# TOPOLOGICAL K-THEORY AND BOTT PERIODICITY

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Abstract. This paper will derive the essential ideas of complex topological K-theory, with the aim of understanding and proving the Bott periodicity theorem.

## 1. Preliminaries

We begin with a few review concepts about vector bundles that will be necesary to construct topological K-theory. Let $X$ be a compact, connected Hausdorff space. Recall that a vector bundle $p : E \to X$ is a surjection of topological spaces such that $p^{-1}(x)$ is has the structure of a vector space for every $x$ with the additional local triviality property. Namely, for every $x \in X$, there exists an open neighborhood $U \subset X$ of $x$ such that $p^{-1}(U) \cong U \times V$ for some vector space $V$. A vector bundle $E$ is called trivial if $E \cong X \times V$.

Standard operations on vector spaces are also valid on vector bundles, such as direct sum and tensoring. We denote $\mathrm{Vect}_k(X)$ to be the set of isomorphism classes of complex $k$ dimensional vector bundles on $X$. The idea of $K$ theory is to make this into a group.

**Proposition 1.1.** *For any vector bundle $p : E \to X$, there exists a bundle $p' : E' \to X$ such that $E \oplus E'$ is trivial.*

*Proof.* Take a partition of unity on a (finite) trivializing cover on $X$ to build maps $g_i : E \to V$ that are linear injections over each trivial neighborhood. Then set $g : E \to V^{\oplus N}$ by $g = (g_i)$ and consider the map $f : E \to X \times V^{\oplus N}$ given by $(p, g)$. Then $p : E \to X$ is a sub-bundle of the trivial bundle $X \times V^{\oplus N}$, and so we can take $p' : E' \to X$ to be the orthogonal complement (with respect to a choice of inner product). See Hatcher [1] §1.1 for more detail. □

In the presence of a nondegenerate inner product, it is also true that any short exact sequence $0 \to E_1 \to E_2 \to E_3 \to 0$ of vector bundles splits. This is because $E_2 \cong E_1 \oplus E_1^{\perp}$ always, but we can also identify $E_3 \cong E_2/E_1 \cong E_1^{\perp}$, so $E_2 \cong E_1 \oplus E_3$.

**Definition 1.2.** Given a vector bundle $p : E \to X$ and a map $f : Y \to X$, the *pullback* of the bundle $p : E \to X$ is:

$$f^*E := \{(v, y) \in E \times Y \mid p(v) = f(y)\}$$

with map $q : f^*E \to Y$ given by projection onto the second component.

An important fact about pullbacks is that the respect homotopy equivalences. That is, if $f, g : Y \to X$ are homotopic, then $f^*E$ and $g^*E$ are isomorphic as vector bundles. As a result, there is a one-to-one correspondence between $\mathrm{Vect}_k(X)$ and $\mathrm{Vect}_k(Y)$ whenever $X$ and $Y$ are homotopically equivalent.

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## 2. Complex K-theory

We now consider vector bundles whose fibers are complex vector spaces, with the goal of proving the Bott Periodicity Theorem.

**Definition 2.1.** Let $M$ be a commutative monoid. The *Grothendieck group* $\overline{M}$ associated to $M$ is the left adjoint to the forgetful functor $\mathcal{F} : \text{AbGrp} \to \text{Mon}$.

More concretely, the Grothendieck group is an abelian group $\overline{M}$ with a homomorphism $i : M \to \overline{M}$ satisfying a universal property. Given any other homomorphism $f : M \to A$, there is a unique map $g : \overline{M} \to A$ such that the following commutes:

$$\begin{array}{ccc} M & \xrightarrow{i} & \overline{M} \\ \big\downarrow{\scriptstyle f} & \swarrow{\scriptstyle g} & \\ A & & \end{array}$$

**Example 2.2.** The Grothendieck group of the natural numbers is the integers: $\overline{\mathbb{N}} \cong \mathbb{Z}$. Indeed, the "smallest" abelian group containing $\mathbb{N}$ is $\mathbb{Z}$.

For $X$ as above, the set of isomorphism classes of vector bundles $\text{Vect}(X)$ has a natural monoid structure given by direct sum. There are also distinguished elements $[\mathcal{E}_k]$, which are the $k$ dimensional trivial bundles. We define $K(X) := \overline{\text{Vect}(X)}$, the Grothendieck group of $\text{Vect}(X)$. We can give an explicit characterization of $K(X)$ as follows. Define an equivalence relation $\sim_s$ on $\text{Vect}(X)$ by:

$$E_1 \sim_s E_2 \iff \exists\, \mathcal{E}_k \text{ such that } E_1 \oplus \mathcal{E}_k \cong E_2 \oplus \mathcal{E}_k$$

If $E_1 \sim_s E_2$, we call $E_1$ and $E_2$ *stably isomorphic.* Now, consider $\text{Vect}(X) \times \text{Vect}(X)$ with equivalence relation $\sim$:

$$(E_1, F_1) \sim (E_2, F_2) \iff E_1 \oplus F_2 \sim_s E_2 \oplus F_1$$

**Proposition 2.3.** *The relations $\sim_s$ and $\sim$ are indeed equivalence relations.*

*Proof.* Symmetry and reflexivity are obvious for both, so we verify transitivity. Suppose $E \sim_s F \sim_s G$. Then there are $k$ and $\ell$ such that:

$$E \oplus \mathcal{E}_k \cong F \oplus \mathcal{E}_k$$
$$F \oplus \mathcal{E}_\ell \cong G \oplus \mathcal{E}_\ell$$

Then:

$$E \oplus F \oplus \mathcal{E}_{k+\ell} \cong F \oplus G \oplus \mathcal{E}_{k+\ell}$$

Let $F'$ be a trivializing bundle for $F$, such that $F \oplus F' \cong \mathcal{E}_j$. Then adding $F'$ to both sides shows $E \sim_s G$. The same can be done to show transitivity of $\sim$. $\square$

There is an abelian group structure on $\mathcal{K}(X) = \text{Vect}(X) \times \text{Vect}(X)/\sim$, with addition given by $(E_1, E_2) + (F_1, F_2) = (E_1 \oplus F_1, E_2 \oplus F_2)$ and identity $0 := (\mathcal{E}_0, \mathcal{E}_0) \sim (E, E)$. Subtraction can be achieved by adding reversed components:

$$(E, F) - (E, F) := (E, F) + (F, E) = (E \oplus F, F \oplus E) \sim (\mathcal{E}_0, \mathcal{E}_0)$$

Then we claim:

**Proposition 2.4.** $K(X) \cong \mathcal{K}(X)$.

*Proof.* First we note that $\mathcal{K}(X)$ is isomorphic to a quotient of the free $\mathbb{Z}$ module with generators $[E] \in \text{Vect}(X)$ via the identification $(E, F) \leftrightarrow [E] - [F]$ subject to the relation $\sim$ above. Using this characterization, we will show that $\mathcal{K}(X)$ satisfies the universal property of the Grothendieck group, which is sufficient to show $K(X) \cong \mathcal{K}(X)$. We define the map $i : \text{Vect}(X) \to \mathcal{K}(X)$ by $[E] \mapsto [E] - [\mathcal{E}_0]$.

Let $f : \mathrm{Vect}(X) \to A$ be a monoid homomorphism, and define $g : \mathcal{K}(X) \to A$ by $[E] - [F] \mapsto f(E)$. Since $i$ is a monoid injection, $g$ is unique. Since $f = gi$, we have found the desired map. □

**Example 2.5.** Let $X = *$ be a point. Then all bundles are trivial, and $\mathrm{Vect}(X) = \{[\mathcal{E}_k]\}_{k\in\mathbb{N}} \cong \mathbb{N}$. We saw earlier that $\overline{N} \cong \mathbb{Z}$, so $K(*) \cong \mathbb{Z}$.

Not only is $K(X)$ a group for any compact $X$, but $K(-)$ has the structure of a functor. To see this, let $f : X \to Y$ be a morphism of topological spaces. Then there is an induced map $f^* : \mathrm{Vect}(Y) \to \mathrm{Vect}(X)$ given by taking pullbacks, which descends to a map $K(Y) \to K(X)$. It is easy to verify that this satisfies all of the necessary functor axioms.

We can also define the reduced $K$-group $\widetilde{K}(X)$ by specifying a basepoint $i : * \to X$. It is defined as the kernel of the induced map $i^* : K(X) \to K(*) \cong \mathbb{Z}$. Explicitly, since $X$ is connected, $i^*$ acts by sending $E$ to it rank, extending linearly to all of $K(X)$. The kernel is generated by elements of the form $F - E$, where $F$ and $E$ have the same rank. The contraction map $g : X \to *$ induces a split short exact sequence:

$$0 \longrightarrow \widetilde{K}(X) \longrightarrow K(X) \underset{g^*}{\overset{i^*}{\rightleftarrows}} \mathbb{Z} \longrightarrow 0$$

Thus $K(X) \cong \widetilde{K}(X) \oplus \mathbb{Z}$.

*Remark* 2.6. There is an alternative definition of $\widetilde{K}(X)$ that is perhaps easier to work with. Two bundles $E, F \in \mathrm{Vect}(X)$ are called *stably equivalent* if there are trivial bundles $\mathcal{E}_k, \mathcal{E}_\ell$ such that $E \oplus \mathcal{E}_k \cong F \oplus \mathcal{E}_\ell$. Then $\widetilde{K}(X)$ can be defined as $\mathrm{Vect}(X)$ modulo stable equivalence. It is not hard to verify that these definitions are compatible.

2.1. **Ring Structure of K-groups.** The tensor operation on $\mathrm{Vect}(X)$ extends to a multiplicative structure on $K(X)$ as follows. For elements $(E_1, F_1)$ and $(E_2, F_2)$, their product is:

$$(E_1, F_1) \cdot (E_2, F_2) = ((E_1 \otimes E_2) \oplus (F_1 \otimes F_2), (F_1 \otimes E_2) \oplus (E_1 \otimes F_2))$$

This is consistent with the familiar rule:

$$(e_1 - f_1)(e_2 - f_2) = (e_1e_1 + f_1f_2) - (f_1e_2 + e_1f_2)$$

Under this definition, the trivial line bundle $\mathcal{E}_1$ is the unit in $K(X)$. All of the unital ring axioms are easy to check, so that $K(X)$ is a commutative ring with unit. We then have $K(-)$ and $\widetilde{K}(-)$ as functors from the homotopy category of compact, Hausdorff spaces to rings.

**Definition 2.7.** The canonical (or tautological) bundle $H^k$ on $\mathbb{CP}^k$ is the bundle with total space $E = \{(\ell, z) \in \mathbb{CP}^k \times \mathbb{C}^{k+1} \mid z \in \ell\}$ and map $p : E \to \mathbb{CP}^k$ given by projecting onto the first component.

**Proposition 2.8.** *Let $H = H^1$ be the canonical bundle on $\mathbb{CP}^1 = S^2$. Then $(H, \mathcal{E}_1)^2 = 0$.*

*Proof.* The key idea is to use the classification of vector bundles on spheres. That is, there is a bijection between homotopy classes of maps $S^{k-1} \to GL_n(\mathbb{C})$ (called clutching functions) and vector bundles of rank $n$ on $S^k$. This bijection in explained and proved in §1.2 of [1]. For our purposes, we will use the fact that the bundles $H \oplus H$ and $(H \otimes H) \oplus \mathcal{E}_1$ have clutching functions:

$$f(z) = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}, \ g(z) = \begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix}$$

We claim that these are homotopic in $GL_2(\mathbb{C})$. To see this, note that the identity and the component-reversing map are homotopic via[1]:

$$F(t) = \frac{1}{2}\begin{pmatrix} e^{i\pi t}+1 & 1-e^{i\pi t} \\ 1-e^{i\pi t} & e^{i\pi t}+1 \end{pmatrix}$$

We can then use this homotopy to switch a factor of $z$ along the diagonal to get $f \sim g$. That is:

$$t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} F(t) \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} F(t)$$

is a homotopy of $f$ and $g$. Then the bundles representing $f$ and $g$, namely $H \oplus H$ and $(H \otimes H) \oplus \mathcal{E}_1$, must be isomorphic. Then:

$$\begin{aligned} H \oplus H \cong (H \otimes H) \oplus \mathcal{E}_1 &\Rightarrow ((H \otimes H) \oplus \mathcal{E}_1, H \oplus H) = 0 \\ &\iff ((H \otimes H) \oplus (\mathcal{E}_1 \oplus \mathcal{E}_1), (H \otimes \mathcal{E}_1) \oplus (H \otimes E_1)) = 0 \\ &\iff (H, \mathcal{E}_1) \cdot (H, \mathcal{E}_1) = 0 \end{aligned}$$

□

This allows us to identify $\mathbb{Z}[H]/(H-1)^2$ with a subring of $K(S^2)$. In fact, that subring is exactly $\widetilde{K}(S^2)$. This is a special case of the Fundamental Product Theorem:

**Theorem 2.9.** *For any Hausdorff space $X$, the rings $K(X) \otimes \mathbb{Z}[H]/(H-1)^2$ and $K(X \times S^2)$ are isomorphic.*

*Remark* 2.10. It is easy to write down what the map should be, but not easy to prove it is an isomorphism. For any $X, Y$, there is a natural map (called the external product) $\mu : K(X) \otimes K(Y) \to K(X \times Y)$ given by $\mu(x \otimes y) = \pi_1^*(x) \cdot \pi_2^*(y)$, where $\pi_1$ and $\pi_2$ are the projections from $X \times Y$. If $i : \mathbb{Z}[H]/(H-1)^2 \to K(S)^2$ is the inclusion, then composing $\mu$ and $1 \otimes i$ gives the desired map. See §2.2 of [1] for a proof that this is an isomorphism.

## 3. Bott periodicity

The Bott periodicity theorem (for the complex vector bundle case) states that the $K-$theory of spheres is periodic with period 2. To prove this, we will need to know some basic information about how $K(-)$ and $\widetilde{K}(-)$ behave under exact sequences of spaces.

**Proposition 3.1.** *Let $X$ be a compact Hausdorff space, and let $A \subset X$ be closed. Denote $i : A \to X$ to be the inclusion and $q : X \to X/A$ to be the quotient by $A$. Then the induced sequence:*

$$\widetilde{K}(X/A) \xrightarrow{q^*} \widetilde{K}(X) \xrightarrow{i^*} \widetilde{K}(A)$$

*is exact.*

*Proof.* To show $i^*q^* = 0$, we note that the composition $qi : A \to X/A$ sends $A$ to the point $A/A$. For any bundle $E \in \mathrm{Vect}(X/A)$ the pullback $(qi)^*(E)$ is the pullback in the diagram:

$$\begin{array}{ccc} E \times_{X/A} A & \longrightarrow & E \\ \downarrow & & \downarrow \\ A & \xrightarrow{qi} & X/A \end{array}$$

But since $qi$ is a constant map, we have $E \times_{X/A} A \cong p^{-1}(A/A) \times A$, which is a trivial bundle. Therefore $i^*q^*(E) = (qi)^*(E)$ is trivial, and therefore equivalent to the zero element of $\widetilde{K}(A)$.

[1] The straight line homotopy is singular at $t = 1/2$, so we use this instead

To show that $\ker(i^*) \subset \operatorname{im}(q^*)$, suppose that $E \in \operatorname{Vect}(X)$ is stably trivial over $A$; that is, when restricted to $A$, we have $E \oplus \mathcal{E}_k \cong \mathcal{E}_\ell$ for some $k, \ell$. Since $(E, \mathcal{E}_0) \sim (E \oplus \mathcal{E}_k, \mathcal{E}_k)$ in $\operatorname{Vect}(X)$, we may assume that $E$ is actually trivial over $A$. Let $\phi : p^{-1}(A) \to \mathbb{C}^\ell \times A$ be a trivialization and let $E' = E/\phi$ be the space formed by identifying $\phi^{-1}(a, v)$ and $\phi^{-1}(b, v)$ for all $a, b \in A$. Then there is a natural map $p' : E' \to X/A$; we claim that $E' \to X/A$ is a vector bundle over $X/A$. Since every point in $X/A$ outside of $A/A$ has a natural trivializing neighborhood inherited from the bundle $p : E \to X$, we only need to check that $A/A$ has a trivializing neighborhood. Take trivializing open sets $\{U_i\}$ covering $A \subset X$; since $p : E \to A$ is trivial over $A$, we can construct a trivialization on $\bigcup U_i$ using partitions of unity. Therefore there is an open neighborhood $U = \bigcup U_i$ of $A$ over which $E$ is trivial, and therefore $U/\phi$ is a trivializing neighborhood of $A/A$. Therefore $E' \in \operatorname{Vect}(X/A)$.

Finally, we claim that $E = q^*(E/\phi)$. To see this, we note that $q^*(E/\phi)$ is, by definition, the pullback:

$$\begin{array}{ccc} q^*(E/\phi) & \longrightarrow & E' \\ \big\downarrow & & \big\downarrow{\scriptstyle p'} \\ X & \xrightarrow{\ q\ } & X/A \end{array}$$

It is easy to check that $E$ satisfies the universal property of the pullback, so that $E \cong q^*(E/\phi)$. Therefore $E \in \operatorname{im}(q^*)$. □

**Corollary 3.2.** *For a sequence $A \to X \to X/A$ as above, there is an induced long exact sequence:*

$$\cdots \longrightarrow \widetilde{K}(SX) \longrightarrow \widetilde{K}(SA) \longrightarrow \widetilde{K}(X/A) \longrightarrow \widetilde{K}(X) \longrightarrow \widetilde{K}(A)$$

*Where $S^nX$ denotes the $n$-fold suspension of $X$.*

*Proof.* For any space $Y$, let $CY$ denote the cone on $Y$. Consider the following sequence of inclusions:

$$A \hookrightarrow X \hookrightarrow X \cup CA \hookrightarrow (X \cup CA) \cup CX \hookrightarrow \cdots$$

Where at each step we glue a cone on the previous object to the current object. We note that $X \cup CA$ is homotopy equivalent to $X/A$ via collapsing $CA$. Similarly, $(X \cup CA) \cup CX$ is homotopy equivalent to $(X \cup CA)/X \cong SA$, and so on. Since $\widetilde{K}(X) \cong \widetilde{K}(Y)$ for $X$ and $Y$ homotopic, we have an induced sequence of maps:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \widetilde{K}((X \cup CA) \cup CX) & \longrightarrow & \widetilde{K}(X \cup CA) & \longrightarrow & \widetilde{K}(X) & \longrightarrow & \widetilde{K}(A) \\ \big\downarrow\cong & & \big\downarrow\cong & & \big\downarrow\cong & & \| & & \| \\ \cdots & \longrightarrow & \widetilde{K}(SA) & \longrightarrow & \widetilde{K}(X/A) & \longrightarrow & \widetilde{K}(X) & \longrightarrow & \widetilde{K}(A) \end{array}$$

□

Further, this is exact by applying the previous proposition at each point in the sequence.

*Remark* 3.3. At this point, things smell very cohomological. For any positive integer $n$, if we define $\widetilde{K}^{-n}(X) := \widetilde{K}(S^nX)$, it is possible to show that $\widetilde{K}^{-n}(-)$ gives rise to a cohomology theory. We have already shown the existence of the long exact sequence, and the remaining axioms are not too hard to check.

**Proposition 3.4.** *The reduced $K$-theory of a product $X \times Y$ decomposes as:*

$$\widetilde{K}(X \times Y) \cong \widetilde{K}(X \wedge Y) \oplus \widetilde{K}(X) \oplus \widetilde{K}(Y)$$

*Proof.* Consider the long exact sequence induced by the sequence $X \vee Y \hookrightarrow X \times Y \to X \wedge Y$. Note that, if $\Sigma(-)$ denotes taking the reduced suspension, we have $\Sigma(X \vee Y) = \Sigma X \vee \Sigma Y$. Therefore, since $\Sigma Z$ is homotopy equivalent to $SZ$, we have:

$$\widetilde{K}(S^n(X \vee Y)) \cong \widetilde{K}(\Sigma^n(X \vee Y)) \cong \widetilde{K}(\Sigma^n X \vee \Sigma^n Y) \cong \widetilde{K}(\Sigma^n X) \oplus \widetilde{K}(\Sigma^n, Y)$$

Where the last isomorphism comes from the restrictions to $\Sigma^n X$ and $\Sigma^n Y$. Then the long exact sequence becomes:

$$\cdots \longrightarrow \widetilde{K}(S(X \times Y)) \xrightarrow{j_1} \widetilde{K}(SX) \oplus \widetilde{K}(SY) \longrightarrow \widetilde{K}(X \wedge Y) \longrightarrow \widetilde{K}(X \times Y) \xrightarrow{j_0} \widetilde{K}(X) \oplus \widetilde{K}(Y)$$

We claim the maps $j_0$ and $j_1$ are surjections. Let $(a,b) \in \widetilde{K}(X) \oplus \widetilde{K}(Y)$ and define $\sigma_0(a,b) = \pi_1^*(a) + \pi_2^*(b)$, where $\pi_1$ and $\pi_2$ are the component projections on $X \times Y$. Then:

$$j_0(\sigma_0(a,b)) = j_0(\pi_1^*(a)) + j_0(\pi_2^*(b)) = (a,0) + (0,b) = (a,b)$$

Therefore $j_0$ is surjective. Similarly, define $\sigma_1(a,b) = (S\pi_1)^*(a) + (S\pi_2)^*(b)$. By the same reasoning $j_1$ is also surjective. In particular, the last three terms comprise a short exact sequence which is split by $\sigma_0$. The desired result follows from the splitting lemma. □

We can now construct the reduced version of the external product $\mu : K(X) \otimes K(Y) \to K(X \times Y)$ from Remark 2.10. Recall that $K(Z) \cong \widetilde{K}(Z) \oplus \mathbb{Z}$, hence by the previous proposition we have:

$$K(X \times Y) \cong \widetilde{K}(X \wedge Y) \oplus \widetilde{K}(X) \oplus \widetilde{K}(Y) \oplus \mathbb{Z}$$

Similarly:

$$\begin{aligned} K(X) \otimes K(Y) &\cong (\widetilde{K}(X) \oplus \mathbb{Z}) \otimes (\widetilde{K}(Y) \oplus \mathbb{Z}) \\ &= \widetilde{K}(X) \otimes \widetilde{K}(Y) \oplus \widetilde{K}(X) \oplus \widetilde{K}(Y) \oplus \mathbb{Z} \end{aligned}$$

Therefore restricting $\mu$ to the $\widetilde{K}(X) \otimes \widetilde{K}(Y)$ component gives us a map to $\widetilde{K}(X \wedge Y)$, which one can check is well-defined and unique. We call this the reduced external product $\tilde{\mu} : \widetilde{K}(X) \otimes \widetilde{K}(Y) \to \widetilde{K}(X \wedge Y)$. We denote $\tilde{\mu}(a \otimes b) \equiv a * b$.

**Theorem 3.5.** *(Bott Periodicity) For $X$ a compact Hausdorff space, there is an isomorphism $\widetilde{K}(X) \to \widetilde{K}(S^2 X)$.*

*Proof.* Recall that $\widetilde{K}(S^2)$ is isomorphic to $\mathbb{Z}$ with generator $(H-1)$. Therefore there is a natural isomorphism:

$$\begin{aligned} \widetilde{K}(X) &\xrightarrow{\cong} \widetilde{K}(S^2) \otimes \widetilde{K}(X) \\ a &\longmapsto (H-1) \otimes a \end{aligned}$$

We can then take the reduced external product $\tilde{\mu}((H-1) \otimes a) \in \widetilde{K}(S^2 \wedge X) \cong \widetilde{K}(S^2 X)$. By Theorem 2.9, the map $\mu$ (and therefore $\tilde{\mu}$) is an isomorphism. Thus $a \mapsto (H-1) * a$ is an isomorphism of $\widetilde{K}(X)$ and $\widetilde{K}(S^2 X)$. □

## References

[1] Allen Hatcher. *Vector Bundles and K-Theory.* Version 2.1, May 2009.
[2] Wen Jiang. *A mini-introduction to topological K-Theory.* Mathematical Institute, Oxford. 2006.