DERIVED FUNCTORS AND HOMOLOGICAL DIMENSION

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Abstract. This paper overviews the basic notions of abelian categories, exact functors, and chain complexes. It will use these concepts to define derived functors, prove their existence, and demonstrate their relationship to homological dimension.

I affirm my awareness of the standards of the Harvard College Honor Code.

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1. Abelian Categories and Homology

The concept of an abelian category will be necessary for discussing ideas on homological algebra. Loosely speaking, an abelian cagetory is a type of category that behaves like modules (R-mod) or abelian groups (Ab). We must first define a few types of morphisms that such a category must have.

Definition 1.1. A morphism $f : X \to Y$ in a category $\mathcal{C}$ is a zero morphism if:

- for any $A \in \mathcal{C}$ and any $g, h : A \to X$, $fg = fh$
- for any $B \in \mathcal{C}$ and any $g, h : Y \to B$, $gf = hf$

We denote a zero morphism as 0_{XY} (or sometimes just 0 if the context is sufficient).

Definition 1.2. A morphism $f : X \to Y$ is a monomorphism if it is left cancellative. That is, for all $g, h : Z \to X$, we have $fg = fh \Rightarrow g = h$. An epimorphism is a morphism if it is right cancellative.

The zero morphism is a generalization of the zero map on rings, or the identity homomorphism on groups. Monomorphisms and epimorphisms are generalizations of injective and surjective homomorphisms (though these definitions don't always coincide). It can be shown that a morphism is an isomorphism iff it is epic and monic.

Definition 1.3. The kernel of a morphism $f \in \mathcal{C}(A, B)$ is the equalizer of f with 0_{AB}. The cokernel of f is the coequalizer of f with 0_{AB}.

Definition 1.4. The image of a morphism is the kernel of its cokernel, and the coimage is the cokernel of its kernel.

As the names suggest, the kernel and cokernel are meant to generalize the concepts well-known in groups. It is easy to show that the kernel of a monomorphism is 0_{AB} and the cokernel of an epimorphism is 0_{AB}; likewise kernels are monic and cokernels are epic.

Definition 1.5. A category $\mathcal{C}$ is called *abelian* if it satisfies the following properties:

(1) It has an object that is both initial and terminal (also called the zero object, denoted 0).
(2) It has all pairwise products and coproducts.
(3) It has all kernels and cokernels.
(4) All monomorphisms (resp. epimorphisms) can be written as the kernel (resp. cokernel) of a morphism.

As mentioned, an abelian category is meant to capture the essential notions of abelian groups and modules over commutative rings. Other examples of abelian categories include topological sheaves and vector spaces over a field. There are a few helpful properties about abelian categories. For example, the image is canonically isomorphic to the coimage[1] and consequently every morphism $f : A \to B$ factors naturally as:

$$\begin{tikzcd}
\ker(f) \arrow[r, "k"] & A \arrow[rr, "f"] \arrow[dr, "u"] & & B \arrow[r, "c"] & \operatorname{coker}(f) \\
& & \operatorname{im}(f) \arrow[ur, "v"] & &
\end{tikzcd}$$

Where u is an epimorphism and v is a monomorphism. Further, any sequence of maps $A \xrightarrow{f} B \xrightarrow{g} C$ with $gf = 0_{AC}$ admits a natural map $\operatorname{im}(f) \to \ker(g)$. To see this, let u and v be

[1] In Grp, this is the First Isomorphism Theorem

as above and let $(k, \ker(g))$ be the kernel of g. Then we have:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "u"] & B \arrow[r, "g"] & C \\
\operatorname{im}(f) \arrow[ur, "v"] & \ker(g) \arrow[u, "k"'] &
\end{tikzcd}$$

But since $gvu = gf = 0_{AC}$ and u is an epimorphism, we have $gv = 0_{AC}$. But by definition of the kernel of g, this means there exists a unique map $\sigma : \operatorname{im}(f) \to \ker(g)$ such that $v = k\sigma$.

Definition 1.6. A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ with $gf = 0_{AC}$ is called exact if the map σ above is an isomorphism.

The dual definition of exactness can be phrased in terms of the image of g and the cokernel of f:

Proposition 1.7. *A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is exact if and only if $\operatorname{im}(g) \cong \operatorname{coker}(f)$.*

Proof. We examine the following decomposition of f and g:

$$\begin{tikzcd}
 & & 0 \arrow[dr] & & 0 \\
 & & \operatorname{coker}(f) \arrow[u] \arrow[r, dashed, "\tau"] & \operatorname{im}(g) \arrow[ur] \arrow[dr] & \\
A \arrow[rr, "f"] \arrow[dr, "u"] & & B \arrow[rr, "g"] \arrow[u, "c"] \arrow[ur, "q"] & & C \\
 & \operatorname{im}(f) \arrow[ur, "v"] \arrow[r, dashed, "\sigma"] \arrow[dr] & \ker(g) \arrow[u, "k"'] & & \\
0 \arrow[ur] & & 0 \arrow[u] & &
\end{tikzcd}$$

Where c and k are the kernel and cokernel maps and q comes from the decomposition of $g : B \to C$. Since $gf = 0$, we obtain the map τ above in a similar manner as we obtained σ. Notice that if $\operatorname{im}(f) \cong \ker(g)$, then both $\operatorname{coker}(f)$ and $\operatorname{im}(g)$ are the cokernel of $k = v$, so they are isomorphic. Similarly, if $\operatorname{coker}(f) \cong \operatorname{im}(g)$, then both $\operatorname{im}(f)$ and $\ker(g)$ are the kernel of $c = q$, and so they are isomorphic. □

Proposition 1.8. *Where τ and σ are as above, we have $\ker(\tau) \cong \operatorname{coker}(\sigma)$.*

Proof. This follows from considering the image and coimage of $ck : \ker(g) \to \operatorname{coker}(f)$ in the prior diagram. On the one hand, we have $\operatorname{im}(ck) = \ker(\tau)$ because τ is its cokernel; on the other hand, we also have $\operatorname{coim}(ck) = \operatorname{coker}(\sigma)$ because σ is its kernel. As we are in an abelian category, the image and coimage are isomorphic and we thus have $\ker(\tau) \cong \operatorname{coker}(\sigma)$. □

This notion of exactness generalizes to sequences of objects of arbitrary length:

Definition 1.9. A sequence $A_\bullet = \cdots A_{i+1} \xrightarrow{\delta_{i+1}} A_i \xrightarrow{\delta_i} A_{i-1} \xrightarrow{\delta_{i-1}} \cdots$ is called a chain complex if $\delta_i \delta_{i+1} = 0$ for all i. Such a complex is called exact if each adjacent pair of maps δ_i, δ_{i+1} is exact.

The failure of a complex to be exact can be characterized by the cokernel of σ_i, the natural map induced between $\operatorname{im}(\delta_i)$ and $\ker(\delta_{i+1})$. Indeed, if a complex is exact and σ_i is an isomorphism, then the cokenel is 0. This is known as the homology of a complex:

Definition 1.10. The nth homology class $H_n(A_\bullet)$ of a chain complex $A_\bullet$ is $\text{coker}(\sigma_i)$. We saw that this is equivalent to $\ker(\tau_i)$, where τ_i is the natural map from the proof of Proposition 1.7. A chain complex is exact iff $H_n(A_\bullet) = 0$ for all n.

Example 1.11. Let $G_\bullet$ be an exact sequence of groups. The map $\sigma : \text{im}(\delta_i) \to \ker(\delta_{i-1})$ can be regarded as inclusion, and so the cokernel is a quotient of the image in the target: $\ker(\delta_{i-1})/\text{im}(\delta_i)$.

A particular type of exact sequence that we will consider is a short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$. Here, exactness implies that f is a monomorphism and g is an epimorphism. As an example of such a sequence, lets return to the situation above:

Proposition 1.12. *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a sequence such that $gf = 0$, and let $\sigma : im(f) \to ker(g)$ be defined as before. Then $0 \longrightarrow im(f) \xrightarrow{\sigma} ker(g) \xrightarrow{c} coker(\sigma) \to 0$ is exact, where $(c, coker(\sigma))$ is the cokernel of σ.*

Proof. Recall the in the above diagram that $v = k\sigma$ is a monomorphism; this means σ is a monomorphism, and therefore $\text{im}(\sigma) \cong \text{im}(f)$. But $\ker(c) = \ker(\text{coker}(\sigma)) = \text{im}(\sigma) = \text{im}(f)$. Since c is epic (as any cokernel is), this sequence is exact at every spot. $\square$

Now we state an important result in homological algebra concerning exact sequences, which will be useful in a later section.

Lemma 1.13 (Snake)**.** *Suppose $A \to B \to C \to 0$ and $0 \to D \to E \to F$ are exact and we have maps $\alpha : A \to D, \beta : B \to E, \gamma : C \to F$ commuting through. Then there is an exact sequence traced out below between the kernels and the cokernels:*

$$
\begin{tikzcd}
 & \ker\alpha \arrow[r] \arrow[d] & \ker\beta \arrow[r] \arrow[d] & \ker\gamma \arrow[d] \arrow[dddll, rounded corners, red, "s"] & \\
 & A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & D \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & F \arrow[d] & \\
 & \text{coker}\alpha \arrow[r] & \text{coker}\beta \arrow[r] & \text{coker}\gamma &
\end{tikzcd}
$$

Biproducts

Another property of abelian categories is that $\mathcal{C}(A, B)$ has an abelian group structure for all A and B. That is, if $f_1, f_2 \in \mathcal{C}(A, B)$, then there is an additive law "+" so that $f_1 + f_2 \in \mathcal{C}(A, B)$, with additive identity 0_{AB}. This additional structure allows us to construct maps that don't necessarily exist otherwise.

Proposition 1.14. *Let A, B be objects in an abelian category $\mathcal{C}$. Suppose that $A \star B$ is another object with the following maps:*

$$\begin{tikzcd} A \arrow[r, bend left, "i_A"] & A \star B \arrow[l, bend left, "\pi_A"] \arrow[r, bend left, "\pi_B"] & B \arrow[l, bend left, "i_B"] \end{tikzcd}$$

Such that:

- $\pi_A i_A = id_A$ *and* $\pi_B i_B = id_B$
- $\pi_B i_A = 0_{AB}$ *and* $\pi_A i_B = 0_{BA}$
- $i_A \pi_A + i_B \pi_B = id_{A \star B}$

Then $A \star B$ is isomorphic to both the product $A \times B$ and coproduct $A \amalg B$.

Proof. To show $A \star B \cong A \times B$, we must show that $(A \star B, \pi_A, \pi_B)$ satisfies the universal property of the product. Thus, let C be any object with maps $p_A : C \to A$ and $p_B : C \to B$. We must exhibit a unique $f : C \to A \star B$ such that the following commutes:

$$\begin{tikzcd} & C \arrow[dl, "p_A"'] \arrow[d, dashed, "f"] \arrow[dr, "p_B"] & \\ A & A \star B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B \end{tikzcd}$$

Let $f = i_A p_A + i_B p_B$. This commutes because:

$$\pi_A f = \pi_A(i_A p_A + i_B p_B) = \pi_A i_A p_A + \pi_A i_B p_B = \mathrm{id}_A p_A + 0 = p_A$$
$$\pi_B f = \pi_B(i_A p_A + i_B p_B) = \pi_B i_A p_A + \pi_B i_B p_B = 0 + \mathrm{id}_B p_B = p_B$$

Now suppose f' is another such map. Then:

$$\begin{aligned} f' &= \mathrm{id}_{A \star B} f' = (i_A \pi_A + i_B \pi_B) f' \\ &= i_A \pi_A f' + i_B \pi_B f' \\ &= i_A p_A + i_B p_B = f \end{aligned}$$

Thus f is unique and $A \star B \cong A \times B$. To show $A \star B \cong A \amalg B$, we proceed with the same universal property argument. That is, let C be any object with maps $q_A : A \to C$ and $q_B : B \to C$. We must then find a unique g such that the following commutes:

$$\begin{tikzcd} & C & \\ A \arrow[ur, "q_A"] \arrow[r, "i_A"'] & A \star B \arrow[u, dashed, "g"] & B \arrow[ul, "q_B"'] \arrow[l, "i_B"] \end{tikzcd}$$

Letting $g = q_A \pi_A + q_B \pi_B$, we have commutativity:

$$g i_A = (q_A \pi_A + q_B \pi_B) i_A = q_A \mathrm{id}_A + 0 = q_A$$
$$g i_B = (q_A \pi_A + q_B \pi_B) i_B = 0 + q_B \mathrm{id}_B = q_B$$

Once again suppose g' is another such map, then:

$$\begin{aligned} g' &= g' \mathrm{id}_{A \star B} = g'(i_A \pi_A + i_B \pi_B) \\ &= g' i_A \pi_A + g' i_B \pi_B \\ &= q_A \pi_A + q_B \pi_B = g \end{aligned}$$

Thus $A \star B \cong A \amalg B$. □

Since any abelian category has both finite products and coproducts, which are unique, we immediately have that finite products and coproducts are canonically isomorphic via $A \star B$. This is often called a biproduct, and is denoted $A \oplus B$.

2. Exact Functors

Now we return to exact sequences with the hopes of understanding their behavior under functors. The only functors we really want to consider are ones that behave (at least somewhat) nicely with abelian categories. From definition 1.5, we might say a well behaved functor preserves biproducts and kernels/cokernels. That is, if $F : \mathcal{C} \to \mathcal{C}'$ is a functor between abelian categories, we would like it to have three properties:

1) F preserves finite biproducts: $F\left(\bigoplus A_i\right) = \bigoplus F(A_i)$ (also called additive)
2) F presereves kernels: $F(\ker(f)) = \ker(F(f))$.
3) F preserves cokernels: $F(\mathrm{coker}(f)) = \mathrm{coker}(F(f))$.

Remark 2.1. It can be shown that all functors preserving kernels (or cokernels) automatically preserve biproducts.

We will show that a functor with these three properties preserves exact sequences. Before we do this, we will give examples of functors satisfying each of the above properties.

Example 2.2. Fix a ring R and a prime ideal $\mathfrak{p}$. Then consider the functor $(-)_\mathfrak{p} : \text{R-}\textsc{mod} \to \text{R-}\textsc{mod}$ given by localizing a module A at $\mathfrak{p}$. Since $A_\mathfrak{p} \cong A \otimes_R R_\mathfrak{p}$, if we have a map $f : A \to B$, we have a corresponding map $f \otimes \mathrm{id}_{R_\mathfrak{p}} : A_\mathfrak{p} \to B_\mathfrak{p}$. Thus we can define $f_\mathfrak{p}$ as $f \otimes \mathrm{id}_{R_\mathfrak{p}}$ and one can easily check that indeed makes this a functor. Further, $(-)_\mathfrak{p}$ preserves biproducts (direct sums) because of the asssociativity of tensors and direct sums:

$$\begin{aligned}
\left(\bigoplus_{i=1}^{n} A_i\right)_\mathfrak{p} &\cong \left(\bigoplus_{i=1}^{n} A_i\right) \otimes_R R_\mathfrak{p} \\
&\cong \bigoplus_{i=1}^{n} A_i \otimes_R R_\mathfrak{p} \\
&= \bigoplus_{i=1}^{n} A_{i\,\mathfrak{p}}
\end{aligned}$$

Example 2.3. Fix a ring R and an R-module A. Then consider the functor $\mathrm{Hom}(A, -) : \text{R-}\textsc{mod} \to \textsc{Ab}$ sending $B \mapsto \mathrm{Hom}(A, B)$. If $f : B \to C$ is a homomorphism, then $f \mapsto \mathrm{Hom}(A, f)$ under this functor, where:

$$\mathrm{Hom}(A, f) : \phi \mapsto f\phi$$

Notice that the kernel of $\mathrm{Hom}(A, f)$ is the set of maps $\phi : A \to B$ that map into $\ker(f) \subset B$. This can be written as the group $\mathrm{Hom}(A, \ker(f))$. Thus we have the equality:

$$\mathrm{Hom}(A, \ker(f)) = \ker(\mathrm{Hom}(A, f))$$

This exactly means that $\mathrm{Hom}(A, -)$ preserves kernels.

Example 2.4. Fix a ring R and an R-module B. Then we consider the functor $(-) \otimes_R B : \text{R-}\textsc{mod} \to \text{R-}\textsc{mod}$ defined by $A \mapsto A \otimes_R B$ and $f \mapsto f \otimes_R 1_B$. To verify that this preserves cokernels, let $f : A \to C$ be a module homomorphism. We must exhibit an isomorphism:

$$C/\mathrm{im}(f) \otimes_R B \cong (C \otimes_R B)/\mathrm{im}(f \otimes_R 1_B)$$

We define a map $\phi : C/\mathrm{im}(f)\otimes_R B \to (C\otimes_R B)/\mathrm{im}(f\otimes_R 1_B)$ on the generating set of the domain (i.e. define on elements of the form $\overline{m}\otimes b$ and extend naturally via the blinear mapping). Choosing a representative $m \in C$ so that m mod $\mathrm{im}(f)$ is $\overline{m}$, we let $\phi(\overline{m}\otimes b) = m\otimes b \bmod \mathrm{im}(f\otimes_R 1_B)$. If $\phi(\overline{m}\otimes b) = 0$, then $\overline{m} = f(m) - m_0$ for $m_0 \in \mathrm{im}(f)$. This means $\overline{m}\otimes b = 0\otimes b = 0$, so ϕ is injective. Now consider generating elements of $(C\otimes_R B)/\mathrm{im}(f\otimes_R 1_B)$, which take the form $\overline{(m\otimes b)}$. Notice that we have:

$$\overline{m\otimes b} = m\otimes b + m_0\otimes b'$$

for any $m_0 \in \mathrm{im}(f)$ and $b' \in \mathrm{im}(1_B) = B$, and where $m\otimes b$ is a lifting of $\overline{m\otimes b}$. Letting $b' = b$ we have:

$$\overline{m\otimes b} = m\otimes b + m_0\otimes b = (m + m_0)\otimes b$$

Therefore the element $m\otimes b \in C/\mathrm{im}(f)\otimes_R B$ maps to $\overline{m\otimes b}$ under ϕ, so we have a surjection and thus an isomorphism. Therefore $(-)\otimes_R B$ is a functor that preserves cokernels.

We now proceed to relating these three properties of functors on abelian categories to the preservation of exact sequences.

Theorem 2.5. *Let $F : \mathcal{C} \to \mathcal{C}'$ be an additive functor between two abelian categories. Then the following are equivalent:*

(1) F preserves exact sequences.
(2) F preserves short exact sequences.
(3) F preserves kernels and cokernels.

Before we prove this, a quick lemma:

Lemma 2.6. $F(0) = 0$ *for an additive functor $F : \mathcal{C} \to \mathcal{C}'$ between abelian categories.*

Proof. Notice that $A \in \mathcal{C}$ is the zero object if and only if $\mathcal{C}(A, A) \cong 0$ as a group[2]. This is equivalent to saying the identity morphism is the zero morphism on A. Since F is a functor we have $F(\mathrm{id}_0) = \mathrm{id}_{F(0)}$, and since it is also a group homomorphism we also have $F(0_{00}) = 0_{F(0)F(0)}$. Since $\mathrm{id}_0 = 0_{00}$, we have $\mathrm{id}_{F(0)} = 0_{F(0)F(0)}$, and so $F(0)$ is the zero object. □

Proof (of Theorem 2.5).

$(1) \to (3)$: Let $f : A \to B$ be a morphism. Then we have an exact sequence $0 \to \ker(f) \xrightarrow{k} A \xrightarrow{f} B$. Since F preserves exact sequences and since $F(0) = 0$, we have another exact sequence:

$$0 \longrightarrow F(\ker(f)) \xrightarrow{F(k)} F(A) \xrightarrow{F(f)} F(B)$$

Exactness means $\ker(F(f)) = \mathrm{im}(F(k))$. But since $F(k)$ is a monomorphism, its image is isomorphic to $F(\ker(f))$. Thus $\ker(F(f)) = F(\ker(f))$ and F preserves kernels. Similarly, we can obtain another exact sequence:

$$F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(c)} F(\mathrm{coker}(f)) \longrightarrow 0$$

By Proposition 1.7, we have $\mathrm{coker}(F(f)) = \mathrm{im}(F(c))$. As $F(c)$ is an epimorphism, its image is isomorphic to $F(\mathrm{coker}(f))$. Thus $\mathrm{coker}(F(f)) = F(\mathrm{coker}(f))$ and so F preserves cokernels.

$(3) \to (2)$: Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be exact. Then by similar reasoning as above we have $A \cong \ker(g)$ and $C \cong \mathrm{coker}(f)$. Applying F gives:

$$0 \longrightarrow F(\ker(g)) \xrightarrow{F(f)} B \xrightarrow{F(g)} F(\mathrm{coker}(f)) \longrightarrow 0$$

[2] Recall all hom-sets in an abelian category have a group structure

Since F preserves kernels and cokernels, which equals:

$$0 \longrightarrow \ker(F(g)) \xrightarrow{F(f)} B \xrightarrow{F(g)} \text{coker}F(f) \longrightarrow 0$$

Which is exact.
(2) $\to$ (1): Let $A_\bullet$ be an exact sequence. Then we can factorize any subsequence $A_{i-1} \to A_i \to A_{i+1}$ and apply the functor F, resulting in:

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \arrow[dr] & & 0 & & \\
& F(\ker(\delta_{i-1})) \arrow[dr] & & & & F(\text{im}(\delta_i)) \arrow[ur] \arrow[dr, "q"] & & & \\
& \dots \arrow[r] & F(A_{i-1}) \arrow[rr, "F(\delta_{i-1})"] \arrow[dr, "u"] & & F(A_i) \arrow[rr, "F(\delta_i)"] \arrow[ur, "p"] & & F(A_{i+1}) \arrow[rr] \arrow[dr] & & \dots \\
& & & F(\text{im}(\delta_{i-1})) \arrow[ur, "v"] \arrow[dr] & & & & F(\text{coker}(\delta_i)) \arrow[dr] & \\
& & 0 \arrow[ur] & & 0 & & & & 0
\end{tikzcd}$$

Since the diagonals were short exact before applying F, they are also short exact after applying F since we are assuming (2). This implies that v, q are monic and u, p are epic and that $\text{im}(v) = \ker(p)$. Now we have:

$$\begin{aligned}
\text{im}(F(\delta_{i-1})) &= \text{im}(vu) \\
&= \text{im}(v) && (v \text{ is epic}) \\
&= \ker(p) \\
&= \ker(qp) && (q \text{ is monic}) \\
&= \ker(F(\delta_i))
\end{aligned}$$

Thus $F(A_\bullet)$ is exact at position i, and therefore everywhere.

□

Now that we see the equivalence of these conditions, we define an exact functor:

Definition 2.7. A functor is called exact if it preserves short exact sequences. If it preserves kernels it is called left exact, and if it preserves cokernels it is called right exact.

Corollary 2.8. *A left exact functor it preserves monomorphisms, and a right exact functor preserves and epimorphisms.*

Proof. This follows from the preceeding proof of (1) $\to$ (3). That is, letting $\ker(f) = 0$, if $F(\ker(f)) = F(0) = 0$ is the kernel of $F(f)$, then it is a monomorphism. Similarly, letting $\text{coker}(f) = 0$, if $F(\text{coker}(f)) = F(0) = 0$ is the cokernel of $F(f)$, then it is an epimorphism. □

Corollary 2.9. *In* R-MOD*, the functor* $Hom(A, -)$ *is left exact, the functor* $(-) \otimes B$ *is right exact, and the functor* $(-)_{\mathfrak{p}}$ *is both left and right exact (i.e. exact).*

3. Injective and Projective Objects

We now generalize to the functors $\mathrm{Hom}(A,-)$ and $\mathrm{Hom}(-,B)$ in any abelian category. This section will characterize what types of objects A and B need to be to make these functors exact. As any hom set in an abelian category is an abelian group, these are functors to Ab. If the former maps from a category $\mathcal{C}$, then the latter can be thought of as mapping from $\mathcal{C}^{op}$. Below we define how they act on morphisms, so let $f : X \to Y$ be a morphism, and let $\overline{f} : Y \to X$ be the corresponding morphism in $\mathcal{C}^{op}$.

$$\begin{aligned} \mathrm{Hom}(A,f) : \mathrm{Hom}(A,X) &\to \mathrm{Hom}(A,Y) \\ \phi &\mapsto f\phi \\ \mathrm{Hom}(f,B) : \mathrm{Hom}(X,B) &\to \mathrm{Hom}(Y,B) \\ \overline{\phi} &\mapsto \overline{\phi}\,\overline{f} \end{aligned}$$

It is not too hard to convince oneself that these are both left exact functors, as we saw in the category of R modules. In the case of $\mathrm{Hom}(A,-)$, we may apply it to a short exact sequence and obtain:

$$0 \longrightarrow \mathrm{Hom}(A,X) \xrightarrow{\mathrm{Hom}(A,f)} \mathrm{Hom}(A,Y) \xrightarrow{\mathrm{Hom}(A,g)} \mathrm{Hom}(A,Z) \longrightarrow 0$$

This is not necessarily exact for arbitrary A; however we may easily characterize when it is. Since $\mathrm{Hom}(A,f)$ is monic, we need only check exactness in the middle and that $\mathrm{Hom}(A,g)$ is epic. Certainly we have $\mathrm{im}(\mathrm{Hom}(A,f)) \subset \ker(\mathrm{Hom}(A,g))$ by the chain condition $gf = 0$, so let $\phi \in \ker(\mathrm{Hom}(A,g))$. By assumption, $g\phi = 0_{AZ}$, so by the universal property of the kernel we have a map $\tilde{\phi} : A \to \ker g$. Since $\ker g \cong \mathrm{im} f \cong X$, we have $\tilde{\phi} : A \to X$ such that $f\tilde{\phi} = \phi$. Therefore we have exactness at the middle for any A. The only place that this fails to be exact is in the surjectivity of $\mathrm{Hom}(A,g)$. Suppose that A has the property that for any $\psi \in \mathrm{Hom}(A,Z)$, there exists $\phi \in \mathrm{Hom}(A,X)$ so that $g\phi = \psi$. Endowing A with this property for any epimorphism g gives the definition for a projective object:

Definition 3.1. An object P of an abelian category is projective if, for any two objects M, N with maps $f : P \to N$ and $g : M \to N$ epic, there is a map h such that the following commutes:

$$\begin{tikzcd} & P \arrow[dl, "\exists h"'] \arrow[d, "f"] & \\ M \arrow[r, "g"] & N \arrow[r] & 0 \end{tikzcd}$$

Dually, there is the notion of an injective object E for which $\mathrm{Hom}(-,E)$ is exact:

Definition 3.2. An object E of an abelian category is injective if, for any two objects N, M with maps $f : N \to E$ and $g : N \to M$ monic, there is a map h such that the following commutes:

$$\begin{tikzcd} & E & \\ M \arrow[ur, "\exists h"] & N \arrow[l, "g"'] \arrow[u, "f"'] & 0 \arrow[l] \end{tikzcd}$$

Projective and injective objects are meant to mimic the property of having a "basis" that can be used to extend or prepend maps (as in the case of, say, a free module). Trivially, it is clear the zero object in any abelian category is both projective and injective.

Examples 3.3.

- In R-MOD, all free modules are projective. This follows from taking a basis and extending by linearity to obtain the desired map h.
- A finite abelian group G is not projective. A proof of this can be found in section 3.1

Proposition 3.4. *Let X be an object in an abelian category. Then:*

(1) X is projective if and only if $Hom(X,-)$ is exact.
(2) Every short exact sequence $0 \to M \xrightarrow{f} N \xrightarrow{g} X \to 0$ splits if X is projective.
(3) X is injective if and only if $Hom(-,X)$ is exact.
(4) Every short exact sequence $0 \to X \xrightarrow{f} M \xrightarrow{g} N \to 0$ splits if X is injective.

Proof.
(1): This is by definition of projectivity and the discussion above.
(2): We may apply projectivity of X to the identity map 1_X; that is, since g is an epimorphism, we have some $h : X \to N$ so that $gh = 1_X$. This means the sequence splits.
$(3),(4)$: Reversing all arrows gives an equivalent proofs as above in $\mathcal{C}^{op}$. □

Having an exact sequence split is helpful because, by the Splitting Lemma, we can write the center object as a biproduct of the outer two and thus applying any (additive) functor preserves exactness.

3.1. **Projective and Injective Modules.**

The above proposition become stronger in the category R-MOD because any module can be written as a quotient of a free module:

Lemma 3.5. *Any R-module M is the quotient of a free module.*

Proof. Let F be the module generated by all elements of M. Then there is a module surjection $f : F \to M$ induced by specifying $f(e_i) = m_i$ for each basis element. Then $F/\ker(f) \cong M$ by the First Isomorphism Theorem. □

Proposition 3.6. *Let X be a module. Then if every short exact sequence $0 \to M \xrightarrow{f} N \xrightarrow{g} X \to 0$ splits, then X is projective. Similarly, if every short exact sequence $0 \to X \xrightarrow{f} M \xrightarrow{g} N \to 0$ splits, then X is injective.*

Proof. We will prove the first statement, and the second follows from the same argument in R-MOD^{op}. By Lemma 3.5, X is a quotient of a free module and therefore we have an exact sequence:

$$0 \longrightarrow L \longrightarrow F \longrightarrow X \longrightarrow 0$$

where F is free. By assumption, this splits, so $F \cong X \oplus L$. In particular, we have a projection $\pi : F \to X$. Now let $g : M \to N$ be any surjection and $f : X \to N$ be any map. Then we have the following diagram:

$$\begin{tikzcd} F \arrow[r, "\pi"] \arrow[d, dashed, "h"] & X \arrow[d, "f"] & \\ M \arrow[r, "g"] & N \arrow[r] & 0 \end{tikzcd}$$

We infer the existence of h from the fact that F is free and therefore projective: we have a map $\pi f : F \to N$ and a surjection $g : M \to N$, so there is a lifting $h : F \to M$ commuting through. Now, since X is a submodule of F, we can restrict h to X and obtain $h_X : X \to M$. This shows that X is projective. □

Theorem 3.7. *A module X over a PID is projective if and only if it is free.*

Proof. One direction has been shown: every free module is projective. Conversely, if X is projective, then it is a summand of a free module (by above) and is therefore a submodule of a free module. Since every submodule of a free module in a PID is also free, we have that X is free. $\square$

Corollary 3.8. *A finite abelian group G is not projective as a $\mathbb{Z}$ module.*

Proof. G cannot be a free module because all elements have finite order, and thus cannot be part of a basis. As $\mathbb{Z}$ is a PID, this also means G is not projective by Theorem 3.7 $\square$

4. The Chain Complex Category

We now revisit the idea of chain complexes to give them their own categorical structure. In this section we will define the category of chain and cochain complexes for an abelian category. It will conclude with the Zig-Zag Lemma, which will be important for future sections.

Definition 4.1. Let $\mathcal{C}$ be an abelian category. Then define the category of chain complexes $\mathrm{Ch}(\mathcal{C})$ by:

- Objects: chain complexes $A_\bullet$ (of any degree) with objects in $\mathcal{C}$.
- A morphism $f_\bullet : A_\bullet \to B_\bullet$ is a collection $f_i : A_i \to B_i$ so that the following commutes for all i:

$$\begin{tikzcd}
A_i \arrow[r, "\alpha_i"] \arrow[d, "f_i"] & A_{i-1} \arrow[d, "f_{i-1}"] \\
B_i \arrow[r, "\beta_i"] & B_{i-1}
\end{tikzcd}$$

The cochain category is the category of chain complexes on $\mathcal{C}^{op}$.

There are two natural functors on this category. One is the natural inclusion: $I : \mathcal{C} \to \mathrm{Ch}(\mathcal{C})$ given by $A \mapsto \ldots \to 0_1 \to A \to 0_{-1} \to \ldots$ and $f \mapsto (..., 0_1, f, 0_{-1}, ...)$. The other is the homology functor $H_n : \mathrm{Ch}(\mathcal{C}) \to \mathcal{C}$ taking a complex to the nth homology class. It is not immediately obvious that this is a functor, since we have not yet specified how to define H_n on morphisms. Below is a diagram that shows, if $f_\bullet : A_\bullet \to B_\bullet$, then there is a natural choice for $H_n(f_\bullet)$.

$$\begin{tikzcd}
\cdots \arrow[r] & A_i \arrow[rrr, "\alpha_i"] \arrow[dr, two heads] \arrow[ddd, "f_i"] & & & A_{i-1} \arrow[rr, "\alpha_{i-1}"] \arrow[ddd, "f_{i-1}"] & & A_{i-2} \arrow[r] \arrow[ddd, "f_{i-2}"] & \cdots \\
& & \mathrm{im}(\alpha_i) \arrow[r, "\sigma_i"] \arrow[urr, hook] \arrow[d, dashed, blue] & \ker(\alpha_{i-1}) \arrow[rr] \arrow[ur, hook] \arrow[d, dashed, blue] & & H_i(A_\bullet) \arrow[d, dashed, red] & & \\
& & \mathrm{im}(\beta_i) \arrow[r, "\tau_i"] \arrow[drr, hook] & \ker(\beta_{i-1}) \arrow[rr] \arrow[dr, hook] & & H_i(B_\bullet) & & \\
\cdots \arrow[r] & B_i \arrow[rrr, "\beta_i"] \arrow[ur, two heads] & & & B_{i-1} \arrow[rr, "\beta_{i-1}"] & & B_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

The blue arrows are inherited maps from the universal property of the kernel, and the red arrow is induced by the blue arrows because the middle two rows are exact. We can now define $H_i(f_\bullet) : H_i(A_\bullet) \to H_i(B_\bullet)$ to be the red arrow.

Definition 4.2. A map $f_\bullet$ of chain complexes is a quasi-isomorphism if $H_i(f_\bullet)$ is an isomorphism for all i.

The quasi-isomorphisms of chain complexes identify complexes that are in some sense similar (i.e. have the same homology). These are not isomorphisms for many reasons; one of which is that being quasi-isomorphic is not symmetric.

Theorem 4.3. *For an abelian category $\mathcal{C}$, the chain complex category is also abelian.*

In particular, we have notions of biproducts, kernels, cokernels and exact sequences in $\mathrm{Ch}(\mathcal{C})$. These will not play a very large role in the discussion to come, so we won't prove this theorem here. One object that we will use is the short exact sequence of chain complexes:

Definition 4.4. A sequence $0 \to A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet \to 0$ is short exact in $\mathrm{Ch}(\mathcal{C})$ if it is exact at each component. That is, the columns in the diagram below are short exact sequences.

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A_{i+1} \arrow[r] \arrow[d, "f_{i+1}"] & A_i \arrow[r] \arrow[d, "f_i"] & A_{i-1} \arrow[r] \arrow[d, "f_{i-1}"] & \cdots \\
\cdots \arrow[r] & B_{i+1} \arrow[r] \arrow[d, "g_{i+1}"] & B_i \arrow[r] \arrow[d, "g_i"] & B_{i-1} \arrow[r] \arrow[d, "g_{i-1}"] & \cdots \\
\cdots \arrow[r] & C_{i+1} \arrow[r] \arrow[d] & C_i \arrow[r] \arrow[d] & C_{i-1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

When we say such a sequence splits, we mean that there exists a map $h_\bullet : C_\bullet \to B_\bullet$ such that $g_i h_i = 1_{C_i}$ for each i.

Proposition 4.5. *If $P_\bullet$ is a complex for which each P_i is projective, then every short exact sequence $0 \to A_\bullet \to B_\bullet \to P_\bullet \to 0$ splits. Similarly, if $E_\bullet$ is a complex for which each E_i is injective, then every short exact sequence $0 \to E_\bullet \to A_\bullet \to B_\bullet \to 0$ splits.*

Proof. This is a consequence of Proposition 3.4 applied to each column. □

Theorem 4.6 (Ziz Zag Lemma). *If $0 \to A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet \to 0$ is a short exact sequence of chain complexes, then there is a natural exact sequence connecting the homology classes:*

$$\cdots \longrightarrow H_i(A_\bullet) \xrightarrow{H_i(f_\bullet)} H_i(B_\bullet) \xrightarrow{H_i(g_\bullet)} H_i(C_\bullet) \xrightarrow{\delta_i} H_{i-1}(A_\bullet) \xrightarrow{H_{i-1}(f_\bullet)} H_{i-1}(B_\bullet) \longrightarrow \cdots$$

Lemma 4.7. *For a chain complex $A_\bullet$ with maps d_i, there is a natural exact sequence:*

$$0 \longrightarrow H_i(A_\bullet) \longrightarrow cokerd_{i+1} \xrightarrow{\zeta} kerd_{i-1} \longrightarrow H_{i-1}(A_\bullet) \longrightarrow 0$$

Proof. See [3]. □

Proof (Of Theorem 4.6). This will be a double application of the Snake Lemma, and is a proof due to [3]. We first have for every i a pair of exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & A_i \arrow[r, "f_i"] \arrow[d, "d_i^A"] & B_i \arrow[r, "g_i"] \arrow[d, "d_i^B"] & C_i \arrow[r] \arrow[d, "d_i^C"] & 0 \\ 0 \arrow[r] & A_{i-1} \arrow[r, "f_{i-1}"] & B_{i-2} \arrow[r, "g_{i-1}"] & C_{i-1} \arrow[r] & 0 \end{tikzcd}$$

By the Snake Lemma, we have two exact sequences: $\mathrm{coker} d_{i+1}^A \to \mathrm{coker} d_{i+1}^B \to \mathrm{coker} d_{i+1}^C \to 0$ and $0 \to \ker d_{i-1}^A \to \ker d_{i-1}^B \to \ker d_{i-1}^C$. Using Lemma 4.7, we can string these together and obtain:

$$\begin{tikzcd} & H_i(A_\bullet) \arrow[r] \arrow[d] & H_i(B_\bullet) \arrow[r] \arrow[d] & H_i(C_\bullet) \arrow[d] & \\ & \mathrm{coker} d_{i+1}^A \arrow[r] \arrow[d] & \mathrm{coker} d_{i+1}^A \arrow[r] \arrow[d] & \mathrm{coker} d_{i+1}^C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \ker d_{i-1}^A \arrow[r] \arrow[d] & \ker d_{i-1}^B \arrow[r] \arrow[d] & \ker d_{i-1}^C \arrow[d] & \\ & H_{i-1}(A_\bullet) \arrow[r] & H_{i-1}(B_\bullet) \arrow[r] & H_{i-1}(C_\bullet) & \end{tikzcd}$$

Using the Snake Lemma once again gives the desired sequence. □

Remark 4.8. There are parts to still be verified in this proof, like checking naturality and checking that that the maps $H_i(A_\bullet) \to H_i(B_\bullet)$ are $H_i(f_\bullet)$. We leave this to [3].

5. Homological dimension

The subject of this section will be connecting our discussion on injective/projective objects with chain complexes. We will define homological dimension and provide a few first order results.

Definition 5.1. We say an abelian category $\mathcal{C}$ has "enough projectives" if for every $A \in \mathcal{C}$ there exists an epimorphism $p : P \to A$ where P is projective. Similarly, we say $\mathcal{C}$ has "enough injectives" if for every $A \in \mathcal{C}$ there exists a monomorphism $e : A \to E$ for E is injective.

Definition 5.2. A projective resolution of an object $A \in \mathcal{C}$ is a quasi-isomorphism $P_\bullet \to I(A)$, where $P_\bullet \in \mathrm{Ch}(\mathcal{C})$ is a chain complex of non-negative degree and where each P_i is projective. An injective resolution of A is a quasi-isomorphism $I(A) \to E_\bullet$, where $E_\bullet$ is a cochain complex of non-negative degree and where each E_i is injective.

Remark 5.3. The quasi-isomorphism $I(A) \cong_q P_\bullet$ is equivalent to providing an exact sequence $\cdots \to P_1 \to P_0 \to A \to 0$.

Definition 5.4. A resolution $P_\bullet$ is said to have length n if $P_i = 0$ for all $i > n$. If no such n exists, we say $P_\bullet$ has infinite length.

Theorem 5.5. *Any abelian category that has enough projectives (resp. injectives) admits a projective (resp. injective) resolution on every object.*

Proof. We will show this in the case of projective resolutions, and the proof for injective resolutions will be dual. Proceed by induction:

- (Base case) By the property of having enough projectives, we have $P_0 \to A \to 0$ exact.
- Now assume $P_n \overset{p_{n-1}}{\to} P_{n-1} \to \cdots \overset{p_0}{\to} P_0 \overset{\epsilon}{\to} A \to 0$ is exact for $n > 0$. Then consider the short exact sequence induced by taking the kernel of p_{n-1}:

$$0 \longrightarrow \ker(p_{n-1}) \xrightarrow{k} P_n \xrightarrow{p_{n-1}} P_{n-1} \longrightarrow 0$$

  Since we have enough projectives, we can find an epimorphism $\ell : P_{n+1} \to \ker(p_{n-1})$, where P_{n+1} is projective. Now we have a map $p_n := k\ell : P_{n+1} \to P_n$. Further, $\mathrm{im}(p_n) = \mathrm{im}(\ell) = \ker(p_{n-1})$ because k is a monomorphism and ℓ is an epimorphism. Thus we have extended the exact sequence:

$$P_{n+1} \xrightarrow{p_n} P_n \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_0} P_0 \xrightarrow{\epsilon} A \longrightarrow 0$$

  And so A admits a projective resolution.

□

Example 5.6. The category of R-modules has enough projectives and injectives. It is easy to see that it has enough projectives from Lemma 3.5: write A as a direct summand of a free module P, which is projective, and we can take the natural surjection $P \to A$. The case of injective modules is not as straight-forward, but is nevertheless true and can be found in [2].

By taking an injecive or projective resolution of an object, we obtain some information about that object and how projective or injective it is. We will see that this can extend to measuring how exact a functor is.

Definition 5.7. Let $\mathcal{C}$ be an abelian category with enough projectives and injectives. The projective dimension $\dim_p(A)$ of an object A is the length of the smallest projective resolution $P_\bullet \to I(A)$. The injective dimension $\dim_i(A)$ of A is the length of the smallest injective resolution $I(A) \to E_\bullet$.

Lemma 5.8. *$dim_p(A) = 0$ if and only if A is projective, and $dim_i(A) = 0$ if and only if A is inejctive.*

Proof. If A is projective, then it admits a trivial resolution $0 \to A \to A \to 0$, which has length 0. Conversely, if A has a length 0 resolution, then we have an exact sequence $0 \to P_0 \to A \to 0$, which means $P_0 \cong A$, so A is projective. The same holds for injectivity. □

Lemma 5.9. *In the category of R modules, where R is a PID, we have $dim_p(M) \leq 1$ for all modules M.*

Proof. We can always find a presentation of M of the form $0 \to L \to P_0 \to M \to 0$, where P_0 is free. Since $L \subset P_0$, and we are working in a PID, we have that L is free and therefore projective. Thus our presentation of M is also a projective resolution of length at most 1. □

Theorem 5.10 (Comparison Theorem). *Let $P_\bullet$ be a projective resolution of A, and let $C_\bullet : \cdots \to C_1 \to C_0 \to B \to 0$ be an exact complex. Then for any $f : A \to B$, there is an extension of f to a map of complexes $f_\bullet : P_\bullet \to C_\bullet$ with $f_{-1} = f$.*

Proof. We must construct f_i so that the following commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r, "p_0"] \arrow[d, dashed, "f_1"] & P_0 \arrow[r, "\epsilon"] \arrow[d, dashed, "f_0"] & A \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & C_1 \arrow[r, "c_0"] & C_0 \arrow[r, "\lambda"] & B \arrow[r] & 0
\end{tikzcd}$$

We proceed by induction.

- (Base case): To construct f_0, we appeal to the projectivity of P_0. Since λ is epic, we have $f_0 : P_0 \to C_0$ induced by $f\epsilon : P_0 \to B$.
- Now assume we have constructed $f_i : P_i \to C_i$. Then we also have the following factorization:

$$\begin{tikzcd} & P_{i+1} \arrow[r, "p_i"] \arrow[d] & P_i \arrow[d, "f_i"] & \\ C_{i+1} \arrow[r, two heads] & \mathrm{im}(c_i) \arrow[r, hook] & C_i \arrow[r, "c_{i-1}"] & C_{i-1} \end{tikzcd}$$

  Notice that $c_{i-1}f_ip_i = f_{i-1}p_{i-1}p_i = 0$. Since $\mathrm{im}(c_i) = \ker(c_{i-1})$, we obtain the red map from the universal property of the kernel. Now since $C_{i+1} \to \mathrm{im}(c_i)$ is an epimorphism, we can use the projectivity of P_{i+1} to obtain a map $f_{i+1} : P_{i+1} \to C_{i+1}$. This completes the construction of $f_\bullet$.

□

Corollary 5.11. *If $0 \to A \to B \to C \to 0$ is a short exact sequence in an abelian category, then for every pair of projective resolutions $P_\bullet^A$ of A and $P_\bullet^C$ of C, there is a projective resolution $P_\bullet^B$ such that $0 \to P_\bullet^A \to P_\bullet^B \to P_\bullet^C \to 0$ is a short exact sequence of chain complexes.*

Proof. The trick is to let $P_i^B = P_i^A \oplus P_i^C$. In order to obtain the connecting maps on $P_i^B \to P_{i-1}^B$, we use the Comparison Theorem. We relegate the details to [1]. □

6. Derived Functors

Now we revisit the question of short exact sequences and how they are preserved under additive functors. The purpose of this section is to use all that we've developed to define exact functors and prove their existence in certain abelian categories.

Let $\mathcal{C}$ be an abelian category with enough projectives, and let $F : \mathcal{C} \to \mathcal{C}'$ be a right exact functor. Consider any short exact sequence:

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

We can projective resolutions of A and C, call these $P_\bullet^A$ and $P_\bullet^C$. By Corollary 5.11, we have a projective resolution $P_\bullet^B$ so that

$$0 \longrightarrow P_\bullet^A \longrightarrow P_\bullet^B \longrightarrow P_\bullet^C \longrightarrow 0$$

is a short exact sequence of complexes. Moreover, this exact sequence is split. Since F is additive, it preserves all split exact sequences. Thus we have another short exact sequence of complexes by applying F:

$$0 \longrightarrow F(P_\bullet^A) \longrightarrow F(P_\bullet^B) \longrightarrow F(P_\bullet^C) \longrightarrow 0$$

Now we apply the Zig-Zag Lemma (Theorem 4.6) and obtain a long exact sequence of homologies:

$$\cdots \longrightarrow H_1(F(P_\bullet^A)) \longrightarrow H_1(F(P_\bullet^B)) \longrightarrow H_1(F(P_\bullet^C)) \longrightarrow H_0(F(P_\bullet^A)) \longrightarrow H_0(F(P_\bullet^B)) \longrightarrow H_0(F(P_\bullet^C)) \longrightarrow 0$$

Now we use the fact that F is right exact to compute the H_0 terms above. Right exactness tells us that the right-most ends of the projective resolutions are still exact after applying F:

$$F(P_2^A) \xrightarrow{F(p_1^A)} F(P_1^A) \xrightarrow{F(p_0^A)} F(P_0^A) \longrightarrow 0$$

Computing $H_0(F(P_\bullet^A))$ via its definition as $\ker(\operatorname{coker}(F(p_0^A)) \to 0)$ gives $F(A)$ itself because $\operatorname{coker}(F(p_0^A)) = F(A)$. The same holds for the chains on B and C, so the long exact sequence becomes:

$$\begin{tikzcd}
\cdots \arrow[r] & H_2(F(P_\bullet^A)) \arrow[r] & H_2(F(P_\bullet^B)) \arrow[r] & H_2(F(P_\bullet^C)) \arrow[dll, rounded corners] \\
& H_1(F(P_\bullet^A)) \arrow[r] & H_1(F(P_\bullet^B)) \arrow[r] & H_1(F(P_\bullet^C)) \arrow[dll, rounded corners] \\
& F(A) \arrow[r] & F(B) \arrow[r] & F(C) \arrow[r] & 0
\end{tikzcd}$$

Now we see that we have extended the left half of our right exact sequence to obtain a long exact sequence. The homology on the left defines what is called the *left derived functor* for F, and is denoted $H_n(F(P_\bullet^A)) := L_nF(A)$. The analogous process using injective resolutions on a left exact functor defines the right derived functor R^nF.

Remark 6.1. There is the important problem of verifying that these functors are well-defined. After all, the projective resolutions chosen at the onset were arbitrary and in no way uniuque. However, it can be shown that they are all the same up to chain homotopy, and therefore have the same homology (before) and after applying F. This is spelled out in [1].

Example 6.2. Recall that, in R-MOD, the functor $\mathrm{Hom}(A, -)$ is left exact and the functor $(-)\otimes B$ is right exact. Since R-MOD has enough injective and projectives, we can take the right and left derived functors of these. The right derived functor for $\mathrm{Hom}(A, -)$ is known as $\mathrm{Ext}^n(A, -)$ and the left derived functor for $(-)\otimes B$ is known as $\mathrm{Tor}_n(-, B)$. We notice in the case of $\mathrm{Hom}(A, -)$ that, if A is a projective module, we can take trivial resolutions and obtain $\mathrm{Ext}^n(A, M) = 0$ for all modules M and integers $n > 0$. However, this means that the subsequent long exact sequence is actually short exact, which coincides with Proposition 3.4.

We conclude this paper by stating the connection between homological dimension and the functors $\mathrm{Hom}(A, -)$ and $\mathrm{Hom}(-, B)$ on abelian categories. For any abelian category with enough injectives and projectives, we denote $\mathrm{Ext}^n(A, -)$ to be the right derived functor of $\mathrm{Hom}(A, -)$. Similarly, we denote $\overline{\mathrm{Ext}}^n(-, B)$ to be the left derived functor for $\mathrm{Hom}(-, B)$.[3]

Proposition 6.3. *The following are equivalent for an object A:*

- $dim_p(A) \leq n$.
- $Ext^i(A, N) = 0$ *for all* $i > n$ *and all objects* N.

Dually, the following are equivalent for an object B:

- $dim_i(B) \leq n$.
- $\overline{Ext}^i(M, B) = 0$ *for all* $i > n$ *and all objects* M.

Proof. See [2]. □

[3]In the case of R-MOD, the left and right derived Hom functors are the same.

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