Notes on Differential Topology

George Torres

Last updated January 4, 2019

Contents

Note to the reader: These are lecture from Harvard's 2014 Differential Topology course Math 132 taught by Dan Gardiner and closely follow Guillemin and Pollack's Differential Topology.

1 Smooth manifolds and Topological manifolds

Definition: A topological manifold X is a locally Euclidean space that is Hausdorff and second countable.

$\triangleright$ To discuss calculus on topological manifolds, they must be equipped with a smooth structure. To define this, we must define the concept of smoothness and smooth manifold.

Definition: A smooth map between Euclidean spaces must first be defined on open sets, then on arbitrary sets:

- A smooth map between open subsets of Euclidean spaces $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ is a function $f : U \to V$ that is C^∞, or infinitely differentiable.
- A smooth map from an arbitrary subset $A \subset \mathbb{R}^n$ is a function $f : A \to R^n$ that extends locally to a smooth function on open sets. Namely, around every $a \in A$ there is an open set $U \subset \mathbb{R}^n$ and a smooth map $F : U \to R^m$ such that:
$$F\Big|_{U \cap A} = f$$

Definition: Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$. A function is a *diffeomorphism* (or smooth equivalence) if f is smooth, bijective and f^{-1} is smooth. The sets X and Y are said to be diffeomorphic if such a function exists between them.

Definition: A *smooth k manifold* is a subset $X \subset \mathbb{R}^n$ such that for every point $p \in X$, there is an open[1] subset $V \subset X$ containing p together with a diffeomorphism $\phi : U \to V$, where $U \subset \mathbb{R}^k$ is some open subset.

The maps $\phi : U \to V$ are called parameterizations, and the maps $\phi^{-1} : V \to U$ are called coordinate systems. Together, the collection of all such triples $\{(U, V, \phi)\}$ is called an *atlas of charts.*

Example (S^1): The 1 sphere $S^1 = \{(x, y) \mid x^2 + y^2 = 1\} \subset R^2$ is a smooth 1 manifold.

To show this, we cover it with four open sets: the top half, the bottom half, the left half and the right half. Each one can be shown to be diffeomorphic to $(-1, 1) \subset \mathbb{R}$. For example, let U_1 be the top half and define:
$$\phi_1 : (-1, 1) \to U_1 : \; x \mapsto \left(x, \sqrt{1 - x^2}\right)$$

This is smooth because it is a composition of continuous and C^∞ functions. The inverse is simply the projection $(x, y) \mapsto x$, which is also smooth (as the projection map on all of $\mathbb{R}^2$ is smooth). Thus ϕ is a diffeomorphism. The same process for all other halves of the circle covers all points in S^1, so it is a 1 manifold.

Definition: A *smooth structure* on a topological k-manifold X is a smooth k-manifold $Y \subset \mathbb{R}^n$ together with a homeomorphism $X \to Y$.

1.1 Smooth Structures

Several questions arise when discussing smooth structures:

Q: Does every topological manifold admit a smooth structure?

A: No. For example, there are 4 manifolds with no smooth structure. These aren't immediately intuitive.

Q: If a topological manifold admits a smooth structure, is it unique (up to diffeomorphism)?

A: No. For example, the 7 sphere has 28 smooth structures, which form a group under connect sums.

[1] Open in the subspace topology

1.2 Product Manifolds

Proposition: Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be smooth manifolds. Then $X \times Y \subset \mathbb{R}^{n+m}$ is also a smooth manifold.

Proof:

Let $(x, y) \in X \times Y$. Since X is a manifold, there is a local parameterization $\phi : U \to X$ around X. There is similarly a local parameterization around Y, say $\psi : V \to Y$. If we take the map $\phi \times \psi : U \times V \to X \times Y$ given by:

$$\phi \times \psi(u, v) = (\phi(u), \psi(v))$$

It is easy to check that this is a parameterization for $X \times Y$ around (x, y).

□

2 Calculus on smooth manifolds

The primary use of smooth manifolds is calculus, and the fundamental component of calculus is the derivative as a linear approximation. Namely, if $f : \mathbb{R}^n \to \mathbb{R}^m$ is a C^∞ function, then the derivative of f at $\vec{c} \in \mathbb{R}^n$, denoted by df_c is a linear map satisfying:

$$f(\vec{x}) = f(\vec{c}) + df_c(\vec{x} - \vec{c}) + \mathrm{err}(\vec{x})$$

where $\mathrm{err}(\vec{x})$ is an error term sufficiently small near $\vec{c}$.

2.1 Derivatives

The goal of this section is to define df, the best linear approximation to a smooth map $f : X \to Y$ between smooth manifolds (possibly of different dimension). This definition will come out of the chain rule between X, Y and their tangent spaces.

2.1.1 Tangent spaces

Definition: The *tangent space* of X at a point x, denoted $T_x(X)$, is the image of $d\phi_{\vec{0}}$, where ϕ is a local paramaterization $\phi : U \to X$ with $\phi(\vec{0}) = x$.

▷ Note that $T_x(X) \subset \mathbb{R}^N$ is a k subspace. It is useful to think of it as a copy of $\mathbb{R}^k$ centered at x on the manifold.

Example: Take $S^1 \subset \mathbb{R}^2$. What is $T_{(0,1)}(S^1)$?

Take the parameterization $\phi : (-1, 1) \to S^1$ that sends $x \mapsto (x, \sqrt{1 - x^2})$. Then:

$$d\phi = \begin{bmatrix} 1 \\ \frac{-x}{\sqrt{1-x^2}} \end{bmatrix}$$

At $\vec{0}$, $d\phi = (1, 0)$, so the tangent space is the subspace spanned by $(1, 0)$.

▷ We now will show that $T_x(M)$ is well defined:

Proposition: The tangent space $T_x(M)$ doesn't depend on the choice of parameterization.

Proof:

Suppose we have distinct parameterizations of X:

$$\phi : U \to X \qquad \phi(0) = x$$

$$\psi : V \to X \qquad \psi(0) = x$$

Since U and V have a nontrivial intersection and they are open, we may shrink them to assume $U = V$. Now define $h = \psi^{-1} \circ \phi$. This is a map $U \to U$. Since ϕ and ψ are diffeomorphisms, so is h. The chain rule now says:

$$d\phi_0 = d\psi_0 \circ dh_0$$

This implies that the image of $d\phi_0$ is restricted to that of $d\psi_0$, or $\mathrm{Im}(d\phi_0) \subset \mathrm{Im}(d\psi_0)$. The same argument can be made for $h' = \phi^{-1} \circ \phi$, so $\mathrm{Im}(d\psi_0) \subset \mathrm{Im}(d\phi_0)$, and we have equality

□

2.1.2 Defining *df*

Now we return to attempting to define the derivative of a smooth map $f : X \to Y$ between manifolds. Intuitively, the derivative at a point x should be a map between the tangent spaces $T_x(X) \to T_x(Y)$.

We can parameterize around x and $f(x)$ in X and Y respectively using:

$$\phi : U \to X \qquad \phi(0) = x$$

$$\psi : V \to Y \qquad \psi(0) = f(x)$$

If we define $h = \psi^{-1} \circ f \circ \phi$, we require the following diagram to commute, so long as U is chosen small enough:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\phi}\uparrow & & {\scriptstyle\psi}\uparrow \\ U & \xrightarrow{h} & V \end{array}$$

Similarly, we require the derivatives to commute. The derivatives of ϕ and ψ also map between tangent spaces $T_0(U) \in \mathbb{R}^k$ and $T_0(V) \in \mathbb{R}^\ell$:

$$\begin{array}{ccc} T_x(X) & \xrightarrow{df_x} & T_{f(x)}(Y) \\ {\scriptstyle d\phi_0}\uparrow & & {\scriptstyle d\psi_0}\uparrow \\ T_0(U) & \xrightarrow{dh_0} & T_0(V) \end{array}$$

Since $d\phi_0$ is an isomorphism (because ϕ is), df_x is uniquely determined because require the derivatives to commute. Thus $df_x = d\psi_0 \circ dh_0 \circ (d\phi_0)^{-1}$.

▷ The derivative also satisfies the chain rule; that is, given:

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

we have $d(f \circ g)x = df_x \circ dg_{(f(x))}$. The proof of this is straightforward.

Example: Let $S^1 \subset \mathbb{C}$ be the set $\{z \in \mathbb{C} \mid |z| = 1\}$. We can define $F : S^1 \to S^1$ by sending $z \mapsto z^2$. What is df at i?

We can associate S^1 with the unit circle in $\mathbb{R}^2$ via $a + bi \sim (a, b)$; in which case the map F is:

$$(x, y) \mapsto (x^2 - y^2, 2xy)$$

Now we seek $df_{(0,1)}$. To do this, we need local parameterizations around $(0, 1)$ and $F(0, 1) = (-1, 0)$. One possibility is:

$$\phi : (-1, 1) \to S^1 \qquad x \mapsto (x, \sqrt{1 - x^2})$$
$$\psi : (-1, 1) \to S^1 \qquad x \mapsto (-\sqrt{1 - x^2}, x)$$

Now we need $h = \psi^{-1} \circ f \circ \phi$. We note that ψ^{-1} is the projection onto the y axis, so:

$$h(x) = 2x\sqrt{1 - x^2}$$

We can easily compute dh_0 to be 2. df takes an element of the tangent space and maps it to another in the tangent space of the image. We already saw that $T_x(S^1)$ is the span of $(1, 0)$. So df takes a vector $(a, 0)$ and returns:

$$df_{(0,1)}\left(\begin{bmatrix} a \\ 0 \end{bmatrix}\right) = d\psi_0 \circ dh_0 \circ (d\phi_0)^{-1} = 2d\psi_0\left((d\phi_0)^{-1}\begin{bmatrix} a \\ 0 \end{bmatrix}\right)$$

We can compute this to be:

$$df_{(0,1)}\left(\begin{bmatrix} a \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2a \end{bmatrix}$$

3 The Inverse Function Theorem

The question naturally arises: What information does the derivative contain? The Inverse Function Theorem tells us that it can be used to determine if f is a local diffeomorphism, which we define below.

Definition: A smooth function $f : X \to Y$ between manifolds is a *local diffeomorphism* at $x \in X$ if $\exists$ an open set $U \subset X$ containing x and an open set $V \subset Y$ containing $f(x)$ such that $f : U \to V$ is a diffeomorphism.

Theorem (Inverse Function): $f : X \to Y$ is a local diffeomorphism at x if and only if df_x is an isomorphism.

In other words, the question of whether f is a diffeomorphism locally comes down to showing $\det(df_x) \neq 0$.

4 Immersions and Submersions

▷ Before, we saw that $f : X \to Y$ is a local diffeomorphism if and only if df_x is an isomorphism. If we pair this with f being bijective, it is enough to claim that f is a (global) diffeomorphism. In this spririt, we seek to characterize functions between spaces that aren't of the same dimension (where df_x can't be an isomorphism).

Definition: $f : X \to Y$ is an immersion if df_x is injective. It is a submersion if df_x is surjective. The former case can only happen when $\dim(X) \leq \dim(Y)$. The latter can only happen when $\dim(Y) \leq \dim(X)$. An immersive submersion is a local diffeomorphism by the Inverse Function Theorem.

4.1 Immersions

▷ There is a canonical immersion defined to be:

$$C : \mathbb{R}^k \to \mathbb{R}^\ell$$

$$(x_1, ..., x_k) \mapsto (x_1, ..., x_k, 0, ..., 0)$$

where $k \leq \ell$.

The Fundamental theorem of immersions tells us about how C relates to general immersions:

Theorem (Immersions): If f is an immersion at x, then f is locally equivalent to the canonical immersion. Specifically, this means there exist parameterizations ϕ, ψ so that $g = C$ in the diagram below.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\phi}\big\uparrow & & {\scriptstyle\psi}\big\uparrow \\ U & \xrightarrow{g} & V \end{array}$$

Proof:

Start with arbitrary parameterizations ϕ and ψ. Define $G : U \times \mathbb{R}^{\ell-k} \to \mathbb{R}^\ell$ as:

$$G(x, z) = g'(x) + h(z)$$

where $g' : U \to \mathbb{R}^\ell$ and $h : \mathbb{R}^{\ell-k} \to \mathbb{R}^\ell$ are:

$$g'(x) = (g(x), 0)$$

$$h(z) = (0, z)$$

As a fact of linear algebra, it is possible to choose coordinates in $\mathbb{R}^\ell$ such that dg'_0 looks like:

$$dg'_0 = \begin{pmatrix} I_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

Where I_k is a k identity matrix, and the 0 are zero matrices of appropriate dimensions. It is easy then to see that dG_0 looks like:

$$dG_0 = dg'_0 + dh_0 = \begin{pmatrix} I_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_\ell \end{pmatrix} = I$$

Therefore dG_0 is bijective, and hence an isomorphism. The Inverse Function Theorem then says G is a local diffeomorphism around 0. This means we have a parameterization $\psi \circ G$ of Y locally, up to shrinking U and V. This is shown below.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\phi"] \arrow[r, "\tilde{g}"] & V \arrow[u, dashed, "\psi \circ G"'] \end{tikzcd}$$

The map $\tilde{g}$ is induced by the other three maps. From our first diagram, we also had:

$$f \circ \phi = \psi \circ g$$

But since $g = G \circ C$, we have $f \circ \phi = (\psi \circ G) \circ C$. Now looking at the second diagram, this tells us $\tilde{g} = C$, as desired.

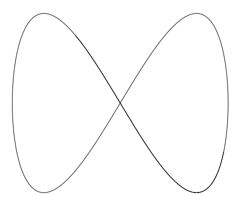
□

▷ Images of immersions have good properties, as we will see below.

Definition: A function is proper if $f^{-1}(K)$ is compact wherever K is compact

Theorem: The image of a proper, injective immersion is a submanifold (such a map is also called an embedding).

Example: Consider the map that sends S^1 to the figure eight shown below.



This is a subset of $\mathbb{R}^2$ and we may ask if the image of this map is a submanifold. It is not because the map is not injective (the origin is mapped to twice). However, even if we remove the redundant point in the domain, the map still wouldn't be proper because the preimage of a closed subset of the image near the origin wouldn't be closed in the domain.

4.2 Submersions

▷ Just as with immersions, there is a canonical submersion defined to be:

$$C_s : \mathbb{R}^k \to \mathbb{R}^\ell$$

$$(x_1, ..., x_k) = (x_1, ..., x_{k-\ell})$$

where $k \geq \ell$. This is simply truncation of the coordinates to $\mathbb{R}^\ell$.

There is also a similar theorem about submersions:

Theorem: If $f : X \to Y$ is a submersion at $x \in X$, then there are coordinates such that f is equivalent to the canonical submersion; that is there exist parameterizations ϕ, ψ such that:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\phi"] \arrow[r, "C_s"] & V \arrow[u, "\psi"] \end{tikzcd}$$

Proof: Found on homework 2.

5 Regular Values and Implicit Manifolds

The submersion theorem above lets us prove useful ways of finding manifolds as the zero locus of a system of equations.

Definition: Given a map $f : X \to Y$, a point $y \in Y$ is a *regular value* if df_x is surjective for all $x \in f^{-1}(y)$. If $y \notin f(X)$, then it is a regular value vaccuously.

Theorem: If $f : X \to Y$ is a smooth map with regular value $y \in Y$, then $f^{-1}(y)$ is a submanifold of X.

Proof:

Let $x \in f^{-1}(y)$ be a point in the preimage. Since y is regular, df_x is surjective, and thereforea submersion. By the Submersion Theorem, we can choose coordinates around x and y such that $y = (0, ..., 0)$ and f is equivalent to the canonical submersion (i.e. $f(x_1, ..., x_k) = (x_1, ..., x_\ell)$). So, in coordinates, $f^{-1}(y)$ looks like:

$$(0, \ldots, \quad \underbrace{x_{\ell+1}, ..., x_k)}_{\text{parameterize } f^{-1}(y) \text{ locally}}$$

Therefore x can be parameterized, making $f^{-1}(y)$ a manifold.

□

This opens up a very useful way of showing objects are a manifold: showing it is the preimage of a point.

Example: Show the sphere S^n is a manifold.

Define the map $f : \mathbb{R}^n \to \mathbb{R}$ that takes $x \mapsto |x|^2$. The derivative at $a \in \mathbb{R}^n$ is:

$$df_a = 2(a_1, ..., a_n)$$

Consider the preimage $f^{-1}(1)$. Since every vector of norm 1 must have a nonzero component, there is some $a_i \neq 0$. This means df_a surjects onto $\mathbb{R}$, and therefore 1 is a regular value. This means:

$$\{x \in \mathbb{R}^n \mid |x|^2 = 1\} = S^{n-1}$$

is a manifold.

Corollary: Any ℓ-manifold M can be written locally as the zero set of a collection of functions. That is, for a subset $U \subset X$, there exist $g_1, .., g_\ell$, with $g_i : U \to \mathbb{R}$, such that $M \cap U = \{x \in X \mid g_1(x) = g_2(x) = ...g_\ell(x) = 0\}$. [2]

5.1 Transversality

▷ We saw that when y is a regular value, $f^{-1}(y)$ is a manifold. This can be generalized to entire subsets Z of the codomain, as long as a condition called transversality is satisfied.

Definition: A smooth function $f : X \to Y$ is transverse to a submanifold $Z \subset Y$ at x if:

$$\mathrm{Im}(df_x) + T_{f(x)}(Z) = T_{f(x)}(Y)$$

Meaning every element of $T_{f(x)}(Y)$ can be written as a sum of elements in $\mathrm{Im}(df_x)$ and $T_{f(x)}(Z)$.

[2] This isn't necessarily true globally, though sometimes it is, as the sphere example showed

Example: Taking $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $Z = \langle 1, 0\rangle$, and $f(t) = (0, t)$, we get a transverse mapping. This transverse because $\operatorname{Im}(df) = \langle 0, 1\rangle$, and $Z + \operatorname{Im}(df) = \operatorname{span}(e_1, e_2) = \mathbb{R}^2$.

▷ An important special case is when $X, Z \subset Y$ are submanifolds, and $f : X \to Y$ is taken to be inclusion. The transversality condition means:

$$T_p(X) + T_p(Z) = T_p(Y)$$

This means that, wherever Z and X intersect, their tangent spaces at that point must span that of Y. We say that two manifolds intersect transversally $Z \pitchfork X$ if this condition is satisfied.

Theorem: If $f : X \to Y$ is transverse to a submanifold $Z \subset Y$, then $f^{-1}(Z)$ is a submanifold.

Proof:

Recall that we can write Z locally as the zero set of functions $g_1, ..., g_\ell$ that are independent. Let U be a neihborhood of $f(x)$ in Y, and define $g : U \to \mathbb{R}^\ell$ as $g = (g_1, ..., g_\ell)$. We can look at the map $g \circ f : X \to \mathbb{R}^\ell$. In particular, the zero set of $g \circ f$ is the set of points that map to Z, which is precisely $f^{-1}(Z)$. Therefore, we need only show that $0 \in \mathbb{R}^\ell$ is a regular value of $g \circ f$ to show that $f^{-1}(Z)$ is a manifold. By the chain rule we have:

$$\begin{tikzcd} T_x(X) \arrow[r, "df_x"] \arrow[rr, bend right, "d(g\circ f)_x"'] & T_{f(x)}(Y) \arrow[r, "dg_{f(x)}"] & \mathbb{R}^\ell \end{tikzcd}$$

We wish to show this is a surjection. Since the g_i were chosen to be independent, dg is surjective. Therefore $d(g \circ f)$ is surjective if and only if df carries its domain beyond the kernel of dg. Not only must the image of df be more than the kernel of dg, it together with the kernel of dg must span $T_{f(x)}(Y)$ (otherwise g wouldn't necessarily hit everything in $\mathbb{R}^\ell$). In other words, we must have:

$$\operatorname{Im}(df_x) + \ker(dg_{f(x)}) = T_{f(x)}(Y)$$

But $\ker(dg) = T_{f(x)}(Z)$, so we must have:

$$\operatorname{Im}(df_x) + T_{f(x)}(Z) = T_{f(x)}(Y)$$

This is guaranteed by the transversality of f.

□

5.2 Stability

▷ An important part of the study of properties of maps is how *stable* these properties are under small deformations. In order to make this rigorous, we define homotopy and stability:

Definition: A pair of maps $f_0, f_1 : X \to Y$ are smoothly homotopic if there is some $F : X \times [0, 1] \to Y$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$.

Definition: A property $\mathcal{P}$ of maps is called stable if it is preserved under small deformation. Specifically, if $f_0 : X \to Y$ satisfies a stable property $\mathcal{P}$, and F is some homotopy with $F(x, 0) = f_0$, then there extists $\epsilon > 0$ such that $F(x, \delta) = f_\delta$ satisfies $\mathcal{P}$ for all $\delta < \epsilon$.

Theorem: The following are stable properties of maps on manifolds:

(a) local diffeomorphism

(b) immersion

(c) submersion

(d) transverse to some submanifold Z

(e) embedding

(f) diffeomorphism

Proof: See G.P. pp 36

6 Sard's Theorem

▷ Recall that the preimage of any regular value y of a smooth mapping $f : X \to Y$ is a smooth manifold. This gives rise to the natural question:

Q: How much of Y is made up of regular values?

A: The answer to this is called Sard's theorem. In order to state this, we need to define the concept of measure on a manifold.

Definition: A subset $A \subset \mathbb{R}^\ell$ has measure zero if A can be covered by a countable number of rectangles with arbitrarily small volume.

Definition: A subset $A \subset M$ of a manifold has measure zero if and only if:

1. For every parameterization $\phi : U \to M$, the preimage $\phi^{-1}(A)$ has measure zero as a subset of $\mathbb{R}^\ell$.
2. There exists a covering of M by charts (ϕ_α, U_α) such that $\phi_\alpha^{-1}(A)$ has measure zero as a subset of $\mathbb{R}^\ell$ for all α.

Theorem (Sard's): If $f : X \to Y$ is a smooth map of manifolds, then almost every point in Y is regular. That is, if C is the set of critcal points of f, then $f(C)$ has measure zero.

6.1 Proof of Sard's Theorem

▷ The proof of Sard's Theorem involves reducing this to a local fact, and treating it like a multivariable calculus problem that uses a version Fubini's Theorem.

Definition: Suppose $n = k + \ell$. Then $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^\ell$. For $c \in \mathbb{R}^k$, let v_c be the vertical slice $\{c\} \times \mathbb{R}^\ell$ in $\mathbb{R}^n$. A subset of v_c has measure zero if it has measure zero on $\mathbb{R}^\ell$.

Theorem (Fubini): If a closed subset $A \subset \mathbb{R}^n$ such that $A \subset v_c$ has measure zero for all c, then A has measure zero.

▷ The proof of Sard's theorem follows from two lemmas:

Lemma: Let $f : U \to \mathbb{R}^p$ be a smooth map on an open set $U \subset \mathbb{R}^n$, and let C be the set of critical points of U. Further, define a filtration of C as:

$$C_1 = \{x \in U \mid df_x = [0]\}$$

$$C_i = \left\{ x \in U \;\middle|\; \frac{\partial^j f}{\partial x_m^j} = \vec{0} \quad \forall m \in \{1, ..., p\},\ \forall j \leq i \right\}$$

Intuitively, C_i is the set of points for which all partials of order $\leq i$ are zero. Also note $C \supset C_1 \supset C_k$. Then, we have:

1. $f(C \setminus C_1)$ has measure 0

2. $f(C_k \setminus C_{k+1})$ has measure 0 for all $k \geq 1$
3. $f(C_i)$ has measure 0 for $k > \frac{n}{p} - 1$.

Proof:

1. Let $x \notin C_1$. Then $df_x \neq [0]$. This means there exists some partial partial derivative $\frac{\partial f_i}{x_j} \neq 0$. Assume without loss of generality that $\frac{\partial f_1}{\partial x_1} = a \neq 0$, and define $h : U \to \mathbb{R}^n$ as:

$$h(x) = (f_1(x), x_2, x_3, ..., x_n)$$

Then dh_x has the form:

$$dh_x = \left(\begin{array}{c|c} \text{a} & \\ \hline & I_{n-1} \end{array}\right)$$

This must be invertible. Therefore, by the Inverse Function Theorem, h is a local diffeomorphism. Specifically, there is some open V containing x which is mapped diffeomorphically by h to some $V' \subset \mathbb{R}^n$. Now we consider $g = f \circ h^{-1} : V' \to \mathbb{R}^p$. Since dh is invertible, the critical values (points whose preimages are critical points) of f and g are the same. Therefore we seek to show that the critical values of g have measure 0.

By construction, g maps points as:

$$(t, x_2, ..., x_n) \mapsto (t, y_2, ..., y_p)$$

Namely, it is the identity on the first component. Therefore for any $t \in \mathbb{R}$, g induces a map $g^t : \{t\} \times \mathbb{R}^{n-1} \cap V' \to \{t\} \times \mathbb{R}^{p-1}$. Now we look at dg and how it relates to dg^t:

$$dg = \left(\begin{array}{c|c} 1 & \mathbf{0} \\ \hline & dg^t \end{array}\right)$$

We see that any point of $\{t\} \times \mathbb{R}^{n-1}$ is critical for g if and only if it is critical for g^t. Therefore if the collection of critical points for g^t is of measure zero in the 'slice' $\{t\} \times \mathbb{R}^{n-1}$, then the entire collection in $\mathbb{R}^n$ has measure zero by Fubini's Theorem. Inducting on n, with the clear base case of $n = 0$, we see that $f(V \cap C)$ has measure zero. Since we can cover $C \setminus C_1$ with countably many such V, we have that the measure of $f(C \setminus C_1)$ is also zero.

2. The proof for this is very similar to that for part 1. See G.P. for complete version.

3. Fix $\delta > 0$, and consider a cube S of side length δ contained in U. We will first show that $f(C_k \cap S)$ has measure 0 for sufficiently large k. To show this, subdivide S into r^n cubelets each of side length δ/r. Let S_1 be a cubelet that intersects C_k. Any point in S_1 can be written as $x + h$ with $|h| < \sqrt{n}\left(\frac{\delta}{r}\right)$ and $x \in C_k$.

Now we use Taylor's Theorem, which states:

$$f(x + h) = f(x) + R(x, h)$$

Where $R(x, h)$ are remainder terms depending on higher derivatives. Since all derivatives are assumed to be zero for $i \leq k$, we have $|R(x, h)| < a|h|^{k+1}$. In this context Taylor's Theorem tells us that $f(S_1)$ is contained in a cube around $f(x)$ with sides of length $\frac{b}{r^{k+1}}$.

Therefore, $f(C_k \cap S)$ is contained in a union of cubelets with total volume $r^n \left(\frac{b}{r^{k+1}}\right)^p = b^p r^{n-(k+1)p}$. If we assume $k > \frac{n}{p} - 1$, then $f(C_k \cap S)$ is contained in a rectangle of volume $b^p r^{-N}$ where N is positive. As $r \to \infty$, this gets arbitrarily small. Therefore $f(C_k \cap S)$ has measure 0.

Now, we cover C_k by countably many such S and we get C_k having measure 0 as well so long as $k > \frac{n}{p} - 1$.

$\square$

Lemma: If $f : U \to \mathbb{R}^p$ is a smooth map on an open set $U \subset \mathbb{R}^p$, and C is the set of critical points of U, then $f(C)$ has measure zero in $\mathbb{R}^p$.

Proof:

By the additivity of measure and the first lemma, we know:

$$m(f(C)) = m\left(f(C \setminus C_1) \cup f(C_1)\right) = m(f(C \setminus C_1)) + m(f(C_1)) = m(f(C_1))$$

By the same reasoning, $m(f(C_1)) = m(f(C_2))$, and so on. Since $m(f(C_k)) = 0$ for $k > \frac{n}{p} - 1$, we have:

$$m(f(C)) = m(f(C_1)) = m(f(C_2) = ... = m(f(C_k)) = 0$$

$\square$

Proof (of Sard's Theorem):

Let C be the set of critical points of $f : X \to Y$. Define the critical values of f to be $f(C)$. For any $x \in X$, parameterize by (ϕ, U), with $U \subset \mathbb{R}^\ell$. Further, we can parameterize around $y \in f(\phi(U))$ so that (ψ, U') is a chart around y and $U' \subset f(\phi(U))$. Composing these mappings gives:

$$U \xrightarrow{\psi^{-1} \circ f \circ \phi} \mathbb{R}^p$$

This is a smooth mapping, and its cricial values are exactly those of $\psi^{-1} \circ f$ because ϕ is a local diffeomorphism. By the lemma above, we know that the crical values of $\psi^{-1} \circ f \circ \phi$ have measure zero. Therefore the critical values of $\psi^{-1} \circ f$ have measure zero in $\mathbb{R}^p$. Since having Lebesque measure 0 is invariant under diffeomorphism, we then have:

$$0 = m(\psi^{-1} \circ f(C)) = m(\psi(\psi^{-1} \circ f(C))) = m(f(C))$$

$\square$

7 Whitney's Embedding Theorem

We showed on homework 1 that every manifold can be embedded into $\mathbb{R}^N$ for N sufficiently large. The natural question then arises:

Q: What is the smallest N such that M embeds into $\mathbb{R}^N$?

A: The answer depends on the manifold, but Whitney's Embedding Theorem gives us an upper bound based on the dimension of M

Theorem (Whitney): If X is a k manifold, then X embeds into $\mathbb{R}^{2k+1}$. In fact, it embeds into $\mathbb{R}^{2k}$, but we won't prove this.

$\triangleright$ The proof of this involves the definition of a tangent bundle, partitions of unity and a lemma.

Definition: The tangent bundle $T(M)$ of a manifold M is the disjoint union of its tangent spaces. In other words:

$$T(M) = \{(x, v) \in X \times \mathbb{R}^k \mid v = T_x(M)\}$$

It can be shown that this is a $2k$ dimensional manifold.

7.1 Partitions of Unity

Definition: Let $X \subset \mathbb{R}^n$ be an arbitrary set. For any covering of X by open sets $\{U_\alpha\}$, a *partition of unity* is a sequence $\{\theta_i\}$ of smooth functions $\theta_i : X \to \mathbb{R}$ satisfying:

1. $0 \leq \theta_i \leq 1$
2. For any x, there extists an open neighborhood of x on which all but finitely many θ_i are zero.
3. Each θ_i is zero except on some closed set contained in U_α for some α.
4. $\sum \theta_i(x) = 1$ for all x.

It can be shown that any set X admits a partition of unity (see GP).

$\triangleright$ We can use partitions of unity to prove a lemma useful in proving Whitney's theorem:

Lemma: On any manifold X, there exists a smooth proper map $\rho : X \to \mathbb{R}$.

Proof:

Let $\{U_\alpha\}$ the collection of open sets in X with compact closure. Take a partition of unity $\{\theta_i\}$ subodinate to $\{U_\alpha\}$. Now define:

$$\rho = \sum_i^\infty i \cdot \theta_i(x)$$

For any x, this is a finite sum because the θ_i are locally finite. Now observe if $\rho(x) \leq j$, then one of $\theta_i, ..., \theta_j \neq 0$ at x. Then the preimage satisfies:

$$\rho^{-1}([-j, j]) \subset \bigcup_{i=1}^j \{x \mid \theta_i(x) \neq 0\}$$

For each i, we see that the set $\{\ x \mid \theta_i(x) \neq 0\}$ is that which is guaranteed for θ_i in property 3 of the definition of partition of unity. Therefore each of these is contained in U_α for some α. Since we picked the cover so that U_α has compact closure, we must have that each set $\{x \mid \theta_i(x) \neq 0\}$ is bounded. Therefore the union is bounded. Since ρ is continuous, the preimage of a closed set $[-j, j]$ is also closed. Therefore it is compact, as it is contained in a bounded set. Hence. ρ is proper because any compact set on $\mathbb{R}$ is contained in some $[-j, j]$.

$\square$

7.2 Proof of Whitney

$\triangleright$ The proof can be broken down into two smaller chunks: first showing that any k manifold has an injective immersion into $\mathbb{R}^{2k+1}$, then showing that this can be a proper map (and hence an embedding).

Proof (of Whitney):

1. Let $X \subset \mathbb{R}^N$ be a manifold. If $f : X \to \mathbb{R}^M$ is an injective immersion with $M > 2k+1$, then we claim we can find $a \neq 0 \in \mathbb{R}^M$ such that $\pi \circ f : X \to H$ is an injective immersion, where H is the orthogonal complement of a. To find such an a, define $h : X \times X \times \mathbb{R} \to \mathbb{R}^M$ and $g : T(X) \to \mathbb{R}^M$ as:

$$h(x, y, t) = t(f(x) - f(y))$$

$$g(x, v) = df_x(v)$$

In particular, the dimensions of the domains of h and g are less than thier codomains (as $M > 2k+1$), so Sard's theorem tells us that there is a nonzero a not contained in $\mathrm{Im}(h)$ and $\mathrm{Im}(g)$. Now we look at $\pi \circ f$, the composition of f and the projection π onto the orthogonal complement of a. If this map wasn't injective, then there are distinct x and y such that

$$\pi \circ f(x) = \pi \circ f(y) \Longrightarrow \pi \circ (f(x) - f(y)) = 0$$

This means $f(x) - f(y)$ is a scalar multiple of a, as its projection onto the orthogonal complement is zero. So:

$$f(x) - f(y) = ta$$

We see that $t \neq 0$ because f is injective. But then this means $h(x, y, 1/t) = a$, which is a contradiction. Therefore $\pi \circ f$ is injective.

Now assume $\pi \circ f$ is not an immersion; this implies there is some $v \neq 0$ in $T_x(X)$ that is contained in the kernel of $d(\pi \circ f)_x$. Namely:

$$d(\pi \circ f)_x(v) = 0 \Longrightarrow \pi \circ df_x(v) = 0 \Longrightarrow df_x(v) = ta$$

But $t \neq 0$ because f is an immersion. This implies $g(x, v/t) = a$, a contradiction. Thus we have found an injective immersion $X \to H \cong \mathbb{R}^{M-1}$.

Now, we proceed inductively. If we start with $f : X \to \mathbb{R}^N$ an embedding, we can compose this with a series of injective immersions constructed as above taking $X \to \mathbb{R}^{N_1} \to \mathbb{R}^{N-2} \to \ldots \to \mathbb{R}^{2k+1}$. Once this point is reached, we cannot use Sard's theorem as above, so the projection concludes. Therefore this composition is an injective immersion of X into $\mathbb{R}^{2k+1}$.

2. Now given an injective immersion $f : X \to \mathbb{R}^{2k+1}$, we can ensure that $|f(x)| < 1$ by composing this with the diffeomorphism of $\mathbb{R}^{2k+1}$ onto its unit ball. Choose a proper function $\rho : X \to \mathbb{R}$ (using the lemma partitions of unity lemma), and define $F : X \to \mathbb{R}^{2k+2}$ given by $F(x) = (f(x), \rho(x))$. This is an injective immersion because f is. As with the previous part, we can compose this with an orthogonal projection π to $\mathbb{R}^{2k+1}$. In other words, we have an injective immersion $\pi \circ F : X \to H$, where H is the orthogonal complement of some choice of $a \neq 0$.

Pick a to be neither of the poles of S^{2k+1} (we can do this because $\pi \circ F$ is an injective immersion at almost every a). We now claim that $\pi \circ F$ is proper. To show this, first we claim that for any c, there is some d such that:

$$|\pi \circ F(x)| \leq c \Longrightarrow |\rho(x)| < d \quad \forall x$$

To see this, assume it is false; that there is some sequence $x_i \in X$ such that $|\pi \circ F(x_i)| \leq c$ but $|\rho(x_i)| \to \infty$. Consider now the vector:

$$w_i = \frac{1}{\rho(x_i)} \left[F(x_i) - \pi \circ F(x_i)\right]$$

This is a vector that takes off the projection of $F(x_i)$ onto H; hence its projective component on H is zero, and so it is a multiple of a. As $i \to \infty$, we have:

$$\frac{F(x_i)}{\rho(x_i)} = \left(\frac{f(x_i)}{\rho(x_i)}, 1\right) \to (0, 0, ..., 1)$$

Further, we also have:

$$\frac{\pi \circ F(x_i)}{\rho(x_i)} \to (0, 0, ..., 0)$$

because $\pi \circ F(x_i)$ is bounded by c. Therefore $w_i \to (0, 0, ..., 1)$. Since a is a multiple of w_i, we have that a is on either of the poles of S^{2k+1}, a contradiction. Therefore $|\rho(x)| \leq d$ for sufficiently large d.

Now, consider the preimage of a closed ball B_c in H, or the set $\{x \mid |\pi \circ F(x)| \leq c\}$. We saw above that this is a subset of $\{x \mid |\rho(x)| \leq d\}$, the preimage under ρ of the closed ball of radius d. Since ρ is proper, we have this preimage being compact. This means $(\pi \circ F)^{-1}(B_c)$ is bounded. It is also closed, as $\pi \circ F$ is continuous. Thus we have the preimage is compact, and so this is a proper map as any compact set is contained in some B_c. So $\pi \circ F : X \to R^{2k+1}$ is injective, proper and is an immersion, so it is an embedding.

□

8 Manifolds with Boundary

We can slightly generalize the definition of a manifold by allowing to have a boundary.

Definition: A subset of $\mathbb{R}^N$ is a manifold with boundary if every point in X has an open neighborhood which is diffeomorphic to an open subset of the upper half plane H^k.

Examples of open subsets in H^2 are shown in the figure. Note how some aren't open in $\mathbb{R}^2$.

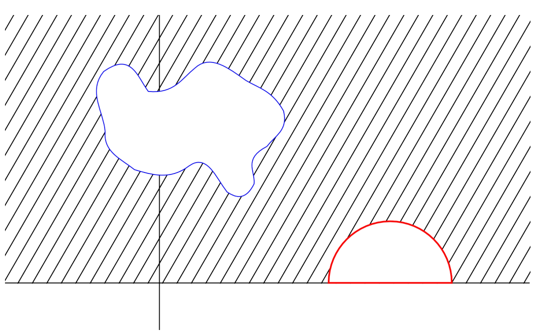


Figure 1: Open sets in H^2

▷ Note that, under this definition, all manifolds are manifolds with boundary. This just means the ones we've studied so far are simply ones with empty boundary.

Terminology:

- The diffeomorphisms from the definition of manifold with boundary are still called local parameterizations.
- The boundary of X, denoted ∂X, is the set of points that land on the boundary hyperplane $x_n = 0$ of H^k undser some coordinate chart.

8.1 Facts about Manifolds with Boundary

▷ Many things about manifolds carry over when they have boundary. In particular, derivatives can be defined in almost the same way, and therefore transversality as well.

▷ The first fact is that ∂X is a manifold with empty boundary.

Derivatives

Let U be an open subset of H^k, and let $g : U \to \mathbb{R}^\ell$ be smooth. If x is in the interior of U, the derivative is defined as it is on an open set of $\mathbb{R}^k$. If $x \in \partial U$, then by the definition of g being smooth, there is an extension $\tilde{g}$ defined on an open neighborhood of x. Then we define $dg_x \equiv d\tilde{g}_x$. This can be shown to be indepdendent of extension $\tilde{g}$, so it is indeed well defined. Because of these extensions, the chain rule still holds.

Definition: The tangent space on a manifold with boundary is defined the same way it is on manifolds without boundary; namely it is the image of dg_x, where g is a chart of X at x. Keep in mind that on the boundary, the tangent space is not a half hyperplane; it is a hyperplane just like everywhere else.

Definition: The derivative between manifolds with boundary is defined in the same way again, namely the unique map induced between the tangent spaces.

Product Manifolds

Beware! The product of two manifolds with boundary isn't necessarily a manifold with boundary. For example, the product $[0, 1] \times [0, 1]$ is the unit square. This square isn't a manifold because the corners aren't diffeomorphic to $\mathbb{H}^k$ (see Homework 4).

However if Y has boundary and X has empty boundary, then $X \times Y$ is a manifold with boundary.

8.2 Transversality

▷ The notion of transversality can be extended to manifolds with boundary, but with a few extra constraints.

Theorem: Let $f : X \to Y$ be a smooth function between X, a manifold with boundary, and Y, a manifold without boundary. Further, let $Z \subset Y$ be a submanifold without boundary. If f is transverse to Z and $\partial f = f\big|_{\partial X} \to Y$ is transverse to Z, then $f^{-1}(Z)$ is a manifold with boundary. Further, we have:

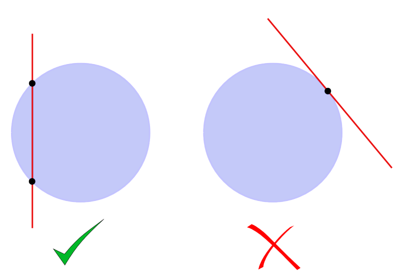
$$\partial(f^{-1}(Z)) = f^{-1}(Z) \cap \partial X$$

$$\operatorname{codim}(X, f^{-1}(Z)) = \operatorname{codim}(Y, Z)$$

Proof: See G.P. Very similar to the proof for the first version.

Example:

Let $Y = \mathbb{R}^2$, X be the closed solid disk of radius 1 (f taken to be inclusion). Lets consider two one dimensional submanifolds Z_1 and Z_1 detailed below.



In both cases, f is transverse to Z, since the tangent space of the disk is all of $\mathbb{R}^2$. Further, ∂f is transverse to Z_1, the vertical line, because the tangent spaces of the ∂X and Z_1 are independent vectors. Thus $Z_1 \cap X$ is a submanifold with boundary (namely, a closed interval). However, ∂f is not transverse to Z_2 because the tangent space of ∂X and Z_2 coincide to only form a 1 dimensional subspace.

8.3 Sard's Theorem for Manifolds with boundary

▷ Sard's theorem has an equivalent formulation as well (with similar additinal constraints as above):

Theorem: Let $f : X \to Y$ be smooth, and X be a manifold with boundary. Further, assume Y has no boundary. Then almost every point in Y is a regular value of f and of ∂f.

8.4 Classification of 1 dimensional Manifolds

Theorem: Every compact, connected, 1-manifold with boundary is diffeomorphic to a circle or an interval.
Proof: See G.P. appendix.

Corollary: The boundary points of any compact 1 manifold with boundary has an even number of points.

▷ An interesting application of this corollary is showing there is no retraction of a manifold to its boundary:

Theorem: If X is a compact manifold with boundary. Then there is no smooth contraction of X to its boundary. Namely, there does not exist $g : X \to \partial X$ such that g restricted to the boundary is the identity.

Proof:

Assume such a g exists, and let $z \in \partial X$ be a regular value (exists by Sard's theorem). By above, we have that $g^{-1}(z)$ is a compact submanifold with boundary. By the transversality theorem, we have that $\operatorname{codim}(X, g^{-1}(z)) = \operatorname{codim}(\partial X, \{z\})$. But the latter is just the dimension of ∂X, which is $n-1$. Therefore $g^{-1}(z)$ is a 1 manifold because its codimension in X is $n-1$. Also by the transversality condition, we have:

$$\partial g^{-1}(z) = g^{-1}(z) \cap \partial X$$

But ∂g is the identity as $z \in \partial X$, so the boundary of $g^{-1}(z)$ is just the point z. This contradicts the corollary above, as this is an odd number. Therefore no such g exists.

□

▷ As a corollary of this, we can prove Brouwer's Fixed Point Theorem:

Corollary: Let B^n be the closed unit ball in $\mathbb{R}^n$. If $F : B^n \to B^n$ is smooth map, then there is a fixed point $x \in B^n$, so that $F(x) = x$.

Proof:

We proceed by contradiction; that its, let $F : B^n \to B^n$ be a map with no fixed point. Then define $g(x)$ to be the intersection of the line from $f(x)$ to x (note that $g(x)$ is necessarily on the boundary of B^n). This is well defined for all x because $F(x) \neq x$ for all x by assumption. The explicit equation for the line along x and $f(x)$ is clearly:

$$L(t) = tx - (1-t)F(x)$$

We note that when $|L(t)| = 1$, we have $L = g(x)$ because g maps to the boundary. If we enforce $L \cdot L = 1$, we get:

$$L \cdot L = 1 \Longrightarrow t^2|x - F(x)|^2 - 2tF(x) \cdot (x - f(x)) + |F(x)|^2 = 1$$

We can invert this relation for t; call it $T(x)$. Since this is a quadratic in t, we note $T(x)$ must be smooth. Then we know $g(x) = L(T(x))$. Since L is smooth and T is smooth, so it g. However, we note that $g : B^n \to \partial B^n$ is a retraction to the boundary because $g(x) = x$ for $x \in \partial B^n$. Therefore we have a contradiction by the above theorem.

□

9 Transversality (Revisited)

▷ Now we revisit the idea of transversality. Specifically, we'll discuss it as a stable property of maps.

Definition: A family of maps parameterized by a boundary-less manifold S is a smooth map $F : X \times S \to Y$, and each family member, denoted f_s, is $F : X \times \{s\} \to Y$.

▷ The parametric transversality theorem (below)[3] tells us about the transversality of families of functions, which will come in handy for a future discussion.

Theorem: Suppose $F : X \times S \to Y$ is a family of maps and $Z \subset Y$ are boundary-less manifolds, and $F, \partial F$ are transverse to Z. Then almost every $f_s, \partial f_s$ is transverse to Z.

Proof:

Since F is transverse, $F^{-1}(Z) = W$ is a submanifold of Y and $\partial W = W \cap \partial(X \times S)$. Now consider the standard projection $\pi : X \times S \to S$ restricted to W. We now claim the following:

1. If s is a regular value of $\pi : W \to S$, then $f_s \pitchfork Z$
2. If s is a regular value of $\pi : \partial W \to S$, then $\partial f_s \pitchfork Z$.

Once we have shown these claims, then the theorem is true by Sard. We will prove (1). [4]. Suppose $F(x, s) = z \in Z$; since F is transverse to Z, we have:

$$\text{Im}(dF_{x,s}) + T_z(Z) = T_z(Y)$$

Namely, for any $a \in T_z(Y)$, there is some $b \in T_{x,s}(X \times S)$ such that:

$$dF_{x,s}(b) - a \in T_z(Z)$$

We note that $T_{x,s}(X \times S) = T_x(X) \times T_s(S)$. This means we can write $b = (w, e)$ for $w \in T_x(X)$ and $e \in T_s(S)$. Now we look at the derivative of $\pi : W \to S$;

$$d\pi_{x,s} : T_x(X) \times T_s(S) \to T_s(S)$$

This derivative is simply $[0, ..., \mathbf{1}]$. When applied to a particular vector, it projects that vector onto its $T_s(S)$ component. Since we assumed a was a regular value of π, this projection is surjective. In particular, we can find some $c = (u, e) \in T_{x,s}(W)$. But since F maps into Z, we have $dF_{x,s}(u, e) \in T_z(Z)$. Now, since $dF_{x,s}(v, 0) = df_s(v)$, we have:

$$df_s(w - u) - a = dF_{x,s}(w - u, 0) - a = dF_{x,s}[b - c] - a = \underbrace{dF_{x,s}(b) - a}_{\in T_x(Z)} - \underbrace{dF_s(c)}_{\in T_z(Z)}$$

Therefore $df_s(w - u) - a \in T_z(Z)$. Thus, we have found a vector in the image of df_s (namely $df_s(w - u)$), and a vector in $T_z(Z)$ (namely $df_s(w - u) - a$) whose difference is $a \in T_z(Y)$. Since this is true for any $a \in T_z(Y)$, we have that f_s is transverse to Z.

□

[3] This is the strongest transversality theorem we have encountered thus far

[4] The proof for (2) is almost identical

9.1 Transversality and Stability

▷ We will see here that we can use the above stated transversality theorem to construct a homotopy of any non-transverse map and a transverse map. To begin, we start with a simplified case:

Simple Case: let $f : X \to Y$ with $Y = \mathbb{R}^M$. Take S to be an open ball in $\mathbb{R}^M$ and define the family $F : X \times S \to \mathbb{R}^M$ by $F(x, s) = f(x) + s$. For any fixed x, the map F simply translates the ball S, which is a submersion. Therefore $\text{Im}(dF_{x,s}) = \mathbb{R}^M$, which means F is transverse to any submanifold of $Y = \mathbb{R}^M$. Appealing to the transversality theorem, we have that $f_s(x) = f(x) + s$ is transverse to Z for almost all $s \in S$. We can take s to be arbitrarily small, so F is a homotopy of f and a transverse map f_s.

▷ To generalize this to any target manifold Y, we need a way to ensure that $f(x) + s$ remains inside Y. The general strategy is to thicken Y by ϵ so that Y^ϵ has zero codimension, and write a similar homotopy F composed with a projection back down onto Y. To do this, we first need to state the ϵ- neighborhood Theorem:

Theorem: For a compact boundary-less manifold $Y \subset \mathbb{R}^M$ and a positive number ϵ, let Y^ϵ be the open set of points of distance $< \epsilon$ from Y. If ϵ is sufficiently small, then each $w \in Y^\epsilon$ has a unique closest point given by the map $\pi : Y^\epsilon \to Y$, which is a submersion.

Remark: There is an equivalent formulation of this for non-compact Y, where ϵ is actually a function of $y \in Y$. Proceeding forward we assume Y is compact implicitly, but rest assured that even if it isn't, the $\epsilon-$neighborhood theorem still applies.

Proof: See G.P. pg 71.

Corollary: Let $f : X \to Y$ be smooth, and Y be boundary-less. Then there is an open ball S in some $\mathbb{R}^M$ and a smooth map $F : X \times S \to Y$ such that $F(x, 0) = f(x)$ and F and ∂F are submersions.

Proof:

Let Y be in $\mathbb{R}^M$ and let S be the unit ball in $\mathbb{R}^M$. Define:

$$F(x, s) = \pi[f(x) + \epsilon s]$$

where ϵ and π are the number/map given by the ϵ-neighborhood theorem. Firstly, note that the map $(x, s) \to f(x) + \epsilon s$ is a submersion because $T_s(S)$ has greater (or equal) dimension than Y. Since π is also a submersion, the composition F is also a sumbersion. The same applies to ∂F.

Further, we see that $F(x, 0) = \pi(f(x)) = f(x)$.

□

▷ Finally, we can state the general Transversality Homotopy Theorem, which follows nearly immediately from the above corollary.

Theorem: For any smooth map $f : X \to Y$ with $Z \subset Y$ both boundary-less, there exists a smooth map $g : X \to Y$ homotopic to f such that $g \pitchfork Z$ and $\partial g \pitchfork Z$.

Proof:

Define the homotopy F as from the corollary:

$$F(x, s) = \pi[f(x) + \epsilon s]$$

Since this is a submersion, the image of dF is all of $T_x(X) \times T_s(S) = T_x(S) \times \mathbb{R}^M$. Since Y has dimension less than or equal to M, we have that F is transverse to any $Z \subset Y$. The transversality theorem states that $f_s \pitchfork Z$ and $\partial f_s \pitchfork Z$ for almost all $s \in S$. Let g be any such f_s. Then $g \sim f$ by the homotopy:

$$G(x, t) = F(x, ts)$$

□

9.2 Extensions

Here we will give a slightly stronger statement of transversality not necessarily everywhere, but on a subset of X.

Definition: A map $f : X \to Y$ is transversal to Z on a subset $C \subset X$ if the transversality condition:

$$\mathrm{Im}(df_x) + T_{f(x)}(Z) = T_{f(x)}(Y)$$

is satisfied on every point $x \in C \cap F^{-1}(Z)$.

Theorem: Let $Z \subset Y$ both be without boundary, and let $C \subset X$ be closed. If $f : X \to Y$ is transverse to Z on C and ∂f is transverse to Z on $\partial X \cap C$, then there is a smooth map g homotopic to f for which $g \pitchfork Z$ and $\partial g \pitchfork Z$. Further, on a neighborhood of C we have $f = g$.

▷ Intuitively, this means we can take a map that is only transverse on a subset of X, and morph it into some g transverse everywhere on X with the condition that $f = g$ locally somewhere.

Proof: See G.P. pp 72.

Corollary: If $f : X \to Y$ is transverse to some Z when restricted to ∂X, then there is a map $g : X \to Y$ that is transverse to Z, homotopic to f and $\partial g = \partial f$.

Proof: This is a special case of the above theorem, since the boundary is always closed in X.

10 Intersection Theory (Mod 2)

▷ We return to manifolds without boundary. Two submanifolds X and Z of Y are of complementary dimension if $\dim(X) + \dim(Z) = \dim(Y)$. Note that this means that X and Z intersect on a set of codimension $\dim(Y)$ (i.e. a zero dimensional set). If X is compact, this is a finite set.

Definition: Let $f : X \to Y$ be smooth, transverse to $Z \subset Y$ which is closed, and X be compact. Further, suppose X and Z have complementary dimension to Y, and suppose X, Y and Z are boundaryless. Define the mod 2 intersection number of f with Z to be the number of points in $f^{-1}(Z)$ mod 2, denoted $I_2(f, Z)$.

The first fact about the intersection number is that it is homotopy invariant:

Theorem 10: If $f_0 \sim f_1$ are transverse to Z, then $I_2(f_0, Z) = I_2(f_1, Z)$.

Proof:

Let $F : X \times I \to Y$ be a homotopy of f_0 and f_1. By the transversality homotopy theorem, we may assume that F is transverse to Z (otherwise we may perturb it a small amount so that it is). Further, since $\partial(X \times I) = X \times \{0\} \cup X \times \{1\}$, and $\partial F = f_0$ on $X \times \{0\}$ and $\partial F = f_1$ on $X \times \{1\}$, we have that $\partial F \pitchfork Z$ since $f_0, f_1 \pitchfork Z$. By the transversality theorem for manifolds with boundary, $F^{-1}(Z)$ is a one dimensional submanifold with boundary and:

$$\partial F^{-1}(Z) = F^{-1}(Z) \cap \partial(X \times I) = f_0^{-1}(Z) \times \{0\} \cup f_1^{-1}(Z) \times \{1\}$$

Our classification of 1 manifolds tells us that $\partial F^{-1}(Z)$ has an even number of boundary points. Then:

$$|\partial F^{-1}(Z)| = 2n = |f_0^{-1}(Z)| + |f_1^{-1}(Z)| \Longrightarrow |f_0^{-1}(Z)| = |f_1^{-1}(Z)| \pmod 2$$

□

This allows us to more generally define the intersection number:

Definition: The intersection number of an arbitarary smooth map $g : X \to Y$ with Z is the interseciton number of f where $f \sim g$ and $f \pitchfork Z$. Since homotopy holds transitively, we have that any two homotopic maps have the same intersection number.

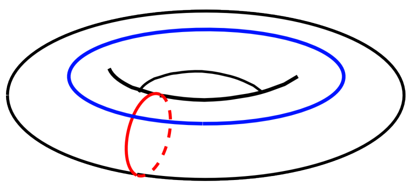
Special Cases

▷ If $X \subset Y$, we can define the intersection number of two manifolds Z and X to be $I_2(X, Z) \equiv I_2(i, Z)$ where $i : X \to Y$ is inclusion. Note that this *isn't* always the number of intersection points of Z and X. For example, if X didn't intersect Z transversely, the intersection number would actually be the intersection number of some deformation of X (given by deforming i homotopically). If

▷ If $I_2(X, Z) \neq 0$, then X cannot be "pulled apart" from Z no matter how it is deformed.

Example:

Consider the torus $\mathbb{T}^2 = S^1 \times S^1$, and let X and Z be the independent circles $S^1 \times \{0\}$ and $\{0\} \times S^1$, shown below.



We see that Z and X intersect transversely, which means their intersection number is the number of intersection points, which is 1. We also see that, no matter how you deform X, there is no way to pull it apart from Z so it has no intersection.

▷ A third special case is when $\dim X = \frac{1}{2} \dim Y$. We can then define the *self intersection* number of X with itself. This tells you how many intersections X has with a (transversely intersecting) deformation of itself.

▷ If X happens to be the boundary of some bigger manifold W, then we have what is known as the boundary theorem:

Theorem: Let X be the boundary of a compact manifold W and let $g : X \to Y$ be a smooth map. If g can be extended to W, then $I_2(g, Z) = 0$ for any closed $Z \subset Y$ of complementary dimension.

Proof:

Let $G : W \to Y$ be an extension of g (that is, $\partial G = g$). From the transversality homotopy theorem, we can find some $F : W \to Y$ homotopic to G that is transverse to Z and has $\partial F \pitchfork Z$. Let $f = \partial F$. Since $G \sim F$, the boundary maps are also homotopic (i.e. $g \sim f$). This means $I_2(g, Z) = |f^{-1}(Z)| \bmod 2$. We also know that $F^{-1}(Z)$ is a manifold in W. The codimension of $F^{-1}(Z)$ is equal to the codimension of Z in Y, which is $\dim(X)$ by assumption. Since W has dimension $\dim(X) + 1$, the dimension of $F^{-1}(Z)$ is 1. Therefore it is a compact 1 manifold with boundary. In particular, from our classification of such objects, we know $|\partial F^{-1}(Z)| = |f^{-1}(Z)| =$ even. Therefore $I_2(f, Z) = I_2(g, Z) = 0$.

□

▷ Finally, the case where $\dim(X) = \dim(Y)$, we get a homotopy invariant of maps between X and Y. Because any submanifold Z in this discussion must be complementary to X, and $\dim(X) = \dim(Y)$, we must have $\dim(Z) = 0$. This leads us to defining:

Definition: The mod 2 *degree* of $f : X \to Y$, where Y is connected and $\dim(X) = \dim(Y)$, is $I_2(f, \{y\})$, denoted $\deg_2(f)$.

Theorem: Mod 2 degree is well defined, and homotopic maps have the same mod 2 degree.

Proof: For the first statement, see G.P. pp 80. The second is a corollary of Theorem 10.

Theorem: If $X = \partial W$ and $f : X \to Y$ may be extended to all of W, then $\deg_2(f) = 0$.

Proof: This is a special case of the theorem before last.

11 Intersection Theory

Mod 2 intersection theory, while helpful, doesn't contain very much information about intersecting manifolds of complementary dimension. Should we wish to develop a more general intersection number, we need to develop the theory of manifold orientation.

11.1 Orientation of Manifolds

To define an orientation of a manifold, we must first define it on vector spaces.

Vector Spaces

Let $\{v_i\}$ and $\{u_i\}$ be ordered bases for a vector space V. The change of basis matrix P maps one basis to another. This induces an equivalence relation on the set of ordered bases of V, namely two bases β, β' are equivalent if $\det(P) > 0$.

Definition: An *orientation* of a vector space is a map $\mathcal{B} \to \{\pm 1\}$, where $\mathcal{B}$ is the set of ordered bases, such that equivalent bases have the same sign. We say an isomorphism A of vector spaces is orientation preserving if $\beta \sim \beta' \Rightarrow A\beta \sim A\beta'$ (otherwise it is orientation reversing).

Manifolds

Orientations of manifolds are based on orientations of vector spaces:

Definition: An *orientation* of a smooth manifold with boundary is a smooth choice of orientations on each $T_x(X)$. 'Smooth' here means around every $x \in X$, there should exist a parameterization $h : U \to X$ such that $dh : \mathbb{R}^k \to T_{h(u)}(X)$ is orientation preserving, where $\mathbb{R}^k$ has the usual orientation.

Remark: Not all manifolds admit an orientation (for example the Möbius Strip). If a manifold X has an orientation, it automatically has another orientation, denoted $-X$, which is the opposite choice of basis at every point.

Induced Orientations

Product Orientation

▷ Let $X \times Y$ be a product of oriented manifolds, with Y boundaryless. This has a natural orientation. Namely, give $T_{(x,y)}(X \times Y)$ the basis $\gamma = \{\alpha \times 0, \beta \times 0\}$, where α is an ordered basis for $T_x(X)$ and β is an ordered basis for $T_y(Y)$. Then the orientation of $X \times Y$ is determined by the orientations of X and Y as:

$$\text{sign}(\gamma) = \text{sign}(\alpha)\text{sign}(\beta)$$

Boundary Orientation

$\triangleright$ The boundary of an oriented manifold also comes with a natural orientation. To define this, we introduce the normal vectors on the boundary. Since ∂X is of codimension 1, there are two unit vectors in $T_x(X)$ for $x \in \partial X$ that are perpendicular to $T_x(\partial X)$. One of these maps into the inward part of $\mathbb{H}^k$, and the other maps out of $\mathbb{H}^k$. Denote η_x to be the outward unit normal. Now, we can define boundary orientation. Let $\{v_1, ..., v_{k-1}\}$ be a basis for $T_x(\partial X)$; its sign under the induced orientation is:

$$\text{sign}(\{v_1, ..., v_{k-1}) = \text{sign}(\{\eta_x, v_1, ...v_{k-1}\})$$

when thought of as a basis of $T_x(X)$.

Example: Let's look at orienting the boundary of a compact 1-manifold:

> If X is a 1 compact manifold, then it is diffeomorphic to a union of intervals and/or circles. The boundary of X is a collection of points, which are the boundary points of the intervals. No matter how X is oriented, every interval's boundary points will be a $+$ and a $-$. Therefore the signed sum of the boundary points is 0, as there are an even number of them.

Preimage Orientation

$\triangleright$ The final instance of an induced orientation is when we have $f : X \to Y$ with f and ∂f transverse to $Z \subset Y$, which are both boundaryless. If X, Y and Z are all oriented, there is an induced orientation for the preimage $f^{-1}(Z)$, which is a manifold. In the same way as before, we first look at this from a linear algebra perspective:

Vector Spaces	Manifolds
Let $V = V_1 \oplus V_2$, and let β_1 and β_2 be bases of V_1 and V_2. Further let $\beta = (\beta_1, \beta_2)$ be a basis of V. Then, if we require $\text{sign}(\beta) = \text{sign}(\beta_1)\text{sign}(\beta_2)$, an orientation of any two determines an orientation of the third.	Let $S = f^{-1}(Z)$, and let $N_x(S, X)$ be the orthogonal complement of $T_x(S)$. Then the following determine the orientation of S: $$df_x(N_x(S, X)) \oplus T_z(Z) = T_z(Y)$$ $$N_x(S, X) \oplus T_x(S) = T_x(X)$$ Namely, an orientation of $T_x(S)$ is induced by the orientations of $N_x(S, X)$ and $T_x(X)$. The former is oriented by the first equation since df_x restricted to $N_x(S, X)$ is an isomorphism. The exact reasoning for this can be found in G.P. pp 100.

One might notice that the boundary of a preimage $S = f^{-1}(Z)$ as formulated above now has two induced orientations: one as the boundary of an oriented manifold S, and one as the preimage of the map $\partial f : \partial X \to Y$ of Z. These (sometimes) differ by a sign. Namely:

Proposition: If $\ell = \text{codim}(Z)$, then $\partial[f^{-1}(Z)] = (-1)^\ell(\partial f)^{-1}(Z)$.

Proof: See G.P. pp 101.

11.2 Oriented Intersection Theory

We can now formulate intersection theory for manifolds of complementary dimension.

Definition: Let $f : X \to Y$ be smooth, $Z \subset Y$ be closed with complementary dimension. Further suppose that X, Y and Z are boundaryless and are oriented. If f is transverse to Z, then $f^{-1}(Z)$ is a finite set, each point having an orientation ± 1. The intersection number of f and Z, denoted $I(f, Z)$ is the signed count of these points.

Proposition: If $X = \partial W$ and $f : X \to Y$ extends to W, then $I(f, Z) = 0$ for any closed manifold of complementary dimension when W is compact.

Proof:

Let F be an extension of f; then $f = \partial F$. Since f is transverse to Z, there is a map G homotopic to F that is transverse to Z by our corollary of the Extension Theorem. This means $F^{-1}(Z)$ is a compact oriented 1 manifold whose boundary is $f^{-1}(Z)$, which is oriented and finite. We showed that the sum of orientation numbers of the boundary is zero in a previous example. Thus $I(f, Z) = 0$.

□

A special case of this proves:

Proposition: If $f_1 \sim f_0$ are transverse to Z, then $I(f_1, Z) = I(f_0, Z)$.

Proof:

Let $F : X \times I \to Y$ be a homotopy between f_0 and f_1. It can be shown that $\partial(I \times X) = X_1 - X_0$. Since $F(x, 0) = f_0$ and $F(x, 1) = f_1$, the preimage $\partial F^{-1}(Z)$ is the union of the preimages of f_1 and f_2 (with proper orientations). Thus:

$$\partial F^{-1}(Z) = f_1^{-1}(Z) - f_0^{-1}(Z)$$

Therefore the intersection number of $F^{-1}(Z)$ is:

$$I(\partial F, Z) = I(f_1, Z) - I(f_0, Z)$$

But from the previous proposition, we have that $I(\partial F, Z) = 0$ because $f_1 \cup f_0 : X_1 - X_0 \to Y$ extends to $F : X \times I \to Y$. Thus $I(f_1, Z) = I(f_0, Z)$.

□

▷ We can also define the intersection number of a nontransverse map to be the intersection number of a homotopic map that is transverse. Further, we can define the intersection number $I(X, Z)$ of two submanifolds as well as the degree of a map when $\dim(X) = \dim(Y)$ in the same way as before in the mod 2 case (except now it is signed count).

11.3 More General Perspective

▷ It will prove useful to to think of intersection theory in the more general setting of having two maps $f : X \to Y$ and $g : Z \to Y$, instead of Z necessarily being a submanifold of Y.

Definition: To maps $f : X \to Y$ and $g : Z \to Y$ of boundaryless manifolds with X, Z compact are transverse if:

$$df_x(T_x(X)) + dg_z(T_z(Z)) = T_y(Y)$$

where $f(x) = g(z) = y$. This is denoted $f \pitchfork g$. When $\dim(X) + \dim(Z) = \dim(Y)$, the derivatives must be injective and so the sum is direct:

$$df_x(T_x(X)) \oplus dg_z(T_z(Z)) = T_y(Y) \tag{1}$$

Since these are injections, they isomorphisms onto their images. This means if any two are oriented, the third can be oriented in the usual way of orienting vector spaces.

$\triangleright$ If X, Y and Z are oriented, we can think of the intersection number of f and g. Their "intersection" will be the set:

$$\mathbb{I} = \{f(x) = g(z)\} \subset X \times Z$$

Any point in this set is oriented via the direct sum in the above definition. That is, a point (x, z) in the intersection will have sign $+$ if the left orientation of (1) agrees with the right orientation of (1), and sign $-$ otherwise.

Definition: The intersection number of transverse maps $f : X \to Y$ and $g : Z \to Y$, where X, Z are of complementary dimension, is the signed count of the intersection set $\mathbb{I}$, denoted $I(f, g)$.

Remark: Two things must be verified for the above definition to make sense: the set $\mathbb{I}$ must be finite, and $I(f, g)$ should be a homotpy invariant (of both f and g). These follow from a proposition:

Proposition: $f \pitchfork g$ if and only if $f \times g \pitchfork \triangle \subset X \times Z$ [5] and:

$$I(f, g) = (-1)^{\dim(Z)} I(f \times g, \triangle)$$

In particular, $I(f, g)$ is well defined and homotopy invariant because $I(f \times g, \triangle)$ is.

Proof: See G.P. pp 114.

Proposition: The intersection number is sometimes antisymmetric (depending on the dimension of X and Z):

$$I(f, g) = (-1)^{\dim(X)\cdot\dim(Z)} I(g, f)$$

Proof: See G.P. pp 115.

$\triangleright$ The intersection number in this formulation has some useful applications in orientability and detecting invariants of Y.

Example: Let $\dim(Y) = 2\dim(X)$ and $\dim(X)$ is odd. Then the intersection of X with itself inside Y, which is $I(i, i) = I(X, X)$, must be zero because $I(X, X) = -I(X, X)$. This means $I_2(X, X) = 0$. Since mod 2 intersection number can be compueted without knowing the orientation of Y, if we find X such that $I_2(X, X) = 1$, then Y must be non-orientable by the contrapositive. This is a helpful way of showing a manifold is not orientable. For example, the diametrical circle in the Möbius strip has self intersection number 1, so the strip must be nonorientable, which is the case.

Example/Definition: Consider a compact boundaryless orientable manifold Y, and consider the submanifold $\triangle \subset Y \times Y$. This has dimension half that of $Y \times Y$, so we can compute $I(\triangle, \triangle)$. We define this to be the Euler Characteristic $\chi(Y)$, an important invariant of Y. This is a bit like the number of intersections that Y has with itself inside of $Y \times Y$. Just as before, we note that if Y is odd dimensional, then $I(\triangle, \triangle) = \chi(Y) = 0$.

[5] the diagonal in $X \times Z$

12 Lefschetz Fixed Point Theory

▷ Let $f : X \to X$ be a map of a compact boundaryless manifold to itself. A particular important question might be: Are there fixed points of f? How many are there? The technique of Lefschetz Fixed Point theory is approaching this question from the perspective of intersection theory.

That is, consider the submanifolds $\triangle$ and $\Gamma(f)$, the graph of f, inside $X \times X$. Then the fixed points of f is the intersection of these manifolds.

Definition: For a map $f : X \to X$ of a compact boundaryless manifold to itself, the Lefschetz number of f, denoted $L(f)$, is $I(\triangle, \Gamma(f))$. This is a homotopy invariant by our discussion of intersection theory.

Example: Suppose $f : X \to X$ has no fixed point and $f \sim id$. Then $\chi(X) = 0$, because $I(\triangle, \triangle) = I(\triangle, \Gamma(f)) = 0$.

▷ The work we've done so far in intersection theory gives us an immediate theorem about fixed points:

Theorem: If $f : X \to X$ is a smooth map on a compact orientable manifold. If $L(f) \neq 0$, then f has a fixed point.

Proof: By contrapositive, if f doesn't have a fixed point, then $\triangle$ and $\Gamma(f)$ have no intersection, and so $L(f) = 0$ trivially.

□

▷ Functions for which $\Gamma(f) \pitchfork \triangle$ are called Lefschetz maps. In this case, since $\Gamma(f)$ and $\triangle$ are of complementary dimension in $X \times X$, their intersection must be finite if they are transverse.

Proposition: Every map $f : X \to X$ is homotopic to a Lefschetz map.

Proof: See G.P. pp 120. This is an indirect consequence of the Transversality Theorem.

Q: What does a fixed point of a Lefschetz map look like?

A: The tangent space of $\Gamma(f)$ can be computed by parameterizing in the obvious way: $\phi(\mathbf{x}) = (\mathbf{x}, f(\mathbf{x}))$. Then the derivative is the diagonal matrix with I_k and df_x as the blocks. Thus the tangent space is $\Gamma(df_x)$. The tangent space of $\triangle$ is the diagonal of $T_x(X) \times T_x(X)$, denoted $\triangle_x$. Then transversality means:

$$\Gamma(df_x) + \triangle_x = T_x(X) \times T_x(X)$$

As these are vector spaces of complementary dimension, they fill out $T_x(X) \times T_x(X)$ if and only if $\Gamma(df_x) \cap \triangle_x = 0$. This means df_x has no fixed point. What is more, it has no eigenvalue $+1$.

▷ The above characterization can sometimes apply to fixed points of non-Lefschetz maps. That is, if a fixed point of any smooth map $X \to X$ has the property that df_x has no $+1$ eigenvalue, then we call it a Lefschetz fixed point. Intuitively, the Lefschetz fixed points are the isolated fixed points of a map.

▷ The Lefschetz points have an orientation, which is either ± 1 depending on the preimage orientation. This is called the local Lefschetz number, denoted $L_x(f)$. A more explicit characterization comes from considering the map $df_x - I$. At a Lefschetz point, this map must have no kernel. Thus it is an isomorphism of $T_x(X)$. Therefore we can consider the sign of the determinant:

Proposition: The local Lefschetz number $L_x(f)$ is the sign of determinant of $df_x - I$.

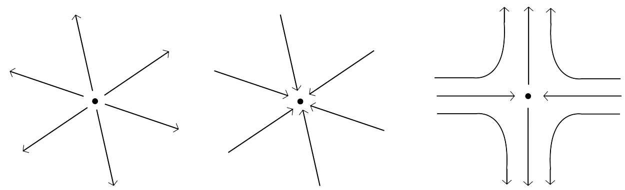
Proof: G.P. pp 121.

12.1 Two dimensional maps and the Euler Characteristic

$\triangleright$ It is instructive to look at the two dimensional case, where $f : \mathbb{R}^2 \to \mathbb{R}^2$. Assuming that f fixes the origin, and let $A = df_0$ so that $f(x) = Ax + \epsilon(x)$ in the usual. Also assuming that A has two independent eigenvectors, we get a diagonalization of A:

$$A = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$$

We can analyze the possible fixed point behavior by looking at different values of α_1, α_2 (when they are positive). These are summarized below.



These correspond to $\alpha_1, \alpha_2 > 1$, $\alpha_1, \alpha_2 < 1$ and $\alpha_1 < 1 < \alpha_2$, respectively. We can use this to characterize the euler characteristic of compact, oriented, boundaryless manifolds.

Proposition: The surface of genus k (torus with k holes) has a Lefshetz map homotopic to the identity, with one source, one sink and $2k$ saddles. Consequently, the Euler characteristic is $2 - 2k$.

The important image to keep in mind here is the "hot fudge" analogy. This map looks like the flow of a fluid poured over the k torus that is vertically oriented. The source fixed point is the top, the sink is the bottom, and the saddles occur near the tops/bottoms of the holes.

12.2 Non-Lefshetz fixed points

Consider a map that is not Lefshetz, but still has isolated fixed points. We can also assign a meaningful value to these points that will in some sense be the same as what we defined for Lefshetz points.

Definition: Let x be an isolated fixed poinf of $f : X \to X$, and let B be a ball around x so that B only contains the one fixed point. Then we have a map $\partial B \to S^{k-1}$ given by:

$$g(x) = \frac{f(x) - x}{|f(x) - x|}$$

The *local Lefshetz number* for x, denoted $L_x(f)$, is then $\deg(g)$.

Proposition: For a Lefshetz fixed point x, our two definitions of $L_x(f)$ agree.

Proof: They key idea is to write $f(x) = df_x(x) + \epsilon(x)$ and look at the homotopy $f_t(x) = df_x(x) + t\epsilon(x)$. Then define:

$$F_t(x) = \frac{f_t(x) - x}{|f_t(x) - x|}$$

We then have $L_x(f) = \deg(F_1) = \deg(F_0)$. Thus, we just need to verify that the degree of the map:

$$F_0 = \frac{df_x - I}{|df_x - I|}$$

is $+1$ whenever $df_x - I$ has positive determinant, and is -1 whenever $df_x - I$ has negative determinant. This follows as a fact of linear algebra (proved on Hw 6).

$\square$

Corollary: The Lefshetz number, as defined below, is well-defined and is a homotopy invariant for smooth maps $f : X \to X$ having finitely many fixed points.

$$L(f) = \sum_{f(x)=x} L_x(f)$$

13 Vector Fields and Poincaré-Hopf

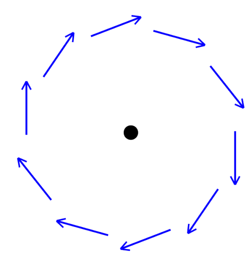
Definition: Let $X \subset \mathbb{R}^n$ be a compact boundaryless manifold. A *vector field* is a smooth map $v : X \to \mathbb{R}^n$ such that $v(x) \in T_x(X)$.

▷ Vector fields carry useful information near points where $v(x) = 0$. We assign a local invariant around an isolated zero of a vector field v called the index. It is defined by taking a ball B around a zero containing no other zeros finding the degree of the map $D : \partial B \to S^{k-1}$ defined by:

$$x \mapsto \frac{v(x)}{|v(x)|}$$

Example:

Consider a vector field $v : \mathbb{R}^2 \to \mathbb{R}^2$ with a zero that looks like the drawing below.



The degree of $D : S^1 \to S^1$ is the number of clockwise rotations the vectors make taken with sign. In this case, this is 1. So $\text{ind}(x) = 1$.

Theorem (Poincaré-Hopf): Let v be a vector field on a compact boundaryless manifold X with finitely many fixed points. Then:

$$\sum_{v(x)=0} \text{ind}(x) = \chi(X)$$

Proof: See G.P. pp 135.

Corolloary: S^k for k even does not admit a non-vanishing vector field.

Proof: If such a vector field existed, then $\sum_{v(x)=0} \text{ind}(x) = 0$ trivially. But we proved on Hw 10 that $\chi(S^k) = 2$ for k even. This is a contradiction by Poincaré-Hopf.

□

▷ Another relevant result named after Hopf is:

Theorem (Hopf degree): Two maps of a compact connected, oriented k manifold X into S^k are homotopic if and only if they have the same degree.

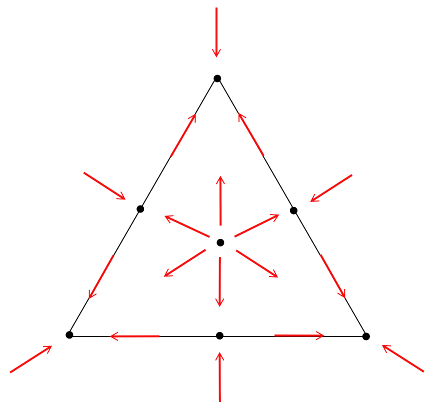
▷ One direction we know to be true as a fact of our definition of degree as an intersection number.

13.1 The Euler Characteristic and Triangulations

▷ We defined before the Euler characteristic of a manifold to the its self intersection number. There is an equivalent formulation using triangulations. Informally, break down a manifold into disjoint simplices (triangles), and look at the number of j dimensional faces. Then:

$$\chi(X) = \sum_j (-1)^j \text{ \# of } j\text{-dimensional faces}$$

We can show this is in fact true by constructing a vector field whose fixed points are in correspondence to the faces of the simplices. For two dimensions, we can visualize such a vector field explicitly (below)



We see that each vertex corresponds to a $+1$ index, each edge corresponds to a -1 index and each face corresponds to a $+1$ inex. Thus in two dimensions the Euler characteristic is $F - E + V$ by Poincaré-Hopf.

14 Integration Theory

▷ Integration on manifolds takes place in the language of differential forms. Namely, we integrate differential forms on manifolds. In order to understand and define integration of differential forms, we must first develop the language of tensors in linear algebra.

14.1 Differential forms

Definition: Let V be a vector space of finite dimension. A p *tensor* on V is a map $T : V \times V \times ... \times V \to \mathbb{R}$ that is multilinear.

▷ For example, a 2 tensor is the dot product. A k tensor can be the determinant of $[v_1, ..., v_k]$. We denote $T^p(V^*)$ to be the vector space of p tensors, where V^* is the dual of V. We can also define a product of two tensors. Namely, let $m \in T^p(V^*)$ and $n \in T^q(V^*)$. Then:

$$m \otimes n(v_1, ..., v_{p+q}) := m(v_1, ..., v_p) \cdot n(v_{p+1}, ..., v_{p+q})$$

Note that $m \otimes n \neq n \otimes m$.

▷ Given a basis of V^*, we can find write a baisis of $T^p(V^*)$, Namely, if $\{\phi_1, ..., \phi_k\}$ is a basis for V^*, then:

$$\left\{\phi_{i_1} \otimes ... \otimes \phi_{i_k} \;\middle|\; = 1 \leq i_1, ..., i_p \leq k\right\}$$

is a basis for $T^p(V^*)$. The justification for this can be found in G.P. pp 154.

$\triangleright$ It is instructive to restrict our scope to alternating tensors, or antisymmetric tensors. These are relevant in the study of differential forms, as we want to keep track of signed areas. We then denote $\Lambda^p(V^*)$ to be the subset of alternating tensors. It is very possible that this is not closed under $\oplus$; so we have to define another law of multiplication (called exterior product, or wedge product).

Definition: We construct a map $\mathrm{Alt} : T^p(V^*) \to \Lambda^p(V^*)$ that, in some sense, forces a p tensor to be alternating:

$$\mathrm{Alt}(T) := \frac{1}{p!}\sum_{\sigma\in S_p} \mathrm{sgn}(\sigma)T^\sigma$$

Where S_n is the symmetric group on n letters and T^σ is T applied to the vectors $\{v_{\sigma(1)}, ..., v_{\sigma(n)}\}$.

Definition: If T and S are alternating p and q tensors, then their wedge product:

$$T \wedge S := \mathrm{Alt}(T \otimes S)$$

is an alternating form satisfying $T \wedge S = (-1)^{pq} S \wedge T$. We note that for $p = q = 1$, we get the important relation $T \wedge T = -T \wedge T = 0$. Just as before, we can take a basis of $\Lambda^1(V^*) = V^*$ and write a basis of $\Lambda^p(V^*)$:

$$\{\phi_{i_1} \wedge ... \wedge \phi_{i_p} \,\big|\, 1 \le i_1 \le ... \le i_p \le k\}$$

From this it follows that $\dim(\Lambda^p(V^*)) = \binom{k}{p}$. In particular, if $k = p$, we have $\dim(\Lambda^p(V^*)) = 1$. This means every multilinear map in this space is a multiple of the determinant. Further, if $p > k$, then one of the indices in the basis elements must be repeated, and thus the tensor must be the zero map. This means that the dimension of $\Lambda^p(V^*)$ terminates at $p = k$; this allows us to form the exterior algebra:

$$\Lambda(V^*) = \Lambda^0(V^*) \oplus \Lambda^1(V^*) \oplus ... \oplus \Lambda^k(V^*)$$

with the convention $\Lambda^0(V^*) = \mathbb{R}$.

$\triangleright$ If $A : W \to V$ is a linear map, there is an induced map $A^* : \Lambda^p(W^*) \to \Lambda^p(V^*)$ given by:

$$A^*(T)(v_1, ..., v_p) = T(Av_1, ..., Av_p)$$

This is sometimes called the transpose of A (when A is a matrix, it is literally the transpose). The determinant theorem tells us more when $W = V$:

Theorem: If $A : V \to V$ is a linear map, then $A^*T = \det(A)T$ for all $T \in \Lambda^k(V)$ where $k = \dim(V)$.

$\triangleright$ All of this work now allows us to define a differential form.

Definition: Let X be a smooth manifold. A differential p form on X is a function $\omega : X \to \Lambda^p(T_x(X)^*)$. That is, ω is a choice of p form on the tangent spaces.

$\triangleright$ In the case of 0 forms, we have $\Lambda^o(T_x(X)) = \mathbb{R}$, and these are just functions $X \to \mathbb{R}$. An example of a 1 form might be $x \mapsto d\phi_x$, where $\phi : X \to \mathbb{R}$ is a smooth 0 form. This is called the differential of ϕ. In this way, the coordinate functions $x_1, ..., x_k$ on $\mathbb{R}^k$ yield 1 forms $dx_1, ..., dx_k$ that project a tangent vector into its individual components.

Proposition: Every p form on an open set $U \subset \mathbb{R}^k$ can be uniquely expressed as a sum $\sum_I f_I dx_I$ over increasing index sequences $I = (i_1 < ... < i_p)$ (here f_I are real valued functions on U), where $dx_I = dx_{i_1} \wedge ... \wedge dx_{i_p}$.

Definition: If $f : X \to Y$ is a smooth map, and ω is a differential form on Y, we define the pullback of ω, denoted $f^*\omega$, to be the form on X given by:

$$f^*\omega(x) = (df_x)^*\omega(f(x))$$

Recall we defined $(df_x)^*$ as the induced map from $\Lambda^p(T_xX) \to \Lambda^p(T_{f(x)}Y)$. The pullback has several properties:

$$f^*(\omega_1 + \omega_2) = f^*\omega_1 + f^*\omega_2$$
$$f^*(\omega \wedge \theta) = (f^*\omega) \wedge (f^*\theta)$$
$$(f \circ h)^*\omega = h^* f^*\omega$$

Definition: Let ω be a p form on $\mathbb{R}^k$; writing it as $\sum_I a_I dx_I$, we say ω is smooth if each f_I is smooth.

Example: What do 2 forms on $\mathbb{R}^3$ look like?

They will be sums of wedge products of the elementary 1 forms:

$$\omega = a(x)dx_1 \wedge dx_2 + b(x)dx_1 \wedge dx_3 + c(x)dx_2 \wedge dx_3$$

Keep in mind these have to be increasing in index.

14.2 Integrating Differential Forms

Integrating k forms on open sets

▷ Let a be a real valued function on an open set $U \subset \mathbb{R}^k$. We can look at the k form $\omega = adx_1 \wedge ...dx_k$, which is in a sense a volume form. Then we define the integral of ω to be:

$$\int_U \omega = \int_U a \ dx_1 \dots dx_k$$

in the usual multivariable sense. With this comes the change of varialbes formula, which makes use of pullback:

Change of Variables: Let $f : V \to U$ is an orientation preserving diffeomorphism of open sets in $\mathbb{R}^k$ (or $\mathbb{H}^k$), and let ω be an integrable k form on U. Then:

$$\int_U \omega = \int_V f^*\omega = \int_V \det(df_x)\ \omega \circ f$$

If f is orientation reversing, then:

$$\int_U \omega = -\int_V f^*\omega$$

.

Integrating k forms on manifolds

▷ Let ω be a k form on a manifold M with compact support (i.e. the closure of the set of points where $\omega(x) \neq 0$ is compact).

Warm-up case: Suppose the support of ω is contained entirely inside a parameterizable open set $W \in M$. Let $\phi : U \to W$ be a diffeomorphism that parameterizes W. Then $\phi^*\omega$ is a differential form on U, an open set in $\mathbb{R}^k$. We know how to integrate forms on open sets in $\mathbb{R}^k$, so we define:

$$\int_M \omega := \int_U \phi^*\omega$$

Is this well defined? Let $\psi : V \to W$ be another parameterization of W. Then $f = \phi^{-1} \circ \psi$ is an orientation preserving diffeomorphism. Then by the change of variables formula, we have:

$$\int_U \phi^*\omega = \int_V f^*\phi^*\omega = \int_V (\phi \circ f)^*\phi^*\omega = \int_V \psi^*\omega \ \checkmark$$

General case: Now we wish to integrate an arbitrary k form whose support may not be entirely contained in a parameterizable open set. To deal with this, we take a partition of unity $\{\rho_i\}$ of M subject to the set of all parameterizable open sets in M. Then local finiteness implies all but finitely many ρ_i are zero on the support of ω. Then finitely many of $\{\rho_i\omega\}$ are nonzero, so we can define the integral of ω:

$$\boxed{\int_M \omega := \sum_i \int_M \rho_i\omega}$$

The right hand side makes sense because it is a finite sum of integrals of functions with support inside a parameterizable open set, which we have already defined. Is this well defined? Let $\{\rho_j'\}$ be another partition of unity; then:

$$\int_M \rho_i\omega = \sum_j \int_M \rho_j'\rho_i\omega$$

$$\int_M \rho_j'\omega = \sum_i \int_M \rho_i\rho_j'\omega$$

So we have:

$$\sum_i \int_M \rho_i\omega = \sum_i\sum_j \int_M \rho_j'\rho_i\omega = \sum_j \int_M \rho_j'\omega \quad \checkmark$$

where we exchanged the summations. Thus this doesn't depend on the partition.

Integrating $\ell < k$ forms on submanifolds

▷ Perhaps we might want to integrate an ℓ form on M, where $\ell < \dim(M)$. We can't integrate over M, because the dimensions don't add up, but we can integrate over an ℓ dimensional submanifold. There is a natrual definition of this via the inclusion map $i : Z \to M$:

$$\int_Z \omega = \int_Z i^*\omega$$

14.3 Familiar examples

▷ Now we look at a few exampeles for $M = \mathbb{R}^3$. If ω is a 1 form, then it looks like:

$$\omega = f_1dx_1 + f_2dx_2 + f_3dx_3$$

Consider integrating ω on a curve $\gamma : [0,1] \to \mathbb{R}^3$, written as $\langle\gamma_1,\gamma_2,\gamma_3\rangle$. Then we compute $\gamma^*\omega$ by computing on the components:

$$\gamma^*dx_i = d\gamma_i = \frac{d\gamma_i}{dt}dt$$

$$\gamma^*f = f\circ\gamma$$

So:

$$\int_C \omega = \sum_{i=1}^3 \int_0^1 f\circ\gamma(t)\frac{d\gamma_i}{dt}dt$$

We recognize this to be the line integral of the vector field $\langle f_1, f_2, f_3\rangle$. If ω is a 2 form, we similarly get the standard flux integral of the same vector field.

14.4 The Exterior Derivative

▷ We will define a map $d : \{k\text{-forms}\} \to \{k+1\text{-forms}\}$, called the exterior derivative. We have already encountered this for $k = 0$. Namely, if f is a 0 form (a function), then the differential df is a 1 form. To generalize this, we start with the Euclidean case.

Euclidean case: Say ω is a p form on $U \subset \mathbb{R}^k$. Then we can write $\omega = \sum a_I dx_I$, where I is defined as in section 14.1. Then, we define $d\omega$ to be exactly what we might expect:

$$d\omega = \sum da_I \wedge dx_I$$

This formulation satisfies three important properties:

1. Linearity: $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$.
2. Product rule: $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$, where α is a p form.
3. $d(d\omega) = 0$.

This last property is a consequence of the equality of mixed partials (see G.P. pp 175). Alternatively, we can define d as the unique operator that satisfies these three properties. Another property of d is that, if $g : V \to U$ is a diffeomorphism, then $d(g^*\omega) = g^*(d\omega)$. Namely, it commutes with pullbacks.

General case: The exterior derivative d for forms on an arbitrary manifold is defined locally in a natural way. That is, if $\phi : U \to V$ is a local parameterization, define $d\omega$ on $\phi(U)$ to be $(\phi^{-1})^* d(\phi^*\omega)$. This formulation of d has the following properties:

1. $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$
2. $d(\omega \wedge \theta) = (d\omega) \wedge \theta + (-1)^p \omega \wedge d\theta$, where ω is a p form
3. $d(d\omega) = 0$
4. If $g : X \to Y$ is a smooth map of manifolds with boundary, then $d(g^*\omega) = g^*(d\omega)$.

Example:

Consider differentiating $0, 1, 2$ and 3 forms on $\mathbb{R}^3$:

- If $\omega = f$ is a 0 form, it is a function and $df = g_1 dx_1 + g_2 dx_2 + g_3 dx_3$, where $g_1 = \frac{\partial f}{\partial x_i}$. We often call $\langle g_1, g_2, g_3 \rangle = \nabla f$ the gradient of f.
- If $\omega = f_1 dx_1 + f_2 dx_2 + f_3 dx_3$ is a 1 form, then $d\omega = g_1 dx_2 \wedge dx_3 + g_2 dx_3 \wedge dx_1 + g_3 dx_1 \wedge dx_2$, where:

$$g_1 = \frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{x_3},\ g_2 = \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1},\ g_3 = \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}$$

  We often call $\langle g_1, g_2, g_3 \rangle = \nabla \times F$ the curl of $F = \langle f_1, f_2, f_3 \rangle$.
- If $\omega = f_1 dx_2 \wedge d_3 + f_2 dx_3 \wedge dx_1 + f_3 dx_1 \wedge dx_2$ is a 2 form, then $d\omega = F dx_1 \wedge dx_2 \wedge x_3$, where:

$$F = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}$$

  This is sometimes called the divergence.
- If ω is any 3 form, then $d\omega = 0$ because at least one of the indices must repeat in each term of the sum.

14.5 Stokes's Theorem

$\triangleright$ One of the most fundamental results of integration theory is Stokes's Theorem, which relates differentiation and integration:

Theorem (Stokes): If X is a k dimensional manifold with boundary, and ω is a smooth $k-1$ form on X (extending to the boundary), then:

$$\int_X d\omega = \int_{\partial X} \omega$$

This has various names for small k. If $k = 1$, this is is the Fundamental Theorem of Calculus. For $k = 2$, this is the Classical Stokes's Theorem (or Green's Theorem). For $k = 3$, this is the divergence theorem.

Proof: See G.P. pp 183-185.