PERIODIC ORBITS AND GLOBAL SURFACES OF SECTION IN HAMILTONIAN SYSTEMS

George Torres
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1. Introduction

A global disk-like surface of section is an embedded disk in phase space of a Hamiltonian system that is transverse to the flow. This paper will overview an important result due to Hofer et al. [1] showing the existence of such a section for energy surfaces satisfying a convexity condition. We will also show that the flow on such a surface can be conjugated to an area perserving map of the unit disk, and, appealing to current knowledge on area preserving maps, we will conclude the existence of either 2 or infinitely many periodic orbits in phase space. We will also look at two physical systems which admit instructive surfaces of section, the Hénon Heiles potential and the restricted 3-body problem.

2. Definitions and Context

Any physical system has a natural association to a manifold in $\mathbb{R}^n$, called the configuration space:

Definition 2.1. The *configuration space* of a system with k degrees of freedom is a k-manifold V consisting of points $(q_1, ..., q_n) \in \mathbb{R}^n$ that are possible values of a physical configuration.

For example, a free particle in 3 dimensions has configuration space $\mathbb{R}^3$; a rotating body has configuration space $\mathbb{R}^3 \times SO(3)$. The dimension of V is the number of degrees of freedom.

Definition 2.2. The *Lagrangian* of a physical system is a convex map of the tangent bundle $TV \times \mathbb{R} \to \mathbb{R}$:

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) = T(\mathbf{q}, \dot{\mathbf{q}}, t) - V(\mathbf{q}, \dot{\mathbf{q}}, t)$$

Where T and V are the kinetic and potential energy of the particle. We think of $\mathbf{q} \in V$ being the coordinates, and $\dot{\mathbf{q}} \in T_{\mathbf{q}}V$ to be the velocity of the particle. Most generally there will be time dependence, hence the t parameter. Unless otherwise stated, we assume no time dependence in the Lagrangian, which means this is an isolated system with no external influence.

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Stationary action and the assumption that the system moves in the optimal physical path leads to the Euler-Lagrange equation, which describes the motion:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

Much can be gleaned from this formulation of the dynamics; however, in some sense, it is more natural to look at the dual notion called the Hamiltonian, which is the Legendre transform of $\mathcal{L}$:

Definition 2.3. The *Hamiltonian* is a map of the cotangent bundle $T^*V \to \mathbb{R}$ given by:

$$H(\mathbf{q}, \mathbf{p}) = \mathbf{p} \cdot \dot{\mathbf{q}} - L(\mathbf{q}, \dot{\mathbf{q}})$$

We think of $\mathbf{p}$ being the generalized momentum. The Hamiltonian represents the energy of a system, which is conserved by our isolation assumption.

The Hamiltonian satisfies the following differential equations of motion, which are equivalent to the Euler-Lagrange equation:

$$\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}$$

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}$$

One of the first advantages of the Hamiltonian formalism is that its equations of motion are ordinary differential equations, whereas the Euler-Lagrange equation is higher order. A solution to these equations is a curve on the cotangent bundle, often called *phase space*, and it is called periodic if after some time it returns to the same point in T^*V.

3. Symplectic structures and Hamiltonian flow

Definition 3.1. A *symplectic manifold* is a manifold M of even dimension equipped with a closed, non-degenerate 2 form ω.

The cotangent bundle, which is the domain of H, is even dimensional and inherits a natural 1 form λ_0, which comes from the derivative of the canonical projection $\pi : T^*V \to V$ taking $(\mathbf{p}, \mathbf{q}) \mapsto \mathbf{q}$. Namely, at a point $(\mathbf{p}, \mathbf{q})$, we get a sequence of maps:

$$T_{(\mathbf{p},\mathbf{q})}(T^*V) \xrightarrow{d\pi} T_{\mathbf{q}}V \xrightarrow{\mathbf{p}} \mathbb{R}$$

Then we define $\lambda_0(\zeta) = \mathbf{p}(d\pi(\zeta))$. Setting $\omega = d\lambda$, we get a 2 form on T^*V. This is closed by the co-cycle condition $d^2 = 0$, and one can check that this is non degenerate. Thus T^*V is a symplectic manifold. The forms λ_0 and ω in coordinates are:

$$\lambda_0 = \frac{1}{2}\sum_{i=1}^{n} q_i dp_i - p_i dq_i$$

$$\omega = d\lambda_0 = \sum_i dp_i \wedge dq_i$$

Any 2 form along with a function $H : T^*V \to \mathbb{R}$ determines a vector field X_H, which can be defined to be the vector field that satisfies:

$$\omega(X_H(\mathbf{p}, \mathbf{q}), \ \cdot) = dH(\cdot)$$

Explicitly, X_H is ΩdH, where $\Omega = \omega^{-1}$ when thought of as an isomorphism of T^*V and TV sending $(\mathbf{p}, \mathbf{q}) \to (\mathbf{q}, \dot{\mathbf{q}})$. In these coordinates, Ω is the symplectic matrix:

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

The vector field X_H induces a one parameter group of diffeomorphisms of phase space $\{g_t\}$ such that:

$$\left.\frac{dg_t(\cdot)}{dt}\right|_{t=0} = X_H(\cdot)$$

It is easy to check that flow lines of $\{g_t\}$ are exactly solutions to Hamilton's equations of motion.

As this derivation shows, the Hamiltonian flow is a natural consequence of the symplectic structure of phase space. An important property of this flow that we will make use of will be its preservation of the symplectic form ω.

Proposition 3.2. *The flow $\{g_t\}$ preserves ω:*

$$g_t^*\omega = \omega \quad \forall t$$

In the case where $\dim(V) = 1$ and ω is an area form, this says that the flow preserves area in T^*V, which is known as Liouville's Theorem.

When H is time independent, the energy of the system is conserved and the orbit of a point under $\{g_t\}$ lies on the constant energy submanifold $S = H^{-1}(c)$ for c a regular value of H. One question is: can we restrict our study to S and still maintain the useful property of pereserving ω?

A flow $\{A_t\}$ from a vector field X on a manifold M induces a flow on any submanifold N; namely $\{A_t|_N\}$. However, it is not generally the case that this inherited flow is due to the restricted vector field $X\big|_N$, as it may not be tangent to N. What is special about the Hamiltonian flow is that this actually *is* the case for the constant energy surface S. Indeed, we note that, for any $\eta \in T^*V$:

$$dH_\eta(X_H(\eta)) = \omega(X_H(\eta), X_H(\eta)) = 0$$

and so $X_H(\eta) \in \ker(dH_\eta) = T_\eta S$. This means X_H is tangent to S, and so the flow of X_H on S coincides with the inherited flow on the parent manifold T^*V. Because of this, the flow on S also preserves ω, which means restricting our study to S keeps the useful properties, and, as we will see, simplifies our search for periodic orbits.

4. Poincaré Sections

An important method through which the Hamiltonian flow can be understood is a Poincaré section.

Definition 4.1. If M is a manifold of dimension n with a smooth flow, a *Poincaré section* is a submanifold $p^{-1}(c)$, where c is a regular value of a map $p : M \to \mathbb{R}$, that is transverse to the flow.

In particular, a Poincaré section is an $n-1$ dimensional submanifold on which the trajectories of orbits are not tangent. Such a surface on phase space equips us to look at the trajectories of particles in a "strobe light" manner by looking at successive intersections of the flow with the surface. To make this rigorous, we define the Poincaré return map:

Definition 4.2. If M has a flow $\{g_t\}$ with Poincaré section $\mathcal{P}$, the *Poincaré return map* $T : \mathcal{P} \to \mathcal{P}$ is:

$$T(\mathbf{x}) = g_{t_0}(\mathbf{x})$$

where t_0 is the first positive time for which $g_t(\mathbf{x}) \in \mathcal{P}$. We will denote $T^n(\mathbf{x}) \equiv T(...(T(\mathbf{x}))...)$. In this language, a periodic orbit is any orbit intersecting a point $\mathbf{x} \in \mathcal{P}$ such that $T^n(\mathbf{x}) = \mathbf{x}$ for some n.

4.1. **Example.** To get a sense of how Poincaré sections work, we look at a particle in two dimensions under the potential:

$$V(x, y) = \underbrace{\frac{x^2}{2} + \frac{y^2}{2}}_{\text{Oscillator}} + \lambda \left(x^2 y - \frac{y^3}{3} \right)$$

This is known as the Hénon Heiles potential. It is a perturbation by small parameter λ of the potential of a harmonic oscillator. The Hamiltonian of the system is:

$$H = \frac{1}{2} \left(p_x^2 + p_y^2 + x^2 + y^2 \right) + \lambda \left(x^2 y - \frac{y^3}{3} \right)$$

Though any particular solution will be bounded, any real pair $(x, y) \in \mathbb{R}^2$ is a possible configuration, so $V = \mathbb{R}^2$ and $T^*V = \mathbb{R}^4$. Another useful property is the homogeneity of the perturbation. Scaling each parameter by a factor of λ yields a Hamiltonian $H' = H(\lambda x, \lambda y, \lambda p_x, \lambda p_y)$ such that:

$$H' = \lambda^2 H$$

And so our choice of λ doesn't really affect the motion, so we choose $\lambda = 1$. The energy surface $S = H^{-1}(c)$ is a 3 manifold, which we can treat as $\mathbb{R}^3$ for solutions near the origin. Additionally, solutions near the origin have the property of being nearly integrable, since the perturbed part of the potential is of degree 3 and therefore small compared to the degree 2 oscillator term. One property of such systems is that the orbits are confined to a surface diffeomorphic to a torus. We can see this in the left of Figure 1[1].

This solution is shown intersecting a Poincaré section $y = 0$ and it is one of many known periodic orbits of the Hénon Heiles system, discussed in [2]. Using `NDSolve` in Mathematica, we can look at the periodic points on the Poincaré surface. This is shown in the right of Figure 1

For solutions near the origin, periodic orbits aren't too hard to find, since the method of action angle variables on a (nearly) integrable system provides precise tools to do so. However, this system further from the origin is no longer constrained to the invariant

[1]Thanks to Prof. Yin for assistance with the graphics

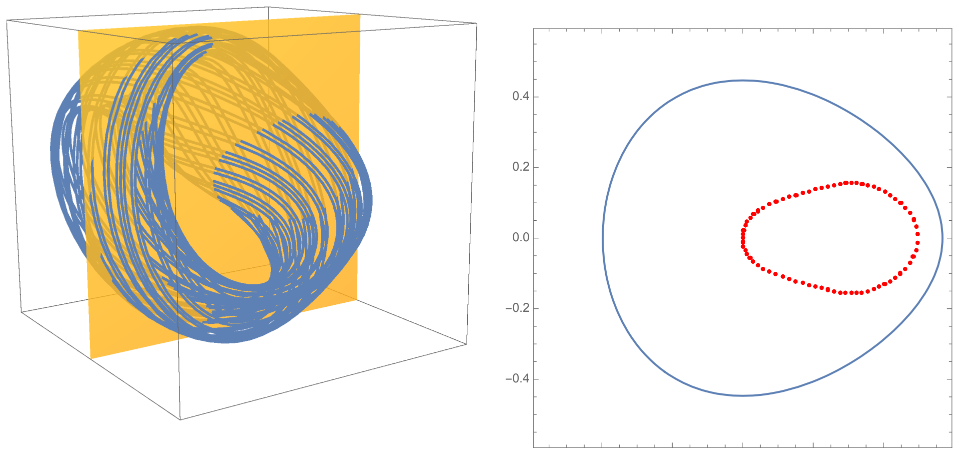

Figure 1. *Left*: A periodic orbit in the Hénon Heiles potential lying on a torus. *Right*: The corresponding Poincaré section.

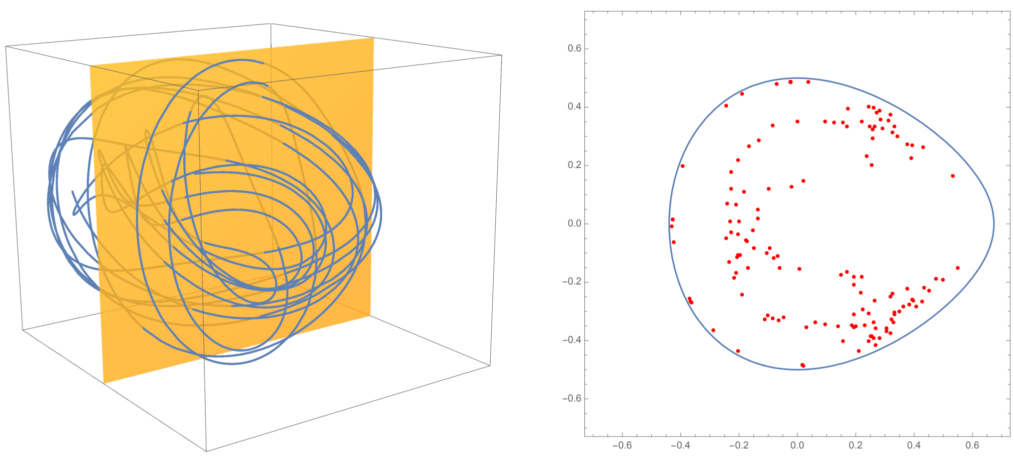

Figure 2. Non periodic orbits unconstrained by tori

tori (see Figure 2), and hence periodic orbits can't be found analytically. This system provides a good example of the difficulty of periodic orbits in non-integrable, and hence arbitrary, Hamiltonians.

5. Convex energy surfaces and Surfaces of Section

Our choice of Poincaré section in the previous example was fairly intuitive, as the Hénon orbits are nearly on tori and a transverse surface is naturally fixing a coordinate. However, it is not always the case that such a section will be easily available. In fact, it is a very nontrivial problem to find such a section for an arbitrary Hamiltonian.

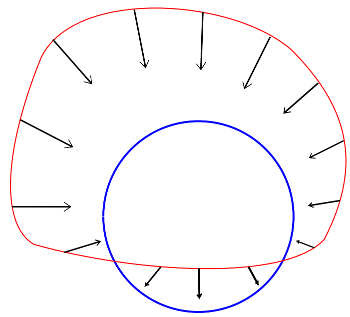

FIGURE 3. Stretching a convex manifold to a sphere radially.

We will outline an important result due to Hofer et al. [1] that, under certain conditions, guarantees such a surface. Once such a surface has been obtained, we will show as in [1] that it conjugates to an area preserving map of of the open unit disk D, which in turn allows us to conclude that there are either 2 or infinitely many periodic orbits. In this discussion, we restrict to the two dimensional particle, where $V = \mathbb{R}^2$ and $T^*V = \mathbb{R}^4$. First, we define the generalized notion of the Poincaré section.

Definition 5.1. If X_H is a smooth Hamiltonian vector field on a three manifold M, a *global surface of section* for X_H is an embedded compact surface $\Sigma \subset M$ for which the flow on $\partial\Sigma$ is a periodic orbit and whose interior $\dot{\Sigma}$ is transverse to the X_H. It also must have the property that every orbit of the flow $\{g_t\}$ intersects $\dot{\Sigma}$ in forward and reverse time.

With such a surface of section, we define the Poincaré return map $\varphi : \dot{\Sigma} \to \dot{\Sigma}$ as before. Turning our attention to the Hamiltonian $H : \mathbb{R}^4 \to \mathbb{R}$, recall the constant energy surface $S = H^{-1}(c) \subset \mathbb{R}^4$. This is a 3 manifold to which the restricted flow $\{g_t\}\big|_S$ of X_H is tangent. We seek a global surface of section on S with which we can understand the orbits of the system. A critical part of the argument in [1] is that S be convex.

Definition 5.2. A set $A \subset \mathbb{R}^n$ is *convex* if, for any $x, y \in A$, the straight line between x and y is entirely contained in A. A manifold X is convex if it bounds a convex set.

If $X = f^{-1}(c)$ for c a regular value of a smooth map $f : \mathbb{R}^n \to \mathbb{R}$, then it has no boundary and hence we can ask if it is convex. It is convex if the Hessian $\nabla^2 f(\mathbf{x})$ is positive definite for all $\mathbf{x} \in \mathbb{R}^n$. A nice property of convex manifolds containing the origin is that they can be "stretched" into the unit sphere (see Figure 3). That is, there is a diffeomorphism $\phi : S^{n-1} \to X$ given by $h(\mathbf{x})\mathbf{x}$, where $h : S^{n-1} \to \mathbb{R}^+$ is smooth[2].

In the case of a convex energy hypersurface $S = H^{-1}(c)$, there is such a diffeomorphism to S^3. Using this particular map, Hofer et al. proceed to consider the so-called Reeb vector field on S^3 induced by the Hamiltonian flow X_H on S via ϕ. There is an induced form λ on S^3 which is pulled back by ϕ from the symplectic structure on S. Namely, if

[2]There are very clearly non convex manifolds that still have this property; these are sometimes called *star-like*.

λ_0 is the canonical 1 form on S, then $\lambda = \phi^*\lambda_0$. Computing:

$$\begin{aligned}\lambda = \phi^*\lambda_0 = \phi^*\left(\frac{1}{2}\sum_{i=1}^{2} q_i dp_i - p_i dq_i\right)\\ = \frac{1}{2}\sum_{i=1}^{2} \phi^*(q_i)\phi^*(dp_i) - \phi^*(p_i)\phi^*(dq_i)\end{aligned}$$

Seeing that $\phi = (hp_1, hp_2, hq_1, hq_2)$, this becomes:

$$\begin{aligned}\phi^*\lambda_0 &= \frac{1}{2}\sum_{i=1}^{2} hq_i d(\phi^* p_i) - hp_i d(\phi^* q_i)\\ &= \frac{h^2}{2}\sum_{i=1}^{2} q_i dp_i - p_i dq_i\\ &= h^2\lambda_0\end{aligned}$$

So, the Reeb flow on S^3 has the natural association to the form $\lambda = h^2\lambda_0$. What is more, the [1] show that Reeb flow X_λ is equivalent to the Hamiltonian flow X_H. We now state the main result proved in [1]:

Theorem 5.3. *Let X_λ be the flow on S^3 from a convex hypersurface S with 1 form $\lambda = f\lambda_0$ for $f : S^3 \to \mathbb{R}^+$ smooth. Then there is a periodic orbit P_0 with the following properties:*

(1) P_0 is the boundary of a global surface of section $\mathcal{D} \subset S^3$ diffeomorphic to a compact disk.
(2) The Poincaré map $\varphi : \dot{\mathcal{D}} \to \dot{\mathcal{D}}$ preserves the symplectic structure: $\varphi^* d\lambda = d\lambda$.

Hofer et al. prove the this by explicitly constructing such a surface of section using the theory of holomorphic curves, which is beyond the scope of this paper. The usefulness of the first part of this theorem should be clear: a convex S has a periodic orbit P_0. However, the more striking fact is that it comes naturally with a surface of section $\mathcal{D}$ on which the Poincaré map is symplectic with respect to λ. We will see that this can be conjugated to an area preserving map of the open unit disk, which in turn will hand us even more periodic orbits.

6. Area preserving maps of the Unit disk and Annulus

We now digress to the problem of understanding the consequence of a symplectic Poincaré map of a disk. The following proposition from [1] shows that any 2 form $f dx \wedge dy$ on the open unit disk D can be conjugated to a map preserving an area form $c dx \wedge dy$ for a constant c:

Proposition 6.1. *Let $f dx \wedge dy$ be a 2 form on D with f is integrable on D. Then there exists a diffeomorphism $\tau : D \to D$ satisfying:*

$$\tau^*(f dx \wedge dy) = c dx \wedge dy$$

If the interior $\dot{\mathcal{D}}$ of the surface of section is diffeomporphic to D via $\sigma : D \to \dot{\mathcal{D}}$, then $\mu = d\lambda$ pulls back to a two form $\sigma^*\mu$, which we write as $f dx \wedge dy$. Then, by the above proposition, we have the following conjugation of maps:

$$\begin{tikzcd} \dot{\mathcal{D}} \arrow[r, "\varphi"] & \dot{\mathcal{D}} \arrow[d, "\sigma^{-1}"] \\ D \arrow[u, "\sigma"] & D \arrow[d, "\tau^{-1}"] \\ D \arrow[u, "\tau"] \arrow[r, "h"] & D \end{tikzcd}$$

Where $h = \tau^{-1}\sigma^{-1}\varphi\sigma\tau$ and $h^*(cdx \wedge dy) = cdx \wedge dy$ for some c. We now have an area perserving map h of the open disk, to which we apply a version of Brouwer's Translation Theorem:

Theorem 6.2. *If h is an orientation preserving homeomorphism of the unit disk with no fixed points, then there is a set $U \subset D$ such that $h^i(U) \cap h^{i+1}(U) = \varnothing$ for all $i \geq 1$.*

Suppose our conjugation h had no fixed points; then by above the union $\mathcal{U}$ of the images of U under h would have measure:

$$m(\mathcal{U}) = m\left(\bigcup_{i\geq 1} h^i(U)\right) = \sum_{i\geq 1} m(h^i(U))$$

But h preserves measure, and U has positive measure so:

$$m(\mathcal{U}) = \sum_{i\geq 1} m(U) = \infty$$

However, $\mathcal{U} \subset D$, which has finite measure, so this is a contradiction. Therefore h has a fixed point ζ, and therefore φ has a fixed point $\mathbf{z} = \sigma(\tau(\zeta))$. Now we appeal to a more recent theorem due to Franks [3]:

Theorem 6.3. *Let $f : A \to A$ be a homeomorphism of the open annulus that preserves the measure m. Then if f has one periodic point, it has infinitely many.*

Restricting h to $D - \{\mathbf{z}\}$ gives us such a map, and therefore we conclude that h on the disk has either 1 or infinitely many periodic points. This shows that the Hamiltonian flow has either 2 or infinitely many periodic orbits. The extra periodic orbits we've obtained are all transverse to $\dot{\mathcal{D}}$, so they are "linked" loops to the spanning orbit P_0.

7. Convex Physical Systems

We return to the Hénon Heiles system. We see that the Hamiltonian has four critical points:

$$\nabla H = \langle x + 2xy, x^2 + y - y^2, p_x, p_y \rangle = [0] \Longrightarrow (x, y) = (0,0), (0,1), \left(\pm\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$$

Three of these occur at energy $c = \frac{1}{6}$ (the other at 0, which isn't interesting). This represents the energy at which the surface $H^{-1}(c)$ splits from 3 into 4 connected components.

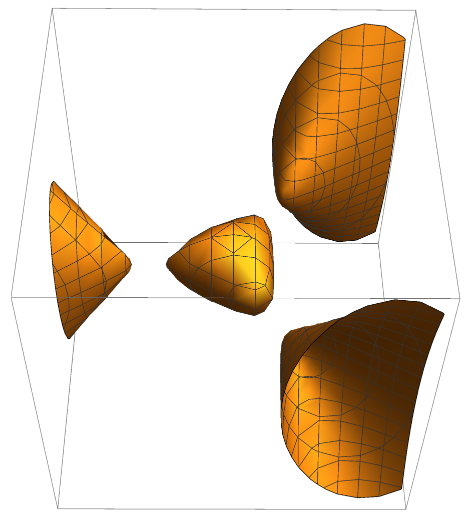

FIGURE 4. Level surface of $H^{-1}(c)$ with $c < \frac{1}{6}$.

As shown in Figure 4, we might expect the component near the origin to be convex. Computing the eigenvalues of $\nabla^2 H$:

$$\nabla^2 H = \begin{pmatrix} 1+2y & 2x & 0 & 0 \\ 2x & 1-2y & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Longrightarrow \lambda_i = \left\{1, 1 - 2\sqrt{x^2+y^2}, 1 + 2\sqrt{x^2+y^2}\right\}$$

These are positive when $x^2+y^2 < \frac{1}{4}$, which indeed is true for the component near the origin at sufficiently small energies. Therefore, for appropriately chosen c, Theorem 5.3 applies to the Hénon Heiles system and we expect periodic orbits (this isn't really necessary, since we were able to find a surface of section without any trouble).

Another interesting physical system is the planar circular restricted 3-body problem. Consider an object orbiting two bodies that are separated by a unit distance in the plane. Further assume that one of the bodies is stationary and the other is orbiting the first circularly, and that the object orbiting both of them is of negligible mass compared to the other two. For example, this describes the Earth-Moon-Sun system to a reasonable approximation. Boosting to a rotating frame so that the only motion is that of the satellite, this system has Lagrangian:

$$\mathcal{L} = \frac{1}{2}\left((\dot{x} - y)^2 + (\dot{y} - x)^2\right) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$$

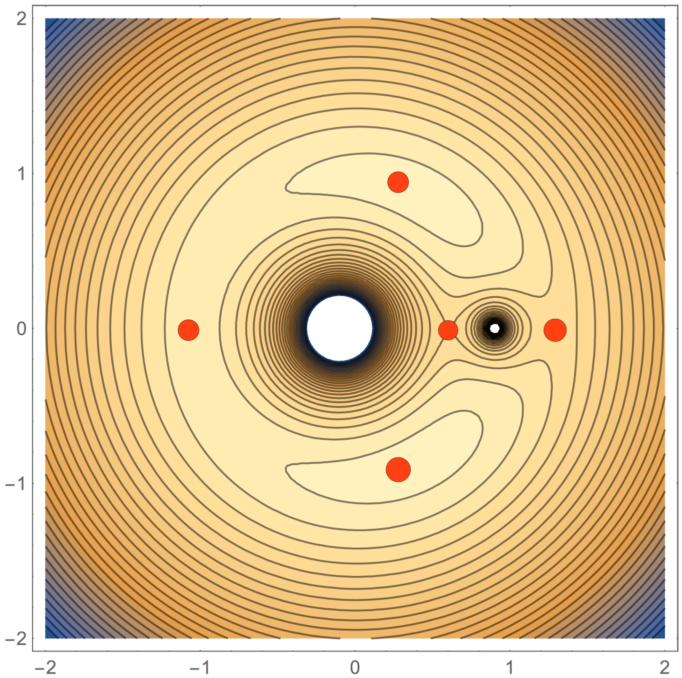

FIGURE 5. Contour of the effective potential V, Lagrange points shown in red.

Where $\mu = \frac{m_1}{m_1+m_2}$ and $r_1^2 = (x+\mu)^2 + y^2$ and $r_2^2 = (x-(1-\mu))^2 + y^2$. The Legendre transform of $\mathcal{L}$ gives the Hamiltonian:

$$\begin{aligned} H &= \frac{1}{2}\left(p_x^2 + p_y^2\right) + p_x y - p_y x - \frac{\mu}{r_1} - \frac{1-\mu}{r_2} \\ &= \underbrace{\frac{1}{2}\left((p_x+y)^2 + (p_y-x)^2\right)}_{T} \underbrace{- \frac{1}{2}\left(x^2+y^2\right) - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}}_{V} \end{aligned}$$

Where T and V are the effective potential and kinetic energies. The critical points of the Hamiltonian are called *Lagrange points*, of which there are 5 (shown in Figure 5).

The energy surface $H^{-1}(c)$ for $c < H(L_1)$, where L_1 is the lowest energy Lagrange point, has three connected components: a sphere near each of the bodies, and surface at infinity. The sphere near the earth, called the hill sphere, is one of great interest. Albers et al. [4] show that, for every energy level $c > \frac{3}{2}$, there exists a minimal mass ratio μ_0 such that a global suface of section of $H^{-1}(c)$ exists for all $\mu \in (\mu_0, 1)$.

Remark 7.1. It should be noted that many results like these can be derived using classical perturbation theory. After all, the Hénon Heiles potential is a perturbation of an oscillator and the restricted 3-body problem is a perturbation of the well-known 2-body problem. However, as I hope we've shown, the surface of section method is very powerful and beautifully demonstrates the generality of these results.

References

[1] H. Hofer, K. Wysocki, E. Zehnder. *The Dynamics of Three-Dimensional Stricly Convex Energy Surfaces*. Annals of Mathematics, Second Series, Vol. 148, No. 1. pp 197-289.

[2] J. Ozaki, S. Kurosaki. Department of Physics, Okayama University. *Periodic Orbits of Hénon Heiles Hamiltonian*. Profress of Theoretical Physics, Vol. 95, No. 3. pp. 519-529.

[3] J. Franks. *Area preserving homeomorphisms of open surfaces of genus zero*. New York Journal of Mathematics. Vol 2 pp. 1-19.

[4] P. Albers, J. Fish, U. Frauenfelder, H. Hofer, O. Van Koert. *Global Surfaces of Section in the Planar Restricted 3-Body Problem*. Archive for Rational Mechanics and Analysis. pp 273-284.