal{O}(\Delta^2 \setminus \{0\})$ is bijective. This is a consequence of Hartog's Lemma.

Lemma 2.4 (Hartogs). *Let ϵ, ϵ' be such that $\epsilon'_i < \epsilon_i$ for all i. Then $\mathcal{O}(D_\epsilon(0)) \to \mathcal{O}(\overline{D_{\epsilon'}(0)})$ is bijective.*

Proof:

> See Huybrechts [Huy05] Proposition 1.1.4. Note that injectivity follows immediately from the identity theorem in one dimension (i.e. if there is an extension, it must be unique). Surjectivity is the substance of this proof.

□

2.1.2 Weierstrass Polynomials

Suppose f is a holomorphic function on a polydisk. The zero locus is very similar to a polynomial zero locus. This is a result of the *Weierstrass polynomial* associated to f.

Definition 2.5. A *Weierstrass polynomial* is a polynomial on one variable w of the form

$$f(z) = w^d + a_1(z) w^{d-1} + ... + a_d(z)$$

where the coefficients $a_i(z)$ are functions in $n-1$ variables which are holomorphic in a disk which vanish at the origin.

In one dimension, recall that if a holomorphic function $f(z)$ vanishes at 0, then $f(z) = z^d g(z)$ for $g(0) \neq 0$ analytic. The polynomial z^d is the Weierstrass polynomial associated to f at zero. The Wieirstrass preparation theorem below is the generaliztion to higher dimensions. For $f : \mathbb{C}^n \to \mathbb{C}$, we denote $f_w(z_1) \equiv f(z_1, z_2, ..., z_n)$, where we see $w = (z_2, ..., z_n)$.

Theorem 2.6 (Weierstrass Preparation). *Let $f : B_\epsilon(0) \to \mathbb{C}$ be holomorphic with $f(0) = 0$ and $f_0(z_1)$ is not identically zero. Then there exists a Weieirstrass polynomial $g(z_1, w) = g_w(z_1)$ and a holomorphic function h on a smaller domain $B_{\epsilon'}(0) \subset B_\epsilon(0)$ such that $f = gh$ and $h(0) \neq 0$. The polynomial g is unique.*

Lemma 2.7. *Let $\epsilon, \epsilon' > 0$ and $f \in \mathcal{O}(D_\epsilon(0))$ and the zeros $\lambda_1, ..., \lambda_d$ of f are contained in $D_{\epsilon'}(0)$ (the zeros are counted with multiplicities). Then for any $k \geq 0$*

$$\sum_{i=1}^{n} \lambda_i^k = \frac{1}{2\pi i} \int z^k \frac{f'(z)}{f(z)}\, dz$$

where each λ_i appears as many times as its multiplicity.

Remark 2.8. The expressions on the left hand side are known as the Newton polynomials in the λ_i. It is a fact that the polynomial ring generated by the elementary symmetric polynomials in variables x_i is the same as the polynomial ring generated by the Newton polynomials in x_i.

Proof:

The residue theorem says that:

$$\frac{1}{2\pi i} \int z^k \frac{f'(z)}{f(z)}\, dz = \sum_{f(w)=0, w \in D_{\epsilon'}(0)} \mathrm{Res}_w \left(z^k \frac{f'(z)}{f(z)} \right)$$

We can compute the residues at each w by writing $f(z) = (z-w)^d h(z)$, where d is the order of the zero and $h(w) \neq 0$.

$$\begin{aligned} \mathrm{Res}_w \left(z^k \frac{f'(z)}{f(z)} \right) &= \frac{1}{(d-1)!} \lim_{z \to w} \left(\frac{d^{d-1}}{dz^{d-1}} (z-w)^d \left(z^k \frac{f'(z)}{f(z)} \right) \right) \\ &= \frac{1}{(d-1)!} \lim_{z \to w} \left(\frac{d^{d-1}}{dz^{d-1}} (z-w)^{d-1} \cdot \frac{z^k (dh(z) + (z-w)h'(z))}{h(z)} \right) \\ &= \lim_{z \to w} z^k \left(d - \frac{h'(z)}{h(z)} (z-w) \right) + (z-w)p(z) \end{aligned}$$

where $p(z)$ is some analytic function. Taking the limit we find $\mathrm{Res}_w(z^k f'(z)/f(z)) = dw^k$. The result now follows.

□

Proof of Weierstrass Preparation Theorem:

Let $U \subset \mathbb{C}^{n-1}$ be open and $\epsilon > 0$ such that $f|_{U \times \Delta_\epsilon}$ has no zeros, which comes from the assumptions of the theorem. For $w \in U$, let $\lambda_1(w), ..., \lambda_{d(w)}(w)$ be the zeros of $f|_{\{w\} \times \Delta_\epsilon}$, where $d(w)$ depends on w *a priori*. By Lemma 2.7 (after shrinking U and ϵ sufficiently) for $k = 0$, we have:

$$d(w) = \sum_{i=1}^{d(w)} \lambda_i^0(w) = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{\partial_z f(z)}{f(z)} dz$$

This varies continuously with w, and so we must have $d(w) = d$ be constant.

Define:

$$g(z, w) = \prod_{i=1}^{d} (z_1 - \lambda_i(w))$$

The coefficients of this polynomial are elementary symmetric polynomials in the roots λ_i. By the remark above, there is a way to express these as polynomials in the Newton polynomials $\sum_{i=1}^{d} \lambda_i(w)$. Therefore, as long as the Newton polynomials are holomorphic, $g(z, w)$ is a Weierstrass polynomial. However, this fact follows again from Lemma 2.7 because the RHS of the Lemma statement is a holomorphic function.

□

Exercise 2.9 (Huybrechts 1.16)**.** Consider the function $f : \mathbb{C}^2 \to \mathbb{C}$, $(z_1, z_2) \mapsto z_2^3 z_1 + z_2 z_1 + z_2^2 z_1^2 + z_1^2 + z_2 z_1^3$ and find an explicit decomposition $f = g_w h$ as claimed by the Weierstrass preparation theorem.

Solution:

Indeed $f_0(z_1) = z_1^2$ is not identically zero, and moreover we see that the degree of g_w must be 2. Let $a_1(w)$ and $a_2(w)$ be the roots of $f_w(z_1)$ in a disk of radius sufficiently small. Writing $f_w(z_1) = z_1(z_2^3 + z_2 + z_2^2 z_1 + z_1 + z_2 z_1^2)$, we see that $z_1 = 0$ is a root always. So we can take $a_1(w) = 0$. To find the second root, we set the other factor to zero and solve for z_1:

$$z_3^3 + z_2 + z_2^2 z_1 + z_2 z_1^2 + z_1 = 0 \iff z_1^2 + \left(z_2 + \frac{1}{z_2}\right) z_1 + (1 + z_2^2) = 0$$

There are two solutions to this given by the quadratic formula:

$$z_1 = -\left(z_2 + \frac{1}{z_2}\right) \pm \sqrt{\left(z_2 + \frac{1}{z_2}\right)^2 - 4(1 + z_2^2)}$$

Notice that, since we are working in a sufficiently small neighborhood of $(0, 0)$ we can use a first order Taylor approximation:

$$z_1 \approx -2\left(z_2 + \frac{1}{z_2}\right) \quad \text{or} \quad z_1 \approx 0$$

The first root is unbounded at 0, and so doesn't exist in the ϵ poly-disk of 0. The second does, so therefore:

$$a_2(w) = a_2(z_2) = -\left(z_2 + \frac{1}{z_2}\right) + \sqrt{\left(z_2 + \frac{1}{z_2}\right)^2 - 4(1 + z_2^2)}$$

This is holomorphic in a sufficiently small poly-disk about $(0, 0)$. Finally, the Weierstrass polynomial is:

$$\begin{aligned} g_w(z_1) &= (z_1 - a_1(w))(z_1 - a_2(w)) \\ &= z_1 \left(z_1 + \left(z_2 + \frac{1}{z_2}\right) - \sqrt{\left(z_2 + \frac{1}{z_2}\right)^2 - 4(1 + z_2^2)}\right) \end{aligned}$$

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Lecture 9/1

2.1.3 Domains of Convergence

Consider the power series:

$$\sum_{k \in \mathbb{N}^n} a_k z^k$$

where k is a multi-index. Where does this converge? A good guess is some poly-disk, but that isn't quite right. To determine the domain of convergence, we will use the following version of Abel's lemma:

Lemma 2.10. *Let $w \in (\mathbb{C}^*)^n$ such that for all $k \in \mathbb{N}^n$, $|a_k w^k| \le C$. Then $\sum a_k z^k$ converges absolutely in $D_w(0)$.*

Proof:

Observe that for $z \in D_w(0)$, we have $|z_j|/|w_j| < 1$ for all j. Then we have:

$$\left|\sum_k a_k z^k\right| \le \sum_k |a_k z^k| = \sum_k |a_k w^k| \frac{|z^k|}{|w^k|} \le C \sum_k \frac{|z^k|}{|w^k|}$$

The RHS sum converges because it converges in each direction in $\mathbb{N}^n$.

□

Definition 2.11. Let $\kappa : \mathbb{C}^n \to \mathbb{R}^n_{\geq 0}$ be given by $z \mapsto (|z_1|, ..., |z_n|)$. A *Reinhardt domain* is a preimage $\kappa^{-1}(\Omega)$ for $\Omega \subset \mathbb{R}^n_{\geq 0}$ connected and open.

Note that the preimage of any point under κ is a torus, and so a Reinhardt domain is a union of tori (not necessarily all of the same dimension). There is also a a Log map $\log : (\mathbb{C}^*)^n \to \mathbb{R}^n$ given by $z \mapsto (\log |z_1|, ..., \log |z_n|)$. If $U \subset \mathbb{C}^n$ is a Reinhardt domain, then it is *logarithmically convex* if $\log(U \cap (\mathbb{C}^*)^n)$ is convex.

Proposition 2.12. *If U is the domain of convergence of $\sum_{k\in\mathbb{N}} a_k z^k$, then U is a logarithmically convex Reinhardt domain.*

Proof:

It is clear that U is a Reinhardt domain because sending $z \mapsto (e^{i\theta_1} z_1, ..., e^{i\theta_n} z_n)$ does not change convergence behavior. Now assume $\sum a_k z^k$ converges at $|z| = \rho$ and $|z| = \sigma$. This implies $|a_k \rho^k|$ and $|a_k \sigma^k|$ are bounded uniformly accross k by some constant C. Let $\chi \in \mathbb{R}^n_{\geq 0}$ be such that $\log(\chi) = \alpha \log(\rho) + \beta \log(\sigma)$, where $\alpha + \beta = 1$. This is a convex combination of $\log(\rho)$ and $\log(\sigma)$. Then for any multi-index $k \in \mathbb{N}^n$:

$$\begin{aligned}
\log(\chi^k) &= \log(\chi_1^{k_1} \cdots \chi_n^{k_n}) \\
&= \sum_{i=1}^{n} k_i \log(\chi_i) \\
&= \alpha \sum k_i \log(\rho_i) + \beta \sum k_i \log(\sigma_i) \\
&= \alpha \log(\rho^k) + \beta \log(\sigma^k) \\
&\leq \alpha \log(C/|a_k|) + \beta \log(C/|a_k|) = \log(C/|a_k|)
\end{aligned}$$

This implies $|a_k \chi^k| < C$, and hence by the Lemma above $\sum a_k z^k$ converges on D_χ. This shows that U is logarithmically convex.

□

2.1.4 The identity and open mapping theorems

Theorem 2.13 (Identity Theorem)**.** *Suppose that f and g are two holomorphic functions on a domain $U \subset \mathbb{C}^n$ which agree on an open neighborhood of a point $z_0 \in U$. Then $f = g$ everywhere in U.*

Proof:

By replacing $f \to f - g$, it suffices to assume $g = 0$. Let $Z \subset U$ be the locus of points where all derivatives of f vanish. This is a closed subset of U. For any $w \in Z$, expand f in a Taylor series that converges in a polydisk $D_\epsilon(w) \subset U$. Since the derivatives of f at w all vanish, the Taylor series is identically on $D_\epsilon(w)$. Therefore $D_\epsilon(w) \subset Z$, which means Z is open in U. Since Z is nonempty (by assumption that f vanishes in an open subset somewhere), it and is both closed and open, we must have $U = Z$.

□

Theorem 2.14 (Open Mapping Theorem)**.** *Any non-constant holomorphic function $f : U \to \mathbb{C}$ is an open mapping, i.e. it maps open sets to open sets.*

Proof:

This follows from the one dimensional case. Let z_0 be in the image of f and let w_0 be such that $f(w_0) = z_0$. There is at least one direction (line) from w_0 in which f is not constant. If we restrict f to this line L, an open neighborhood V of z_0 intersects L at an open subset of that line. Therefore $f(V \cap L) \subset f(V)$ is open by the one dimensional open mapping theorem and hence z_0 has an open neighborhood contained in $f(V)$.

□

Lecture 9/3 and 9/8

2.2 Complex Vector Spaces

A complex vector space can be seen as a $2n$ dimensional real vector space together with a *complex structure* I, which is an endomorphism such that $I^2 = -\mathrm{id}$. Every real vector space V can be complexified by tensoring with $\mathbb{C}$:

$$V \mapsto V \otimes_{\mathbb{R}} \mathbb{C}$$

This complexified vector space is isomorphic to $V \oplus V$, where the first component is understood to be the real part of $V \otimes \mathbb{C}$ and the second is the imaginary part. With this identification, the complex structure is $(v, w) \mapsto (-w, v)$.

Conversely, a real structure on a complex vector space W is a conjugate-linear involution $\kappa : W \to W$. The fixed locus $V = W^\kappa$ of this involution is a subspace such that there is a cononical isomorphism $W \cong V \otimes_{\mathbb{R}} \mathbb{C}$. In other words, complexifying is compatible with choosing a real structure. The maps:

$$\mathrm{Re} : W \to W^\kappa, \qquad w \mapsto \frac{1}{2}(w + \kappa(w))$$

$$\mathrm{Im} : W \to iW^\kappa, \qquad w \mapsto \frac{1}{2i}(w - \kappa(w))$$

Provide two projections $W \to V$.

If V is now a *complex* vector space, we can see it as a real vector space and hence complexify it to $V \otimes_{\mathbb{R}} \mathbb{C}$. Doing so, we get a canonical isomorphism of *complex* vector spaces $V \otimes_{\mathbb{R}} \mathbb{C} \cong V \oplus \overline{V}$, where $\overline{V}$ is the complex vector space with the same underlying real vector space as V but with the complex structure $-I$. It is also isomorphic to $V \oplus V$. Each of these three ways to write it comes with a complex structure, which we summarize below.

Vector Space	$V \otimes_{\mathbb{R}} \mathbb{C}$	$V \oplus V$	$V \oplus \overline{V}$
Iso. to $V \otimes_{\mathbb{R}} \mathbb{C}$ maps $v \otimes (a + bi)$ to	$v \otimes (a + bi)$	(av, bv)	$(av + ibv, av - ibv)$
Complex Structure	$v \otimes \alpha \mapsto v \otimes \bar{\alpha}$	$(v, w) \mapsto (v, -w)$	$(u, \overline{u}) \mapsto (\bar{u}, u)$

2.3 The local ring $\mathcal{O}_{\mathbb{C}^n,0}$

Definition 2.15. Given $U \subset \mathbb{C}^n$ open, a set $Z \subseteq U$ is *analytic* if for all $w \in Z$, there exists $V \subseteq U$ open containing w and $f_1, \ldots, f_n \in \mathcal{O}(V)$ such that $Z \cap V = \{z \in V \mid f_1(z) = \ldots = f_n(z) = 0\}$. If we make reference to an analytic subset without a corresponding U, it is understood that $U = \mathbb{C}^n$.

Note that the vanishing locus of a collection of holomorphic functions $f_1, \ldots, f_n$ is automatically analytic. Every analytic subset is locally the vanishing locus of a finite collection of holomorphic functions.

Definition 2.16. A *germ* is an equivalence class of sets $X \subset \mathbb{C}^n$ containing 0, where $X \sim X'$ if and only if there exists an open set $0 \in U$ such that $U \cap X = U \cap X'$. A germ X is called *analytic* if there exists an open $U \supset X$ and holomorphic functions $f_1, \ldots, f_k \in \mathcal{O}(U)$ such that $X \sim Z(f_1, \ldots, f_k)$.

The functions that cut out the analytic germs are exactly the set of germs of functions at 0, otherwise known as $\mathcal{O}_{\mathbb{C}^n,0}$:

Definition 2.17. The functor $\mathcal{O}_{\mathbb{C}^n}$ sending $U \mapsto \mathcal{O}_{\mathbb{C}^n}(U)$ is a sheaf and we denote $\mathcal{O}_{\mathbb{C}^n,z}$ to be its stalk at z. This is the set of equivalence classes of pairs (also called germs) (U, f) for U an open neighborhood of z and $f : U \to \mathbb{C}$. Two pairs (U, f) and (U', f') are equivalent if there exists $V \subset U \cap U'$ such that $f|_V = f'|_V$.

Remark 2.18. The ring $\mathcal{O}_{\mathbb{C}^n,0}$ is also the ring of convergent power series $\mathbb{C}\{z_1, \ldots, z_n\} \subset \mathbb{C}[[z_1, \ldots, z_n]]$. Moreover, $\mathcal{O}^\times_{\mathbb{C}^n,0} = \mathcal{O}_{\mathbb{C}^n,0} \setminus \mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal $\mathfrak{m} = \{f \in \mathcal{O}_{\mathbb{C}^n,0} \mid f(0) = 0\}$. For any $f \in \mathcal{O}_{\mathbb{C}^n,0}$, we let $Z(f)$ be the analytic germ cut out by f about 0.

Proposition 2.19. *The ring $\mathcal{O}_{\mathbb{C}^n,0}$ is:*

1. *An integral domain and a UFD.*
2. *Noetherian.*
3. *A local ring with maximal ideal $\mathfrak{m} = \{f \in \mathcal{O}_{\mathbb{C}^n,0} \mid f(0) = 0\}$.*

Proof:

1. Suppose $f \cdot g = 0$ for $f, g \in \mathcal{O}_{\mathbb{C}^n,0}$. Then without loss of generality $f, g \in \mathcal{O}(D_\epsilon)$. If $f \not\equiv 0$, then there exists $z \in D_\epsilon$ with $f(z) \neq 0$, and hence we can shrink D_ϵ so that $f|_{D_\epsilon(z)} \neq 0$. This implies $g|_{D_{\epsilon'}(z)} = 0$, and so by the identity theorem $g = 0$.
   The fact that it is a UFD comes from an application of the Weierstrass Preparation Theorem. Using the fact that R UFD $\Rightarrow R[x]$ UFD, we find $\mathcal{O}_{\mathbb{C}^{n-1},0}[z]$ is a UFD by induction on n. Then proceed with the Weierstrass Preparation Theorem. For details, see [Huy05].
2. We will prove this in subsection 2.5.
3. Exercise.

□

Corollary 2.20. *Given a germ of a hypersurface $Z(f)$ with $f \in \mathcal{O}_{\mathbb{C}^n,0} \setminus \{0\}$, it decomposes uniquely into germs of irreducible hypersurfaces.*

Lemma 2.21. *Let $f, g \in \mathcal{O}_{\mathbb{C}^n,0}$ and $Z(g) \subset Z(f)$. Then g divides f.*

Proof of Corollary 2.20:

Decompose $f = g_1 \cdots g_r$ for g_r irreducible, by the UFD property. Thus $Z(f) = Z(g_1) \cup \cdots \cup Z(g_r)$. Since g_i is irreducible and $\mathcal{O}_{\mathbb{C}^n,0}$ is a UFD, it is prime. By Lemma 1.1.28 of [Huy05], the germ $Z(g_i)$ is irreducible. By Lemma 2.21, this is a unique decomposition.

□

Lemma 2.22. *Let X be an analytic germ about $0 \in \mathbb{C}^n$. Then X is irreducible if and only if $I(X) \subset \mathcal{O}_{\mathbb{C}^n,0}$ is a prime ideal.*

2.3.1 The Nullstellensatz

The reason we localize all of this discussion to a point $0 \in \mathbb{C}^n$ is because there are very useful ring properties of $\mathcal{O}_{\mathbb{C}^n,0}$ that are very nearly identical to those of a coordinate ring of an affine variety. If X is a germ about the origin, we denote $I(X) \subset \mathcal{O}_{\mathbb{C}^n,0}$ to be the set of elements f such that $X \subset Z(f)$.

Lemma 2.23. *For any germ X, the set $I(X) \subset \mathcal{O}_{\mathbb{C}^n,0}$ is an ideal. If $(A) \subset \mathcal{O}_{\mathbb{C}^n,0}$ denotes the ideal generated by a subset $A \subset \mathcal{O}_{\mathbb{C}^n,0}$, then $Z(A) = Z((A))$ and $Z(A)$ is analytic.*

Definition 2.24. Given an ideal I in a ring R, the *nilradical* is $\sqrt{I} = \{a \in R \mid a^k \in I \text{ for some } k\}$.

There is a basic correspondence:

$$\{I \subset \mathcal{O}_{\mathbb{C}^n,0} \mid I = \sqrt{I}\} \leftrightarrow \{\text{Analytic germs about } 0 \in \mathbb{C}^n\}$$

$$I \mapsto Z(I)$$

$$I(Z) \leftarrow\!\shortmid Z$$

This is the analytic Nullstellensatz.

2.4 Inverse and Implicit Function Theorems ❖

Lecture 9/8

If $U \subset \mathbb{C}^m$ is open, a holomorphic map $f : U \to \mathbb{C}^n$ is one whose components are each holomorphic. Identifying $\mathbb{C}^m$ with $\mathbb{R}^{2m}$ via $z_i = x_i + iy_i$ and $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ via $w_j = u_j + iv_j$, we get a decomposition $f_j = g_j + ih_j$. The real Jacobian is:

$$J_{\mathbb{R}} f(z) = \begin{pmatrix} (\partial_{x_j} g_i) & (\partial_{y_j} g_i) \\ (\partial_{x_j} h_i) & (\partial_{y_j} h_i) \end{pmatrix}, \qquad i = 1, ..., m, j = 1, ..., n$$

seen as a real $2n$ by $2m$ matrix, which we see as an element of $\mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^n)$, where $\mathbb{C}^n$ and $\mathbb{C}^m$ are seen as real vector spaces. The complexified version of this linear map is:

$$J_{\mathbb{C}} f(z) = J_{\mathbb{R}} f(z) \otimes \mathrm{id}_{\mathbb{C}}$$

This now is a complex linear map in $\mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^n)$. Rewrite this with $z_j, \bar{z}_j, f_j, \bar{f}_j$:

$$J_{\mathbb{C}} f(z) = \begin{pmatrix} (\partial_{z_j} f_i) & (\partial_{\bar{z}_j} f_i) \\ (\partial_{z_j} \bar{f}_i) & (\partial_{\bar{z}_j} \bar{f}_j) \end{pmatrix}, \qquad i = 1, ..., m, j = 1, ..., n$$

Exercise 2.25. Show that this version of $J_{\mathbb{C}}$ is a conjugate to the first one and compute the conjugation matrix.

Since f is holomorphic, the off-diagonal blocks vanish and it takes the form:

$$J_{\mathbb{C}} f(z) = \begin{pmatrix} (\partial_{z_j} f_i) & 0 \\ 0 & \overline{(\partial_{z_j} f_i)} \end{pmatrix}$$

We define $Jf(z) := \partial_{z_j} f_i$. Note that $\det J_{\mathbb{C}} f(z) = \det J_{\mathbb{R}} f(z) = (\det Jf(z))^2$. In particular, the maps $J_{\mathbb{C}} f(z)$ and $J_{\mathbb{R}} f(z)$ are automatically orientation preserving.

Definition 2.26. A map $f : U \to V$, where $U \subset \mathbb{C}^m$ and $V \subset \mathbb{C}^n$ are open, is *biholomorphic* if there exists $g : f(V) \to V$ holomorphic such that $f \circ g = \mathrm{id}_{f(v)}$ and $g \circ f = \mathrm{id}_V$.

Theorem 2.27 (Inverse Function Theorem). *Suppose $f : U \to \mathbb{C}^n$ is a holomorphic map with $Jf(z_0) \neq 0$ for some $z_0 \in U$. Then there exists $U' \subset U$ open containing z_0 such that $f|_{U'} : U' \to f(U')$ is a biholomorphism.*

Proof:

The existence of the local inverse $g = f^{-1}$ follows from standard multivariable analysis. We must show that g is holomorphic. Let $w_1, ..., w_n$ be coordinates on $\mathbb{C}^n$. By the chain rule and the Cauchy-Riemann equations for f, we have:

$$\begin{aligned} 0 &= \partial_{\bar{z}_j}(f^{-1} \circ f) \\ &= \sum_{k=1}^{n} \partial_{w_k} f^{-1} \cdot \partial_{\bar{z}_j} f_k + \partial_{\bar{w}_k} f^{-1} \cdot \partial_{\bar{z}_j} \bar{f}_k \\ &= \sum_{k=1}^{n} \overline{\partial_{z_j} f_k} \cdot \partial_{\bar{w}_k} f^{-1} \\ &= \overline{Jf(z)} \cdot (\partial_{\bar{w}_k} f^{-1}) \end{aligned}$$

Since $Jf(z)$ is invertible in a sufficiently small neighborhood, we must have $\partial_{\bar{w}_k} f^{-1} = 0$ for all k in this neighborhood.

□

Proposition 2.28. *If $f : U \to V$ is a holomorphic bijection, then it is biholomorphic.*

Remark 2.29. Note that replacing "holomorphic" with "smooth" makes this not necessarily true. For example $y = x^3$.

Theorem 2.30 (Implicit Function Theorem)**.** *Let $U \subset \mathbb{C}^m \times \mathbb{C}^n$ be open and $f : U \to \mathbb{C}^n$ be holomorphic. Let (z, w) be coordinates on $\mathbb{C}^m \times \mathbb{C}^n$. Assume $Jf = (\partial_{w_j} f_i)$ is invertible at some $(z_0, w_0) \in U$ and $f(z_0, w_0) = 0$. Then there exists $U_1 \subset \mathbb{C}^m, U_2 \subset \mathbb{C}^n$ open with $U_1 \times U_2 \subset U$ and $(z_0, w_0) \in U_1 \times U_2$ together with a function $g : U_1 \to U_2$ such that:*

$$\{(z, w) \in U_1 \times U_2 \mid f(z, w) = 0\} = \Gamma_g = \{(z, g(z)) \mid z \in U_1\}$$

Proof:

Once again, the existence of such a g is given to us by the real version of this theorem, so we must only show that such a g is holomorphic. By construction, $f_i(z, g(z)) = 0$ and hence by the chain rule:

$$\begin{aligned} 0 &= \partial_{\bar{z}_j}(f_i(z, g(z))) \\ &= \sum_{k=1}^{n} \partial_{w_k} f_i(z, g(z)) \cdot \partial_{\bar{z}_j} g \\ &= Jf(z, g(z)) \cdot (\partial_{\bar{z}_j} g) \end{aligned}$$

Since $Jf(z, g(z))$ is invertible for z in a sufficiently small neighborhood of z_0, we have $\partial_{\bar{z}_j} g = 0$ for all j.

□

Lecture 9/10

2.5 Analytic Sets: Local Finite Mappings and Dimension ❖

Let R be a commutative ring with unit. Given $f, g \in R[t]$ of degrees d and e, respectively. Define:

$$\phi : R[t]^{d-1} \oplus R[t]^{e-1} \to R[t]^{d+e-1}$$

$$(p, q) \mapsto pg + qf$$

This is an R-linear homomorphism of free R modules of rank $d+e$. We can take the determinant of this mapping, which is called the *resultant*:

$$\text{Res}(f, g) := \det(\phi)$$

This is a polynomial in the coefficients of f and g

Remark 2.31. If $f = a_0 \prod_i (t - \lambda_i)$ and $g = b_0 \prod_j (t - \mu_j)$, then the resultant is $a_0 b_0 \prod_{i,j} (\lambda_i - \mu_j)$.

The resultant has the property that $\text{Res}(f, g) = 0$ when f and g share a common factor. Indeed, if $f = a\bar{f}$ and $g = a\bar{g}$, then $\phi(\bar{f}, -\bar{g}) = \bar{f}g - \bar{g}f = 0$ and hence the determinant is zero. Moreover, the familiar discriminant Δf is $\text{Res}(f, f')$.

Proposition 2.32. *If R is a UFD, and if $f, g \in R[t]$ with f monic, then f and g being relatively prime is equivalent to* $\text{Res}(f, g) = 0$.

We will use this with the ring $\mathcal{O}_{\mathbb{C}^n,0}$, which is a UFD and we want to study analytic germs $X = Z(f)$. Without loss of generality, f is a Weierstrass polynomial $g(w, z) = w^d + a_1(z) w^{d-1} + \cdots + a_d(z)$ where $a_i \in \mathcal{O}(D_\epsilon)$. This requires us to work in a polydisk $D_\epsilon \subset \mathbb{C}^{n-1}$ for the z variables. For $\mu \in \mathbb{N}$, define:

$$A_\mu = \{(z, w) \in D_\epsilon \times \mathbb{C} \mid \partial_w^i g(z, w) = 0 \ i = 0, ..., \mu\}$$

Note that $A_0 = Z(g)$. The higher critical loci A_μ for $\mu \geq 1$ are the "bad" regions around where $\pi : X \subset \mathbb{C}^n \to \mathbb{C}^{n-1}$ is not a local homeomorphism. In particular, observe that $X \setminus A_1 \to D_\epsilon \setminus \pi(A_1)$ is a covering space. This might be a problem, however, if $\pi(A_\mu) = D_\epsilon$.

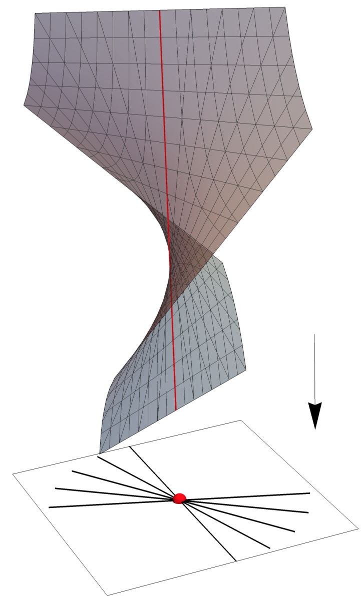

Figure 2.1: The mapping from Example 2.38 which is not finite at the origin.

Lemma 2.33. *If g is irreducible, for all $\mu \in \mathbb{N}$, $\pi(A_\mu)$ is a proper analytic subset of D_ϵ.*

Proof:

Note that $\pi(A_0) = D_\epsilon$, $\pi(A_1) = Z(\mathrm{Res}(g, \partial_w g))$. In fact, $\pi(A_\mu) = Z(\mathrm{Res}(g, \partial_w), \cdots, \mathrm{Res}(g, \partial_w^\mu g))$. This is well-defined because the resultants are polynomials in elements of $\mathcal{O}_{\mathbb{C}^n,0}$. This exhibits $\pi(A_\mu)$ as an analytic subset. Moreover, it is proper because if $\pi(A_1) = D_\epsilon$, then $\mathrm{Res}(g, g') = 0$ and so g and g' have a common factor. This contradicts the fact that g is irreducible.

□

Remark 2.34. Since $A \subset D_\epsilon$ is analytic and properly contained, then it is nowhere dense. This follows from the identity theorem.

Exercise 2.35. Using the implicit function theorem, prove that $X \setminus A_1 \to D_\epsilon \setminus \pi(A_1)$ is a covering space as claimed.

Example 2.36. Consider the elliptic curve $w^2 = z(z-1)(z-\lambda)$ with $\lambda \in \mathbb{C} \setminus \{0, 1\}$. We are taking $f = w^2 - z(z-1)(z-\lambda)$. Fix $p \in \mathbb{C} \setminus \{0, 1, \lambda\}$ and let $X^* \subset X = Z(f)$ be the subset away from the branch locus. The unbranched covering $X^* \to \mathbb{C} \setminus \{0, 1, \lambda\}$ outside of the points $0, 1$ and λ is the same as a representation of $\pi_1(\mathbb{C} \setminus \{0, 1, \lambda\}) \to S(\pi^{-1}(p)) \cong S_2$. Any generator γ_i maps to the nonzero element of S_2.

Lecture 9/15

The following geometric version of Noether's Normalization Lemma shows how, locally, analytic sets project to a lower dimensional complex space with finite fibers. This is a generalization of the above discussion, where we just considered sets of the form $X = Z(f)$.

Theorem 2.37 (Local Finite Mapping / Noether Normalization)**.** *Let $X \subseteq \mathbb{C}^n$ be an analytic germ about 0. Then there exists $d \leq n$ and a linear coordinate system $(z_1, ..., z_d, z_{d+1}, ..., z_n)$ such that the projection $\pi : \mathbb{C}^n \to \mathbb{C}^d$, $(z_1, ..., z_n) \to (z_1, ..., z_d)$ induces a proper and open surjection $X \to \mathbb{C}^d$ of germs with finite fibers.*

There is an algebraic proof of this theorem by observing that $\mathcal{O}_{\mathbb{C}^n,0} \hookrightarrow \mathcal{O}_{X,0}$ is an integral ring extension of finite type.

The following is a *non-example* of such a projection. Let $\pi : \mathbb{C}^3 \to \mathbb{C}^2, (x, y, z) \mapsto (x, y)$ and $X = Z(y - zx)$. For fixed $z = c$, this hypersurface is a line containing $(c, 0, 0)$. Therefore X is a family of lines (see Figure 2.1); it is also irreducible. However, π is not finite because $\pi^{-1}(0, 0)$ is an infinite set. One can check that $\mathcal{O}_{\mathbb{C}^2,0} \to \mathcal{O}_{X,0}$ is not an integral ring extension. However, if we change the projection slightly, we can ensure that the fibers are finite.

We present two applications of Theorem 2.37.

1. **Dimension:** The first is simply that we can define $\dim(X) := d$. How is this well-defined? One way to show that is to use the branched covering picture and the inverse function theorem to write a homeomorphism $\mathbb{C}^d \to \mathbb{C}^{d'}$, and conclude that $d = d'$. Another way is to show that it coincides with the Krull dimension (maximal length of chains of prime ideals) of $\mathcal{O}_{X,0}$.

2. **Stratification:** If X is an analytic subset of $U \subset \mathbb{C}^n$, then we can consider the singular locus X_{sing} which is analytic. This gives rise to a stratification:

$$X \supset X_{\text{sing}} \supset (X_{\text{sing}})_{\text{sing}} \supset \cdots$$

   Then the dimension of X can also be defined as $\dim(X) := \dim_{\mathbb{C}}(X \setminus X_{\text{sing}})$, which is the dimension in the usual differential geometry sense outside of the singular locus.

3. Complex Manifolds

❖

A complex manifold is a real, even dimensional manifold with the additional condition that we require that the atlas be holomorphic.

Definition 3.1. A *holomorphic atlas* of a smooth manifold is an atlas $\{(U_i, \phi_i)\}$, where each chart $\phi : U_i \to \mathbb{C}^n$ is a diffeomorphism to its image, such that the transition functions $\phi_{ij} := \phi_i \circ \phi_j^{-1}$ are holomorphic. Two holomorphic atlases $\{(U_i, \phi_i)\}$ and $\{(U'_j, \phi'_j)\}$ are called equivalent if all maps $\phi_i \circ \phi_j'^{-1}$ are holomorphic.

Definition 3.2. A complex manifold X of complex dimension n is a real differentiable manifold of dimension $2n$ equipped with an equivalence class of holomorphic atlases.

The holomorphic functions on a complex manifold X are maps $f : X \to \mathbb{C}$ such that $f \circ \phi_i^{-1} : \phi_i(U_i) \to \mathbb{C}$ is holomorphic for any chart (U_i, ϕ_i) of X. This allows us to define the sheaf of holomorphic functions $\mathcal{O}_X(U) := \{f : U \to \mathbb{C} \text{ holomorphic}\}$ and germs $\mathcal{O}_{X,p}$ for $p \in X$. Moreover there is the notion of holomorphic maps between complex manifolds.

Definition 3.3. A continuous map $f : X \to Y$ of complex manifolds is called holomorphic if for any chart (U, ϕ) on X and (U', ϕ') on Y, the map $\phi' \circ f \circ \phi^{-1}$ is holomorphic. THe map f is called biholomorphic if it is also a homeomorphism.

Let $U \subset \mathbb{C}^n$ be open and let $f \in \mathcal{O}(U) \setminus \{0\}$. Let $X = Z(f)$, which we assume to be nonempty. This is a complex manifold if $\nabla f|_X \neq 0$, i.e. for all $x \in X$, there exists i such that $\frac{\partial f}{\partial z_i}(x) \neq 0$. The implicit function theorem shows that X is an $n-1$ dimensional submanifold of $\mathbb{C}^n$.

Example 3.4. Let $X = Z(zu^2 - v^2) \subset \mathbb{C}^3$, where the coordinates on $\mathbb{C}^3$ are (z, u, v). This is the Whitney Umbrella. The singular locus is the z axis $(z, 0, 0)$, which we can find by computing the partial derivatives. The locus of points where X is locally reducible (i.e. reducible as a germ) is a subset of this locus. At 0, $zu^2 - v \in \mathcal{O}_{\mathbb{C}^3,0}$ is irreducible. However, away from $z = 0$, locally we can take a square root h of z and so the function $zu^2 - v^2 \in \mathcal{O}_{\mathbb{C}^3,p}$ can be factored $(hu - v)(hu + v)$ and hence the germ of X at $p \neq 0$ is reducible. Therefore the reducible locus is the origin.

Lecture 9/17

3.1 Projective Space ❖

The most important example of a (compact) complex manifold is complex projective space:

$$\mathbb{CP}^n = \{\text{lines } L \text{ through } 0 \ \subset \mathbb{C}^{n+1}\} \cong (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$$

where the $\mathbb{C}^*$ action is given by component wise multiplication. We will hereafter denote $\mathbb{P}^n := \mathbb{CP}^n$. This inherits the quotient topology via $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$. Since $S^{2n+1} \subset \mathbb{C}^{n+1} \setminus \{0\}$ surjects on to $\mathbb{P}^n$ via π, complex projective space is compact.

Remark 3.5. For any finite dimensional complex vector space V, we can repeat this construction to get its projectivization $\mathbb{P}(V)$. Any complex linear map $T : V \to W$ can also be projectivized naturally, denoted $\mathbb{P}(T)$, and hence $\mathbb{P}(-)$ defines a functor from complex vector spaces to complex manifolds.

We use homogenous coordinate notation on $\mathbb{P}^n$:

$$\mathbb{P}^n \ni [z_0, ..., z_n] := \text{Equivalence class of } (z_0, ..., z_n) \text{ under } \pi$$

In other words, $[z_0, ..., z_n] = [\lambda z_0, ..., \lambda z_n]$ for any nonzero $\lambda \in \mathbb{C}$. The standard atlas on $\mathbb{P}^n$ is given by $U_i = \{[z_0, ..., z_n] \in \mathbb{P}^n \mid z_i \neq 0\}$ for all $i = 0, ..., n$. These form an open cover of $\mathbb{P}^n$. Moreover U_i is biholomorphic to $\mathbb{C}^n$ by taking

$$[z_0, ..., z_n] \mapsto (z_0/z_i, ..., \widehat{z_i/z_i}, ..., z_n/z_i)$$

where the hat denotes omitting the variable $z_i/z_i = 1$. The change of charts $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is given by:

$$(w_0, ..., \hat{w}_i, ..., w_n) \mapsto \left(\frac{w_0}{w_j}, ..., \frac{w_{i-1}}{w_j}, \frac{1}{w_j}, \frac{w_{i+1}}{w_j}, ..., \frac{\widehat{w_j}}{w_j}, ..., \frac{w_n}{w_j} \right)$$

This is visibly biholomorphic on its domain, and hence $\mathbb{P}^n$ is a complex manifold.

Exercise 3.6. The projectivization functor defines a group homomorphism $F : \mathrm{GL}(n+1, \mathbb{C}) \to \text{Bihol}(\mathbb{P}^n)$ given by $T \mapsto \mathbb{P}(T)$. Show that $\ker(F) = \mathbb{C}^* \cdot E$, where E is the set of diagonal matrices. We call $\mathbb{P}GL(n+1, \mathbb{C})$ to be the quotient of $\mathrm{GL}(n+1, \mathbb{C})$ by this kernel.

3.1.1 Submanifolds of $\mathbb{P}^n$

A homogenous polynomial $f \in \mathbb{C}[z_0, ..., z_n]$ of degree d satisfies $f(\lambda z_0, ..., \lambda z_n) = \lambda^d f(z_0, ..., z_n)$ for all $\lambda \in \mathbb{C}^*$. This makes the zero locus $Z(f)$ is well-defined on $\mathbb{P}^n$:

$$Z(f) := \{[z_0, ..., z_n] \in \mathbb{P}^n \mid f(z_0, ..., z_n) = 0\}$$

It is important to note that such a polynomial is *not* a function on $\mathbb{P}^n$. However, it does define a zero set.

Definition 3.7. A *complex hypersurface* in $\mathbb{P}^n$ is the zero locus $Z(f)$ for some non-constant homogenous $f \in \mathbb{C}[z_0, ..., z_n]$. For any subset S of homogenous polynomials, we define $Z(S) = \bigcap_{f \in S} Z(f)$ and call such a submanifold *algebraic*.

Complex hypersurfaces and more generally intersections of complex hypersurfaces are submanifolds, as one can verify using the implicit function theorem. More importantly, they are the *only* analytic submanifolds of $\mathbb{P}^n$.

Theorem 3.8 (Chow 1947)**.** *Any analytic subset $X \subset \mathbb{P}^n$ is of the form $Z(S)$ for some subset S of homogenous polynomials.*

3.2 Non Algebraic Manifolds ❖

A natural question following Chow's theorem is: do there exist complex manifolds that are not algebraic? The answer is yes, and we can witness this by considering the field of meromorphic functions on a complex manifold.

Definition 3.9. A *meromorphic function* on a complex manifold X is an equivalence class of pairs (U, f) where $U \subseteq X$ is open and dense, $X \setminus U$ is analytic and nowhere dense, and $f \in \mathcal{O}(U)$ such that for all $x \in X$ there exists $V \subseteq X$ containing x and $g, h \in \mathcal{O}(V)$ such that $f|_{V \cap U} = \frac{g}{h}\big|_{V \setminus U}$. Two such pairs (U, f) and (V, g) are equivalent iff $f|_{U \cap V} = g|_{U \cap V}$.

These are the functions which are locally quotients of holomorphic functions. What is hopefully clear is that the set of meromorphic functions form a field, which we denote $K(X)$.

Example 3.10. Let $f = z/w$ on $\mathbb{C}^2$ with coordinates (z, w). This has a pole at $w = 0$ if you approach along the w axis, but approaching along, say, $z = w^2$, you get 0 at $w = 0$. So we should be careful to note that meromorphic functions are not necessarily just maps to $\mathbb{CP}^1$ (i.e. allowing infinity). Some can be undefined at certain points in their domain.

Definition 3.11. The *algebraic dimension* of a compact complex manifold X is the transcendence degree $a(X) := \text{trdeg}_{\mathbb{C}} K(X)$.

Remark 3.12. Note that for $X \subseteq \mathbb{P}^n$ a submanifold, any meromorphic function on X is a rational function. This shows $K(X)$ is the field of rational functions on X and hence $a(X) = \dim(X)$.

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Theorem 3.13 (Remmert-Siegel-Thimm). *If X is a connected, compact, complex manifold, then $a(X) \leq \dim(X)$.*

Proof:

Let $\dim(X) = n$ and let $f_1, ..., f_{n+1} \in K(X)$. We need to show that there exists $F \in \mathbb{C}[x_1, ..., x_{n+1}] \setminus \{0\}$ such that $F(f_1, ..., f_{n+1}) = 0$. The key idea is to construct this locally so that it vanishes to a large degree and use the Schwarz Lemma.

Step 0: For all $x \in X$, there exists $U_x \subseteq X$ open such that $f_i|_{U_x} = g_{i,x}/h_{i,x}$ for some $g_{i,x}, h_{i,x} \in \mathcal{O}_{X,x}$ with $h \neq 0$. Without loss of generality assume that $g_{i,x}$ and $h_{i,x}$ are relatively prime. Proposition 1.1.35 of [Huy05] says that (after perhaps shrinking U_x), the functions $g_{i,x}, h_{i,x}$ are relatively prime in $\mathcal{O}_{X,y}$ for any $y \in U_x$. Choose neighborhoods $W_x \subseteq V_x \subseteq \overline{V}_x \subseteq U_x$ all containing x, where V_x is a neighborhood that maps to $D_1(0) \subseteq \mathbb{C}^n$ under a chart and W_x maps to $D_{1/2}(0)$. Since X is compact, we can ensure that X is covered by finitely many (say N) such $W_k := W_{x_i}$. We also write $g_{i,k} := g_{i,x_k}, h_{i,k} := h_{i,x_k}$.

Step 1: Now we construct $F \in \mathbb{C}[x_1, ..., x_{n+1}]$ satisfying:

$$\partial_z^I F\left(\frac{g_{1,k}}{h_{1,k}}, ..., \frac{g_{n+1,k}}{h_{n+1,k}}\right) = 0 \qquad \forall |I| < m', \forall k$$

On $U_k \cap U_\ell$, we have $g_{i,k}/h_{i,k} = g_{i,\ell}/h_{i,\ell}$ and hence $h_{i,k} = h_{i,\ell} \cdot \phi_{i,k\ell}$ with $\phi_{i,k\ell} = g_{i,k}/g_{i,\ell} \in \mathcal{O}^\times(U_k \cap U_\ell)$. In other words, $\phi_{i,k\ell}$ has no zeros or poles, which follows from the relatively prime assumption. Define:

$$\phi_{k\ell} := \prod_{i=1}^{n+1} \phi_{i,k\ell}$$

$$C := \max\{\|\phi_{k\ell}\|\}_{k,\ell=1,...,N}$$

where $\|\cdot\|$ is the sup norm on $\overline{V}_k \cap \overline{V}_\ell$. Note that by symmetry $\phi_{k\ell} \cdot \phi_{\ell k} = 1$ and hence $C \geq 1$. With this, for any polynomial F of degree m, define G_k by clearing denominators:

$$F\left(\frac{g_{1,k}}{h_{1,k}}, ..., \frac{g_{n+1,k}}{h_{n+1,k}}\right) = \frac{G_k}{\left(\prod_{i=1}^{n+1} h_{i,k}\right)^m}$$

Symmetry implies $G_k = \phi_{\ell k}^m G_\ell$. Suppose we have m, m' such that:

$$\binom{m+n+1}{m} > N\binom{m'-1+n}{m'-1} \qquad (*)$$

We claim that $(*)$ implies that there exists F of degree m with $\partial^I(g_{i,k}/h_{i,k})(x_k) = 0$ for $k = 1, ..., N$ and $|I| < m'$. This follows by dimension counts:

$$\binom{m+n-1}{m} = \dim\{F \in \mathbb{C}[x_1, ..., x_{n+1}] \mid \deg(F) \leq m\}$$

$$\binom{m'-1+n}{m'-1} = |\{I \subset \mathbb{N}^n \mid |I| < m'\}| = \dim\{\text{Differential operators } \partial^I \text{ for } |I| < m'\}$$

Requiring that the differential operator vanishes at x_k for $k = 1, ..., N$ imposes $N \cdot \binom{m'-1+n}{m'-1}$ linear conditions on the space of degree at most m polynomials F. The assumption $(*)$ ensures that there exists at least one polynomial F with the claimed property.

Step 3: Now we proceed to the application of the Schwartz Lemma (see [Huy05] Proposition 1.1.36). Since $\partial^I F(\cdots)(x_k) = 0$, we have that $\partial^I G_k(x_k) = 0$ for all k and for all $|I| < m'$. Then for all $x \in \overline{W}_k$ we have the estimate $|G_k(x)| \leq \left(\frac{1}{2}\right)^{m'} C'$, where $C' = \|G_k|_{W_k}\|$. Fix k and let $x \in \overline{V}_k$. Write $C' = |G_k(x)|$. Since $X = \bigcup W_\ell$, there exists ℓ such that $x \in W_\ell$. We thus have:

$$C' = |G_k(x)| = |G_\ell(x)| \cdot |\phi^m_{\ell k}(x)| \leq \frac{C'}{2^{m'}} C^m$$

In other words, we have $\left(1 - \frac{C^m}{2^{m'}}\right) C' \leq 0$. Now assuming we can find m, m' such that:

$$\frac{C^m}{2^{m'}} < 1 \tag{†}$$

that would imply $C' = 0$ and hence $G_{\ell k} = 0$. Thus $F(\cdots) = 0$, and we are done.

Step 4: The final piece is to show that there exist choices of m and m' satisfying both $(*)$ and $(\dagger)$. Write $C = 2^\lambda$ for some non-negative λ. Then the inequality $C^m/2^{m'} < 1$ holds iff $m' > \lambda m$. Since the LHS (and resp. RHS) of inequality $(*)$ is a polynomials of degree $n+1$ (resp. n) in m (resp. m'), there exist a, b such that:

$$\binom{m+n+1}{m} > a m^{n+1} \qquad \text{for } m \gg 0$$

$$N \cdot \binom{m'-1+n}{m'-1} < b(m')^n \qquad \text{for } m' \gg 0$$

We can take $m > \frac{b}{a}(2\lambda)^n$ and $m' = 2\lambda m$ (which satisfies $(\dagger)$). We see it also satisfies $(*)$ because of the polynomial estimates we have.

□

3.3 Complex Tori ❖

The standard real n-torus is the quotient of the real numbers by the standard lattice $\pi : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$. For complex tori, we can identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ in the standard way and to get a torus $\pi : \mathbb{C}^n \to \mathbb{C}^n/\mathbb{Z}^{2n}$ as well, which is given the quotient topology as usual. Since π is a local homeomorphism, we get the structure of a complex manifold on $\mathbb{C}^n/\mathbb{Z}^{2n}$. More generally, if V is a finite dimensional complex vector space and $\Gamma \subset V$ is a discrete free abelian group of maximal rank, fix an isomorphism $V \to \mathbb{C}^n$. Then there are induced isomorphisms $\Gamma \to \mathbb{Z}^n + \mathbb{Z}\tau_1 + \ldots + \mathbb{Z}\tau_n$ where $\tau_i \in \mathbb{C}^n$ parameterize the elliptic curves that we can get.

Proposition 3.14. *With above, $V/\Gamma \cong (S^1)^{2n}$. Moreover, $V/\Gamma \cong V'/\Gamma' \iff \exists A \in \mathrm{GL}(n, \mathbb{C})$ such that $A(\Gamma) = \Gamma'$.*

3.3.1 Elliptic Curves

Elliptic curves are complex tori of (complex) dimension $n = 1$. Given a lattice $\Gamma = \mathbb{Z}z_1 + \mathbb{Z}z_2$ and the corresponding elliptic curve $E = \mathbb{C}/\Gamma$, by the above proposition we can assume without loss of generality that $z_1 = 1$ and $\mathrm{Re}(z_2) > 0$. This means that elliptic curves are parameterized by $\tau \in \mathbb{H}$, the upper half plane, and are all of the form $E = E_\tau := \mathbb{C}/Z + \mathbb{Z}\tau$.

Exercise 3.15. Show that E_τ is biholomorphic to $E_{\tau'}$ if and only if there exists $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ such that $\tau' = \frac{a\tau+b}{c\tau+d}$.

As a result of the exercise, we have that:

$$\{\text{Elliptic curves}\}/\text{Biholomorphism} \longleftrightarrow \mathbb{H}/\,\mathrm{SL}_2(\mathbb{Z})$$

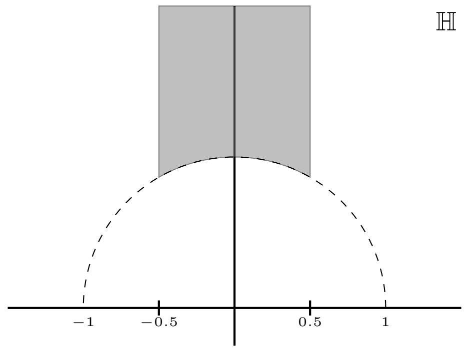

Figure 3.1: A fundamental domain for $\mathbb{H}/\operatorname{SL}_2(\mathbb{Z})$, or the moduli space of elliptic curves. It is the set of z such that $|\operatorname{Re}(z)| \leq \frac{1}{2}$ and $|z| \geq 1$

Note that $\operatorname{SL}_2(\mathbb{Z}) = \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \rangle$. Therefore the space of elliptic curves is in bijection with the upper half plane modulo two operations. The first operation sends $\tau \mapsto \tau + 1$ and the other maps $\tau \mapsto -\tau^{-1}$. A fundamental domain for $\mathbb{H}/\operatorname{SL}_2(\mathbb{Z})$ is shown in Figure 3.1.

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For higher dimensions $n > 1$, this analysis generalizes where the lattices can be assumed to be of the form $\Gamma = \mathbb{Z}^n + A\mathbb{Z}^n$, where $A \in \mathbb{H}_n \subseteq \operatorname{GL}_n(\mathbb{C})$ is an element of a space called the Siegel upper half plane.

3.3.2 Plane Cubic Curves

While there are no non-constant holomorphic functions on the torus, there are interesting meromorphic functions. One important example is the Weierstrass $\wp$ function associated to a lattice $\Gamma = \mathbb{Z} + \mathbb{Z}\tau$

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Gamma \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right), \qquad z \notin \Gamma$$

This is a meromorphic function on $\mathbb{C}$ with double poles in points of Γ. It is also Γ-periodic, i.e. $\wp(z+\omega) = \wp(z)$ for all $\omega \in \Gamma$.[1] It follows that $\wp \in K(E)$ where $E = \mathbb{C}/\Gamma$. By Siegel's theorem (Theorem 3.13), there must be an algebraic relation on $\wp$ and $\wp'$. It is:

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3$$

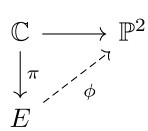
where:

$$g_2 = g_2(\tau) = 60 \sum_{(m,n) \neq (0,0)} \frac{1}{(m+n\tau)^4} \in \mathcal{O}(\mathbb{H})$$

$$g_3 = g_3(\tau) = 140 \sum_{(m,n) \neq (0,0)} \frac{1}{(m+n\tau)^6} \in \mathcal{O}(\mathbb{H})$$

These are examples of *elliptic modular forms* of weight 4 and 6. They are also examples of Eisenstein series G_k ($g_2 = 60G_4$ and $g_3 = 140G_6$).

We define a map $\mathbb{C} \setminus \Gamma \to \mathbb{C}^2$ by $z \mapsto (\wp(z), \wp'(z))$ and regard it a map $\mathbb{C} \mapsto \mathbb{P}^2$ given by $z \mapsto [\wp(z), \wp'(z), 1]$. Since $\wp$ and $\wp'$ are Γ periodic, this map factors through E:



[1] This can be seen by first checking that the derivative $\wp'$ is Γ-periodic

Let $X = \mathrm{im}(\phi)$. Notice that the points in X satisfy $Y^2 = 4X^3 - g_2X - g_3$. We can homogenize this to $Y^2Z = 4X^3 - g_2Z^2 - g_3Z^3$, and so we have $X \subseteq V(Y^2Z - 4X^3 - g_2XZ^2 - g_3Z^3)$. In fact, this is an equality:

Theorem 3.16. *A one dimensional complex torus $E = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$ for $\tau \in \mathbb{H}$ is biholomorphic to the plane cubic curve $C \subset \mathbb{P}^2$ with $C \cap \mathbb{C}^2_{x,y} = Z(y^2 - 4x^3 - g_2(\tau)x - g_3(\tau))$.*

This shows how elliptic curves are all algebraic and take the form of the vanishing locus of a regular cubic polynomial in $\mathbb{P}^2$.

3.4 Holomorphic Group Actions ❖

Definition 3.17. A (complex) *Lie group* is a differentiable (complex) manifold G together with a group structure such that $G \times G \to G, (x, y) \mapsto xy^{-1}$ is differentiable (holomorphic).

Example 3.18. Some examples are:

- Linear groups: $\mathrm{GL}_n(\mathbb{C}), \mathrm{SL}_n(\mathbb{C}), \mathrm{Sp}_n(\mathbb{C})$ but *not* $U_n(\mathbb{C})$. For $n \geq 1$ these are not compact and $n > 1$ these are not abelian.
- A complex torus $X = \mathbb{C}^n/\Gamma$ is abelian and compact.
- Finite/discrete groups.

Proposition 3.19. *If G is a connected, compact, complex Lie group then G is abelian.*

Proof:

For $g \in G$ consider:

$$\Phi_g : G \to G, \qquad h \mapsto g^{-1}hg$$

Then G abelian if and only if $\forall g \in G$ the map $\Phi_g = \mathrm{id}$. We apply the maximum principle to the components of the adjoint action:

$$\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g}) \cong \mathrm{GL}_n(\mathbb{C}), \qquad g \mapsto (\Phi_g)_*|_e$$

where $\mathfrak{g} = T_eG$ is the Lie algebra of G. Since this is a holomorphic map, we must conclude that Ad is constant and hence must be the identity. Finally, we use the basic fact of Lie theory that the exponential map $\exp : \mathfrak{g} \to G$ given by $\xi \mapsto \gamma(1)$, where $\gamma : \mathbb{R} \to G$ is the unique homomorphism with $\gamma'(0) = \xi$, commutes with Φ_g. Thus:

$$\Phi_g(\exp(\xi)) = \exp((\Phi_g)_*\xi) = \exp \xi$$

Therefore Φ_g is the identity.

□

3.4.1 Group Actions and Quotients

Let X be a topological space and suppose a group G acts on X continuously, i.e. the map $G \times X \to X$ given by $(g, x) \mapsto g \cdot x$ is continuous. Generally, the quotient space X/G won't always be a nice space if the action of G doesn't meet certainc conditions. We will consider the case where X is a complex manifold, G is a complex Lie group, and the action is holomorphic. The two condition we need on the action in order to get a nice quotient are free-ness and proper-ness.

Definition 3.20. A group action G on a topological space is called *free* if for all $g \neq e \in G$, and all X, we have $g \cdot x \neq x$. The action is called *proper* if the map $G \times X \to X \times X, (g, x) \mapsto (g \cdot x, x)$ is a proper map.

Theorem 3.21. *Let G be a complex Lie group which acts freely and properly on a complex manifold X. Then the quotient X/G is a complex manifold in a natural way and the quotient map $\pi : X \to X/G$ is holomorphic. Moreover, $\pi : X \to X/G$ is a principale G-bundle.*

Some examples of such quotients we have already seen are:

1. The $\mathbb{C}^*$ action on $\mathbb{C}^{n+1} \setminus \{0\}$ that gives rise to $\mathbb{P}^n$.
2. The action of a full-rank lattice Γ on $\mathbb{C}^n$ gives rise to an elliptic curve.

Example 3.22 (The Versal Family of Elliptic Curves)**.** TODO

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Proposition 3.23. *If G is discrete and X is locally compact, then a group action is proper and free if and only if G acts properly discontinuously, i.e.*

i. For all $x \in X$ there exists $U = U(x)$ such that $\forall g \in G \setminus e$ such that $g(U) \cap U = \emptyset$.

ii. For all $x, y \in X$ with $y \notin G \cdot x$, there exists $U = U(x), V = V(y)$ such that $G \cdot U \cap G \cdot V = \emptyset$.

3.4.2 Ball Quotients

An important class of quotient manifolds are ball quotients. Let Σ be a Riemann surface whose universal cover is isomorphic to $\Delta \cong \mathbb{H}$. Then $\Sigma = \mathbb{H}/\Gamma$ where $\Gamma = \pi_1(\Sigma)$. Here Γ can be seen as a discrete subgroup of $\mathrm{Aut}(D) \cong \mathrm{SU}(1,1) \cong \mathrm{SL}(2,\mathbb{R})/\{\pm 1\}$. For example, if Σ is a closed genus 2 surface, then $\Gamma = \langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} = e\rangle$. This surface admits a hyperbolic structure, and hence Σ can be seen as a quotient of the hyperbolic disk by geodesics which are lifts of generators of Γ (see Figure 3.2). This is an example of a Ball quotient in complex dimension 1.

More generally, if $n > 1$, then the ball $B_1(0) \subset \mathbb{C}^n$ can be mapped into $\mathbb{P}^n$ via a chart as a subset $U_0 \subset \mathbb{P}^n$ of the form:

$$U_0 = \left\{ [z_0, ..., z_n] \in \mathbb{P}^n \;\middle|\; |z_0|^2 - \sum_{i \geq 1} |z_i|^2 > 0 \right\}$$

This is invariant under $\mathrm{SU}(1,n) = \mathrm{SU}(\langle \cdot, \cdot \rangle)$, where $\langle \cdot, \cdot \rangle$ is the metric:

$$\langle z, w \rangle := \bar{z}_0 w_0 - \sum_{i \geq 1} \bar{z}_i w_i$$

In fact, $\mathrm{Aut}(B_1(0)) = \mathrm{SU}(1,n)$. A *ball quotient* is a quotient of $U_0 \cong B_1(0)$ by any discrete subgroup $\Gamma \subset \mathrm{SU}(1,n)$ acting freely on D.

3.4.3 Finite Quotients of Products of Curves

Let C be a closed Riemann surface and $E = \mathbb{C}/\Gamma$ an elliptic curve. Let $G \subset (E, +)$ a finite group acting on C. Then we get a free, proper action on $C \times E$ given by $g \cdot (x, y) = (g \cdot x, y + g)$. The quotient $X := (C \times E)/G$ is called a bielliptic or hyperelliptic surface. There is a map $\pi : X \to C/G$ which is holomorphic, and if G acts freely, then the fibers $\pi^{-1}([z])$ are isomorphic to E. If not, then $\pi^{-1}([z]) \cong E/\mathrm{Stab}(z)$.

A hyperelliptic curve is a Riemann surface that arises as a 2:1 cover of $\mathbb{P}^1$. There is a $\mathbb{Z}/2$ action given by swapping branches. There is also a $\mathbb{Z}/2$ action on an elliptic curve given by $z \mapsto -z$.

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3.4.4 Hopf Manifolds

Let $\lambda \in \mathbb{C}^*$ with $|\lambda| < 1$ and define a $\mathbb{Z}$ action $\mathbb{Z} \times \mathbb{C}^n \setminus \{0\} \to \mathbb{C}^n \setminus \{0\}$ given by $k \cdot (z_1, ..., z_n) := (\lambda^k z_1, ..., \lambda^k z_n)$. This is a proper and free action, and hence $X = (\mathbb{C}^n \setminus \{0\})/\mathbb{Z}$ is a manifold, called a *Hopf manifold*. It is diffeomorphic to $S^1 \times S^{2n-1}$, and hence the latter can be endowed with a complex structure.

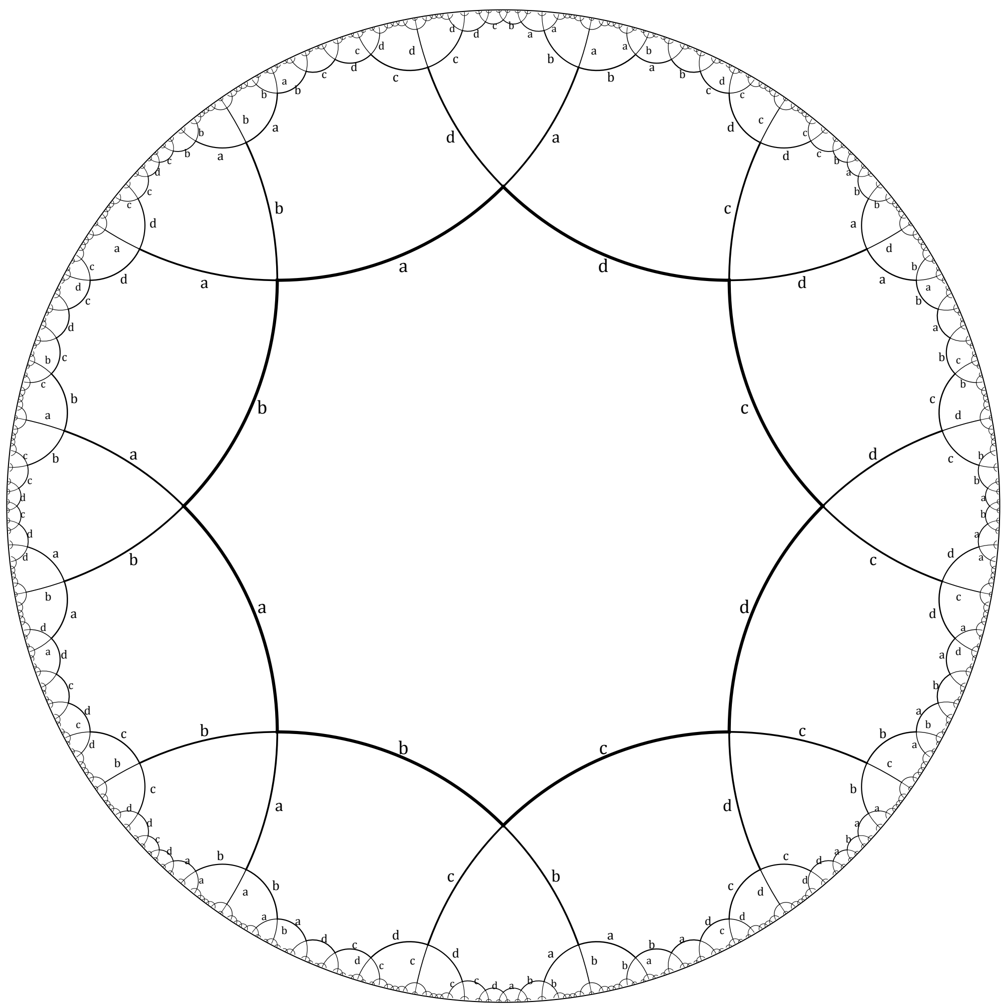

Figure 3.2: Cayley graph of the fundamental group Γ of a genus 2 Riemann surface, seen as a quotient of the Hyperbolic disk by Γ. The fundamental domain is the hyperbolic octagon in the center. Code to generate this image can be found at `https://github.com/geodavic/hyperdraw`.

3.4.5 Iwasawa Manifolds

Consider the Heisenberg group:

$$G = \left\{ \begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; z_i \in \mathbb{C} \right\} \subset \mathrm{GL}_3(\mathbb{C})$$

This is isomorphic to $\mathbb{C}^3$. The set $\Gamma = G \cap \mathrm{GL}_3(\mathbb{Z}+\mathbb{Z}i)$ acts on G properly and freely. Then the quotient $X = G/\Gamma$ is called the *Iwasawa Manifold*. Note that there is a natural map $\pi : X \to E \times E$, where $E = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}i)$ given by $(z_1, z_2, z_3) \mapsto (z_1, z_3)$. This makes X a holomorphic fiber bundle with fibers E, but $X \ncong E \times E \times E$.

3.4.6 Grassmannians

Recall the Grassmannian:

$$\mathrm{Gr}(k, n) = \{W \subset \mathbb{C}^n \text{ linear} \mid \dim_{\mathbb{C}} W = k\}$$

Note that $\mathrm{Gr}(1, n) = \mathbb{P}^{n-1}$ and $\mathrm{Gr}(n-1, n) = \mathbb{P}^{n-1}$. The group $\mathrm{GL}(n, \mathbb{C})$ acts transitively on $\mathrm{Gr}(k, n)$. We can produce a chart of $\mathrm{Gr}(k, n)$ at $W = \mathbb{C}^k \times \{0\}$ by seeing the nearby planes to W as graphs:

$$\mathrm{Mat}((n-k) \times k, \mathbb{C}) \to \mathrm{Gr}(k, n)$$

$$A \mapsto \mathrm{im}\left(\begin{bmatrix} E \\ A \end{bmatrix} : \mathbb{C}^k \to \mathbb{C}^n \right)$$

Unlike the previous examples, Grassmannians are actually algebraic manifolds. This can be proven by using the Plücker embedding:

$$L : \mathrm{Gr}(k, n) \to \mathbb{P}(\Lambda^k \mathbb{C}^n) \cong \mathbb{P}^{\binom{n}{k}-1}$$

which takes a basis $\{v_1, ..., v_k\}$ of W and maps it to $[v_1 \wedge \cdots \wedge v_k]$. This is well-defined because a change of basis will mutliply the wedge product by a scalar. One can show that there are algebraic relations on the image of this embedding, called the Plücker relations. For example, $\mathrm{Gr}(2, 4)$ is a hypersurface in $\mathbb{P}^5$ with the relation $Z_1 Z_2 - Z_3 Z_4 + Z_5 Z_6 = 0$, where Z_i are the homogenous coordinates on $\mathbb{P}^5$.

3.5 Calculus on Complex Manifolds ❖

Let M be a complex manifold. The real tangent bundle $T_{\mathbb{R}} M$ has local coordinates $\partial_{x_1}, ..., \partial_{x_n}, \partial_{y_1}, ..., \partial_{y_n}$. This can be complexified to $T_{\mathbb{C}} M = T_{\mathbb{R}} M \otimes_{\mathbb{R}} \mathbb{C} = TM \oplus \overline{TM}$. Locally, the coordinates become:

$$\partial_{z_\mu} = \frac{1}{2}\left(\partial_{x_\mu} - i\partial_{y_\mu}\right) \quad (\text{spans } TM)$$

$$\partial_{\bar{z}_\mu} = \frac{1}{2}\left(\partial_{x_\mu} + i\partial_{y_\mu}\right) \quad (\text{spans } \overline{TM})$$

We call TM the holomorphic tangent bundle. Dually, we can look at the cotangent bundle $T^*_{\mathbb{R}} M$ and get a complexification $T^*_{\mathbb{C}} M = T^*_{\mathbb{C}} M \otimes_{\mathbb{R}} \mathbb{C} = T^* M \oplus \overline{T^* M}$. We will also sometimes write this as $T^{1,0} M \oplus T^{0,1} M$ (note that this notation differs from Huybrechts). More generally, we set $T^{p,q} M := \Lambda^p T^* M \otimes_{\mathbb{C}} \Lambda^q \overline{T^* M}$. The basis elements of $T^{p,q} M$ are locally of the form:

$$dz_{i_1} \wedge \cdots dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$$

We denote $\mathcal{A}^k_X$ to be the sheaf of k forms (sections of $\Lambda^k T^* M$) and $\mathcal{A}^{p,q}_X$ to be the sheaf of p, q forms. There is a natural decomposition:

$$\mathcal{A}^k_X = \bigoplus_{p+q=k} \mathcal{A}^{p,q}_X$$

with $\overline{\mathcal{A}^{p,q}}_X = \mathcal{A}_X^{q,p}$. We let $\pi^{p,q} : \mathcal{A}_X^* \to \mathcal{A}_X^{p,q}$ be the projection. The operators $\partial, \bar{\partial}$ can be defined on $\mathcal{A}^{p,q}(X)$ by projecting the exterior derivative; that is, $\partial := \pi^{p+1,q} \circ d$ and $\bar{\partial} := \pi^{p,q+1} \circ d$. Integration of forms can be defined on complex manifolds by integrating real and imaginary parts separately as real forms.

The Dolbeault cohomology of a complex manifold is the cohomology of the chain complex $(\mathcal{A}^{p,\bullet}, \bar{\partial})$. It is thus:

$$H^{p,q}(X) := H^q(\mathcal{A}^{p,\bullet}, \bar{\partial}) = \frac{\ker(\bar{\partial} : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q+1}(X))}{\operatorname{im}(\bar{\partial} : \mathcal{A}^{p,q-1}(X) \to \mathcal{A}^{p,q}(X))}$$

We will see in a future section that this is the same as computing the sheaf cohomology of the sheaf of holomorphic p forms Ω_X^p.

4. Vector Bundles and Sheaves

❖

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4.1 Holomorphic Vector Bundles ❖

A holomorphic vector bundle is a vector bundle in the holomorphic category. In other words, it is a map $\pi : E \to X$ with local trivializations $\Phi_U : \pi^{-1}(U) \xrightarrow{\cong} U \times \mathbb{C}^r$ such that $\Phi_V \circ \Phi_U^{-1}$ is a complex linear transformation, where defined. Holomorphic vector bundles define a comlplex vector space fiberwise over X, and the dimension of that space (in this case r) is called the rank. The fibers $\pi^{-1}(x)$ we denote by E_x. A section of a holomorphic vector bundle is a map $s : X \to E$ such that $\pi \circ s = \mathrm{id}$. The zero section is defined by $s_0 : X \to E$ given by sending $x \mapsto 0 \in \pi^{-1}(x)$.

A morhpism of vector bundles $\pi : E \to X$ and $\pi' : E' \to X$ is given by a holomorphic map $\phi : E \to E'$ such that the restriction to a fiber $\phi_x : E_x \to E'_x$ is complex linear and the rank of ϕ_x is constant over $x \in X$. It must also satisfy the commutative diagram below:

$$\begin{array}{ccc} E & \xrightarrow{\phi} & E' \\ \downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle \pi'} & \\ X & & \end{array}$$

They are isomorphic if ϕ is a biholomorphism. Given two holomorphic vector bundles E and E' over a complex manifold X of rank r and r', there are several vector bundles that can be constructed from them in the same way as with real vector bundles:

i. The direct sum $E \oplus E'$ is the holomorphic vector bundle whose fiber is isomorphic to $E_x \oplus E'_x$. This has rank $r + r'$.

ii. The tensor product $E \otimes E'$ is the holomorphic vector bundle whose fiber is isomorphic to $E_x \otimes E'_x$. This has rank rr'.

iii. The exterior powers $\Lambda^i E$ and symmetric powers $S^i E$ are the holomorphic vector bundles over X whose fibers are isomorphic to $\Lambda^i(E_x)$ and $S^i(E_x)$, respectively.

iv. The dual bundle E^* is the holomorphic vector bundle whose fiber over x is $(E_x)^*$.

v. The determinant line bundle of a holomorphic vector bundle of rank $r$is $\det(E) = \Lambda^r E$.

vi. The projectivization of E is the bundle $\mathbb{P}(E)$ whose fibers are $\mathbb{P}(E_x)$. It can als be defined as:

$$\mathbb{P}(E) = (E \setminus s_0(X))/\mathbb{C}^*$$

vii. Given a morphism $\phi : E \to E'$, the kernel $\ker(\phi)$ and cokernel $\mathrm{coker}(\phi)$ are vector bundles whose fibers are the kernels $\ker(\phi_x)$ and cokernels $\mathrm{coker}(\phi_x)$.

viii. A short exact sequence of holomorphic vector bundles is:

$$0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0$$

where $\ker(E \to F) = 0$ and $\mathrm{coker}(E \to F) \cong G$.

ix. Given a holomorphic map $f : Y \to X$, we can form the pullback bundle $f^*E = Y \times_X E = \{(y, u) \in Y \times E \mid f(y) = \pi(u)\}$. This is a vector bundle over Y whose fiber over y is $E_{f(x)}$. When f is the inclusion of Y as a submanifold into X, then we call $f^*E := E|_Y$ the restriction of E to y.

Remark 4.1. It is important to note that in the category of holomorphic vector bundles (unlike with differentiable vector bundles), short exact sequences don't necessarily split.

Given a holomorphic map $f : Y \to X$ and a holomorphic vector bundle $\pi : E \to X$, we can form the pullback bundle $f^*E = Y \times_X E = \{(y, u) \in Y \times E \mid f(y) = \pi(u)\}$.

Example 4.2. The first interesting line bundle one comes across is the tautological line bundle on $\mathbb{P}(V)$ for a complex vector space V. It is $\mathcal{O}(-1) := \{(\ell, v) \in \mathbb{P}(V) \times V \mid v \in \ell\}$. In other words, over each point in $\mathbb{P}(V)$ is the line represented by that point.

Exercise 4.3 (Huybrechts 2.2.9)**.** Show that $\mathcal{O}(-1) \setminus s_0(\mathbb{P}^n)$ can be naturally identified with $\mathbb{C}^{n+1} \setminus \{0\}$. Use this to construct a submersion $S^{2n+1} \to \mathbb{P}^n$ with fiber S^1. For $n = 1$, this is the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$.

Solution:

Recall that $\mathcal{O}(-1) = \{(z, \ell) \in \mathbb{C}^{n+1} \times \mathbb{P}^n \mid z \in \ell\}$. This has a projection onto $\mathbb{C}^{n+1}$ given by taking the first factor. Away from $0 \in \mathbb{C}^{n+1}$, there is an obvious inverse given by $z \mapsto (z, \mathrm{span}(z))$. Therefore we have an isomorphism of $\mathbb{C}^{n+1} \setminus \{0\}$ and $\mathcal{O}(-1)/s_0(\mathbb{P}^n)$. The desired fibration comes from taking the inclusion $S^{2n+1} \hookrightarrow \mathbb{C}^{n+1} \cong \mathcal{O}(-1)/s_0(\mathbb{P}^n)$ followed by the projection onto the second factor to $\mathbb{P}^n$.

■

4.1.1 Line Bundles

Line bundles have the interesting property that tensoring them produces a new line bundle. Since tensor products are associative and $L \otimes L' \cong L' \otimes L$, we get an abelian group called the Picard group:

$$\mathrm{Pic}(X) := \{L \to X \text{ line bundles}\}/\cong$$

The Picard group of $\mathbb{C}^n$ is trivial and the Picard group of $\mathbb{P}^n$ is isomorphic to $\mathbb{Z}$ and is generated by the tautological bundle $\mathcal{O}(-1)$.

Proposition 4.4. *There is a natural isomorphism* $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^*)$.

Proof:

See [Huy05] Corollary 2.2.10.

□

Remark 4.5. Note that a vector bundle being topologically trivial is *not* the same as being holomorphically trivial. This is why $\mathrm{Pic}(\mathbb{C}^n) = \{0\}$ is not quite a trivial fact.

Exercise 4.6 (Huybrechts 2.2.7)**.** Let L_1 and L_2 be two holomorphic line bundles on a complex manifold X. Suppose that $Y \subset X$ is a submanifold of codimension at least two such that L_1 and L_2 are isomorphic on $X \setminus Y$. Show that $L_1 \cong L_2$.

Theorem 4.7 (Birkhoff - Grothendieck)**.** *Every vector bundle of rank r on* $\mathbb{P}^1$ *decomposes as a direct sum of line bundles* $E = \bigoplus_i \mathcal{O}(-1)^{\otimes a_i}$ *where* $a_i \in \mathbb{Z}$.

Theorem 4.8 (Atiyah)**.** *Given* $M = \mathbb{C}/\Lambda$ *an elliptic curve, there exist indecomposable bundles* $E_1, E_2, \ldots$ *of rank rank*$(E_r) = r$ *such that if* $E \to M$ *is an indecomposable (of degree 0) then* $E = \varphi^* E_i$ *for some* i *and* $\varphi \in \mathrm{Aut}(M)$.

4.1.2 Tangent, Cotangent and Normal Bundles

The tangent bundle and cotangent bundle are natural holomorphic vector bundles associated to any complex manifold. There is also the canonical line bundle $K_X = \Lambda^n(T^*X) = \det(T^*X)$. Now, given a complex submanifold $Y \subset X$, we get an inclusion $TY \hookrightarrow TX$.

Definition 4.9. The holomorphic normal bundle to a submanifold $Y \subset X$ is the quotient $N_{Y/X} = TX|_Y/TY$.

Proposition 4.10 (Adjunction Formula)**.** *Let $Y \subset X$ be a submanifold. Then $K_Y = K_X|_Y \otimes \det(N_{Y/X})$.*

This follows from applying the following lemma to the short exact sequence:

$$0 \longrightarrow TY \longrightarrow T_X|_Y \longrightarrow N_{Y/X} \longrightarrow 0$$

Lemma 4.11. *Suppose we have a short exact sequence of holomorphic vector bundles $0 \to E \to F \to G \to 0$, then $\det(F) = \det(E) \otimes \det(G)$ canonically.*

4.2 Sheaves ❖

We briefly remind the reader of the essential notions of sheaves.

Definition 4.12. A *presheaf* (of abelian groups) on a topological space X is a contravariant functor $\mathcal{F}$ from the category of open sets in X to the category of abelian groups. In other words, to every open set we have an association of U to an abelian group $\mathcal{F}(U)$ and the inclusion maps $i : U \to V$ give us restriction morphisms $r = \mathcal{F}(i) : \mathcal{F}(V) \to \mathcal{F}(U)$. Moreover $\mathcal{F}(\emptyset) = 0$.

Definition 4.13. A *sheaf* is a presheaf $\mathcal{F}$ that satisfies the vanishing and gluing properties. That is, for all $U = \bigcup V_i$, $V_i \subset X$ open, we have:

1. If $s \in \mathcal{F}(U)$ and $s|_{V_i} = 0$ for all i, then $s = 0$.
2. If $s_i \in \mathcal{F}(V_i)$ and for all i, j we have $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there exists an extension $s \in \mathcal{F}(U)$ such that $s|_{V_i} = s_i$.

The use of abelian groups here is not special; there are analogous notions of sheaves over other categories, such as sets, rings, etc (though the definition has to change slightly for each).

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Example 4.14. The following are three common examples of sheaves that we use.

- Given a holomorphic vector bundle $E \to M$, the sheaf of sections of this bundle is $\xi(U) = \{\text{holomorphic sections } s : U \to E|_U\}$.
- Given an analytic subset $A \subseteq M$, the ideal sheaf $\mathcal{I}_A$ is the sheaf sending $U \mapsto \{f \in \mathcal{O}_M(U) \mid A \subseteq Z(f)\}$.
- Given an abelian group A, the constant presheaf $\underline{A}$ sends U to A itself. This won't in general be a sheaf; however, a locally constant sheaf $\mathcal{F}$ is one for which every point $x \in M$ has a neighborhood U such that $\mathcal{F}|_U$ is a constant sheaf.
- Given an abelian group A, the skyscraper sheaf at a fixed point $x \in M$ is the sheaf sending $U \mapsto A$ iff $x \in U$ and $U \mapsto 0$ otherwise.

Morphisms of sheaves are natural transformations. In other words, a morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ consists of the data of a morphism $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for every open set U such that, for every pair $V \subset U$ open, we have the commutative square:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ \downarrow r & & \downarrow r \\ \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V) \end{array}$$

The stalks of sheaves at points $p \in X$ are:

$$\mathcal{F}_p = \varinjlim_{p \in U} \mathcal{F}(U)$$

We think of the stalk at p to be the set of local functions near p that agree on sufficiently small neighborhood (called *germs* of sections of $\mathcal{F}$). Every morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ determines a morphism $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$. The stalks allow us to work with sheaves locally and, often, it is sufficient to prove things on stalks to get global results. For example:

Exercise 4.15. Show that $\varphi : \mathcal{F} \to \mathcal{G}$ is an isomorphism if and only if $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ is an isomorphism.

A critial operation in the theory of sheaves is *sheafification*. Often one tries to define a sheaf, but it fails to meet the gluing properties. In such a case, one "sheafifies" and produces a natural sheaf from the presheaf that one started with. The *Étalé space* of a presheaf $\mathcal{F}$ is:

$$\mathrm{Et}(\mathcal{F}) := \coprod_{p \in X} \mathcal{F}_p$$

There is a natural projection $\Phi : \mathrm{Et}(\mathcal{F}) \to X$ which sends a germ to the point in X over which it lies. One can topologize $\mathrm{Et}(\mathcal{F})$ and hence one can talk about continuous sections s of Φ. The sheafification of $\mathcal{F}$, denoted $\widetilde{\mathcal{F}}$ or $\mathcal{F}^{\sim}$, is the sheaf of continuous sections of Φ.

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Given a morhpism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$, the kernel $\ker(\varphi)$ is a well-defined sheaf that sends $U \mapsto \ker(\mathcal{F}(U) \to \mathcal{G}(U))$. The analogue for the cokernel $\mathrm{coker}(\varphi)$, however, only gives us a presheaf and it fails to be a sheaf. We thus have to sheafify to get a proper sheaf: $\mathrm{coker}(\varphi)^{\sim}$. The same holds for the image sheaf. This allows one to talk about exact sequences of sheaves just as one would for any abelian category. Moreover:

Lemma 4.16. *Given a morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$, we have:*

$$(\ker \varphi)_p = \ker(\varphi_p)$$

$$(\mathrm{im}\, \varphi)_p = \mathrm{im}(\varphi_p)$$

$$(\mathrm{coker}\, \varphi)_p = \mathrm{coker}(\varphi_p)$$

Example 4.17. Let M be a complex manifold and $Z \subset M$ a complex submanifold. Then we get an exact sequence of sheaves:

$$0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

The first nonzero map is inclusion of the ideal sheaf and the second is restriction. Note: we are seeing $\mathcal{O}_Z$ as a sheaf on M via $U \subseteq M \mapsto \mathcal{O}_Z(Z \cap U)$. We can check that this is exact stalkwise.

Example 4.18 (The Exponential Sequence)**.** Given a complex manifold M, the exponential sequence is:

$$0 \longrightarrow \mathbb{Z}_M \longrightarrow \mathcal{O}_M \xrightarrow{\exp} \mathcal{O}_M^{\times} \longrightarrow 1$$

The first map is the inclusion of locally constant functions and the second map is the exponential map $f \mapsto e^{2\pi i f}$. The sheaf $\mathcal{O}_M^{\times}$ is the sheaf sending $U \mapsto \{f \in \mathcal{O}_M(U) \mid Z(f) = \emptyset\}$. Once again, we can check this is exact by checking at stalks. Locally, we can take logarithms and hence the second map is surjective with kernel exactly the first map.

4.3 Sheaf Cohomology ❖

We will discuss the Čech cohomology version of sheaf cohomology (as opposed to the derived functor approach, in for example Hartshorne §3.1 and §3.2).

Proposition 4.19. *Given an exact sequence of sheaves $0 \to \mathcal{F}' \to \mathcal{F} \xrightarrow{\varphi} \mathcal{F}'' \to 0$, then for all U the following is exact:*

$$0 \longrightarrow \mathcal{F}'(U) \longrightarrow \mathcal{F}(U) \xrightarrow{\varphi(U)} \mathcal{F}''(U)$$

However, generally $\varphi(U)$ is not surjective.

For a concrete example for when $\varphi(U)$ is not surjective, consider the ideal sheaf exact sequence associated to $Z = \{0, 1\} \subset \mathbb{P}^1$. We know that $\mathcal{O}_{\mathbb{P}^1} \cong \mathbb{C}$, and by consequence any function which is zero on Z must vanish everywhere. Thus $\mathcal{I}_Z = 0$. The sheaf $\mathcal{O}_Z$ is a sum of two skyscraper sheaves, at 0 and 1. Let U be an open set containing 0 and 1; then the ideal sequence evaluated at U is:

$$0 \longrightarrow 0 \longrightarrow \mathbb{C} \xrightarrow{d} \mathbb{C} \oplus \mathbb{C} \longrightarrow 0$$

The nonzero map is the diagonal map $z \mapsto (z, z)$, which is clearly not surjective.

Definition 4.20. A sheaf $\mathcal{F}$ is called *flasque* or *flabby* if for all $U \subseteq X$ open, the map $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective. It is called *soft* if for all $A \subseteq X$ closed, $\mathcal{F}(X) \to \mathcal{F}(A) := \varinjlim_{U \supset A} \mathcal{F}(U)$ is surjective.

The sheaves of continuous and smooth functions $\mathcal{C}^0_X, \mathcal{C}^\infty_M$ are soft.

Proposition 4.21. *Given an exact sequence of sheaves $0 \to \mathcal{F}' \to \mathcal{F} \xrightarrow{\varphi} \mathcal{F}'' \to 0$ with $\mathcal{F}'$ is flasque, then for all U the following is exact:*

$$0 \longrightarrow \mathcal{F}'(U) \longrightarrow \mathcal{F}(U) \xrightarrow{\varphi(U)} \mathcal{F}''(U) \longrightarrow 0$$

4.3.1 Flasque and Soft Resolutions

Proposition 4.22. *Any abelian sheaf $\mathcal{F}$ has a canonical resolution by flasque sheaves:*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^0 \xrightarrow{\varphi^0} \mathcal{F}^1 \xrightarrow{\varphi^1} \mathcal{F}^2 \xrightarrow{\varphi^2} \cdots$$

where $\mathcal{F}^i$ is flasque for $i \geq 0$. Similarly, $\mathcal{C}^0_X$ modules for X paracompact Hausdorff space have (non-canonical) soft resolutions.

Proof:

For the first assertion, recall that every sheaf is isomorphic to it sheafification, which we can regard as a subsheaf of the sheaf of *discontinuous* sections of the Étalé space. We can thus set $\mathcal{F}^0$ to be the latter, which is a flasque sheaf. To obtain $\mathcal{F}^1$, we simply set $\mathcal{F}^1 := \mathcal{F}^0/\mathcal{F}$. We proceed analogously for all $\mathcal{F}^i$ to obtain the desired resolution.

□

Applying global sections $\Gamma(-, X)$ to the resolution above we get a complex which may not be exact anymore. The failure to be exact is measured by the *cohomology* groups $H^i(\mathcal{F}) := \ker(\Gamma\varphi_i)/\operatorname{im}(\Gamma\varphi^{i-1})$. This is well-defined because, given two flasque resolutions, there is an induced quasi morhpism of complexes.

Definition 4.23. A sheaf $\mathcal{F}$ is Γ-acyclic (or just *acyclic*) if $H^i(\mathcal{F}) = 0$ for $i > 0$.

Since any flasque or soft sheaf has trivial flasque/soft resolution, it is easy to see that:

Proposition 4.24. *Any soft or flasque sheaf is acyclic.*

Note that $H^0(\mathcal{F})$ is always easy to compute, and it is $\Gamma(\mathcal{F}, X) \equiv \mathcal{F}(X)$ (the global sections).

Proposition 4.25. *Given* $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ *exact, there is an induced long exact sequence in cohomology:*

$$\begin{array}{ccccccc} \cdots \longleftarrow & H^2(X,\mathcal{F}'') & \longleftarrow & H^2(X,\mathcal{F}) & \longleftarrow & H^2(X,\mathcal{F}') & \longleftarrow \\ & H^1(X,\mathcal{F}'') & \longleftarrow & H^1(X,\mathcal{F}) & \longleftarrow & H^1(X,\mathcal{F}') & \longleftarrow \\ & \mathcal{F}''(X) & \longleftarrow & \mathcal{F}(X) & \longleftarrow & \mathcal{F}'(X) & \longleftarrow 0 \end{array}$$

4.3.2 Application: Singular Cohomology

Let X be a locally contractible space that is paracompact and let G be an abelian group. Define $\mathcal{S}^p$ to be the sheafification of the presheaf sending $U \mapsto \{\text{singular } p\text{-chains on } U \text{ with coefficients in } G\}$. There is a differential map $\mathcal{S}^p \to \mathcal{S}^{p+1}$ in the usual way with singular chains. Then $0 \to G_X \to \mathcal{S}^\bullet$ is an acyclic resolution. It follows that $H^i(X, G_X) = H^i(\Gamma\mathcal{S}^\bullet) = H^i(X, G)$; that is, sheaf cohomology with locally constant coefficients recovers singular cohomology.

4.4 Divisors and Line Bundles ❖

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Definition 4.26. A *divisor* D on a complex manifold is a formal linear combination:

$$D = \sum a_i[Y_i]$$

where $Y_i \subset X$ are irreducible hypersurface and $a_i \in \mathbb{Z}$. Such a divisor is called *effective* if $a_i \geq 0$, in which case we write $D \geq 0$. The group of divisors is called $\mathrm{Div}(X)$.

Every $f \in K(X)$ has an associated divisor (f), which is roughly the divisors minus the poles of f (counted with multiplicity). Any such divisor is called *principal*. Two divisors D, D' are called linearly equivalent if $D - D'$ is principal.

Every divisor D has an associated sheaf $\mathcal{O}(D)$ whose sections are the functions in $f \in K(X)$ such that $(f) + D \geq 0$. This is a locally free sheaf of rank one, and so it can also be seen as a line bundle.

Proposition 4.27. *The mapping* $\mathrm{Div}(X) \to \mathrm{Pic}(X)$ *sending* $D \mapsto \mathcal{O}(X)$ *is a group homomorphism.*

Lemma 4.28. *A divisor* D *is principal if and only if* $\mathcal{O}(D) \cong \mathcal{O}_X$.

The group of divisors modulo linear equivalence is called the Divisor class group $\mathrm{Cl}(X)$. As a consequence of the Lemma above, the map induced map $\mathrm{Cl}(X) \to \mathrm{Pic}(X)$ is injective. The image of $\mathrm{Div}(X)$ in $\mathrm{Pic}(X)$ is generated by the line bundles that admit a global section.

Recall that given a complex submanifold $Z \subset X$, there is an associated exact sequence of sheaves $0 \to \mathcal{I}_Z \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0$. A generalization for codimension 1 subvarieties with multiplicity (effective divisors) is as follows. Given an effective divisor $D \in \mathrm{Div}(X)$, we get $s \in \Gamma(X, L_D) \setminus \{0\}$ with $(s) = D$. This gives us a morphism of line bundles $\phi : X \times \mathbb{C} \to L_D$ given by $(z, a) \mapsto as(z)$. The restriction to the fiber $\phi_z : \mathbb{C} \to (L_D)_z$ is an isomorphism when $s(z) \neq 0$ and 0 otherwise. Moreover, there is an associated homomorphism of sheaves of $\mathcal{O}_X$ modules $\psi : \mathcal{O}_X \to \mathcal{O}_X(D)$ given by taking $f \in \mathcal{O}_X(U)$ to $f \cdot s|_U \in \mathcal{O}_X(D)(U)$. This is injective as a morphism of $\mathcal{O}_X$ modules because the set $\{z \in X \mid s(z) = 0 \in (L_D)_z\}$ is nowhere dense. However, since s has zero values, this map cannot be surjective.

Example 4.29. Suppose $X = \mathbb{C}^2$ and $D = [V(x)] + [V(y)] = (xy)$. The cokernel of $\mathcal{O}_{X,0} \to (\mathcal{O}_X(D))_0$, which is a map $\mathbb{C}\{x,y\} \to \mathbb{C}\{x,y\}$ given by multiplicaiton by xy, is thus $\mathbb{C}\{x,y\}/(xy) \cong (\mathbb{C}\{x\} \times \mathbb{C}\{x\})/(1,-1)$. These are germs at 0 of holomorphic functions on D.

The dual of ψ is $\eta : \mathcal{O}_X(-D) \to \mathcal{O}_X$ (alternatively, tensor ψ by $\mathcal{O}_X(-D)$) and is injective.

Lemma 4.30. *If $D = [Y]$ for $Y \subseteq X$ an irreducible hypersurface, then $\operatorname{im}(\eta) = \mathfrak{I}_Y$ and $\operatorname{coker}(\eta) = i_*\mathcal{O}_Y$. Thus there is an exact sequence $0 \to \mathcal{O}_X(-D) = \mathfrak{I}_Y \to \mathcal{O}_X \to i_*\mathcal{O}_Y \to 0$, which is the sequence we sought to generalize above.*

Given $D \in \operatorname{Div}(X)$, we can define $\mathcal{O}_D := \mathcal{O}_X/\mathcal{O}_X(-D)$, which sees multiplicities.

Example 4.31. Let $X = \mathbb{C}$ and $D = 2 \cdot [0] \in \operatorname{Div}(X)$. Then $\mathcal{O}_{D,0} = \mathbb{C}\{z\}/(z^2)$. This is a non-reduced ring.

4.5 Linear Systems and Maps to $\mathbb{P}^n$ ❖

Let $L \in \operatorname{Pic}(X)$ and $V \subseteq H^0(X, L)$ a nonzero linear subspace of finite dimension. The *linear system* $|V|$ associated to V is the set $\{(s) \in \operatorname{Div}(X) \mid s \in V\} \subset \operatorname{Div}(X)$. The linear system associated to L itself is denoted $|L|$ is the linear system associated to the whole space $H^0(X, L)$.

Definition 4.32. The *base locus* of a linear system $Bs(|V|)$ is $\{z \in X \mid \forall s \in V : s(z) = 0\}$.

Example 4.33. Let $X = \mathbb{P}^2$ and $H = Z(z_0)$. Then $\mathcal{O}_{\mathbb{P}^2}(-H)$ is the tautological bundle, and its dual $\mathcal{O}_{\mathbb{P}^2}(H)$, which is the sheaf of sections of the dual of the tautological line bundle. Recall that this has a canonical section σ. Then $H^0(\mathcal{O}_{\mathbb{P}^2}(X)) = \langle s_0, s_1, s_2\rangle \cong \mathbb{C}^3$, where $(s_i) = Z(z_i)$. The corresponding linear system to H is $|H| = \{\ell \subset \mathbb{P}^2\}$, which is the set of projective lines in $\mathbb{P}^2$. This has an empty base locus.

Now consider $V := \{s \in H^0(\mathcal{O}_{\mathbb{P}^2}(H)) \mid s([0,0,1]) = 0\}$, the subspace of sections that vanish at $[0,0,1]$. It is spanned by s_0 and s_1. The linear system of V is the set of all lines in $\mathbb{P}^2$ going through $[0,0,1]$, and the base locus is $[0,0,1]$. This is known as a *pencil* of lines.

Every linear system has an associated map to $\mathbb{P}^n$. That is, given $L \in \operatorname{Pic}(X)$ and $V \subseteq H^0(X, L)$, we get a map $\phi_{|V|} : X \setminus Bx(|V|) \to \mathbb{P}(V^*)$. It is given by $x \mapsto \mathbb{C} \cdot (V \to L_X \cong \mathbb{C}, s \mapsto s(x))$ and it is holomorphic. If $s_0, ..., s_N$ is a basis of V, then we get dual basis elements s_i^* of V^*. Then the map $\phi_{|V|}$ is given in this basis as:

$$\phi_{|V|}(x) = \left[\sum_{i=0}^{N} s_i(x) \cdot s_i^*\right] = [s_0(x) : ... : s_N(x)$$

This map pulls back the dual of the canonical bundle of $\mathbb{P}^N$ to L itself:

Proposition 4.34. $\phi_{|V|}^*(\mathcal{O}_{\mathbb{P}^n}(H)) \cong \mathcal{L}|_{X\setminus Bs(|V|)}$*, where $\mathcal{L}$ is the sheaf of sections of L.*

As a consequence, we get a map of X into $\mathbb{P}^n$ as long as the base locus is empty.

Definition 4.35. A line bundle $L \in \operatorname{Pic}(X)$ is called *ample* if if $\phi_{|L^k|}$ is an embedding (injective immersion) for k sufficiently large (and $Bs(|L^k|) = \emptyset$).

Example 4.36. Let $X = \mathbb{P}^2$ and $\mathcal{L} = \mathcal{O}_X(H)$, with $V = \operatorname{span}(s_0, s_1)$ as in the previous example. The base locus is $p = [0,0,1]$. Then $\phi_{|V|} : \mathbb{P}^2 \setminus \{p\} \to \mathbb{P}^2$ is given by $[z_0, z_1, z_2] \mapsto [z_0, z_1]$.

5. Kähler Geometry

❖

The story of Kähler manifolds starts with Hermitian manifolds, which are complex manifolds with Hermitian metrics. Recall that a Hermitian inner product $h : V \times V \to \mathbb{C}$ on a complex vector space V is a nondegenerate sesqui-linear map. In other words $\forall v, w : h(v,w) = \overline{h(w,v)}$ and $\forall v \neq 0, h(v,v) > 0$. It is $\mathbb{C}$-linear in the first entry and anti-linear in the second entry:

$$h(\lambda v, w) = \lambda h(v,w) = h(v, \bar{\lambda} w)$$

Example 5.1. If $V = \mathbb{C}^n$ and $A \in \mathrm{GL}(n,\mathbb{C})$ then we can define a Hermitian metric $h(v,w) = (\bar{A}v)^t A\bar{w} = v^t \bar{A}^t A \bar{w} = v^t A^* \bar{w}$. Sometimes

Sometimes the inner product is denoted $h(v,w) \equiv \langle v, w \rangle$. Notice that taking the real part $g := \Re(h)$, we get a scalar inner product on V as an $\mathbb{R}$ vector space. Further taking the imaginary part $\alpha = \Im(h)$ gives a symplectic form on V as a real vector space. Notice that:

$$g(v,w) = \alpha(v, iw) = g(iv, iw)$$

Conversely, given a real inner product g on V as a real vector space, then define $h(v,w) := g(v,w) - ig(iv,w)$, which is a Hermitian inner product if and only if g is compatible with the $\mathbb{C}$-structure, i.e. $\forall v, w : g(iv, iw) = g(v,w)$.

5.1 Hermitian Manifolds ❖

Definition 5.2. A *Hermitian manifold* is a complex manifold X with a pointwise Hermitian inner product $h \in \mathbb{C}^\infty(X, T^*X \otimes \overline{T^*X})$. In other words, for all $x \in X$ we have a Hermitian inner product $T_xX \times T_xX \to \mathbb{C}$. This is called a Hermitian metric on X.

Given a Hermitian manifold, with Hermitian metric h, then we get $g = \Re(h)$ a Riemannian metric and $\alpha = \Im(h)$ a (pointwise!) symplectic form. Note that α is not necessarily a global symplectic form, since it might not be closed.

The associated (1,1) form to a Hermitian metric h on X is $\omega := \frac{i}{2}(h - \bar{h})$, which is a differential form. Alternatively, Locally, if $h = \sum h_{ij} dz_i \otimes d\bar{z}_j$, then $\omega = \frac{i}{2} \sum h_{ij} dz_i \wedge d\bar{z}_j$. As a result, we can also get a volume form

$$\frac{1}{n!}\omega^{\wedge n} = \left(\frac{i}{2}\right)^n \det(h_{ij}) = dz_1 \wedge d\bar{z}_i \wedge \ldots \wedge dz_n \wedge d\bar{z}_n$$

This reflects the fact that every complex manifold is orientalble and hence must have a nonzero volume form.

Remark 5.3. We often call ω the Kähler “metric", even though it isn't a metric. The reason for this is that the tensor h and form ω are essentially interchangeable (locally both determined by h_{ij} as we see above). The Hermitian metric is the symmetrized Kähler form, and the Kähler form is the anti-symmetrized Hermitian metric.

Example 5.4. If $X = \mathbb{C}^n$ and h is the standard Hermitian metric, then $\omega = \frac{i}{2}\sum_i dz_i \wedge d\bar{z}_i$ and then $\frac{1}{n!}\omega^{\wedge n} = dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n$, where $z_i = x_i + iy_i$. This induces the standard orientation of $\mathbb{C}^n$.

Lecture 10/29

5.1.1 Lefshetz and Hodge $*$ Operators

The map $L : \Lambda^k T^*_{\mathbb{C}} X \to \Lambda^{k+2} T^*_{\mathbb{C}} X$ gien by $L(\alpha) = \alpha \wedge \omega$ is called the *Lefshetz operator*. Recall also the Hodge star operator $\star : \Lambda^k T^*_{\mathbb{C}} X \to \Lambda^{2n-k} T^*_{\mathbb{C}} X$ defined uniquely by the property:

$$\alpha \wedge \star\bar{\beta} = \langle \alpha, \beta \rangle \frac{1}{n!}\omega^{\wedge n}$$

The *dual Lefshetz operator* is defined to be $\Lambda := \star^{-1} \circ L \circ \star : \Lambda^{k+2} T^*_{\mathbb{C}} X \to \Lambda^k T^*_{\mathbb{C}} X$.

Locally, we can choose $e_1, ..., e_n \in T^*_x X$ a unitary basis with dual basis $e_1^*, ..., e_n^*$. Then $h(e_i^*, e_j^*) = \delta_{ij}$ and $\omega = \frac{i}{2} \sum_{i=1}^n e_i \wedge \bar{e}_i$, and then we can compute:

$$\frac{1}{n!} \omega^{\wedge n} = \left(\frac{i}{2}\right)^n (-1)^{n(n-1)/2} e_1 \wedge .... \wedge e_n \wedge \bar{e}_i \wedge ... \wedge \bar{e}_n$$

To express the $\star$ operator in this basis, we introduce multi-index notation $I = \{i_1, ..., i_a\} \subset \{1, ..., n\}$ where $i_i < ... < i_a$, and $e_I := e_{i_1} \wedge ... \wedge e_{i_a}$. The vectors $e_I \wedge \bar{e}_J$ with $|I| + |J| = k$ form a basis of $\Lambda^k T^*_{\mathbb{C}} X = \bigoplus_{a=0}^k \Lambda^a T^* X \otimes_{\mathbb{C}} \Lambda^{k-a} \overline{T^* X}$. Then the metric on these vectors acts as $\langle e_K \wedge \bar{e}_L, e_I \wedge \bar{e}_J \rangle = 1$ iff $K = I, L = J$ and zero otherwise. The Hodge star operator then acts as:

$$(e_K \wedge \bar{e}_L) \wedge * \overline{(e_I \wedge \bar{e}_J)} = \langle e_K \wedge \bar{e}_L, e_I \wedge \bar{e}_J \rangle \frac{1}{n!} \omega^{\wedge n} = \begin{cases} \left(\frac{i}{2}\right)^n (-1)^{n(n-1)/2} e_1 \wedge .... \wedge e_n \wedge \bar{e}_i \wedge ... \wedge \bar{e}_n & K = I, L = J \\ 0 & \text{otherwise} \end{cases}$$

Now comparing sides, we see that we must have:

$$\star \overline{(e_I \wedge \bar{e}_J)} = \pm \left(\frac{i}{2}\right)^n e_K \wedge \bar{e}_L, \quad K = I^c, L = J^c$$

The sign $\pm$ is the same sign as the one below:

$$e_I \wedge \bar{e}_J \wedge e_K \wedge \bar{e}_L = \pm (-1)^{n(n-1)/2} e_1 \wedge ... \wedge e_n \wedge \bar{e}_i \wedge ... \wedge \bar{e}_n$$

Some other properties about the star operator are:

1. $\langle \alpha, \star \beta \rangle = (-1)^k \langle \star \alpha, \beta \rangle$, i.e. it is self-adjoint up to a sign.
2. $\star 1 = \frac{1}{n!} \omega^{\wedge n}$.
3. On $\Lambda^k T^* X$, we have $\star^2 = (-1)^k$.

The dual Leftshetz operator Λ we claim is adjoint to L:

$$\begin{aligned} \langle L\alpha, \beta \rangle \frac{1}{n!} \omega^{\wedge n} &= L\alpha \wedge \star \bar{\beta} \\ &= \omega \wedge \alpha \wedge \star \bar{\beta} \\ &= \alpha \wedge \omega \wedge \star \bar{\beta} \\ &= \alpha \wedge (L \star \bar{\beta}) \\ &= \langle \alpha, (\star^{-1} \circ L \circ \star) \beta \rangle \frac{1}{n!} \omega^{\wedge n} = \langle \alpha, \Lambda \beta \rangle \frac{1}{n!} \omega^{\wedge n} \end{aligned}$$

A form that is in the kernel of Λ is called primitive. The vector bundle $\Lambda^k X := \Lambda^k T^*_{\mathbb{C}} X$ admits a direct sum decomposition by primitive sub-bundles:

Proposition 5.5. *Let $P^m X := \ker(\Lambda : \Lambda^m X \to \Lambda^{m-2} X)$ be the bundle of primitive m-forms. Then:*

$$\Lambda^k X = \bigoplus_{i \geq 0} L^i (P^{k-2i} X)$$

5.1.2 d^* and Δ Operators

As with any dimension m Riemannian manifold, we can define the (formally) adjoint operator $d^* : \mathcal{A}^k(M) \to \mathcal{A}^{k-1}(M)$ by $d^* := (-1)^{m(k+1)+1} \star d \star$ and the Laplace operator $\Delta = d^* d + d d^*$. Since we are considering even-dimensional manifolds, we have $d^* = - \star d \star$. The adjoint d^* has the property that:

$$(d\alpha, \beta) := \int_X \langle d\alpha, \beta \rangle \frac{\omega^n}{n!} = \int_X \langle \alpha, d^* \beta \rangle \frac{\omega^n}{n!} =: (\alpha, d^* \beta)$$

for all $\alpha \in \mathcal{A}^k(X)$ and $\beta \in \mathcal{A}^{k+1}(X)$ where the support of β is compact.

Example 5.6. If $X = \mathbb{C}^n$ with $\omega = \frac{i}{2}\sum dz_i \wedge d\bar{z}_i$ and $k = 0$, then $\Delta = -2\sum \partial_{x_i}^2 + \partial_{y_i}^2$.

The exterior and dual exterior differential operators decompose via the complex structure as:

$$d = \partial + \bar{\partial}$$

$$d^* = \partial^* + \bar{\partial}^*$$

where $\partial^* = -\ast \circ \bar{\partial} \circ \ast$ and $\bar{\partial}^* = -\ast \circ \partial \circ \ast$. These satisfy $\partial^2 = 0$ and $\bar{\partial}^2 = 0$. We can also define $\Delta_\partial := \partial^*\partial + \partial\partial^*$ and $\Delta_{\bar{\partial}} := \bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*$. It should be noted, however, that $\Delta \neq \Delta_\partial + \Delta_{\bar{\partial}}$ in general. However this will be true for Kähler manifolds.

Lemma 5.7. *If (X, h) is a Hermitian manifold, then $(\partial^*)^2 = (\bar{\partial}^*)^2 = 0$.*

5.2 Kähler Manifolds ❖

Definition 5.8. A Hermitian manifold (X, h) is called *Kähler* if $d\omega = 0$ (i.e. the associated (1,1) form is closed). In this case, ω is called a Kähler form.

There are some other equivalent definitions of a Kähler manifold make it clear why this is a good condition. The first is that the parallel transport is $\mathbb{C}$-linear on X. Another is that, pointwise (X, h) can be associated to $(\mathbb{C}^n, \sum dz_i \otimes d\bar{z}_i)$ to second order, i.e. for all $p \in X$ there exist holomorphic coordinates $z_1, .., z_n$ with $z_i(p) = 0$ such that $h_{ij} = \delta_{ij} + O(|z|^2)$.

Proposition 5.9. *Let X be a compact complex manifold. Then the set of Kähler forms on X is a (possibly empty) open, convex cone in $C^{1,1}_{\mathbb{R}}(X) = \{\alpha \in \mathcal{A}^{1,1}(X) \mid \bar{\alpha} = \alpha, d\alpha = 0\}$.*

Proof:

It is clear that all Kähler forms are closed and real, by how they are defined from h. Moreover, positivity of ω is an open condition becuase it is locally an open condition on the determinant of h_{ij}. To see that it is a cone, note that the sum of two Kähler forms is again Kähler, which follows from the fact that the sum of two Hermitian forms is also a Hermitian form. Notice also that multiplying by a positive constant preserves the Kähler condition.

□

5.2.1 Fubini-Study Metric

The most important example of a Kähler structure comes from the Fubini-Study metric. It can be characterized as the unique $U(n+1)$ invariant Kähler metric h_{FS} on $\mathbb{P}^n$ whose associated (1,1) form ω_{FS} on satisfying $\int_{\mathbb{P}^1} \omega_{FS} = 1$, where $\mathbb{P}^1 \subset \mathbb{P}^n$ is projective line.

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Definition 5.10. On $U_i = \mathbb{P}^n \setminus V(z_i)$, the form ω_{FS} is $\omega_{FS}|_{U_i} = \omega_i = \frac{1}{2\pi}\partial\bar{\partial}\log\left(\sum_{\ell=0}^n \left|\frac{z_\ell}{z_i}\right|^2\right)$.

We can check that these glue together to give a form $\omega_{FS} \in \mathcal{A}^{1,1}(\mathbb{P}^n)$ by writing:

$$\log\left(\sum_{\ell=1}^n \left|\frac{z_\ell}{z_i}\right|^2\right) = \log\left(\left|\frac{z_j}{z_i}\right|^2 \sum_{\ell=0}^n \left|\frac{z_\ell}{z_j}\right|^2\right) = \log\left(\left|\frac{z_j}{z_i}\right|^2\right) + \log\left(\sum_{\ell=0}^n \left|\frac{z_\ell}{z_j}\right|^2\right)$$

Taking $\partial\bar{\partial}$ of both sides, we get:

$$\omega_i = \partial\bar{\partial}\log\left(\left|\frac{z_j}{z_i}\right|^2\right) + \omega_j$$

We are restricting to $U_i \cap U_j$ here, so $w_j := z_j/z_i$ is the jth coordinate function on U_i. Thus we have:

$$\partial\bar{\partial}\log|w_j|^2 = \partial\left(\frac{1}{w\bar{w}}\bar{\partial}(w\bar{w})\right) = \partial\left(\frac{wd\bar{w}}{w\bar{w}}\right) = 0$$

and so $\omega_i = \omega_j$ on $U_i \cap U_j$. From the definition and the fact that $d = \partial + \bar{\partial}$, it is clear that $d\omega = 0$.

Exercise 5.11. Show that, on U_0 with coordinates $w_i = z_i/z_0$, the form $\omega_0 = \omega_{FS}|_{U_0}$ takes the form:

$$\omega_0 = \sum_{ij} h_{ij} dw_i \wedge d\bar{w}_i$$

$$h_{ij} = \frac{i}{2\pi}\frac{\left(1+\sum|w_i|^2\right)\delta_{ij} - \bar{w}_i w_j}{\left(1+\sum|w_i|^2\right)^2}$$

From there, show that ω_{FS} is positive definite by showing h_{ij} is positive definite.

A more intrinsic definition of h_{FS} is that it is the pushforward of the round metric on S^{2n+1} via the Hopf fibration $S^1 \to S^{2n+1} \to \mathbb{P}^n$. Explicitly, the second map is quotienting by the $U(1)$ action $x \mapsto e^{i\theta}x$ which is invariant under the round metric. Therefore we have a well-defined pushforward metric on $\mathbb{P}^n$. It can be described explicitly as:

$$\|\xi\|^2_{FS} = \inf\left\{\|\tilde{\xi}\|_{S^{2n+1}} \mid \pi_*\tilde{\xi} = \xi\right\}$$

5.2.2 Examples of Kähler Manifolds

Some examples of Kähler manifolds are:

- Complex tori $X = \mathbb{C}^n/\Gamma$. In this case, since $T_xX = \mathbb{C}^n$ canonically, we have h a constant metric.
- Riemann surfaces Σ. In this case, any Hermitian metric is Kähler because $d\omega \in \mathcal{A}^3(\Sigma) = 0$.
- Ball quotients $B_1(0)/\Gamma$, where the Kähler form on $B_1(0) = D^n \subset \mathbb{C}^n$ is:

$$\omega = \frac{i}{2}\partial\bar{\partial}\log(1-\|z\|^2).$$

Proposition 5.12. *Given (X,ω) Kähler and $Y \subset X$ a complex submanifold, then $(Y, \omega|_Y)$ is Kähler.*

Proof:

> From the inclusion $i : Y \to X$, we find $\omega|_Y = i^*\omega$. It thus follows that $\omega|_Y$ is positive (i.e. the Hermitian metric is positive) and that $d(i^*\omega) = i^*(d\omega) = 0$.

□

As a corollary, complex projective manifolds are Kähler. However, it should be noted that the Kähler form itself depends on the embedding into $\mathbb{P}^n$.

5.2.3 Properties of Kähler Forms

Since a Kähler form ω is closed, we must have $[\omega] \in H^2_{dR}(X)$. Is it a nonzero class?

Proposition 5.13. *If (X,ω) is a Kähler manifold, then $[\omega] \neq 0$.*

Proof:

Consider the homomorphism $H^{2n}_{dR}(X) \to \mathbb{R}$ given by $\alpha \mapsto \int_X \alpha$. Since ω^n is a volume form $\frac{1}{n!}[\omega^n] \mapsto \mathrm{vol}(X) > 0$, we have $[\omega^n] \neq 0$ and hence $[\omega] \neq 0$.

□

Corollary 5.14. *If $b_{2k}(X) = 0$ for any $1 \leq k \leq n-1$, then X is not Kähler.*

This allows us to come up with an example of a complex manifold which is not Kähler: set $X = (\mathbb{C}^n \backslash \{0\})/\mathbb{Z} \cong S^1 \times S^{2n-1}$, which is a Hopf manifold. By the Künneth formula, we find that if $n \geq 2$, then $b_2(X) = 0$.

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5.3 The Kähler Identities ❖

The following is a collection of very important properties about the operators $*, L, \Lambda, \partial, \bar{\partial}, \partial^*, \bar{\partial}^*$, and Δ on a Kähler manifold, known as the Kähler identities:

Proposition 5.15. *Let (X, ω) be Kähler. Then:*

i. $[\bar{\partial}, L] = [\partial, L] = 0$ *and* $[\bar{\partial}^*, \Lambda] = [\partial^*, \Lambda] = 0$.

ii. $[\bar{\partial}^*, L] = i\partial$, $[\partial^*, L] = -i\bar{\partial}$, $[\Lambda, \bar{\partial}] = -i\partial^*$, $[\Lambda, \partial] = i\bar{\partial}^*$.

iii. $\Delta_\partial = \Delta_{\bar{\partial}} = \frac{1}{2}\Delta$ *and* Δ *commutes with* $*, L, \Lambda, \partial, \bar{\partial}, \partial^*, \bar{\partial}^*$.

Proof:

For $i.$, we have:

$$[\bar{\partial}, L](\alpha) = \bar{\partial}(\omega \wedge \alpha) - \omega \wedge \bar{\partial}\alpha = (\bar{\partial}\omega) \wedge \alpha = 0$$

And similarly for $[\partial, L]$. Now for $[\bar{\partial}^*, \Lambda]$, we get:

$$\begin{aligned}
[\bar{\partial}^*, \Lambda](\alpha) &= - * \partial * *^{-1} L * \alpha - *L * (- * \partial *)\alpha \\
&= - * \partial L * \alpha - (-1)^k *^{-1} L\partial * \alpha \\
&= - * [\partial, L] * \alpha = 0
\end{aligned}$$

and the final equality in $i.$ comes by conjugating what we just showed. We will not prove $ii.$ (see [Huy05]. For $iii.$, note that:

$$i(\partial\bar{\partial}^* + \bar{\partial}^*\partial) = \partial[\Lambda, \partial] + [\Lambda, \partial]\partial = \partial\Lambda\partial - \partial\Lambda\partial = 0$$

by $ii.$, and so therefore $\partial\bar{\partial}^* + \bar{\partial}^*\partial = 0$. To see that $\Delta_\partial = \Delta_{\bar{\partial}}$, we use $ii.$ again:

$$\begin{aligned}
\Delta_\partial = \partial^*\partial + \partial\partial^* &= i[\Lambda, \bar{\partial}]\partial + i\partial[\Lambda, \bar{\partial}] \\
&= i(\Lambda\bar{\partial}\partial - \bar{\partial}\Lambda\partial + \partial\Lambda\bar{\partial} - \partial\bar{\partial}\Lambda) \\
&= i(\Lambda\bar{\partial} - (\bar{\partial}[\Lambda, \partial] + \bar{\partial}\partial\Lambda) + ([\partial, \Lambda]\bar{\partial} + \Lambda\partial\bar{\partial}) - \partial\bar{\partial}\Lambda) \\
&= i(\Lambda\bar{\partial}\partial - i\bar{\partial}\bar{\partial}^* - \bar{\partial}\partial\Lambda - i\bar{\partial}^*\bar{\partial} + \Lambda\partial\bar{\partial} - \partial\bar{\partial}\Lambda) \\
&= \Delta_{\bar{\partial}}
\end{aligned}$$

Now, the laplacian Δ is:

$$\begin{aligned}
\Delta &= (\partial + \bar{\partial})(\partial^* + \bar{\partial}^*) + (\partial^* + \bar{\partial}^*)(\partial + \bar{\partial}) \\
&= \Delta_\partial + \Delta_{\bar{\partial}} + (\partial\bar{\partial}^* + \bar{\partial}^*\partial) + \overline{(\partial\bar{\partial}^* + \bar{\partial}^*\partial)} \\
&= 2\Delta_\partial
\end{aligned}$$

Finally, to show that these commute with everything else, we make use of $i.$ and $ii.$ liberally. For example, to show that $[\Lambda, \Delta] = 0$, define $d^c = -i(\partial - \bar{\partial})$ and $d^{c*} = -\star d^c \star$. Then observe that $dd^c = 2i\partial\bar{\partial} = -d^c d$. Using this, we can show:

$$\begin{aligned}\Lambda\Delta &= \Lambda dd^* + \Lambda d^* d \\ &= d\Lambda d^* - id^{c*}d^* + d^*\Lambda d \\ &= dd^*\Lambda + id^*d^{c*} + d^*d\Lambda - id^*d^{c*} = \Delta\Lambda\end{aligned}$$

□

5.4 Hodge Theory ❖

A basic analytic fact for a Hermitian manifold (X, h) is that Δ, Δ_∂, and $\Delta_{\bar{\partial}}$ are elliptic partial differential operators. If X is compact, any elliptic operator is Fredholm, which means $\ker \Delta, \ker \Delta_\partial$ and $\ker \Delta_{\bar{\partial}}$ are finite dimensional.

Definition 5.16. For a Hermitian manifold (X, h), the space of *harmonic forms* is:

$$\mathcal{H}^k(X, h) = \{\alpha \in \mathcal{A}^k(X) \mid \Delta\alpha = 0\}$$

$$\mathcal{H}^{p,q}(X, h) = \mathcal{H}^k(X, h) \cap \mathcal{A}^{p,q}(X)$$

These are finite dimensional. Similarly, we define $\mathcal{H}^k_\partial, \mathcal{H}^k_{\bar{\partial}}$ and $\mathcal{H}^{p,q}_\partial, \mathcal{H}^{p,q}_{\bar{\partial}}$.

Remark 5.17. Note that if X is Kähler, these all coincide because $\Delta_\partial = \Delta_{\bar{\partial}} = \frac{1}{2}\Delta$.

Lemma 5.18. *If X is compact and $\alpha \in \mathcal{A}^k(X)$, then*

$$\Delta_{\bar{\partial}}\alpha = 0 \iff \bar{\partial}\alpha = 0 \text{ and } \bar{\partial}^*\alpha = 0$$

Proof:

One direction is immediate from the formula $\Delta_{\bar{\partial}} = \bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*$. For the other direction, we have:

$$\begin{aligned}0 = \langle \Delta_{\bar{\partial}}\alpha, \alpha\rangle = \int_X \Delta_{\bar{\partial}}\alpha \wedge *\bar{\alpha} &= \langle \bar{\partial}\alpha, \bar{\partial}\alpha\rangle + \langle \bar{\partial}^*\alpha, \bar{\partial}^*\alpha\rangle \\ &= \|\bar{\partial}\alpha\| + \|\bar{\partial}^*\alpha\|\end{aligned}$$

Since these are norms, we must have $\bar{\partial}\alpha = \bar{\partial}^*\alpha = 0$.

□

Therefore harmonic forms are closed, and we have projections $\mathcal{H}^k(X, h) \to H^k(X)$ and $\mathcal{H}^{p,q}(X, h) \to H^{p,q}(X)$.

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Proposition 5.19. *Suppose (X, h) is Hermitian. Then:*

$$\mathcal{H}^k_{\bar{\partial}}(X, h) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\bar{\partial}}$$

and similarly for $\mathcal{H}^k_\partial$.

Proof:

Let $\alpha \in \mathcal{A}^k(X)$ be any form and let $\alpha = \sum \alpha^{p,q}$ be its bidegree decomposition. Then if each component $\alpha^{p,q}$ is harmonic, then α is harmonic. This shows that $\bigoplus \mathcal{H}^{p,q}_{\bar\partial} \to \mathcal{H}^k_{\bar\partial}(X,h)$ induced by the bidegree decomposition is injective. Given any harmonic k form α, we have $0 \sum \Delta_{\bar\partial}(\alpha^{p,q})$ once again by the bidegree decomposition. However, this is a direct sum decomposition, so $\Delta_{\bar\partial}(\alpha^{p,q}) = 0$ and hence the map is surjective.

□

Note that this decomposition isn't necessarily true for $\mathcal{H}^k(X,h)$ unless X is Kähler.

5.4.1 Hodge Symmetry and Serre duality

Note that $\star : \mathcal{H}^{p,q}_{\bar\partial}(X,h) \to \mathcal{H}^{n-q,n-p}_{\partial}(X,h)$ is an isometry, and this isometry depends on h. There is a similar isometry if we exchange $\bar\partial$ and ∂.

Suppose X is compact. Then the bilinear form:

$$\mathcal{H}^{p,q}_{\bar\partial}(X,h) \times \mathcal{H}^{n-p,n-q}_{\bar\partial}(X,h) \to \mathbb{C}$$

$$(\alpha,\beta) \mapsto \int_X \alpha \wedge \beta$$

is nondegenerate. We can see this by taking any nonzero $\alpha \in \mathcal{H}^{p,q}_{\bar\partial}(X,h)$ and noting that $\star\bar\alpha \in \mathcal{H}^{n-p,n-q}_{\bar\partial}$ and:

$$\int_X \alpha \wedge \star\bar\alpha = \int_X \langle \alpha, \alpha \rangle \frac{\omega^n}{n!} = \|\alpha\|^2 > 0$$

This is nondegenerate pairing is called Serre duality, and it induces an isomorphism:

$$\mathcal{H}^{p,q}_{\bar\partial}(X,h) \cong \mathcal{H}^{n-p,n-q}_{\bar\partial}(X,h)^*$$

5.4.2 Hodge Decompositions

A very deep and important result to Hodge theory is the following decomposition on forms:

Theorem 5.20. *Let (X,h) be a compact Hermitian manifold. Then we have two decompoisitions:*

$$\begin{aligned}\mathcal{A}^{p,q}(X) &= \mathcal{H}^{p,q}_{\partial}(X,h) \oplus \partial\mathcal{A}^{p-1,q}(X) \oplus \partial^*\mathcal{A}^{p+1,q}(X)\\ &= \mathcal{H}^{p,q}_{\bar\partial}(X,h) \oplus \bar\partial\mathcal{A}^{p-1,q}(X) \oplus \bar\partial^*\mathcal{A}^{p+1,q}(X)\end{aligned}$$

Remark 5.21. There is an analogous result for a Riemannian manifold (M,g):

$$\mathcal{A}^k(X) = \mathcal{H}^k(M,g) \oplus d\mathcal{A}^{k-1}(X) \oplus d^*\mathcal{A}^{k+1}(X)$$

Corollary 5.22. *Let (X,h) be a compact Hermitian manifold. Then the canonical projection:*

$$\mathcal{H}^{p,q}_{\bar\partial}(X,h) \to H^{p,q}(X)$$

is an isommetry, and similarly for the projection from $\mathcal{H}^{p,q}_{\partial}(X,h)$.

Proof:

Consider the kernel of $\bar\partial : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q-1}(X)$. Using the $\bar\partial$ decomposition, we know this kernel contains the first two components of $\mathcal{A}^{p,q}(X)$, namely $\bar\partial(A^{p,q-1}(X)) \oplus \mathcal{H}^{p,q}_{\bar\partial}(X,h)$, because $\bar\partial^2 = 0$ and harmonic p,q forms are $\bar\partial$ closed. Moreover, any nonzero element $\bar\partial^*\beta$ of $\bar\partial^* A^{p,q+1}(X)$ is not in the kernel because:

$$0 = (\bar\partial\bar\partial^*\beta, \beta) = (\bar\partial\beta, \bar\partial\beta) = \|\bar\partial^*\beta\|^2 > 0$$

Thus $\ker(\bar\partial) = \bar\partial(A^{p,q-1}(X)) \oplus \mathcal{H}^{p,q}_{\bar\partial}(X,h)$. Quotienting by the image of $\bar\partial$ gives the result.

□

Corollary 5.23. $\overline{H^{p,q}(X)} \cong H^{q,p}(X)$

Proof:

$$\overline{H^{p,q}(X)} = \overline{\mathcal{H}^{p,q}_{\bar\partial}(X,h)} = \mathcal{H}^{q,p}_{\partial}(X) = H^{q,p}(X)$$

□

We will not prove the Hodge decomposition theorem, but will provide the motivating idea for how it is done. Let V be a Hermitian vector space, $d : V \to V$ be an automorphism with $d^2 = 0$, and d^* its adjoint. Then define $\Delta = dd^* + d^*d$ and let $\mathcal{H} := \ker \Delta$. Note that if $\Delta h = 0$, then $dh = d^*h = 0$ because:

$$0 = \langle \Delta h, h\rangle = \langle dd^*h, h\rangle + \langle d^*dh, h\rangle = \|dh\| + \|d^*h\|$$

The subspaces $\mathcal{H}, dV$ and d^*V are all mutually orthogonal, as one can verify directly. For example, if $h \in \mathcal{H}$ and $\beta \in V$, then:

$$\langle h, d\beta\rangle = \langle d^*h, \beta\rangle = 0$$

Now we must show that every $v \in V$ can be decomposed as a sum $h + d\alpha + d^*\beta$. Note that Δ induces an invertible map $\Delta_{\mathcal{H}^\perp} : \mathcal{H}^\perp \to \mathcal{H}^\perp$. Then we define:

$$G = \begin{pmatrix} 0 & 0 \\ 0 & (\Delta_{\mathcal{H}^\perp})^{-1} \end{pmatrix}$$

If $H : V \to \mathcal{H}$ is projection, then we claim $G\Delta + H = I$, as can be easily verified. Moreover, G, Δ, and H all commute with each other and d, d^* each commute with Δ, G, and H. Then the next step is to use these facts to show that $V \cong \mathcal{H} \oplus dV \oplus d^*V$ via $\alpha \mapsto (H\alpha, dd^*G\alpha, d^*dG\alpha)$ is an isomorphism. This is rougly the roadmap to prove the Hodge decomposition on forms, which requires some theory of pseudo-differential operators beause our Hermitian vector space V is no longer finite dimensional and we no longer know how to define G. See [Wells]

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Another important decomposition in Hodge theory is the following, known as the Hodge theorem:

Theorem 5.24. *If (X, h) is compact and Kähler, then:*

a. $H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$ and this decomposition is independent of h.

b. The isommetry $\overline{H^{p,q}(X)} \cong H^{q,p}(X)$ is induced by conjugation on $H^k(X, \mathbb{C})$.

c. $H^{p,q}(X) = H^{n-p,n-q}(X)^*$.

The hodge numbers are $h^{p,q}(X) := \dim_{\mathbb{C}} H^{p,q}(X)$. The array of these numbers forms a diamond, known as the *Hodge diamond* (see Figure 5.1). The horizontal reflection symmetry is given by conjugation, the vertical reflection symmetry is Hodge symmetry, and Serre duality is the symmetry under rotation by 180 degrees. In particular, all information about the hodge numbers is contained in one quadrant of the diamond.

$$\begin{array}{ccccc} & & h^{2,2} & & \\ & h^{2,1} & & h^{1,2} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{1,0} & & h^{0,1} & \\ & & h^{0,0} & & \end{array}$$

Figure 5.1: A hodge diamond for a complex dimension 2 Kähler manifold.

5.5 The $\partial\bar{\partial}$-Lemma ❖

The following is very useful for understanding Kähler manifolds.

Proposition 5.25. *Let (X, h) be compact and Kähler, $\alpha \in \mathcal{A}^{p,q}(X)$ with $d\alpha = 0$. Then the following are equivalent:*

i. $\exists \beta \in \mathcal{A}^{p+q-1}(X)$ *such that* $\alpha = d\beta$.

ii. $\exists \beta \in \mathcal{A}^{p-1,q}(X)$ *such that* $\alpha = \partial\beta$.

iii. $\exists \beta \in \mathcal{A}^{p,q-1}(X)$ *such that* $\alpha = \bar{\partial}\beta$.

iv. $\exists \beta \in \mathcal{A}^{p-,q-1}(X)$ *such that* $\alpha = \partial\bar{\partial}\beta$.

v. For any Kähler metric h'*:* $\alpha \perp \mathcal{H}^{p,q}(X, h')$.

For example, this gives us some properties about the space of Kähler forms:

Corollary 5.26. *Let X be a compact Kähler manifold. Then:*

a. If ω is the Kähler form, then locally $\omega = i\partial\bar{\partial} f$ for some $C^\infty(U, \mathbb{R})$. The function f is known as the Kähler potential.

b. If ω, ω' are Kähler forms on X with $[\omega] = [\omega'] \in H^2_{dR}(X)$ then $\omega = \omega' + i\partial\bar{\partial} f$ for some $f \in \mathbb{C}^\infty(X, \mathbb{R})$.

c. The Kähler cone:

$$\mathcal{K}_X = \{[\omega] \in H^{1,1}(X, \mathbb{R}) \mid \omega \text{ Kähler form}\}$$

is an open convex cone in $H^{1,1}(X, \mathbb{R}) := H^2(X, \mathbb{R}) \cap H^{1,1}(X)$. Moreover, $\mathcal{K}_X$ does not contain any line $[\omega] + \mathbb{R}\alpha, [\omega] \in \mathcal{K}_X$ and if $\alpha \in H^{1,1}(X, \mathbb{R}), [\omega] \in \mathcal{K}_X$ then $\alpha + \lambda[\omega] \in \mathcal{K}_X$ for $\lambda \gg 0$. In other words, $\mathcal{K}_X$ is open, full-dimensional, and has no lineality.

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5.6 The Hard Lefschetz Theorem ❖

Theorem 5.27. *Let (X, h) be Kähler and $n = \dim(X)$. Then for $0 \le k = p + q \le n$. Then $L^{n-k} : \mathcal{H}^{p,q}(X, h) \to \mathcal{H}^{n-p,n-q}(X, h)$ given by $\alpha \mapsto \alpha \wedge \omega^{n-k}$ is an isomorphism.*

Proof:

Recall the identities $[L, \Delta] = [\Lambda, \Delta] = 0$. This means L, Λ induce an $\mathfrak{sl}(2, \mathbb{C})$ representation on $\bigoplus_{p,q} \mathcal{H}^{p,q}(X, h)$. Then using $\Delta\alpha = 0$, we get $\Delta L\alpha = \Delta\Lambda\alpha = 0$. The statement now follows from $\mathfrak{sl}(2, \mathbb{C})$ representation theory.

□

On a compact Kähler manifold, L and Λ induce morphisms:

$$L : H^{p,q}(X) \to H^{p+1,q+1}(X)$$

$$\Lambda : H^{p+1,q+1}(X) \to H^{p,q}(X)$$

via the identification $H^{p,q}(X) = \mathcal{H}^{p,q}_{\bar\partial}(X)$. These only depend on the Kähler class $[\omega]$ of X.

Theorem 5.28 (Hard Lefschetz)**.** *If (X, h) is compact Kähler, then:*

a. $L^{n-k} : H^k(X, \mathbb{R}) \to H^{2n-k}(X, \mathbb{R})$ *is an isomorphism for* $0 \le k \le n$.

b. $H^k(X, \mathbb{R}) = \bigoplus_{i \ge 0} L^i H^{k-2i}(X, \mathbb{R})_p$*, where:*

$$H^j(X, \mathbb{R})_p = \ker(\Lambda : H^j(X, \mathbb{R}) \to H^{j-2}(X, \mathbb{R}))$$

which we call the primitive cohomology*.*

5.6.1 Hodge Conjecture

Let (X, h) be compact Kähler and $Y \subset X$ a complex submanifold of codimension k. Then the fundamental class $[Y]$ is Poincaré dual to $\alpha \in H^{k,k} \cap H^{2k}(X, \mathbb{C})_{\mathbb{Z}}$, where:

$$H^{2k}(X, \mathbb{C})_{\mathbb{Z}} = \operatorname{im}(H^{2k}(X, \mathbb{Z}) \to H^{2k}(X, \mathbb{C}))$$

Indeed, $\beta \in H^{p,q}$ with $p \ne q$ and $p + q = 2(n - k)$, then $\int_Y \beta = 0$. The Hodge conjecture states that if X is projective, then $H^{k,k}(X) \cap H^{2k}(X, \mathbb{Q})$ is generated by such classes.

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5.6.2 Lefschetz Theorem on (1,1)-classes

Theorem 5.29. *Let X be compact and Kähler. Then $c_1 : \operatorname{Pic}(X) \to H^{1,1}(X)_{\mathbb{Z}} = H^{1,1}(X) \cap H^2(X, \underline{\mathbb{Z}})$ is surjective. That is, every integral (1,1)-class is the first Chern class of some line bundle.*

Lemma 5.30. *Let X be compact and Kähler; then the following two maps agree:*

$$\phi : H^k(X, \mathbb{C}) \to H^k(X, \mathcal{O}_X) = H^{0,k}(X)$$

$$\psi : H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X) \to H^{0,k}(X)$$

where the first map comes from the inclusion $\underline{\mathbb{C}} \to \mathcal{O}_X$ and the second is a projection.

Proof:

Consider the de Rham and Dolbeault complexes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \underline{\mathbb{C}} & \longrightarrow & \mathcal{A}^0_{\mathbb{C}}(X) & \xrightarrow{d} & \mathcal{A}^1_{\mathbb{C}}(X) & \xrightarrow{d} & \cdots \\
 & & \downarrow & & \downarrow{=} & & \downarrow{\pi^{0,1}} & & \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{A}^{0,0}(X) & \xrightarrow{\bar\partial} & \mathcal{A}^{0,1}(X) & \xrightarrow{\bar\partial} & \cdots
\end{array}$$

These are both acyclic resolutions, and so they compute cohomology. Moreover each square is commutative by the fact that $d = \partial + \bar\partial$. For any cohomology class $[\alpha] \in H^k(X, \mathbb{C})$, we can choose a representative $\alpha \in \mathcal{H}^k(X, h)$ for some Kähler metric h on X. Its image in $H^k(X, \mathcal{O}_X)$ is $[\pi^{0,k}(\alpha)]$, which is the projection onto the $0, k$ part of α.

□

To prove the Lefschetz theorem on $(1,1)$ classes, consider the following. The space $H^2(X,\mathbb{C})$ decomposes as $H^{2,0}(X)\oplus H^{1,1}(X)\oplus H^{0,2}(X)$. Given $\alpha \in H^2(X,\underline{\mathbb{Z}})$ which we think of as an element of $H^2(X,\mathbb{C})$, it decomposes as $\alpha = \alpha^{2,0}+\alpha^{1,1}+\alpha^{0,2}$. Since α is real, $\alpha^{0,2} = \overline{\alpha^{2,0}}$. This implies that $\alpha \in H^{1,1}(X)_{\mathbb{Z}}$ if and only if $\alpha^{2,0} = 0$.

Now we turn to the exponential sequence:

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^{\times} \longrightarrow 1$$

The long exact sequence in cohomology yields $H^1(X,\mathcal{O}_X^{\times}) \cong \mathrm{Pic}(X) \to H^2(X,\underline{\mathbb{Z}}) \to H^2(X,\mathcal{O}_X)$. We also have the map $H^2(X,\underline{\mathbb{Z}}) \to H^2(X,\mathbb{C})$ induced by $\mathbb{Z}\subset\mathbb{C}$, which factors through the LES:

$$\begin{array}{ccccccc} \cdots \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & H^2(X,\underline{\mathbb{Z}}) & \longrightarrow & H^2(X,\mathcal{O}_X) & \longrightarrow \cdots \\ & & & \downarrow & \nearrow_{\pi} & & \\ & & & H^2(X,\mathbb{C}) & & & \end{array}$$

By Lemma 5.30, the map π is $\alpha \mapsto \alpha^{0,2}$, and hence $\ker(\pi) = H^{1,1}(X)\cap H^2(X,\underline{\mathbb{Z}}) \subset H^2(X,\mathbb{C})$. Since the image of $c_1 : \mathrm{Pic}(X) \to H^2(X,\mathbb{C})$ is contained in $H^{1,1}(X)$, exactness of the above sequence then implies that $\mathrm{im}(c_1) = \ker(\pi) = H^{1,1}(X)_{\mathbb{Z}}$.

5.7 Formality of Kähler Manifolds ❖

Recall a CDGA, which is a type of chain complex with the structure of a differential algebra:

Definition 5.31. A (*graded commutative*) *differential graded algebra* (or (C)DGA) over a field k is a triple $(A^\bullet,\wedge,d)$ such that:

- $A^\bullet = \bigoplus_{i\in\mathbb{N}} A^i$ is a graded k-algebra with $k\subset A^0$.
- $(A^\bullet, d)$ is a chain complex.
- $d(a\wedge b) = (da)\wedge b + (-1)^{|a|} a\wedge (db)$.
- It is graded commutative: $a\wedge b = (-1)^{|a||b|} b\wedge a$.

For example, if M is a smooth manifold, then $(A^*(M),\wedge,d)$ is a CDGA. Taking cohomology of any CDGA $H^i(A^\bullet) = (\ker d\cap A^i)/dA^{i-1}$, we get another DGA with $d=0$. We call a morphism of CDGA's a quasi-isormorphism if it induces an isomorphism on cohomology.

Real homotopy theory tells us that if $\pi_1(M)=0$, then $A^*(M)$ knows everything about $\pi_n(M)\otimes_{\mathbb{Z}}\mathbb{R}$. Unfortunately, $A^*(M)$ is very big (it is infinite-dimensional). Griffiths-Morgan, "Rational Homotopy Theory and Differential Forms"

Definition 5.32. A (C)DGA $A^\bullet/k$ is *connected* if $k\to A$ induces an isomorphism on H^0. It is called *simply connected* if it connected and $H^1(A^\bullet)=0$.

Note that for the deRham CDGA, it is connected if and only if M is connected and it is simply connected if and only if $b_1(M)=0$.

5.7.1 Minimal Model of a CDGA I

Definition 5.33. A CDGA $M^\bullet/k$ is *minimal* if there exist $x_i\in M^{d_i}$, $1\le d_1\le d_2\le\cdots$ such that:

- $M^\bullet = \langle x_i\rangle_{i\in\mathbb{N}} = \Lambda^*\bullet(\bigoplus k\cdot x_i)$ (i.e. it is the free CDGA over k generated by the x_i's)
- For all i, then $dx_i = \langle x_1,\ldots,x_{i-1}\rangle^+$. (where $+$ denotes the degree >0 part).

Lemma 5.34. *If $M^\bullet$ is a minimal CDGA, then $M^\bullet$ is simply connected if and only if $M^1=0$.*

Definition 5.35. A *minimal model* of a CDGA $A^\bullet$ is a quasi-isomorphism $M^\bullet\to A^\bullet$ for some minimal $M^\bullet$.

Proposition 5.36. *Every simply connected CDGA has a minimal model, which is unique up to isomorphism (and the quasi-isomorphism is unique up to homotopy).*

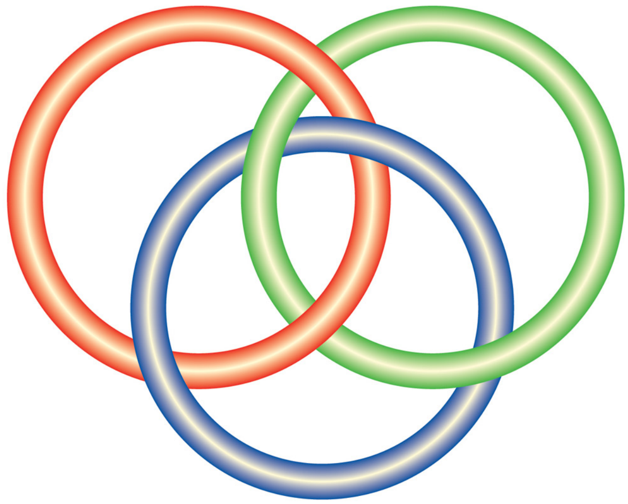

Figure 5.2: The complement of the Borromean rings has a nontrivial Massey product. Image source: Math SE user wonderich.

5.7.2 Formality

Definition 5.37. A CDGA $A^\bullet$ is *formal* if $A^\bullet$ is equivalent (connected by a series of quasi-isomorphisms) to a CDGA $B^\bullet$ with $d_B = 0$. We call a smooth manifold formal if $A^*(M)$ is formal as a CDGA.

Note that a CDGA is formal if and only if its minimal model is formal.

Example 5.38. Consider the sphere S^n for $n > 1$. If n s odd, then $n = 2\ell + 1$, the minimial model is $M^\bullet = \langle x \rangle = k \oplus kx$ where $\deg x = 2\ell + 1$. Note that $x^2 = 0$ because it is odd degree. Clearly $d = 0$ and so S^n is formal for n odd.

If $n = 2\ell$ is even, the minimal model is $M^\bullet = \langle x, y \rangle = k[x] \oplus k[x]y$. The degree of x is 2ℓ and the degree of y is $4\ell - 1$. Also $dx = 0$ and $dy = x^2$, and it is not formal. While this is not formal, there do exist formal models for n even, so you can take it on our word that S^n is formal for n even[2]

Proposition 5.39. *If $A^\bullet$ is formal, then higher Massey products on $H^*(A^\bullet)$ vanish.*

Example 5.40. Let $X = S^3 \setminus (S^1 \sqcup S^1 \sqcup S^1)$ be the complement of the Borromean rings (see Figure 5.2). There is a well-known non-zero Massey product on X, and so X is not formal.

Theorem 5.41. *A compact Kähler manifold is formal.*

[2]This wasn't checked, but Rok was "pretty sure".

6. Chern-Weil Theory

❖

Lecture 12/1

6.1 Connections and Curvature ❖

Let E be a complex vector bundle of rank k over a real manifold X. Let $\mathcal{A}^i(E)$ be the sheaf of i forms on X with values in E. A connection gives a splitting of the sequence of vector bundles $0 \to \pi^*E \to TE \to \pi^*TX \to 0$, which amounts to writing an isomorphism $TE \cong \pi^*TX \oplus \pi^*E$. Equivalentlly, a connection is a first order differential operator $\nabla : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ with $\nabla(fs) = df \otimes s + f\nabla(s)$, where s is a section of E and f is a function on X.

Locally, $E = U \times \mathbb{R}^k$ and any section $s : U \to \mathbb{R}^k$ is a collection of functions $s_1, ..., s_k$. Then $\nabla s = ds + A \otimes s$, where A is a matrix of 1-forms ($A \in \mathcal{A}^1(\operatorname{End} E)$). So locally a connection is always of the form $d + A$, where A is a 1-form valued matrix. To see why this is true, we need to check that $\nabla - d$ is a $\mathcal{A}^0$ linear operator. This follows from the Leibniz rule above:

$$(\nabla - d)(f \cdot s) = (df \otimes s + f\nabla(s)) - (df \otimes s + f \cdot ds) = f(\nabla - d)s$$

A more intrinsic way to express this result is:

Proposition 6.1. *The set of connections on a vector bundle E is an affine space over $\mathcal{A}^1(\operatorname{End} E)$.*

Remark 6.2. For any connection and $x_0 \in X$, there always exists a local trivialization with $\nabla = d + A$ and $A(x_0) = 0$.

6.1.1 Induced Connections

Given two connections ∇_1, ∇_2 on E, then there are several natural induced connections that we can form.

- On $E = E_1 \oplus E_2$, there is a diagonal connection ∇ defined by $\nabla(s_1 \oplus s_2) = \nabla_1 s_1 \oplus \nabla_2 s_2$. In general, connections on E are of the form:
$$\nabla_E = \begin{pmatrix} \nabla_1 & b_2 \\ b_1 & \nabla_2 \end{pmatrix}$$
where $b_1 \in \mathcal{A}^1(\operatorname{Hom}(E_1, E_2))$ and $b_2 \in \mathcal{A}^1(\operatorname{Hom}(E_2, E_1))$. These are the second fundamnetal forms for E_1, E_2 seen as sub-bundles of E.
- On $E_1 \otimes E_2$, there is a connection ∇ defined by $\nabla(s_1 \otimes s_2) = \nabla s_1 \otimes s_2 + s_1 \otimes \nabla s_2$.
- On E_1^*, there is a connection ∇^* defined by $\nabla^*(f)(s) = d(f(s)) - f(\nabla s)$, where $f \in \mathcal{A}^0(E^*)$ and $s \in \mathcal{A}^0(E)$.
- On $\operatorname{Hom}(E_1, E_2) = E_1^* \otimes E_2$, we can combine the above constructions to get a connection defined by $\nabla(f)(s) = \nabla_2(f(s)) - f(\nabla_1(s))$. Locally, we write $\nabla_i = d + A_1$. Given $f : U \to M(r_2 \times r_1, \mathbb{R})$, then $\nabla f = df + A_1 \cdot f - f \cdot A_2$.

6.1.2 Hermitian Connections

Now, we let $E \to X$ be a complex vector bundle (over a real manifold). A *hermitian structure* h on E is a hermitian inner product h_x on each fiber E_x which depends differentiably on x. A pair (E, h) is called a *hermitian vector bundle*. Connections on hermitian vector bundles are just usual connections that are compatible with the hermitian metric. For sections s_1, s_2, write $\langle s_1, s_2\rangle_h = h(s_1 \otimes \bar{s}_2) = h(s_1, s_2)$.

Definition 6.3. A connection ∇ is *hermitian* if for all local sections s_1, s_2 we have:

$$d\langle s_1, s_2\rangle_h = \langle \nabla s_1, s_2\rangle_h + \langle s_1, \nabla s_2\rangle_h$$

Locally, write a hermitian connection as $\nabla = d + A$ and define $H = \langle e_i, e_j \rangle_h$ for e_i the basis vectors of $\mathbb{C}^r$. Then:

$$dH = (d\langle e_i, e_j\rangle)_{ij} = (\langle \nabla e_i, e_j\rangle_h + \langle e_i, \nabla e_j\rangle_h)_{ij} = ((Ae_i)^T H\bar{e}_j + e_i H\overline{Ae_j})_{ij} = A^T H + H\bar{A}$$

Thus ∇ is hermitian if and only if $dH = A^T H + H\bar{A}$ locally.

Proposition 6.4. *The set of hermitian connections with respect to h is an affince space over real vector space $\mathcal{A}^1(X, \mathrm{End}(E,h))$.*

6.1.3 Chern Connections

Up to this point, X was a real manifold. If we now consider a holomorphic vector bundle $E \to X$ for X a complex manifold, we have a decomposition $\mathcal{A}^1(E) = \mathcal{A}^{0,1}(E) \oplus \mathcal{A}^{1,0}(E)$. As a consequence, any connection ∇ on E decomposes into two components $\nabla = \nabla^{0,1} \oplus \nabla^{1,0}$, where:

$$\nabla^{0,1} : \mathcal{A}^0(E) \to \mathcal{A}^{0,1}(E), \qquad \nabla^{1,0} : \mathcal{A}^0(E) \to \mathcal{A}^{1,0}(E)$$

We can see that $\nabla^{0,1}(f \cdot s) = \bar{\partial}(f) \otimes sf \cdot \nabla^{0,1}(s)$, a property shared by $\bar{\partial}$. This motivates the definition:

Definition 6.5. A connnection ∇ on a holomorphic vector bundle is *compatible with the holomorphic structure* if $\nabla^{0,1} = \bar{\partial}$.

Just as with other connections we've seen:

Proposition 6.6. *The space of connections on a holomorphic vector bundle compatible with the holomorphic structure forms an affine space over the complex vector space $\mathcal{A}^{0,1}(X, \mathrm{End}(E))$.*

For a given hermitian structure h, which hermitian connections ∇ are *also* compatible with the holomorphic structure? It turns out there is only one such connection and it is called the *Chern connection* on (E, h). If we assume that one exists, to show uniqueness we use the property that $dH = A^T H + H\bar{A}$, where $\nabla = d + A$ locally and $H = \langle e_i, e_j\rangle_h$. Since ∇ is compatible with the complex structure, the matrix A is of type $(1,0)$. Comparing both sides:

$$(\partial + \bar{\partial})H = \underbrace{A^T H}_{(1,0)} + \underbrace{H\bar{A}}_{(0,1)}$$

We see that $\bar{\partial}H = H\bar{A}$, from which we find $A = \bar{H}^{-1}\partial\bar{H}$. Thus A is uniquely determined. Since it is uniquely determined locally, it is uniqely determined globally.

6.1.4 Curvature

For any connection ∇ on a vector bundle $E \to X$, there is a natural way to extend it to a map $\nabla : \mathcal{A}^k(E) \to \mathcal{A}^{k+1}(E)$. If α is a k-form on X and s is a local section of E, then it is defined by:

$$\nabla(\alpha \otimes s) = d\alpha \otimes s + (-1)^k \alpha \wedge \nabla s$$

For $k = 0$ is is the usual Leibniz rule. Moreover, the Leibniz rule generalizes to:

$$\nabla(\beta \wedge t) = d\beta \wedge t + (-1)\beta \wedge \nabla t$$

where t is a section of $\mathcal{A}^\ell(E)$ and β is a k form.

Definition 6.7. The *curvature* F_∇ of a connection ∇ on a vector bundle E is the composition $F_\nabla := \nabla \circ \nabla : \mathcal{A}^0(E) \to \mathcal{A}^2(E)$.

One can show that the curvature is $\mathcal{A}^0$-linear, and hence we can think of it as a global section of $\mathcal{A}^2(\mathrm{End}(E))$.

Lecture 12/3

6.2 Chern Classes ❖

Consider the map $M(r,\mathbb{C}) \to \mathbb{C}$ given by $B \mapsto \det(\mathrm{id}+B)$. It is a polynomial in the entries b_{ij} of B, i.e. $\det(\mathrm{id}+B) \in \mathbb{Z}[b_{ij} \mid 1 \leq i,j \leq r]$. We can write this into homogenous pieces:

$$\det(\mathrm{id}+B) = 1 + P_1(B) + P_2(B) + ... + P_r(B)$$

where $P_k \in \mathbb{Z}[b_{ij}]$ is homogenous of degree k. Then we claim these polynomials are invariant under conjugaction of the matrix. Given a connection ∇ on a complex vector bundle $E \to M$ of rank r, which we see as a smooth manifold. Then $P_k(F_\nabla) \in \mathcal{A}^{2k}(M)$ is well-defined. These actually define cohomology classes independent of ∇:

Proposition 6.8. *a)* $dP_k(F_\nabla) = 0$.

b) $[P_k(F_\nabla)] \in H^{2k}_{dR}(M,\mathbb{C})$ *is independent of* ∇.

Definition 6.9. The k-th *Chern class* is of $E \to M$ is $c_k(E) := [P_k(iF_\nabla/2\pi)]$. The *total Chern class* is:

$$c(E) := c_0(E) + c_1(E) + ... + c_r(E) \in H^*(M)$$

One can check that $c_0(E) = 1$ and that $c_\ell(E) = 0$ if l exceeds the rank of E. One can also compute that $c_r(E) = c_1(\det E)$.

The map $B \to \det(\mathrm{id}+B)$ was an Ad-invariant, symmetric k-multilinear form, the space of which objects is denoted $(S^k M(r,\mathbb{C}))^{\mathrm{GL}(r,\mathbb{C})}$. The same argument as above for any such form induces a homomorphism:

$$(S^k M(r,\mathbb{C}))^{\mathrm{GL}(r,\mathbb{C})} \to H^{2k}(M,\mathbb{C})$$

which is known as the *Chern-Weil homomorphism*. This produces other characteristic classes, such as:

- $B \mapsto \mathrm{tr}(e^B)$ gives rise to the Chern Characters.
- For sufficiently small t, the map $B \mapsto \frac{\det(tB)}{\det(\mathrm{id}-e^{-tB})}$ gives rise to the Todd classses.

Remark 6.10. Note that the Chern characters and Todd classes don't necessarily vanish when you go past the rank of E. In addition, they can both be expressed in terms of polynomials in the Chern classes $c_i(E)$.

6.3 Interpretations of c_1 on line bundles ❖

Proposition 6.11. *For* $L \in \mathrm{Pic}(X)$*, the following agree in* $H^2(X,\mathbb{C})$.

a) $\delta(-[L])$ *for* $\delta : H^1(X,\mathcal{O}_X^\times) \to H^2(X,\mathbb{Z})$ *from the exponential sequence.*

b) $c_1(L,\nabla) = [iF_\nabla/2\pi] \in H^2_{dR}(X,\mathbb{R})$ *(Chern-Weil).*

c) $A(L) \in H^1(X,\Omega_X) = H^{1,1}(X)$ *(Atiyah class).*

d) The Poincaré dual of $[D]$*, where* $L = \mathcal{O}_X(D)$.

6.4 The Hirzebruch-Riemann-Roch Formula ❖

Let $E \to X$ be a holomorphic vector bundle of rank r with X compact of dimension n. The holomorphic Euler characteristic is:

$$\chi(E) := \sum_{i=0}^{n} (-1)^i h^i(X,E)$$

Theorem 6.12 (Hirzebruch-Riemann-Roch)**.** *If* $\mathrm{ch}(E)$ *is the Chern character of* E *and* $td(X) = td(TX)$ *is the Todd class of of* TX*, then:*

$$\chi(E) = \int_X \mathrm{ch}(E) \cdot td(X)$$

The striking part of this theorem is that the left hand side is holomorphic and the right hand side is topological. Moreover, it gives famous formulas such. For example, with $r = 1, n = 1$ (a line bundle on a Riemann surface), we get the classical Riemann-Roch formula:

$$h^0(L) - h^1(L) = \deg(L) + 1 - g(c)$$

For any vector bundle of rank r on a Riemann surface, we get:

$$h^0(L) - h^1(L) = \deg(L) + r(1 - g(c))$$

This is used to compute expected dimensions of moduli spaces of holomorphic curves in Gromov-Witten theory.

Remark: lectures have officially ended for this class, but I will be (hopefully) finishing all the TODO sections marked above in the next few weeks.

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