Wavelets and Signal Processing

John E. Gilbert

Mathematics in Science Lecture

April 30, 2002.
Motivation for wavelets and contributions to the theory of wavelets came from seismology, physics, and signal processing as much as from mathematics itself. In turn, this means that there are many varied applications of the mathematics underlying wavelets. This talk will describe some basic wavelet ideas and how they can be used in signal analysis and data detection.

Tuesday, April 30, 4pm
RLM 9.166
Math isn’t about “real” things in daily life

versus: way to hear music or speech; read EKG’s; transmit data; image enhancement.
Misconceptions

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  versus: way to hear music or speech; read EKG’s; transmit data; image enhancement.

- Math results are abstract
  versus: can see results, e.g. compression; hear them, e.g. de-noising; reconstruct images, e.g. extrapolation.
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versus: way to hear music or speech; read EKG’s; transmit data; image enhancement.

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versus: can see results, e.g. compression; hear them, e.g. de-noising; reconstruct images, e.g. extrapolation.

■ Math amounts to finding right formula or right theorem.

versus: iterative series of explorations, asking questions, getting back answers, refining techniques, approximations.
Mathematically, a signal is a function

- of continuous variable(s) = analogue
- of discrete variable(s) = digital
Signal = Function

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one variable: audio signals, sensor outputs, heart beats, ...

two variables: video images (still, moving), MRI, ...
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Fundamental concerns:

- Compression for storage or transmission, de-noising
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- Data-mining, Edge-detection in (possibly) noisy signals
Audio signals

Caruso is just a few mouse clicks away

- original
- denoised
- noise
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Represent noisy signal graphically:

\[ \text{original signal} = \text{clear signal} + \text{noise} \]
Some basic features of images are shown in

- Barbara
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ridges, edges. Same for fingerprints.

In images transients may depend on direction (think fingerprints).
Compress fingerprints

- each one $\sim 1$ MB. storage, transmission, security
Compress fingerprints

- each one \( \sim 1 \text{ MB} \). storage, transmission, security

Can probably do much better with specialized software.
real data. Need to analyze distance between peaks.

Monitor patients; immediate detection onset heart irregularities = transients.

Basic problem: edge detection.
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Monitor patients; immediate detection onset heart irregularities = transients.

Basic problem: edge detection.

tumor

Tumor malignant when spikes present.

Basic problem: (directional) edge detection.
Standard idea: given mathematical object, represent it with respect to carefully chosen basis.
Bases, Representations

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Vectors in $\mathbb{R}^3$: usual Cartesian basis $i, j, k$

$$v = a_1i + a_2j + a_3k \sim (a_1, a_2, a_3).$$

Measure size ($= \text{‘energy’}$) by ‘dot product’ of vectors.
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In high dimensions achieve compression by projecting onto low-dimensional subspace. Loss of energy?
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Maybe need to change basis depending on how subspace lies.
Bases from Calculus

Need bases having good diff and integral properties. Choose monomials.
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Dot product

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f.g = f \left( \frac{d}{dx} \right) g(x) \big|_{x=0}
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Orthonormal basis

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\frac{1}{\sqrt{n!}} x^n, \quad n = 0, 1, 2, \ldots.
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**Orthonormal basis**

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**Energy**

\[
\text{Energy}(f) = \sum_{n=0}^{\infty} \frac{1}{n!} (f^{(n)}(0))^2.
\]
Taylor polynomials

\[ P_N f(x) = \sum_{n=0}^{N} \frac{1}{n!} f^{(n)}(0) x^n \]

provide compression with little loss of energy if ‘tail’

\[ \sum_{n=N+1}^{\infty} \frac{1}{n!} (f^{(n)}(0))^2 \]

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of energy is small.

Usually don’t measure error

\[ f(x) - P_N f(x) = \sum_{n=N+1}^{\infty} \frac{1}{n!} f^{(n)}(0) x^n \]

by loss of energy, but by pointwise maximum.
Simple linear transformation \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \). In matrix form with respect to usual basis of plane

\[
T \sim \begin{bmatrix}
a & b \\
c & d
\end{bmatrix}, \quad T : \begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} a + 2b \\ c + 2d \end{bmatrix}.
\]
Compression of matrices

Simple linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. In matrix form with respect to usual basis of plane

$$T \sim \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad T : \begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} a + 2b \\ c + 2d \end{bmatrix}. $$

$$\text{Energy}(T) = a^2 + b^2 + c^2 + d^2 = \text{trace} \left( T T^{\text{trans}} \right).$$

(independent of basis).
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Energy ($T$) $= a^2 + b^2 + c^2 + d^2 = \text{trace} (TT^{trans})$.

(independent of basis).

■ compression when diagonalized by orthogonal matrix

$T \sim \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

by changing to new basis of eigenvectors.

No loss of energy!!
Example $T$

Usual basis: $T \sim \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$  

New basis: $T \sim \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$

Compress: $T \approx \begin{bmatrix} 9 & 0 \\ 0 & 0 \end{bmatrix}$

compression of 75% with less than 1.25% loss of energy.
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Basic wavelet ideas

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Building blocks = bases: \[ f = \sum_n \gamma_n \psi_n \]
Basic wavelet ideas

Most data sets have **correlation in time (space) and frequency.**

- **Wavelets:** building blocks to quickly decorrelate data.

- **Building blocks = bases:** \( f = \sum_n \gamma_n \psi_n \)

- **Decorrelate data:** wavelets to resemble given data.
Basic wavelet ideas

Most data sets have correlation in time (space) and frequency.

Wavelets: building blocks to quickly decorrelate data.

Building blocks = bases: \[ f = \sum_n \gamma_n \psi_n \]

Decorrelate data: wavelets to resemble given data.

signal: \[ \left\{ f \left( \frac{n}{256} \right) \right\}_{n=0}^{255} \]

256 sample values
Wavelets = ‘little waves’

Simplest wavelet functions introduced by: Haar (1910)

\[
\int_{-\infty}^{\infty} \phi(x) \, dx = 1, \\
\int_{-\infty}^{\infty} \psi(x) \, dx = 0.
\]

scaling function $\phi$  wavelet function $\psi$
Wavelets = ‘little waves’

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Scaling function \( \phi \)  
Wavelet function \( \psi \)

Basic relations:

\[
\phi(x) = \phi(2x) + \phi(2x - 1), \quad \psi(x) = \phi(2x) - \phi(2x - 1)
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scaling function \( \phi \) \hspace{1cm} \text{wavelet function} \( \psi \)

Basic relations:

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\[
\phi(2x) = \frac{1}{2} \left( \phi(x) + \psi(x) \right), \quad \phi(2x - 1) = \frac{1}{2} \left( \phi(x) - \psi(x) \right)
\]
Decompose into hierarchical set of approximations and details
Decompose into **hierarchichal** set of approximations and details

 hierarchy: decreasing resolution = increasing coarseness.
wavelet analysis, synthesis

Decompose into **hierarchichal** set of approximations and details

- **Hierarchichal:** decreasing resolution = increasing coarseness.

**Digitized signal**

\[ \sum_{n=0}^{255} f \left( \frac{n}{256} \right) \phi(2^8 x - n) \]

**Level 8 resolution**
Coarse + detail idea

Use relation

\[ \phi(2x) = \frac{1}{2}(\phi(x) + \psi(x)), \quad \phi(2x - 1) = \frac{1}{2}(\phi(x) - \psi(x)) \]

- algebraically:

\[ a\phi(2x) + b\phi(2x - 1) = \frac{1}{2}(a + b)\phi(x) + \frac{1}{2}(a - b)\psi(x) \]

- graphically:

![Diagram showing the addition of original, coarse, and fine details]

original  \hspace{1cm} coarse detail  \hspace{1cm} +  \hspace{1cm} fine detail
Hunt the Transients

simple signal

noisy signal

transients

transients in noise
Digitized signal \[ \left\{ f\left( \frac{n}{256} \right) \right\}_{n=0}^{255} \]

- **Split signal:** at resolution 7

- **Now split level 7 coarse detail;**

- **then split level 6 coarse detail.**
From level 8 down to level 5

course detail level 5

fine detail level 6

fine detail level 7

fine detail level 5
Wavelet functions are related by scaling equations.

In general: filter coefficients $h_0, h_1, \ldots, h_N$

\[
\phi(x) = \sum_{k=0}^{N} h_n \phi(2x - k), \quad \psi(x) = \sum_{k=0}^{N} (-1)^k h_{1-2k} \psi(2x - k).
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Coarse + detail:

$$f(x) = \sum_n a_n^{(1)} \phi(2x - n) = \sum_n a_n^{(0)} \phi(x - n) + \sum_n d_n^{(0)} \psi(x - n)$$
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\]

where $a_n^{(0)}$ running average, $d_n^{(0)}$ running difference, of $a_n^{(1)}$

using $h_0, h_1, \ldots, h_N$. 
Ingrid Daubechies (1988)

- **Constructed** smooth, compactly supported wavelets
  - with \( N = 3, 5, 7, \ldots \) (Haar \( \sim N = 1 \))
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![db2 scaling function](image1)

$h_0 = \frac{1 + \sqrt{3}}{4}, \ h_1 = \frac{3 + \sqrt{3}}{4}, \ h_2 = \frac{3 - \sqrt{3}}{4}, \ h_3 = \frac{1 - \sqrt{3}}{4}$
Ingrid Daubechies (1988)

**Constructed smooth, compactly supported wavelets with** $N = 3, 5, 7, \ldots$ (Haar $\sim N = 1$)

$\begin{align*}
  h_0 &= \frac{1+\sqrt{3}}{4}, \quad h_1 = \frac{3+\sqrt{3}}{4}, \quad h_2 = \frac{3-\sqrt{3}}{4}, \quad h_3 = \frac{1-\sqrt{3}}{4}
\end{align*}$

**How does one arrive at these numbers?**
Image restoration

Done by Maia Croft as Honors Thesis at McQuarie University.
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Wavelets can be used to add to, fill-in images. Extend ideas to image fusion, morphing?
Complex signals; speech

General Fourier advises that excessive use of wavelets is dangerous to your success: need more sophisticated decompositions.

- At each stage choose to split coarse detail or fine detail.

- Need ‘cost function’ to determine which is ‘cheaper’.
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  • 10%wavelet
  • 10%wavepacket
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- Wavelets exploit: scaling (resolution), translation.
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- Wavelets exploit: scaling (resolution), translation.
- Wavelet packets add: frequency.
In Summer 2002, Professor John Gilbert will present a six-week intensive course on Fourier series, discrete Fourier transforms, spectrograms and wavelets.

The course will discuss Haar wavelets, and use them as a means of analyzing and also compressing data. The course will also introduce the Daubechies wavelets, and apply them to reducing noise or extracting music from LP recordings. We also give applications to image compression (the FBI fingerprint databank) and image reconstruction.

The course is open to all UT Austin undergraduates. In addition, funds are available to support 13 undergraduates for the six weeks, at $2400 per person.

Applications, topics, and reading lists are all online at:

http://www.ma.utexas.edu/vigre/reu