TOPOLOGICAL K-THEORY AND BOTT PERIODICITY

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Abstract. This paper will derive the essential ideas of complex topological K-theory, with the aim of understanding and proving the Bott periodicity theorem.

1. Preliminaries

We begin with a few review concepts about vector bundles that will be necessary to construct topological K-theory. Let $X$ be a compact, connected Hausdorff space. Recall that a vector bundle $p : E \to X$ is a surjection of topological spaces such that $p^{-1}(x)$ is has the structure of a vector space for every $x \in X$ with the additional local triviality property. Namely, for every $x \in X$, there exists an open neighborhood $U \subset X$ of $x$ such that $p^{-1}(U) \cong U \times V$ for some vector space $V$. A vector bundle $E$ is called trivial if $E \cong X \times V$.

Standard operations on vector spaces are also valid on vector bundles, such as direct sum and tensoring. We denote $\text{Vect}_k(X)$ to be the set of isomorphism classes of complex $k$ dimensional vector bundles on $X$. The idea of K theory is to make this into a group.

Proposition 1.1. For any vector bundle $p : E \to X$, there exists a bundle $p' : E' \to X$ such that $E \oplus E'$ is trivial.

Proof. Take a partition of unity on a (finite) trivializing cover on $X$ to build maps $g_i : E \to V$ that are linear injections over each trivial neighborhood. Then set $g : E \to V^\oplus N$ by $g = (g_i)$ and consider the map $f : E \to X \times V^\oplus N$ given by $(p, g)$. Then $p : E \to X$ is a sub-bundle of the trivial bundle $X \times V^\oplus N$, and so we can take $p' : E' \to X$ to be the orthogonal complement (with respect to a choice of inner product). See Hatcher [1] §1.1 for more detail.

In the presence of a nondegenerate inner product, it is also true that any short exact sequence $0 \to E_1 \to E_2 \to E_3 \to 0$ of vector bundles splits. This is because $E_2 \cong E_1 \oplus E_3$ always, but we can also identify $E_3 \cong E_2/E_1 \cong E_1^\perp$, so $E_2 \cong E_1 \oplus E_3$.

Definition 1.2. Given a vector bundle $p : E \to X$ and a map $f : Y \to X$, the pullback of the bundle $p : E \to X$ is:

$$f^*E := \{(v, y) \in E \times Y \mid p(v) = f(y)\}$$

with map $q : f^*E \to Y$ given by projection onto the second component.

An important fact about pullbacks is that the respect homotopy equivalences. That is, if $f, g : Y \to X$ are homotopic, then $f^*E$ and $g^*E$ are isomorphic as vector bundles. As a result, there is a one-to-one correspondence between $\text{Vect}_k(X)$ and $\text{Vect}_k(Y)$ whenever $X$ and $Y$ are homotopically equivalent.

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2. Complex K-theory

We now consider vector bundles whose fibers are complex vector spaces, with the goal of proving the Bott Periodicity Theorem.

**Definition 2.1.** Let $M$ be a commutative monoid. The Grothendieck group $\overline{M}$ associated to $M$ is the left adjoint to the forgetful functor $\mathcal{F}: \text{AbGrp} \Rightarrow \text{Mon}$. More concretely, the Grothendieck group is an abelian group structure given by direct sum. There are also distinguished elements $[\text{“smallest” abelian group containing } N]$, which are the $k$ dimensional trivial bundles. We define $K$-theory as follows. Define an equivalence relation $\sim$ on $\text{Vect}(X)$ by:

$$E_1 \sim E_2 \iff \exists \mathcal{E}_k \text{ such that } E_1 \oplus \mathcal{E}_k \cong E_2 \oplus \mathcal{E}_k$$

If $E_1 \sim E_2$, we call $E_1$ and $E_2$ stably isomorphic. Now, consider $\text{Vect}(X) \times \text{Vect}(X)$ with equivalence relation $\sim$:

$$(E_1, F_1) \sim (E_2, F_2) \iff E_1 \oplus F_1 \sim E_2 \oplus F_2$$

**Example 2.2.** The Grothendieck group of the natural numbers is the integers: $\mathbb{N} \cong \mathbb{Z}$. Indeed, the “smallest” abelian group containing $\mathbb{N}$ is $\mathbb{Z}$.

For $X$ as above, the set of isomorphism classes of vector bundles $\text{Vect}(X)$ has a natural monoid structure given by direct sum. There are also distinguished elements $[\mathcal{E}_k]$, which are the $k$ dimensional trivial bundles. We define $K(X) := \text{Vect}(X)$, the Grothendieck group of $\text{Vect}(X)$. We can give an explicit characterization of $K(X)$ as follows. Define an equivalence relation $\sim_s$ on $\text{Vect}(X)$ by:

$$E_1 \sim_s E_2 \iff \exists \mathcal{E}_k \text{ such that } E_1 \oplus \mathcal{E}_k \cong E_2 \oplus \mathcal{E}_k$$

If $E_1 \sim_s E_2$, we call $E_1$ and $E_2$ stably isomorphic. Now, consider $\text{Vect}(X) \times \text{Vect}(X)$ with equivalence relation $\sim$:

$$(E_1, F_1) \sim (E_2, F_2) \iff E_1 \oplus F_1 \sim_s E_2 \oplus F_2$$

**Proposition 2.3.** The relations $\sim_s$ and $\sim$ are indeed equivalence relations.

*Proof.* Symmetry and reflexivity are obvious for both, so we verify transitivity. Suppose $E \sim_s F \sim_s G$. Then there are $k$ and $\ell$ such that:

$$E \oplus \mathcal{E}_k \cong F \oplus \mathcal{E}_k$$

$$F \oplus \mathcal{E}_\ell \cong G \oplus \mathcal{E}_\ell$$

Then:

$$E \oplus F \oplus \mathcal{E}_{k+\ell} \cong F \oplus G \oplus \mathcal{E}_{k+\ell}$$

Let $F'$ be a trivializing bundle for $F$, such that $F \oplus F' \cong \mathcal{E}_j$. Then adding $F'$ to both sides shows $E \sim_s G$. The same can be done to show transitivity of $\sim$. $\square$

There is an abelian group structure on $K(X) = \text{Vect}(X) \times \text{Vect}(X)/\sim$, with addition given by $(E_1, E_2) + (F_1, F_2) = (E_1 \oplus F_1, E_2 \oplus F_2)$ and identity $0 := (\mathcal{E}_0, \mathcal{E}_0) \sim (E, E)$. Subtraction can be achieved by adding reversed components:

$$(E, F) - (E, F) := (E, F) + (F, E) = (E \oplus F, F \oplus E) \sim (\mathcal{E}_0, \mathcal{E}_0)$$

Then we claim:

**Proposition 2.4.** $K(X) \cong K(X)$.

*Proof.* First we note that $K(X)$ is isomorphic to a quotient of the free $\mathbb{Z}$ module with generators $[E] \in \text{Vect}(X)$ via the identification $(E, F) \leftrightarrow [E] - [F]$ subject to the relation $\sim$ above. Using this characterization, we will show that $K(X)$ satisfies the universal property of the Grothendieck group, which is sufficient to show $K(X) \cong K(X)$. We define the map $i : \text{Vect}(X) \to K(X)$ by $[E] \mapsto [E] - \mathcal{E}_0$.
Let \( f : \text{Vect}(X) \to A \) be a monoid homomorphism, and define \( g : K(X) \to A \) by \([E] - [F] \mapsto f(E)\).
Since \( i \) is a monoid injection, \( g \) is unique. Since \( f = gi \), we have found the desired map. \( \square \)

**Example 2.5.** Let \( X = * \) be a point. Then all bundles are trivial, and \( \text{Vect}(X) = \{[\mathcal{E}_k]\}_{k \in \mathbb{N}} \cong \mathbb{N}. \)
We saw earlier that \( \overline{\mathbb{N}} \cong \mathbb{Z} \), so \( K(*) \cong \mathbb{Z}. \)

Not only is \( K(X) \) a group for any compact \( X \), but \( K(-) \) has the structure of a functor. To see this, let \( f : X \to Y \) be a morphism of topological spaces. Then there is an induced map \( f^* : \text{Vect}(Y) \to \text{Vect}(X) \) given by taking pullbacks, which descends to a map \( K(Y) \to K(X) \). It is easy to verify that this satisfies all of the necessary functor axioms.

We can also define the reduced \( (K(X) \oplus \mathbb{Z}) \) by specifying a basepoint \( i : * \to X \). It is defined as the kernel of the induced map \( i^* : K(X) \to K(*) \cong \mathbb{Z}. \) Explicitly, since \( X \) is connected, \( i^* \) acts by sending \( E \) to \( \pi_1 \) rank, extending linearly to all of \( K(X) \). The kernel is generated by elements of the form \( F - E \), where \( F \) and \( E \) have the same rank. The contraction map \( g : X \to * \) induces a split short exact sequence:

\[
0 \longrightarrow \tilde{K}(X) \longrightarrow K(X) \longrightarrow \mathbb{Z} \longrightarrow 0
\]

Thus \( K(X) \cong \tilde{K}(X) \oplus \mathbb{Z}. \)

**Remark 2.6.** There is an alternative definition of \( \tilde{K}(X) \) that is perhaps easier to work with. Two bundles \( E, F \in \text{Vect}(X) \) are called stably equivalent if there are trivial bundles \( \mathcal{E}_k, \mathcal{E}_t \) such that \( E \oplus \mathcal{E}_k \cong F \oplus \mathcal{E}_t \). Then \( \tilde{K}(X) \) can be defined as \( \text{Vect}(X) \) modulo stable equivalence. It is not hard to verify that these definitions are compatible.

### 2.1. Ring Structure of K-groups

The tensor operation on \( \text{Vect}(X) \) extends to a multiplicative structure on \( K(X) \) as follows. For elements \((E_1, F_1)\) and \((E_2, F_2)\), their product is:

\[
(E_1, F_1) \cdot (E_2, F_2) = ((E_1 \otimes E_2) \oplus (F_1 \otimes F_2), (F_1 \otimes E_2) \oplus (E_1 \otimes F_2))
\]

This is consistent with the familiar rule:

\[
(e_1 - f_1)(e_2 - f_2) = (e_1e_2 + f_1f_2) - (f_1e_2 + e_1f_2)
\]
Under this definition, the trivial line bundle \( \mathcal{E}_1 \) is the unit in \( K(X) \). All of the unital ring axioms are easy to check, so that \( K(X) \) is a commutative ring with unit. We then have \( K(-) \) and \( \tilde{K}(-) \) as functors from the homotopy category of compact, Hausdorff spaces to rings.

**Definition 2.7.** The canonical (or tautological) bundle \( H^k \) on \( \mathbb{CP}^k \) is the bundle with total space \( E = \{(\ell, z) \in \mathbb{CP}^k \times \mathbb{C}^{k+1} | z \in \ell\} \) and map \( p : E \to \mathbb{CP}^k \) given by projecting onto the first component.

**Proposition 2.8.** Let \( H = H^1 \) be the canonical bundle on \( \mathbb{CP}^1 = S^2 \). Then \( (H, \mathcal{E}_1)^2 = 0 \).

**Proof.** The key idea is to use the classification of vector bundles on spheres. That is, there is a bijection between homotopy classes of maps \( S^{k-1} \to GL_n(\mathbb{C}) \) (called clutching functions) and vector bundles of rank \( n \) on \( S^k \). This bijection in explained and proved in §1.2 of [1]. For our purposes, we will use the fact that the bundles \( H \oplus H \) and \( (H \oplus H) \oplus \mathcal{E}_1 \) have clutching functions:

\[
f(z) = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}, \quad g(z) = \begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix}
\]
We claim that these are homotopic in $GL_2(\mathbb{C})$. To see this, note that the identity and the component-reversing map are homotopic via:

$$F(t) = \frac{1}{2} \begin{pmatrix} e^{i\pi t} + 1 & 1 - e^{i\pi t} \\ 1 - e^{i\pi t} & e^{i\pi t} + 1 \end{pmatrix}$$

We can then use this homotopy to switch a factor of $z$ along the diagonal to get $f \sim g$. That is:

$$t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} F(t) \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} F(t)$$

is a homotopy of $f$ and $g$. Then the bundles representing $f$ and $g$, namely $H \oplus H$ and $(H \otimes H) \oplus E_1$, must be isomorphic. Then:

$$H \oplus H \cong (H \otimes H) \oplus E_1 \Rightarrow ((H \otimes H) \oplus E, H \oplus H) = 0$$

$$\iff ((H \otimes H) \oplus (E \oplus E_1), (H \otimes E_1) \oplus (H \otimes E_1)) = 0$$

$$\iff (H, E_1) \cdot (H, E_1) = 0$$

This allows us to identify $\mathbb{Z}[H]/(H - 1)^2$ with a subring of $K(S^2)$. In fact, that subring is exactly $\tilde{K}(S^2)$. This is a special case of the Fundamental Product Theorem:

**Theorem 2.9.** For any Hausdorff space $X$, the rings $K(X) \otimes \mathbb{Z}[H]/(H - 1)^2$ and $K(X \times S^2)$ are isomorphic.

**Remark 2.10.** It is easy to write down what the map should be, but not easy to prove it is an isomorphism. For any $X, Y$, there is a natural map (called the external product) $\mu : K(X) \otimes K(Y) \to K(X \times Y)$ given by $\mu(x \otimes y) = \pi_1(x) \cdot \pi_2(y)$, where $\pi_1$ and $\pi_2$ are the projections from $X \times Y$. If $i : \mathbb{Z}[H]/(H - 1)^2 \to K(S^2)$ is the inclusion, then composing $\mu$ and $1 \otimes i$ gives the desired map. See §2.2 of [1] for a proof that this is an isomorphism.

3. **Bott Periodicity**

The Bott periodicity theorem (for the complex vector bundle case) states that the $K$–theory of spheres is periodic with period 2. To prove this, we will need to know some basic information about how $K(\cdot)$ and $\tilde{K}(\cdot)$ behave under exact sequences of spaces.

**Proposition 3.1.** Let $X$ be a compact Hausdorff space, and let $A \subset X$ be closed. Denote $i : A \to X$ to be the inclusion and $q : X \to X/A$ to be the quotient by $A$. Then the induced sequence:

$$\tilde{K}(X/A) \xrightarrow{i^*} \tilde{K}(X) \xrightarrow{q^*} \tilde{K}(A)$$

is exact.

**Proof.** To show $i^* q^* = 0$, we note that the composition $q i : A \to X/A$ sends $A$ to the point $A/A$. For any bundle $E \in \text{Vect}(X/A)$ the pullback $(q i)^* (E)$ is the pullback in the diagram:

$$\begin{array}{ccc}
E \times_{X/A} A & \longrightarrow & E \\
\downarrow & & \downarrow \\
A & \xrightarrow{q i} & X/A
\end{array}$$

But since $q i$ is a constant map, we have $E \times_{X/A} A \cong p^{-1}(A/A) \times A$, which is a trivial bundle. Therefore $i^* q^* (E) = (q i)^* (E)$ is trivial, and therefore equivalent to the zero element of $\tilde{K}(A)$.

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1The straight line homotopy is singular at $t = 1/2$, so we use this instead
To show that \( \ker(i^*) \subseteq \text{im}(q^*) \), suppose that \( E \in \text{Vect}(X) \) is stably trivial over \( A \); that is, when restricted to \( A \), we have \( E \oplus E_k \cong E \) for some \( k \in \mathbb{Z} \). Since \((E, E_0) \sim (E \oplus E_k, E_k)\) in \( \text{Vect}(X) \), we may assume that \( E \) is actually trivial over \( A \). Let \( \phi : p^{-1}(A) \to \mathbb{C}^\ell \times A \) be a trivialization and let \( E' = E/\phi \) be the space formed by identifying \( \phi^{-1}(a, v) \) and \( \phi^{-1}(b, v) \) for all \( a, b \in A \). Then there is a natural map \( p' : E' \to X/A \); we claim that \( E' \to X/A \) is a vector bundle over \( X/A \). Since every point in \( X/A \) outside of \( A/A \) has a natural trivializing neighborhood inherited from the bundle \( p : E \to X \), we only need to check that \( A/A \) has a trivializing neighborhood. Take trivializing open sets \( \{ U_i \} \) covering \( A \subset X \); since \( p : E \to A \) is trivial over \( A \), we can construct a trivialization on \( \bigcup U_i \) using partitions of unity. Therefore there is an open neighborhood \( U = \bigcup U_i \) of \( A \) over which \( E \) is trivial, and therefore \( U/\phi \) is a trivializing neighborhood of \( A/A \). Therefore \( E' \in \text{Vect}(X/A) \).

Finally, we claim that \( E = q^*(E/\phi) \). To see this, we note that \( q^*(E/\phi) \) is, by definition, the pullback:

\[
\begin{array}{ccc}
q^*(E/\phi) & \longrightarrow & E' \\
\downarrow & & \downarrow p' \\
X & \longrightarrow & X/A
\end{array}
\]

It is easy to check that \( E \) satisfies the universal property of the pullback, so that \( E \cong q^*(E/\phi) \). Therefore \( E \in \text{im}(q^*) \).

**Corollary 3.2.** For a sequence \( A \to X \to X/A \) as above, there is an induced long exact sequence:

\[
\cdots \longrightarrow \widetilde{K}(S^nX) \longrightarrow \widetilde{K}(SA) \longrightarrow \widetilde{K}(X/A) \longrightarrow \widetilde{K}(X) \longrightarrow \widetilde{K}(A)
\]

Where \( S^nX \) denotes the \( n \)-fold suspension of \( X \).

**Proof.** For any space \( Y \), let \( CY \) denote the cone on \( Y \). Consider the following sequence of inclusions:

\[
A \hookrightarrow X \hookrightarrow X \cup CA \hookrightarrow (X \cup CA) \cup CX \hookrightarrow \cdots
\]

Where at each step we glue a cone on the previous object to the current object. We note that \( X \cup CA \) is homotopy equivalent to \( X/A \) via collapsing \( CA \). Similarly, \((X \cup CA) \cup CX \) is homotopy equivalent to \((X \cup CA)/X \cong SA\), and so on. Since \( \widetilde{K}(X) \cong \widetilde{K}(Y) \) for \( X \) and \( Y \) homotopic, we have an induced sequence of maps:

\[
\begin{array}{cccc}
\cdots & \longrightarrow & \widetilde{K}((X \cup CA) \cup CX) & \longrightarrow \\
\downarrow & & \downarrow & \cong \\
\cdots & \longrightarrow & \widetilde{K}(SA) & \longrightarrow \\
\downarrow & & \downarrow & \cong \\
\cdots & \longrightarrow & \widetilde{K}(X/A) & \longrightarrow \\
\downarrow & & \downarrow & \cong \\
\cdots & \longrightarrow & \widetilde{K}(X) & \longrightarrow \\
\downarrow & & \downarrow & \\
\cdots & \longrightarrow & \widetilde{K}(A) & \longrightarrow \\
\end{array}
\]

Further, this is exact by applying the previous proposition at each point in the sequence.

**Remark 3.3.** At this point, things smell very cohomological. For any positive integer \( n \), if we define \( \widetilde{K}^{-n}(X) := \widetilde{K}(S^nX) \), it is possible to show that \( \widetilde{K}^{-n}(\cdot) \) gives rise to a cohomology theory. We have already shown the existence of the long exact sequence, and the remaining axioms are not too hard to check.

**Proposition 3.4.** The reduced \( K \)-theory of a product \( X \times Y \) decomposes as:

\[
\widetilde{K}(X \times Y) \cong \widetilde{K}(X \wedge Y) \oplus \widetilde{K}(X) \oplus \widetilde{K}(Y)
\]
Proof. Consider the long exact sequence induced by the sequence \( X \vee Y \to X \times Y \to X \land Y \). Note that, if \( \Sigma(-) \) denotes taking the reduced suspension, we have \( \Sigma(X \vee Y) = \Sigma X \vee \Sigma Y \). Therefore, since \( \Sigma Z \) is homotopy equivalent to \( S Z \), we have:

\[
\widetilde{K}(S^n(X \vee Y)) \cong \widetilde{K}(\Sigma^n(X \vee Y)) \cong \widetilde{K}(\Sigma^n X \vee \Sigma^n Y) \cong \widetilde{K}(\Sigma^n X) \oplus \widetilde{K}(\Sigma^n Y)
\]

Where the last isomorphism comes from the restrictions to \( \Sigma^n X \) and \( \Sigma^n Y \). Then the long exact sequence becomes:

\[
\cdots \to \widetilde{K}(S(X \times Y)) \xrightarrow{j_1} \widetilde{K}(SX) \oplus \widetilde{K}(SY) \to \widetilde{K}(X \wedge Y) \to \widetilde{K}(X \times Y) \xrightarrow{j_0} \widetilde{K}(X) \oplus \widetilde{K}(Y)
\]

We claim the maps \( j_0 \) and \( j_1 \) are surjections. Let \((a, b) \in \widetilde{K}(X) \oplus \widetilde{K}(Y)\) and define \(\sigma_0(a, b) = \pi_1(a) + \pi_2(b)\), where \(\pi_1\) and \(\pi_2\) are the component projections on \(X \times Y\). Then:

\[
j_0(\sigma(a, b)) = j_0(\pi_1(a)) + j_0(\pi_2(b)) = (a, 0) + (0, b) = (a, b)
\]

Therefore \(j_0\) is surjective. Similarly, define \(\sigma_1(a, b) = (S\pi_1)^*(a) + (S\pi_2)^*(b)\). By the same reasoning \(j_1\) is also surjective. In particular, the last three terms comprise a short exact sequence which is split by \(\sigma_0\). The desired result follows from the splitting lemma.

We can now construct the reduced version of the external product \(\mu : K(X) \otimes K(Y) \to K(X \times Y)\) from Remark 2.10. Recall that \(K(Z) \cong \widetilde{K}(Z) \oplus \mathbb{Z}\), hence by the previous proposition we have:

\[
K(X \times Y) \cong \widetilde{K}(X \wedge Y) \oplus \widetilde{K}(X) \oplus \widetilde{K}(Y) \oplus \mathbb{Z}
\]

Similarly:

\[
K(X) \otimes K(Y) \cong (\widetilde{K}(X) \oplus \mathbb{Z}) \otimes (\widetilde{K}(Y) \oplus \mathbb{Z})
\]

\[
= \widetilde{K}(X) \otimes \widetilde{K}(Y) \oplus \widetilde{K}(X) \otimes \widetilde{K}(Y) \oplus \mathbb{Z}
\]

Therefore restricting \(\mu\) to the \(\widetilde{K}(X) \otimes \widetilde{K}(Y)\) component gives us a map to \(\widetilde{K}(X \wedge Y)\), which one can check is well-defined and unique. We call this the reduced external product \(\tilde{\mu} : \widetilde{K}(X) \otimes \widetilde{K}(Y) \to \widetilde{K}(X \wedge Y)\). We denote \(\tilde{\mu}(a \otimes b) = a \ast b\).

Theorem 3.5. (Bott Periodicity) For \(X\) a compact Hausdorff space, there is an isomorphism \(\widetilde{K}(X) \to \widetilde{K}(S^2 X)\).

Proof. Recall that \(\widetilde{K}(S^2)\) is isomorphic to \(\mathbb{Z}\) with generator \((H - 1)\). Therefore there is a natural isomorphism:

\[
\widetilde{K}(X) \xrightarrow{\cong} \widetilde{K}(S^2) \otimes \widetilde{K}(X)
\]

\[
a \longmapsto (H - 1) \otimes a
\]

We can then take the reduced external product \(\tilde{\mu}((H - 1) \otimes a) \in \widetilde{K}(S^2 \wedge X) \cong \widetilde{K}(S^2X)\). By Theorem 2.9, the map \(\mu\) (and therefore \(\tilde{\mu}\)) is an isomorphism. Thus \(a \mapsto (H - 1) \ast a\) is an isomorphism of \(\widetilde{K}(X)\) and \(\widetilde{K}(S^2 X)\).

\[
\square
\]


References
