# THE GROTHENDIECK SPECTRAL SEQUENCE IN ALGEBRAIC GEOMETRY

GEORGE TORRES

Abstract. The purpose of this paper is to prove the existence of the Grothendieck spectral sequence (GSS), which is the chain rule for derived functors. On the way, we will develop some necessary tools in the homological algebra of bicomplexes in abelian categories. At the end, we will give some applications of GSS to sheaf cohomology in Algebraic Geometry. The primary reference for this paper is Daniel Murfet's notes on Spectral Sequences [1] and various nLab articles.

## 1. Spectral Sequences in Abelian Categories

In this section we will begin with providing a brief overview of spectral sequences in abelian categories. Some familiarity with these ideas will be assumed, so some details will be omitted for brevity. Our goal in this section will be to introduce and explain the canonical spectral sequences associated to filtered cochain complexes and bicomplexes, which will be important for deriving the Grothendieck spectral sequence.

**Definition 1.1.** Let $\mathcal{A}$ be an abelian category. A *spectral sequence* is a collection of objects $E_r^{pq} \in \mathcal{A}$ (called *pages*) for $p, q \in \mathbb{Z}$ and $r \in \mathbb{Z}^+$ with morphisms $d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}$ satisfying $d^2 = 0$. If we set $Z_{r+1}(E_r^{pq}) = \ker(d_r^{pq})$ and $B_{r+1}(E_r^{pq}) = \mathrm{im}(d_r^{p-r,q+r-1})$, then we also require that $Z_{r+1}(E_r^{pq})/B_{r+1}(E_r^{pq}) \cong E_{r+1}^{pq}$. That is, the $r+1$ page is the cohomology of the $r$ page at each point $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ with respect to $d_r^{pq}$.

We care most about spectral sequences that converge to something useful. That is, we would like $E_r^{pq}$ to eventually stabilize for $r$ sufficiently large. To be more precise, suppose that for all $r \geq R$, we have isomorphisms:

$$E_r^{pq} \cong E_R^{pq} \quad \forall p, q \in \mathbb{Z}$$

Then we call $E_\infty := E_\infty^{pq} := E_R^{pq}$ the *limiting term* of the spectral sequence. Often, the limiting term takes the form of an associated graded complex. We call such spectral sequences weakly convergent:

**Definition 1.2.** A spectral sequence $E_r^{pq}$ is *weakly convergent* if it has a limiting term $E_\infty$ that is the associated graded complex of a graded object. That is, there is a graded object $H^* \in \mathcal{A}$ with a filtration $F^p H^*$ such that:

$$E_\infty^{pq} \cong F^p H^{p+q} / F^{p+1} H^{p+q}$$

This sequence is said to converge to $H^*$, and is denoted $E_r^{pq} \Rightarrow H^*$.

If the spectral sequence is in the first quadrant, meaning $E_r^{pq} = 0$ for $\neg(p, q \geq 0)$, then they automatically have a limiting term because eventually the maps $d_r$ extend into the zero regions and hence become zero.

---

*Date*: May 12, 2017.

1.1. **The Filtered Chain Complex.** One of the first examples of the usefulness of spectral sequences comes from their ability to compute the cohomology of a filtered cochain complex.

**Definition 1.3.** Let $\mathcal{A}$ be an abelian category and let $(C^\bullet, d)$ be a cochain complex in $\mathcal{A}$. A *filtration* on $C^\bullet$ is a collection of subcomplexes $F^pC^\bullet$ for $p \in \mathbb{Z}$ such that:

$$F^{p+1}C^q \subseteq F^pC^q \quad \forall p, q \in \mathbb{Z}$$
$$d(F^pC^q) \subseteq F^pC^{q+1}$$

We say this filtration is *separated* if $\bigcap_p F^pC^\bullet = 0$ and *exhaustive* if $\bigcup_p F^pC^\bullet = C^\bullet$.

If $F^pC^\bullet$ is a filtered cochain complex (which we will assume to be in the first quadrant, though this is only to ensure convergence properties), we can approximate the cocycles and coboundaries of $C^\bullet$ in the following way:

$$\tilde{Z}_r^{pq} = \left\{x \in F^pC^{p+q}/F^{p+1}C^{p+1} \mid d(x) \in F^{p+r}C^{p+q+1}\right\}$$
$$\tilde{B}_r^{pq} = d\left(F^{p-r+1}C^{p+q-1}\right)$$

The first object is the module of $(pq)$ cochains whose differentials vanish up to order $p+r$ in the grading) and the second object is the set of $(pq)$ cochains that are in the image up to order $p-r+1$. Now we can define the $r$-"almost cohomology" as:

$$E_r^{pq} = \tilde{Z}_r^{pq}/\tilde{B}_r^{pq}$$

The cochain differentials $d$ restrict to morphisms $d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}$ on each of these objects.

**Proposition 1.4.** *The objects $E_r^{pq}$ and maps $d_r^{pq}$ constructed above form a spectral sequence starting at page 0 such that:*

$$E_0^{pq} = F^pC^{p+q}/F^{p+1}C^{p+q}$$
$$E_r^{pq} \Rightarrow H^{p+q}(C^\bullet)$$

*Proof.* See [1]. □

Given how the approximate cocycles and coboundaries were constructed, it shouldn't be surprising that this sequence converges to the cohomology of $C^\bullet$. As such, the proof of this Proposition is fairly straightforward.

1.2. **Bicomplexes.** A particularly common object in computing spectral sequences is the bicomplex, which is a lattice of objects in $\mathcal{A}$. Formally:

**Definition 1.5.** In an abelian category $\mathcal{A}$, a bicomplex $(C^{\bullet,\bullet}, d_1, d_2)$ is a collection of objects $C^{ij}$ for every $i, j \in \mathbb{Z}$ with maps $d_1^{ij} : C^{ij} \to C^{i+1,j}$ and $d_2 : C^{ij} \to C^{i,j+1}$ satisfying $d_1^2 = d_2^2 = 0$ and $d_1d_2 = d_2d_1$.

*Remark* 1.6. We often will denote such a bicomplex as $C$, with all other data implicit. Bicomplexes can alternatively be defined as objects in $\textsc{Ch}^2(\mathcal{A})$, the category of cochain complexes of cochain complexes.

**Definition 1.7.** For a bicomplex $C$, the *totalization* of $C$ is the complex $\mathrm{tot}(C)^\bullet$ defined by:

$$\mathrm{tot}(C)^n = \bigoplus_{i+j=n} C^{ij}$$

$$d^n\big|_{C^{ij}} = d_1^{ij} + (-1)^i d_2^{ij} \qquad (n = i + j)$$

where $d^n\big|_{C^{ij}}$ is the restriction of $d^n$ to the $C^{ij}$ component of its domain. In this definition of the differential we are implicitly composing the maps $d_1^{ij}$ and $d_2^{ij}$ with the inclusions $C^{i,j+1}, C^{i+1,j} \hookrightarrow \mathrm{tot}(C)^{i+j+1}$ so that $d^n : \mathrm{tot}(C)^n \to \mathrm{tot}(C)^{n+1}$.

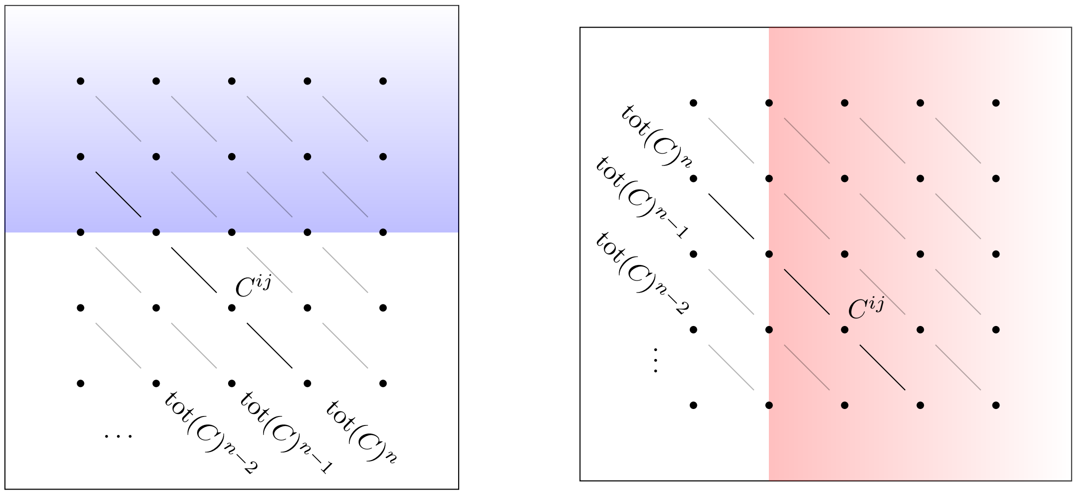


FIGURE 1. The vertical and vertical filtrations $F^p_{(2)}\mathrm{tot}(C)^n$ and $F^p_{(1)}\mathrm{tot}(C)^n$ .

There are two canonical filtrations on the totalization complex $\mathrm{tot}(C)^\bullet$:

$$F^p_{(1)}\mathrm{tot}(C)^n = \bigoplus_{r\geq p} C^{r,n-r}$$

$$F^p_{(2)}\mathrm{tot}(C)^n = \bigoplus_{r\geq p} C^{n-r,r}$$

These correspond to "filling in" from the right and from the top (see Figure 1). Since there are two filtrations on $\mathrm{tot}(C)^\bullet$, there are two associated spectral sequences ${}^{(1)}E^{pq}_r$ and ${}^{(2)}E^{pq}_r$ that converge to the cohomology of $\mathrm{tot}(C)^\bullet$. To state this convergence, it will be convenient to define the cohomology complexes:

$$H^{\bullet,j}_{(1)} = H^j(C^{\bullet,*}) = \ker(d_2^{ij})/\mathrm{im}(d_2^{i,j-1})$$

$$H^{i,\bullet}_{(2)} = H^i(C^{*,\bullet}) = \ker(d_1^{ij})/\mathrm{im}(d_1^{i-1,j})$$

The first is the complex that arises from taking the $j$th vertical cohomology on the columns, and the second is the complex that arises from taking the $i$th horizontal cohomology on the rows.

**Proposition 1.8.** *For a bicomplex $C$ with associated spectral sequences ${}^{(1)}E$ and ${}^{(2)}E$ as above, we have:*

*a)* ${}^{(1)}E^{pq}_0 \cong C^{pq}$ *and* ${}^{(2)}E^{pq}_0 \cong C^{qp}$.
*b)* ${}^{(1)}E^{pq}_1 \cong H^{pq}_{(1)}(C)$ *and* ${}^{(2)}E^{pq}_1 \cong H^{qp}_{(2)}(C)$.
*c)* ${}^{(1)}E^{pq}_2 \cong H^p(H^{\bullet,q}_{(1)})$ *and* ${}^{(2)}E^{pq}_2 \cong H^p(H^{q,\bullet}_{(2)})$.

*Additionally, $C$ is a first quadrant bicomplex, then:*

$${}^{(1)}E^{pq}_r, {}^{(2)}E^{pq}_r \Rightarrow H^{p+q}(tot(C))$$

*Proof.* It suffices to prove these statements for ${}^{(1)}E$, by planar symmetry. It is immediate from Proposition 1.4 that the zero page ${}^{(1)}E^{pq}_0$ is the associated graded of $\mathrm{tot}(C)^\bullet$:

$${}^{(1)}E^{pq}_0 = F^p_{(1)}\mathrm{tot}(C)^{p+q}/F^{p+1}_{(2)}\mathrm{tot}(C)^{p+q} = \bigoplus_{r\geq p} C^{r,p+q-r}/\bigoplus_{r\geq p+1} C^{r,p+q-r} = C^{pq}$$

Recall also that ${}^{(1)}E_r^{pq} = \tilde{Z}_r^{pq}/\tilde{B}_r^{pq}$ by definition. In particular, if $r = 1$, then the approximate cocycles $\tilde{Z}_r$ and approximate coboundaries $\tilde{B}_r$ are:

$$\tilde{Z}_1^{pr} = \left\{x \in C^{pq} \mid d(x) \in F_{(1)}^{p+1}\mathrm{tot}(Cr^{p+q+1}\right\} = \{x \in C^{pq} \mid d_2^{pq}(x) = 0\}$$
$$\tilde{B}_1^{pr} = \mathrm{im}(d_2^{p,q-1})$$

These are exactly the cocycles and coboundaries of the associated graded of the filtration $F_{(1)}$ (i.e. $C^{pq}$). Therefore ${}^{(1)}E_1^{pq} = H_{(1)}^{pq}(C)$. Part *c*) follows from the fact that ${}^{(1)}E_2$ is the cohomology of ${}^{(1)}E_1$ with respect to the differential $d = d_1 \pm d_2$; but since $d_2$ acts trivially on ${}^{(1)}E_1$ as we showed above, it is just the cohomology with respect to $d_1$, which is exactly $H^p(H_{(1)}^{\bullet,q})$.

Finally, Proposition 1.4 tells us that any filtration on the cochain complex $\mathrm{tot}(C)^\bullet$ converges to $H^{p+q}(\mathrm{tot}(C))$ provided it is a first quadrant bicomplex. □

## 2. Cartan-Eilenberg Resolutions

The second ingredient for deriving the Grothendieck spectral sequence is proving the existence of a particular class of injective resolutions in the cochain category $\textsc{Ch}(\mathcal{A})$, called Cartan-Eilenberg Resolutions.

**Definition 2.1.** An injective resolution of a cochain complex $C^\bullet$ is a upper half-plane bicomplex $\{I^{ij}\}_{i,j\geq 0}$ of injective objects with maps $C^i \to I^{i,0}$ such that we have a commutative diagram:

$$\begin{array}{ccccccccc}
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & I^{p-1,1} & \longrightarrow & I^{p,1} & \longrightarrow & I^{p+1,1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & I^{p-1,0} & \longrightarrow & I^{p,0} & \longrightarrow & I^{p+1,0} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & C^{p-1} & \longrightarrow & C^{p} & \longrightarrow & C^{p+1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We also require that the columns be exact.

*Remark* 2.2. If $d_1$ and $d_2$ are the horizontal and vertical differentials of the bicomplex $I$ and $d$ is the differential of $C^\bullet$, we denote:

$$Z^p(C^\bullet) = \ker(d^p), \quad B^p(C^\bullet) = \mathrm{im}(d^{p+1})$$
$$Z^p(I^{\bullet,q}) = \ker(d_1^{pq}), \quad B^p(I^{\bullet,q}) = \mathrm{im}(d_1^{p+1,q})$$

By restricting the vertical maps to these cocycles and coboundaries, any injective resolution $I \to C^\bullet$ yields two sequences:

$$\begin{array}{l}
0 \longrightarrow Z^p(C^\bullet) \longrightarrow Z^p(I^{\bullet,0}) \longrightarrow Z^p(I^{\bullet,1}) \longrightarrow \cdots \\
0 \longrightarrow B^p(C^\bullet) \longrightarrow B^p(I^{\bullet,0}) \longrightarrow B^p(I^{\bullet,1}) \longrightarrow \cdots
\end{array} \tag{2.3}$$

These are clearly complexes, since the restricted maps still satisfy $d_2^2 = 0$. It is easy to check that these are also in fact exact sequences. As a result, these induce an exact sequence in cohomology groups:

$$0 \longrightarrow H^p(C^\bullet) \longrightarrow H^p(I^{\bullet,0}) \longrightarrow H^p(I^{\bullet,1}) \longrightarrow \cdots \tag{2.4}$$

**Definition 2.5.** We say a resolution $I \to C^\bullet$ is a *Cartan-Eilenberg resolution* (also called *fully injective resolution*) if the sequences in (2.3) and (2.4) are injective resolutions.

**Proposition 2.6.** *Let $\mathcal{A}$ be an abelian category with enough injectives. Then every cochain complex $C^\bullet \in \mathrm{Ch}(\mathcal{A})$ admits a Cartain-Eilenberg resolution.*

*Proof.* We will construct such a resolution explicitly. We begin by taking injective resolutions $I^{n,\bullet} \to B^n(C^\bullet)$ and $J^{n,\bullet} \to H^n(C^\bullet)$ for every $n$. Now consider the natural exact sequence:

$$0 \to B^n(C^\bullet) \to Z^n(C^\bullet) \to H^n(C^\bullet) \to 0$$

By the horseshoe lemma, we can use this exact sequence to construct an injective resolution $K^{n,\bullet} \to Z^n(C^\bullet)$ such that we have the following diagram:

$$\begin{array}{ccccccccc}
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I^{n,1} & \dashrightarrow & K^{n,1} & \dashrightarrow & J^{n,1} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I^{n,0} & \dashrightarrow & K^{n,0} & \dashrightarrow & J^{n,0} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & B^n(C^\bullet) & \longrightarrow & Z^n(C^\bullet) & \longrightarrow & H^n(C^\bullet) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Now consider the exact sequence:

$$0 \to Z^n(C^\bullet) \to C^n \to B^{n+1}(C^\bullet) \to 0$$

Again by the horseshoe lemma, the resolutions $K^{n,\bullet}$ and $I^{n+1,\bullet}$ on either side induce an injective resolution $L^{n,\bullet} \to C^n$ in the middle that fits into the exact sequence above for all $n$.

Now we string these all together: by the universal property of injective objects, we can construct lateral morphisms between the $L$ resolutions to obtain an injective resolution of the complex $C^\bullet$:

$$
\begin{array}{ccccccccc}
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \dashrightarrow & L^{p-1,1} & \dashrightarrow & L^{p,1} & \dashrightarrow & L^{p+1,1} & \dashrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \dashrightarrow & L^{p-1,0} & \dashrightarrow & L^{p,0} & \dashrightarrow & L^{p+1,0} & \dashrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & C^{p-1} & \longrightarrow & C^{p} & \longrightarrow & C^{p+1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Indeed, this is actually a fully injective resolution by construction, since all the above resolutions of cocycles, coboundaries, and cohomologies were injective by construction. □

*Remark* 2.7. Cartan-Eilenberg resolutions are in some sense the "most correct" type of injective resolution in $\text{Ch}(\mathcal{A})$. Not only are they important in the construction of the Grothendieck spectral sequence, as we will see, but they are also the type of resolution that one uses to compute hyper-derived functors (derived functors on the cochain category $\text{Ch}(\mathcal{A})$).

## 3. The Grothendieck Spectral Sequence

Using the tools developed above, we can now construct the Grothendieck spectral sequence (GSS). The idea of GSS is, given two functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ with right derivations $R^nF$ and $R^nG$, under certain assumptions we can construct a spectral sequence that converges to $R^n(G \circ F)$, the right derivation of their composition.

**Definition 3.1.** Let $\mathcal{A}$ be an abelian category with enough injectives, and let $T : \mathcal{A} \to \mathcal{B}$ be an additive functor. A object $Q \in \mathcal{A}$ is called *right $T$-acyclic* if $R^iT(Q) = 0$ for all $i > 0$.

*Remark* 3.2. Since any injective object admits a trivial injective resolution, it follows that any injective object is right $T$-acyclic.

**Theorem 3.3.** *(GSS) Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be abelian categories and assume that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives and that $\mathcal{C}$ is co-complete. If $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ are additive functors such that $F$ sends injective objects to right $G$-acyclic objects, then there is a canonical spectral sequence $E_r^{pq}$ such that, for any $A \in \mathcal{A}$:*

$$E_2^{pq} = R^pG(R^qF(A)) \Rightarrow R^{p+q}(G \circ F)(A)$$

*Proof.* Let $C^\bullet \to A$ be an injective resolution and consider the complex $FC^\bullet$. By the previous proposition, we can take a fully injective resolution $I \to FC^\bullet$. Since $C^i = 0$ for $i < 0$, the resulting bicomplex $I$ is in the first quadrant, and so the resolution $I \to FC^\bullet$ takes the form:

$$\begin{array}{ccccccccc}
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I^{0,1} & \longrightarrow & I^{1,1} & \longrightarrow & I^{2,1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I^{0,0} & \longrightarrow & I^{1,0} & \longrightarrow & I^{2,0} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & FC^0 & \longrightarrow & FC^1 & \longrightarrow & FC^2 & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We would like to understand the behavior of $G \circ F$ on injective resolutions in order to compute $R^n(G \circ F)(A)$. To do this, we apply $G$ to the diagram above to get a first quadrant bicomplex $GI$. In particular, from our discussion of bicomplexes, we find that there are two canonical filtrations on the totalization $\mathrm{tot}(GI)^\bullet$:

$$F^p_{(1)}(\mathrm{tot}(GI))^n = \bigoplus_{r \geq p} GI^{r,n-r}$$

$$F^p_{(2)}(\mathrm{tot}(GI))^n = \bigoplus_{r \geq p} GI^{n-r,r}$$

Following Proposition 1.8, we have two spectral sequences ${}^{(1)}E$ and ${}^{(2)}E$. The first of these has second page:

$${}^{(1)}E_2^{pq} = H^p(H^{\bullet,q}_{(1)}(GI)) \cong H^p(R^qG(FC^\bullet))$$

where we used the fact that $I^{\bullet,q}$ is an injective resolution. Recall that, since $C^i$ is injective for all $i$, the objects $FC^i$ are right G-acyclic, and so the above quantity vanishes for $q > 0$. Similarly, for $q = 0$, the right derivation is trivial: $R^0G \cong G$. Then we have:

$${}^{(1)}E_2^{pq} = \begin{cases} 0 & q > 0 \\ R^p(G \circ F)(A) & q = 0 \end{cases}$$

Since all but one row of ${}^{(2)}E_2$ is zero, all differentials $d_2^{pq}$ must be zero, and therefore the sequence stabilizes at this page. In particular, we have:

$$H^n(\mathrm{tot}(GI)^\bullet) \cong {}^{(1)}E_\infty^{n0} \cong R^n(G \circ F)(A)$$

This was, in a sense, the "trivial" spectral sequence that immediately told us what both sequences should limit to.

Now we claim that ${}^{(2)}E_r^{pq}$ is the desired Grothendieck spectral sequence. Since we just showed that it abuts to $R^{p+q}(G \circ F)(A)$, we need only verify that the second page is $R^pG(R^q(F(A))$. Consider the exact sequence:

$$0 \to Z^p(I^{\bullet,q}) \to I^{pq} \to B^{p+1}(I^{\bullet,q}) \to 0$$

Since we started with a fully injective resolution, the object $Z^p(I^{\bullet,q})$ is injective and therefore, by the Splitting Lemma, the above exact sequence splits. Since $G$ is additive, we also get a split exact sequence by applying $G$:

$$0 \to G(Z^p(I^{\bullet,q})) \to G(I^{pq}) \to G(B^{p+1}(I^{\bullet,q})) \to 0$$

Exactness tells us that $G(Z^p(I^{\bullet,q}))$ is canonically isomorphic to $\ker(G(I^{pq}) \to G(B^{p+1}(I^{\bullet,q}))) = Z^p(GI^{\bullet,q})$ for all $p$ and $q$. Similarly for the image, we have $B^{p+1}(GI^{\bullet,q}) \cong G(B^{p+1}(I^{\bullet,q}))$ for all $p, q$.

Now consider the image of $G$ under the cohomology short exact sequence:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B^p(I^{\bullet,q}) & \longrightarrow & Z^p(I^{\bullet,q}) & \longrightarrow & H^p_{(2)}(I^{\bullet,q}) & \longrightarrow & 0 \\
 & & & & \big\downarrow{\scriptstyle G(-)} & & & & \\
0 & \longrightarrow & G(B^p(I^{\bullet,q})) & \longrightarrow & G(Z^p(I^{\bullet,q})) & \longrightarrow & G(H^p_{(2)}(I^{\bullet,q})) & \longrightarrow & 0 \\
 & & \big\downarrow\cong & & \big\downarrow\cong & & & & \\
0 & \longrightarrow & B^p(GI^{\bullet,q}) & \longrightarrow & Z^p(GI^{\bullet,q}) & \longrightarrow & H^{pq}_{(2)}(GI) & \longrightarrow & 0
\end{array}$$

Therefore we have a canonical isomorphism $G(H^p_{(2)}(I^{\bullet,q})) \cong H^{pq}_{(2)}(GI)$ for all $p$ and $q$. We have thus demonstrated an isomorphism of complexes:

$$H^{p,\bullet}_{(2)}(GI) \cong G(H^{p,\bullet}_{(2)}(I))$$

The second page of the ${}^{(2)}E$ sequence is then, by Proposition 1.8:

$${}^{(2)}E_2^{pq} \cong H^p(H^{q,\bullet}_{(2)}(GI)) \cong H^p(G(H^{q,\bullet}_{(2)}(I))) \tag{3.4}$$

Recall that our assumption that $I$ was fully injective gives us an injective resolution:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^q_{(2)}(FC) & \longrightarrow & H^{q,0}_{(2)}(I) & \longrightarrow & H^{q,1}_{(2)}(I) & \longrightarrow & \cdots \\
 & & \| & & & & & & \\
 & & R^qF(A) & & & & & &
\end{array}$$

Therefore applying $G$ to the above resolution and computing the $p$th cohomology gives us $R^pG(R^qF(A))$; but this is exactly equal to the right hand side of (3.4). Therefore ${}^{(2)}E_2^{pq} \cong R^pG(R^qF(A))$ as desired. □

## 4. Applications of GSS

The GSS is a generalization of many well-known spectral sequences in geometry and topology. In this section, we will give some examples of these and will demonstrate how the GSS can be a useful tool for computing sheaf cohomology.

### 4.1. **The Leray Spectral Sequence.**

Let $X$ and $Y$ be topological spaces, and let $f : X \to Y$ be a continuous map. Recall that, for any sheaf $\mathcal{F} \in \mathrm{Sh}(X)$, we have a direct image sheaf $f_*\mathcal{F} \in \mathrm{Sh}(Y)$ defined by $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$. The functor $f_*(-) : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ sends injective sheaves to flabby sheaves (which are acyclic). In the context of the above theorem, if we let $F = f_*(-)$ and $G = \Gamma(Y, -)$, then $F$ and $G$ satisfy the necessary conditions so that we have a spectral sequence $E_r^{pq}$ with:

$$E_2^{pq} = H^p(Y, R^qf_*(\mathcal{F})) \Rightarrow R^{p+q}(\Gamma(Y, f_*(-)))(\mathcal{F})$$

for any sheaf $\mathcal{F}$. On the left hand side, we see what are known as the *higher direct images* $R^qf_*(\mathcal{F})$. We can simplify the right hand side by noting that the sheaves $\Gamma(Y, f_*(-))$ and $\Gamma(X, -)$ are canonically isomorphic because $f_*\mathcal{F}(Y) = \mathcal{F}(X)$. Therefore:

$$E_2^{pq} = H^p(Y, R^qf_*(\mathcal{F})) \Rightarrow H^{p+q}(X, \mathcal{F})$$

This is known as the Leray spectral sequence. It shows how we can approximate the sheaf cohomology on $X$ by looking at the sheaf cohomology on $Y$ with respect to the higher direct images.

### 4.2. **The Čech-to-derived functor Spectral Sequence.**

Čech cohomology is an important tool in sheaf cohomology, since it is one of the main practical ways to actualy compute it. The relationship that Čech cohomology has with sheaf cohomology can be captured by a Grothendieck spectral sequence.

Let $X$ be a topological space and fix a sheaf $\mathcal{F} \in \textsc{Sh}(X)$. Given an open cover $\mathcal{U} = \{U_i\}$ of $X$, recall that the Čech cohomology $\check{H}^p(\mathcal{U}, \mathcal{P})$ is defined for any presheaf $\mathcal{P}$. We can then consider the functor $F : \textsc{Sh}(X) \to \textsc{PSh}(X)$ given by inclusion. Further, if we take $G : \textsc{PSh}(X) \to \textsc{AbGrp}$ given by $\mathcal{P} \mapsto \check{H}^0(\mathcal{U}, \mathcal{P})$, the composition $G \circ F$ is the regular global sections functor $\Gamma(X, -)$, whose right derivation is ordinary sheaf cohomology. We can then apply the GSS to get:

$$E_2^{pq} = \check{H}^p(\mathcal{U}, H^q(\,\cdot\,, \mathcal{F})) \Rightarrow H^{p+q}(X, \mathcal{F})$$

Where $\mathcal{H}^q(\,\cdot\,, \mathcal{F})$ is the sheaf sending $U \subset X \mapsto H^q(U, \mathcal{F})$. Here we have used the fact that Čech cohomology can be presented as the right derived functor of $\check{H}^0(\mathcal{U}, -)$.

An important case for Algebraic Geometry is when $X$ is a scheme and the cover $\mathcal{U}$ is affine. If we also assume $X$ is quasicompact, we can take this cover to be finite. In particular, since higher cohomology groups of affines are always zero, we have:

$$E_2^{pq} = \begin{cases} 0 & q > 0 \\ \check{H}^p(\mathcal{U}, \mathcal{F}) & q = 0 \end{cases}$$

This means that all differentials $d_2^{pq}$ must vanish, since either their target or source is zero. The sequence then terminates at this page, and we get the important result:

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \cong H^p(X, \mathcal{F})$$

Namely, Čech cohomology computes sheaf cohomology when we take affine covers.

## References

[1] D. Murfet. “Spectral Sequences.” University of Melbourne.
http://therisingsea.org/notes/SpectralSequences.pdf.