Chapter 4

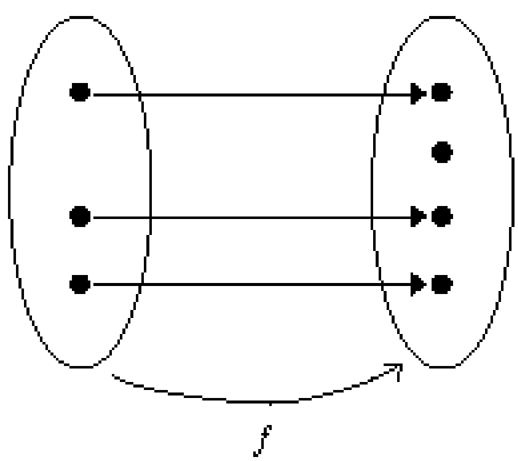
TRANSFORMATIONS

4.1 TRANSFORMATIONS, ISOMETRIES. The term *transformation* has several meanings in mathematics. It may mean any change in an equation or expression to simplify an operation such as computing a derivative or an integral. Another meaning expresses a functional relationship because the notion of a function is often introduced in terms of a mapping

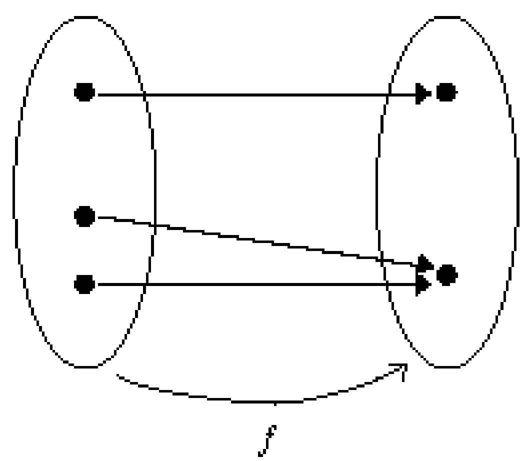
$$f : A \to B$$

between sets A and B; for instance, the function $y = x^2$ can be thought of as a mapping $f : x \to x^2$ of one number line into another. On the other hand, in linear algebra courses a *linear* transformation maps vectors to vectors and subspaces to subspaces. When we use the term transformation in geometry, however, we have all of these interpretations in mind, plus another one, namely the idea that the transformation should map a geometry to a geometry. A formal definition makes this precise.

Recall first that if $f : A \to B$ is a mapping such that every point in the range of f has a unique pre-image in A, then f is said to be *one to one* or *injective*. If the range of f is all of B, then f is said to be *onto* or *surjective*. When the function is both one to one and onto, it is called a *bijection* or is said to be *bijective*. The figures below illustrate these notions pictorially.



one to one



onto

4.1.1 Definition. Let $\mathcal{G}_1 = (\mathcal{P}_1, \mathcal{L}_1)$ and $\mathcal{G}_2 = (\mathcal{P}_2, \mathcal{L}_2)$ be two abstract geometries, and let f : $\mathcal{P}_1 \to \mathcal{P}_2$ a function that is bijective. Then we say that f is a *geometric transformation* if f also maps $\mathcal{L}_1$ onto $\mathcal{L}_2$.

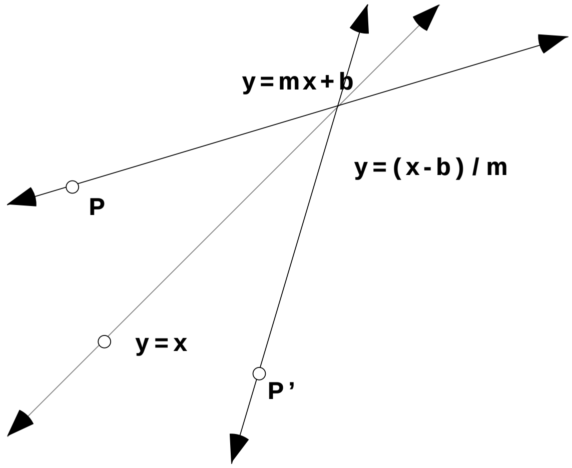
In other words, a 1-1 transformation $f : \mathcal{P}_1 \to \mathcal{P}_2$ is *geometric* if takes the set $\mathcal{P}_1$ of all points in $\mathcal{G}_1$ onto the set $\mathcal{P}_2$ of all points in $\mathcal{G}_2$, and takes the set $\mathcal{L}_1$ of all lines in $\mathcal{G}_1$ onto the set $\mathcal{L}_2$ of all lines in $\mathcal{G}_2$. It is this last property that distinguishes geometric transformations from more general transformations. A more sophisticated way of formulating definition 4.1.1 is simply to say that $f : \mathcal{G}_1 \to \mathcal{G}_2$ is bijective. Notice that the definition makes good sense for models of both Euclidean and hyperbolic geometries. For instance, we shall see later that there is geometric bijection from the model $\mathbf{H}^2$ of hyperbolic geometry in terms of lines and planes in three space and the Poincaré disk model **D** in terms of points and arcs of circles.

Some simple examples from Euclidean plane geometry make the formalism much clearer. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ both be models of Euclidean plane geometry so that $\mathcal{P}_1$ and $\mathcal{P}_2$ can be identified with all the points in the plane. For $f : \mathcal{P}_1 \to \mathcal{P}_2$ to be geometric it must map the plane onto itself, and do so in a 1-1 way, as well as map any straight line in the plane to a straight line. It will be important to see how such transformations can be described both algebraically and geometrically. It is easy to come up with functions mapping the plane onto itself, but it is much more restrictive for the function to map a straight line to a straight line. For example, $(x,y) \to (x,y^3)$ maps the plane onto itself, but it maps the straight line $y = x$ to the cubic $y = x^3$.

4.1.2 Examples. (a) Let

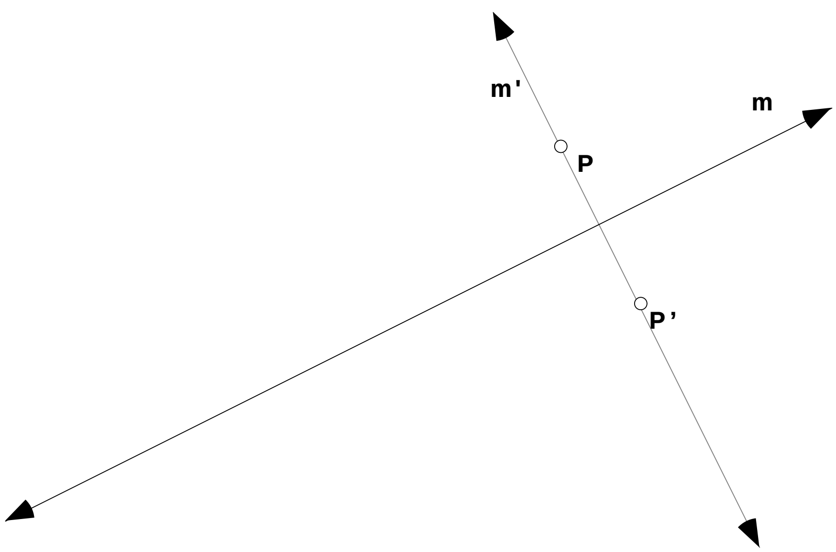
$$f : (x,y) \to (y,x)$$

be the function mapping any point $P = (x,y)$ in the plane to its reflection $P' = (y,x)$ in the line $y = x$. Since successive reflections $P \to P' \to P$ maps P back to itself, this mapping is 1-1 and maps the plane onto itself. But does it map a straight line to a straight line? Well the equation of a non-vertical straight line is $y = mx + b$. The mapping f interchanges x and y, so f maps the straight line $y = mx + b$ to the straight line $y = (x-b)/m$. Algebraically, f maps a non-vertical straight line to its *inverse*. Geometrically, f maps the graph of the straight line $y = mx + b$ to the graph of its straight line inverse $y = (x-b)/m$ as the figure below shows



One can show also that f maps any vertical straight line to a horizontal straight line, and conversely. Hence f maps the family of all lines in Euclidean plane geometry onto itself - hence f is a geometric transformation of Euclidean plane geometry.

(b) More generally than in (a), given any fixed line m, let f be the mapping defined by *reflection* in the line m. In other words, f maps any point in the plane to its 'mirror image' with respect to the *mirror line* m. For instance, when m is the x-axis, then f takes the point $P = (x, y)$ in the plane to its mirror image $P' = (x, -y)$ with respect to the x-axis. In general it is not so easy to express an arbitrary reflection in algebraic terms (see **Exercise Set 4.3**), but it is easy to do so in geometric terms. Given a point P, let m' be the straight line through P that is perpendicular to m. Then P' is the point on m' on the opposite side of m to P that is equidistant from m. Again a figure makes this much clearer



What is important to note here is that all these geometric notions make sense in hyperbolic geometry, so it makes good sense to define reflections in a hyperbolic line. This will be done in Chapter 5 where we will see that this hyperbolic reflection can be interpreted in terms of the idea of inversion as hinted at in the last section of Chapter 3.

(c) Let *f* be a *rotation* through 90˚ counter-clockwise about the some fixed point in the plane. In algebraic terms, when the fixed point is the origin, *f* is given algebraically by $f:(x,y) \to (-y,x)$. So *f* is 1-1 and maps the plane onto itself. What does *f* do to the straight line $y = mx + b$? (see **Exercise Set 4.3**)

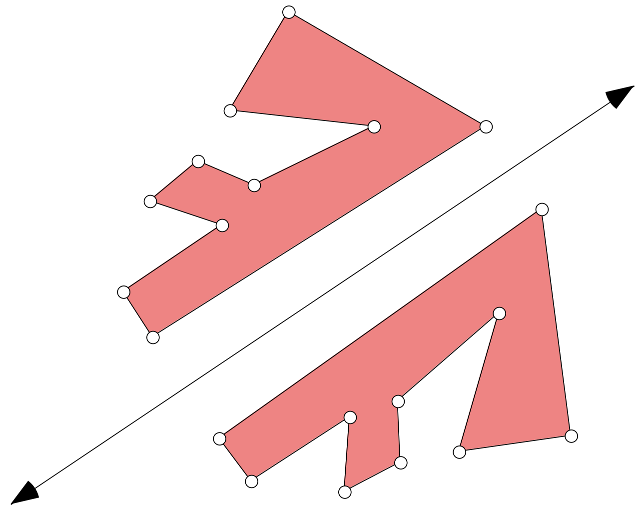
(d) Let *f* be a *translation* of the plane in some direction. Then *f* is given algebraically by $f:(x,y) \to (x+a, y+b)$ for some real numbers *a* and *b*. Again, it is clear that *f* is 1-1 and maps the plane onto itself.

Sketchpad is particularly useful for working with transformations because the basic transformations are all built into the program. We can use Sketchpad to look at the properties of reflections, rotations, and translations.

4.1.2a Demonstration.

- Open a new sketch on Sketchpad and draw a line. This will be the mirror line. Construct a polygon in the general shape of an “F”. Color its interior.
- To reflect the polygon across the mirror line, select the line and use the **Transform** menu to select “Mark Mirror”. Under the **Edit** menu, select “Select All”. Then under the **Transform** menu, select “Reflect”.
- Try dragging some of the vertices of the polygon to investigate the properties of reflection in the mirror line. What happens when the mirror line is dragged?

Your figure should look like the following:



The *orientation* of the reflected“F” is said to be *opposite* to that of the original “F” because the clockwise order of the vertices of the image is the reverse of the clockwise order of the vertices of the pre-image. In other words, a reflection *reverses* orientation.

- Measure the area of each image polygon and its pre-image. Measure corresponding side lengths. Measure corresponding angles. Check what happens to your measurements as the vertices of the pre-image are dragged. What happens to the measurements when the mirror line is dragged? Now, complete **Conjecture 4.1.3.**

End of Demonstration 4.1.2a.

4.1.3 Conjecture. Reflections ________________ distance, angle measure and area.

4.1.4 Definition. A geometric transformation f of the Euclidean plane is said to be an *isometry* when it preserves the distance between any pair of points in the plane. In other words, f is an isometry of the Euclidean plane, when the equality $d(f(a), f(b)) = d(a,b)$ holds for every pair of points a, b in the plane.

By using triangle congruences one can prove the following.

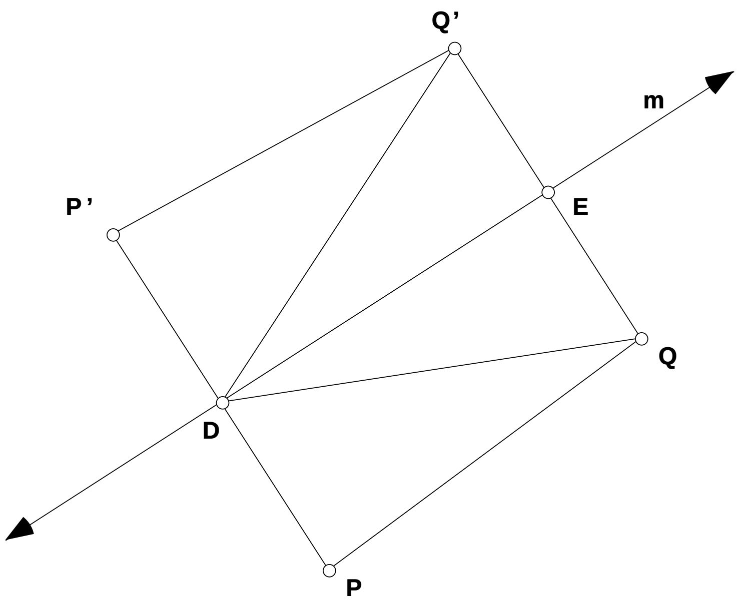
4.1.5 Lemma. Any isometry preserves angle measure.

The earlier Sketchpad activity supports the conjecture that every reflection of the Euclidean plane is an isometry. A proof of this can be given using congruence properties.

4.1.6 Theorem. Every reflection of the Euclidean plane is an isometry.

Proof. In the figure below P and Q are arbitrary points, while P' and Q' are their respective images with respect to reflection in the mirror line *m*. D and E are the intersection points between the mirror line and the segments $\overline{PP'}$ and $\overline{QQ'}$. For convenience we have assumed that P, Q lie on the same side of the mirror line. Use the definition of a reflection to show first that ΔEDQ is congruent to $\Delta EDQ'$, and hence that $\overline{DQ}$ is congruent to $\overline{DQ'}$. Now use this to show that ΔPDQ is congruent to $\Delta P'DQ'$. Hence $\overline{PQ}$ is congruent to $\overline{P'Q'}$.

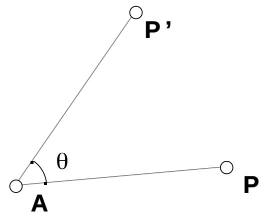
QED



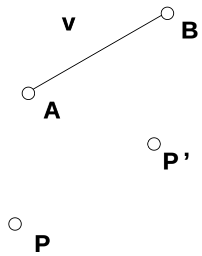
How would this proof have to be modified if P, Q lie on opposite sides of the mirror line? Notice by combining Lemma 4.1.5 with Theorem 4.1.6 we now have a proof of Conjecture 4.1.3.

Two other very familiar transformations of the Euclidean plane are rotations through a given angle about a given fixed point, and translation in a given direction by a fixed amount.

The most precise definition of these are terms of compositions of reflections (as we'll see in the next section), but direct geometric definitions can be given.



Rotation



Translation

Formally, a rotation $\rho_{A,\theta}$ about the point *A* through a directed angle θ is the transformation that fixes *A* and otherwise sends a point *P* to the point P' such that $\overline{AP}$ is congruent to $\overline{AP'}$ and θ is the directed angle measure of PAP'. A translation T_v is the transformation that sends every point *P* the same distance direction, as determined by a given vector *v*. Again, Sketchpad makes the idea clear.

4.1.6a Demonstration.

- Open a new sketch and draw an "".
- First we'll look at rotations. Construct a point and label it *A*. This will be the 'center' of the rotation, *i.e*., the fixed point. Select the point *A* and then use the **Transform** menu to select "Mark Center A".
- Under the **Edit** menu, select "Select All". Then under the **Transform** menu select "Rotate". The rotate screen will pop up with the angle of rotation θ selected. You can change the degrees in a positive or negative direction.
- Investigate if rotation preserves distance, angle measure and area. Does rotation preserve or reverse orientation?
- Now for translations. Open a new sketch and draw an "". Construct a line segment in a corner of your sketch and label the endpoints *A* and *B*. First select the endpoints in that order and the use the **Transform** menu to "Mark Vector A->B".

- Using the Marquee (Arrow Tool) select the “F”. Under the **Transform** menu select “Translate”. The translate screen will pop up with “By Marked Vector” selected. Click on “OK”.
- Investigate if translation preserves distance, angle measure and area. Does rotation preserve or reverse orientation? Now, complete **Conjecture 4.1.7.**

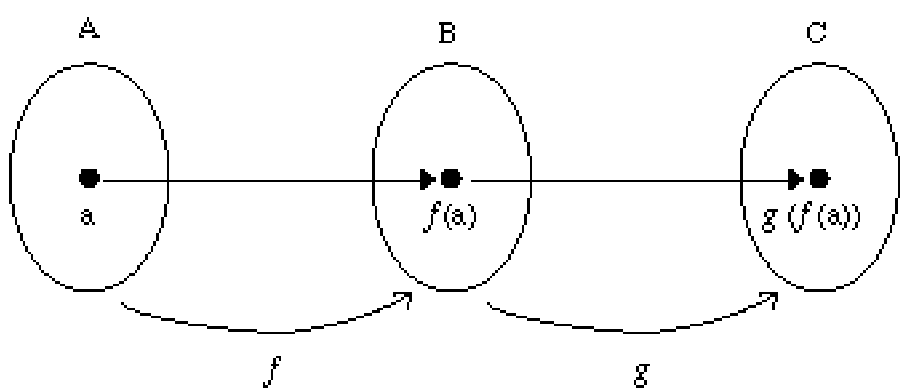
End of Demonstration 4.1.6a.

4.1.7 Conjecture. The rotation $\rho_{A,\theta}$ is ___________________ and also ___________________orientation. The translation $T_{A,B}$ is ___________________ and also ___________________orientation.

4.2 COMPOSITIONS. The usual composition of functions plays a very important role in the theory of transformations. Recall the general idea of composition of functions. Given functions *f*: $A \to B$ and *g*: $B \to C$, mapping a set *A* into a set *B* and *B* into a set *C* respectively, then the *composition*

$$(g \circ f)(a) = g(f(a)), \qquad (a \quad A)$$

maps *A* into *C*. Pictorially, composition can be represented by the figure below



Notice that if *f*: $A \to B$ and *g*: $B \to C$ are bijective, then the composition will also be bijective.

4.2.1 Exercise. Show that if *f*: $\mathcal{G}_1 \to \mathcal{G}_2$ and *g*: $\mathcal{G}_2 \to \mathcal{G}_3$ are bijective, then the composition $g \circ f$ is bijective from $\mathcal{G}_1$ onto $\mathcal{G}_3$. In other words, the composition of geometric transformations is again geometric.

The concept of geometric transformation is very general. What we do is impose restrictions on a transformation *f*: $\mathcal{G}_1 \to \mathcal{G}_2$ by imposing extra structure on $\mathcal{G}_1$ and $\mathcal{G}_2$ and then requiring that *f preserve* this extra structure. For instance, when a distance function is defined on $\mathcal{G}_1$ and $\mathcal{G}_2$, we can focus on geometric transformations f: $\mathcal{G}_1 \to \mathcal{G}_2$ that preserve the distance between points - what we called isometries in the case of Euclidean geometry. If a notion of angle measure is defined on $\mathcal{G}_1$ and $\mathcal{G}_2$, then we could focus on geometric transformations that preserve the angle between lines; such transformations are called *conformal* transformations. A complex-valued function $f : \Omega \to \Omega$ which is 1-1 and invertible on a set Ω in the complex plane is conformal whenever f is analytic. This is one reason why analytic function theory is closely connected with geometry. (There are many interesting ideas for semester projects here if one knows something about complex numbers and analytic function theory.)

4.2.2 Theorem. Let f and g be isometric transformations of the Euclidean plane. Then the composition $g \circ f$ of f and g also is an isometric transformation of the Euclidean plane.

Proof. Let P and Q be arbitrary points in the plane. Since f is an isometry,

$$dist(P,Q) = dist(f(P), f(Q)) .$$

But g also is an isometry, so

$$dist(f(P), f(Q)) = dist(g(f(P)), g(f(Q))) .$$

Combining these two results we see that

$$dist(P,Q) = dist((g \circ f)(P),(g \circ f)(Q)).$$

Hence the composition $g \circ f$ preserves lengths and so is an isometry. **QED**

This theorem shows why there are close connections between geometry and group theory. For if f: $\mathcal{G} \to \mathcal{G}$ is a geometric transformation, then f will have an inverse $f^{-1}: \mathcal{G} \to \mathcal{G}$Ā and f^{-1} will be a geometric transformation; in addition, if f is an isometry, then f^{-1} will be an isometry. Thus the set of all geometric transformations f: $\mathcal{G} \to \mathcal{G}$ is a group under composition, while the set of all isometries is a *subgroup* of this group. Now let's look more closely at the set of all isometries of the Euclidean plane - in more elaborate language, we are going to study the *Isometry Group* of the Euclidean plane. In the previous section we saw that any reflection is an isometry. Theorem 4.2.2 ensures that the composition of two reflections will be an isometry, and hence the composition of three, four or more reflections will be isometries as well. But how can we describe the composition of reflections in geometric terms? Let’s first use Sketchpad to see what happens for the composition of two reflections.

4.2.2a Demonstration. The Composition of Two Reflections.

- Open a new sketch and draw two mirror lines *l* and *l'*. Draw an "F" somewhere in the plane.
- Now reflect this "F" first in the mirror line *l* and then in the mirror line *l'*, producing a new image of "F".
- Describe carefully the position of the final image "F" in relation to the first "F".
- What happens if the lines *l* and *l'* are parallel. What if they are not parallel?
- You should now be able to complete **Conjecture 4.2.3.**

End of Demonstration 4.2.2a.

4.2.3 Conjecture. The composition of reflections in two mirror lines is a ____________________ when the mirror lines are parallel. The composition of reflections in two mirror lines is a ____________________ when the mirror lines intersect.

To investigate this more carefully, let's go once more to Sketchpad.

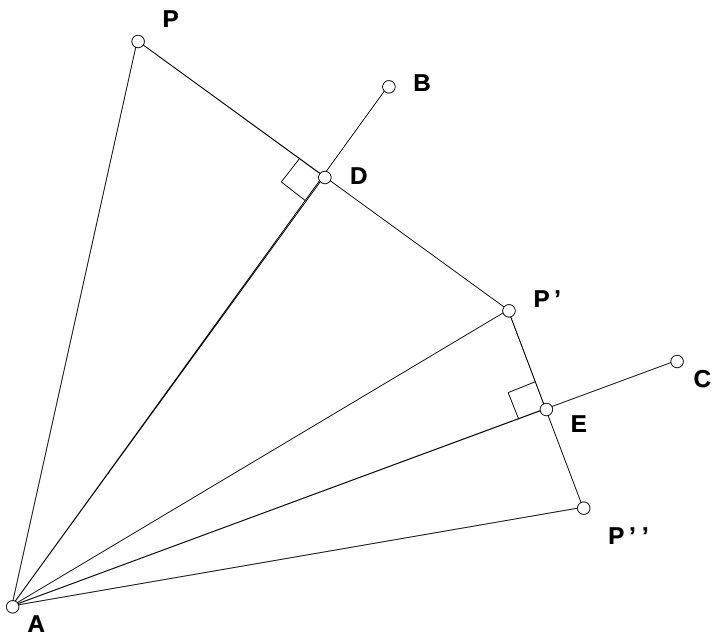
4.2.3a Demonstration.

- Open a new sketch and draw intersecting lines by first choosing three points *A*, *B*, and *C* then drawing two line segments *AB* and *AC*. The reason for constructing the mirror lines in this way is that dragging on *B* or *C* changes the angle between the mirror lines by rotating one of them about the vertex *A*.
- Now draw an "F" on one side of a mirror line and then reflect it successively in the two mirror lines, producing a new image "F" which should appear to be a rotation of the first "F". Measure first the angle between the mirror lines and then measure the angle by line segments joining the vertex A to corresponding points on the first "F" and its image. Compare the two values. This suggests that **Theorem 4.2.4** is true.

End of Demonstration 4.2.3a.

4.2.4 Theorem. Successive reflection in two intersecting mirror lines produces a rotation about the point of intersection through twice the angle between the mirror lines.

Proof. Consider the following figure, where P is first reflected in the mirror line AB with image P'. Then P' is reflected in the mirror line AC with image P''. There are two pairs of congruent triangles. By construction $PD = DP''$, so ΔPAD is congruent to $\Delta P'AD$ by the SAS criterion. Thus $PAD = P'AD$. By a similar argument $P'AE = P''AE$. Combining these two equalities we see that $PAP'' = 2\ DAE$. **QED**



Now lets go back to Sketchpad and look at the case of parallel mirror lines.

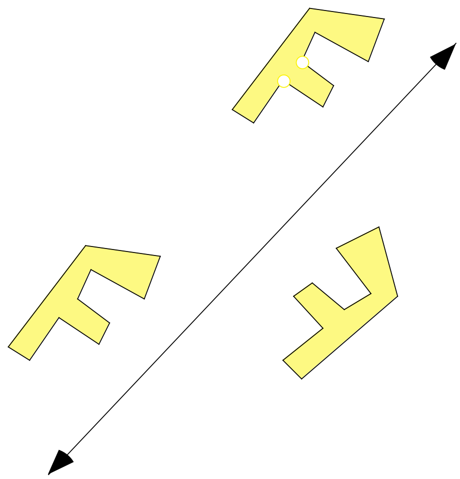
- Open a new sketch and draw two parallel lines. On one side of these lines draw an “F” and then reflect this successively in the two mirror lines. Drag one of the mirror lines so that it remains parallel to the other mirror line - you can do this by grabbing the line and then dragging. The image “F”should then appear to be a translate of the first one.
- Measure the distance between the parallel mirror lines and then measure the distance between corresponding points on the first “F” and the image “F”. Compare the two values.

4.2.5 Theorem. Successive reflection in parallel mirror lines produces a translation in a direction perpendicular to the mirrors through a distance equal to twice the distance between the mirrors.

Proof. See **Exercise Set 4.3.**

Next it makes sense to look at the composition of three reflections and see if we can describe the result in terms of rotations and translations as well. First we need to introduce one more Euclidean motion of the plane.

4.2.6 Definition. A glide reflection is the composition of a reflection with a translation parallel to the line of reflection.



We should note that sketchpad does not have the glide reflection transformation built into the program. But we could easily build our own using scripts or custom transformations. We'll see how to use custom transformations in the next section.

A transformation in the plane has **direct orientation** if it preserves the orientation of any triangle. If the transformation does not preserve the orientation but reverses it then it has **opposite orientation**. Thus if a motion is the product of an even number of reflections then it will have direct orientation. If a motion is the product of an odd number of reflections then it will have opposite orientation. Rotations and translations are examples of ____________ orientation while reflections and glide reflections show _______________ orientation. This observation will help us when trying to describe the results of composing three reflections.

There are different cases that need to be considered when looking the possible outcomes of reflecting in three mirror lines.

4.2.6a Demonstration.

- **Three Parallel Lines**: What do you get when you reflect something in three parallel lines? Draw three parallel lines and a simple polygonal figure. Reflect the figure successively about the 3 lines. (Hide intermediate figures to avoid confusion)
  What sort of transformation is this? What do the connected midpoints create?
  Draw at least 3 segments joining corresponding points on the pre-image to the final image. For each adjoining segment construct a midpoint and connect them together. Ignoring the three original lines what does this line suggest? How does your answer depend on the order of the lines? Investigate what happens when you change the order of reflection. (Drag the lines, say from #1 to #2)

- **Two Parallel Lines and One Non Parallel**: What is this a composition of?
  Draw two parallel lines and one that crosses them both. Now draw a simple figure on the outside of the parallel line and below the transverse line. Reflect it about the parallel line, then again about the other parallel line. What kind of motion is this? Now reflect it in the transversal. What is this motion called and what is the result of the two combined? Does it make any difference where the figure ends if you reflect it in another sequence, say reflecting it in the transversal first? Does it matter if the transversal is perpendicular to the parallel lines?

- **No Parallel Lines**: What sort of transformation does this case result in?
  Draw three lines that only intersect each other in one place. They should look like a triangle with its sides extended. Pick a place and draw yourself a small figure. Begin reflecting over the lines. What is the end result?

- **Three Concurrent Lines**: What is the line of reflection for this case? To construct concurrent lines make sure the lines intersect at one point. Draw such lines. Draw a small figure between two of the lines. (It will be contained in a V shaped segment) Begin your reflections here. What sort of transformation is this? If you reflect a point all the way around the six lines what do you get? Start with a point where you had drawn your figure. Reflect it around each of the lines until you get back to the start. Is the last point is the same place as the first?

What happens if two of the mirror lines are identical? What happens if all three are identical?

You should be able to complete the following:

Product of Two Reflections	If the 2 lines of the reflection are parallel then the motion is a ______________________________.
Product of Two Reflections	If the 2 lines of the reflection are not parallel then the motion is a ___________________________.
Product of Three Reflections	If all 3 of the lines of the reflection are parallel then the motion is a ____________________________.
Product of Three Reflections	If 2 of the lines of the reflection are parallel th the motion is a _______________________________.
Product of Three Reflections	If the 3 lines of the reflection are concurrent then the motion is a ________________________________.
Product of Three Reflections	If the 3 lines of reflection intersect each other only once then the motion is a ____________________.

End of Demonstration 4.2.6a.

With these notes in mind we can realize two of the most important theorems in the theory of isometric transformations of the Euclidean plane.

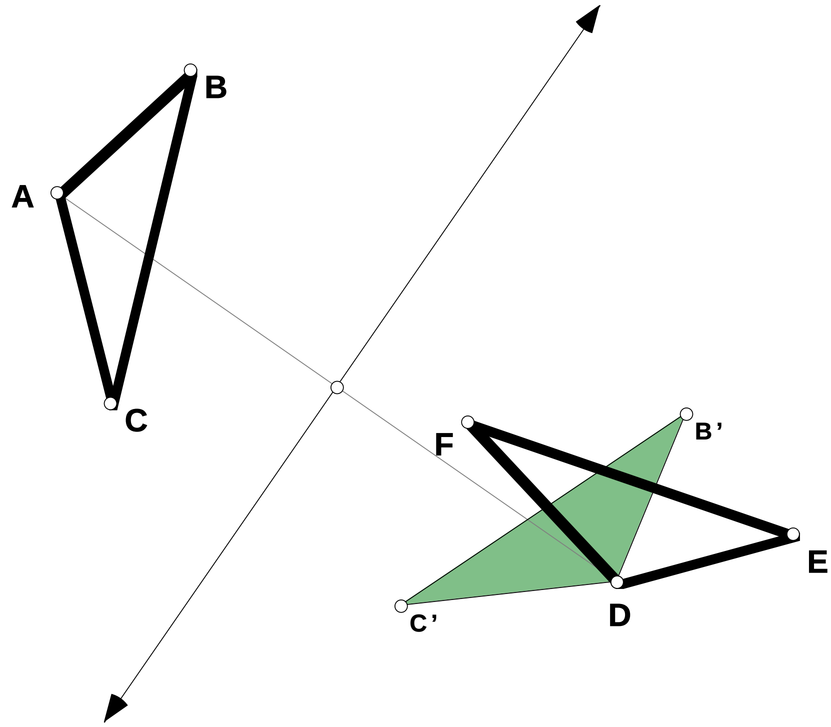
4.2.7 Theorem. Any isometry of the Euclidean plane can be written as a composition of no more than 3 reflections.

As a consequence of our exploration on composition of reflections we get the following as well.

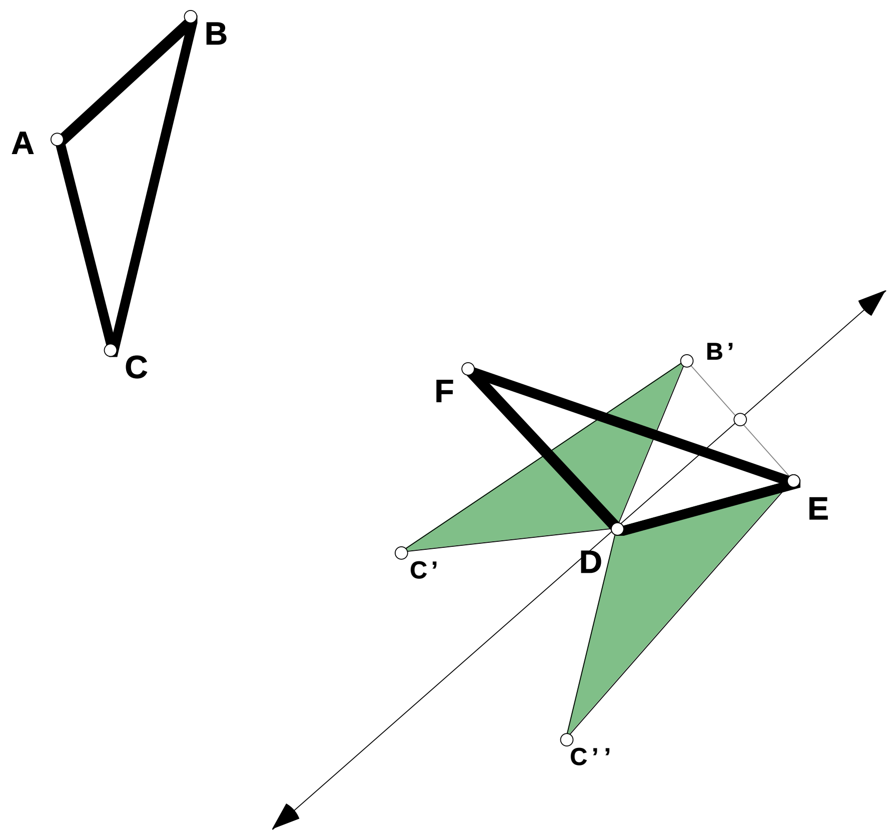
4.2.8 Theorem. Any isometry of the Euclidean plane can be written as one of the following transformations: reflection, rotation, translation or glide reflection.

Crucial to the proof of Theorem 4.2.7 will be the following. To show if we are given an isometry and three points *A*, *B*, and *C* with image points *D*, *E*, and *F* we can take the composition of (at most) three reflections and also map *A*, *B*, and *C* to *D*, *E*, and *F* respectively. If the orientation of the points is preserved it will take two reflections, and otherwise it will take three reflections.

- Open a new sketch and draw two congruent triangles, ΔABC and ΔDEF. We will find a transformation which maps ΔABC to ΔDEF.
- Draw a line segment between A and D and find the midpoint. Construct the line *l* perpendicular to the line segment and through the midpoint. Reflect ΔABC in *l* and A will be mapped onto D. So there is one point in the correct position and one reflection.



- If B and C also land on E and F then you would be done. If this is not this case, then we are to map B′ to E by reflecting through the perpendicular bisector of $B'E$ where B' is the image of B under the first reflection. This maps B' to E and keeps D fixed. Why does D stay fixed?



· This leaves you with only C'' (from the original C) to be mapped. If it falls on F after the second reflection then you would be done, but if it does not, map C'' to map to F by reflecting about the line $\overleftrightarrow{DE}$. Why is $\overleftrightarrow{DE}$ the perpendicular bisector of $\overline{FC''}$? Now you are done and it has taken 3 reflections to get from the pre-image to the final image.

Before proving Theorem 4.2.7 we need to establish another property of isometries.

4.2.9 Lemma. An isometry maps any three non-collinear points into non-collinear points.

Proof. Let A, B, and C be non-collinear points. Then by the triangle inequality the non-collinearity means that

$$dist(A,B) + dist(B,C) > dist(A,C).$$

Now let A', B', and C' be the images of A, B, and C. Since the isometry preserves distances,

$$dist(A',B') + dist(B',C') > dist(A',C').$$

But this ensures that A', B', and C' cannot be collinear, proving the lemma. **QED**

Proof of Theorem 4.2.7. Given an isometry F, choose a set of non-collinear points A, B, and C. Let $A' = F(A), B' = F(B)$, and $C' = F(C)$ be their images. Suppose that F has preserved orientation of ΔABC. Then the Sketchpad activity on 'Composition of reflections' shows that there exist reflections S_1 and S_2 so that their composition $S_1 \circ S_2$ has the properties

$$(S_1 \circ S_2)(A) = A', \ (S_1 \circ S_2)(B) = B', \ (S_1 \circ S_2)(C) = C'.$$

We will prove that

$$(S_1 \circ S_2)(P) = F(P).$$

holds for every point P. So set

$$(S_1 \circ S_2)(P) = P', \ F(P) = P''.$$

We have to show that $P' = P''$. Because $S_1 \circ S_2$ and F are isometries,

$$dist(A',P') = dist(A',P''), \ dist(B',P') = dist(B',P''), \ dist(C',P') = dist(C',P'').$$

Thus A', B', and C' will all lie on the perpendicular bisector of the segment $\overline{P'P''}$ if $P' \neq P''$. But this can happen only if A', B', and C' are collinear. But A, B, and C are not collinear, so A', B', and C' are not collinear. Hence $P' = P''$ showing that $S_1 \circ S_2 = F$. If F does not preserve the orientation of ΔABC then the same proof will show that F can be written as the composition of either one reflection or three reflections. This completes the proof. **QED**

4.3 Exercises. The problems in this assignment are a combination of algebraic and geometric ones.

Exercise 4.3.1. Show that the function $f:(x,y)\to(-y,x)$ maps the straight line $y=mx+b$ to the straight line $y=-(x+b)/m$. Explain the relationship between the slopes of these two lines in terms of the transformation in 4.1.2 (c).

Exercise 4.3.2. Show that reflection in the line $y=mx$ is given by

$$f:(x,y)\to\left(\left(\frac{2m}{m^2+1}\right)y-\left(\frac{m^2-1}{m^2+1}\right)x,\left(\frac{2m}{m^2+1}\right)x+\left(\frac{m^2-1}{m^2+1}\right)y\right).$$

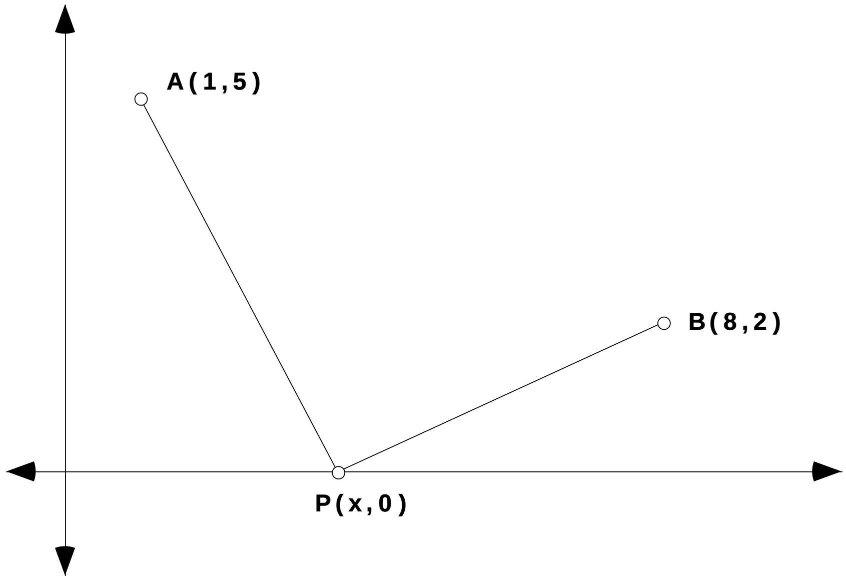
Hint: Let the reflection of the point $P=(x,y)$ be $P'=(x',y')$. You need to find two equations and then solve for x', y'. Let Q be the midpoint of PP'; so what are its coordinates? The point Q also lies on the mirror line $y=mx$; so what does this say about the coordinates of Q'? Use this to get the first equation for x', y'. The line PP' is perpendicular to the mirror line $= mx$. How can we use this to get a second equation for x', y'? Now solve the two equations you have obtained.

Exercise 4.3.3. Prove synthetically that every rotation $\rho_{A,\theta}$ is an isometry.

Exercise 4.3.4. Prove that successive reflections in parallel mirror lines produce a translation in a direction perpendicular to the mirrors through a distance equal to twice the distance between the mirrors.

Exercise 4.3.5. Suppose you wish to join the two towns $A(1,5)$ and $B(8,2)$ via a pipeline. A pumping station is to be placed along a straight river bank (the x-axis). Determine the location of a pumping station, $P(x,0)$, that minimizes the amount of pipe used? Solve this

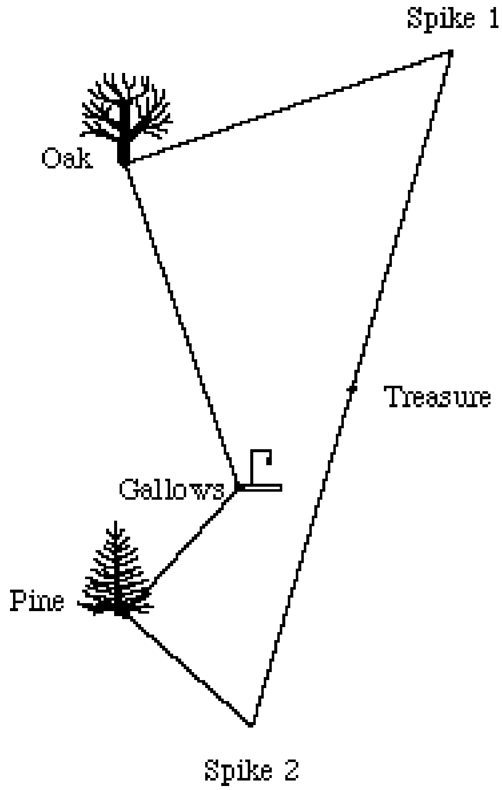
- by transformations.
- by calculus.



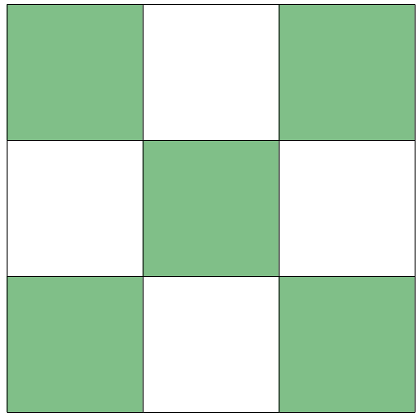
Exercise 4.3.6 Buried Treasure. Among his great-grandfather's papers, José found a parchment describing the location of a hidden treasure. The treasure was buried by a band of pirates on a deserted island which contained an oak tree, a pine tree, and a gallows where the pirates hanged traitors. The map looked like the accompanying figure and gave the following directions.

"Count the steps from the gallows to the oak tree. At the oak, turn 90° to the right. Take the same number of steps and then put a spike in the ground. Next, return to the gallows and walk to the pine tree, counting the number of steps. At the pine tree, turn 90° to the left, take the same number of steps, and then put another spike in the ground. The treasure is buried halfway between the spikes."

José found the island and the trees but could not find the gallows or the spikes, which had long since rotted. José dug all over the island, but because the island was large, he gave up. Devise a plan to help José find the treasure.



4.4 TILINGS REVISITED. To illustrate further the idea of reflections, rotations, translations, and glide reflections we want to begin the geometric analysis of 'wallpaper' designs. A wallpaper design is a tiling of the plane that admits translational symmetry in two directions. That is the design can be "moved" in two different directions and coincide with itself. The checkerboard below would produce a wallpaper design if continued indefinitely.



First we notice that certain rotations are admissible. For the checkerboard we can rotate by $90°$ (quarter-turn) about the center of any green or white square and repeat the same figure. Also we can rotate by $180°$ (half-turn) about the vertex of any square and repeat the same figure. There are wallpaper designs that admit $60°$ (sixth-turn) rotations and $120°$ (third-turn) rotations. What is more remarkable is that these are the only rotations allowed in any wallpaper design! A simple argument shows why. (See Crowe) To get you started on the fifth-turn case, try the following. Choose one center of rotation P and then choose another center of rotation that is closest to Q. Next argue why this cannot happen. The n-th turn case is even easier.

This restriction on rotations provides a convenient way to analyze wallpaper patterns. In fact, it can be shown that there are only 17 different types of wallpaper designs!

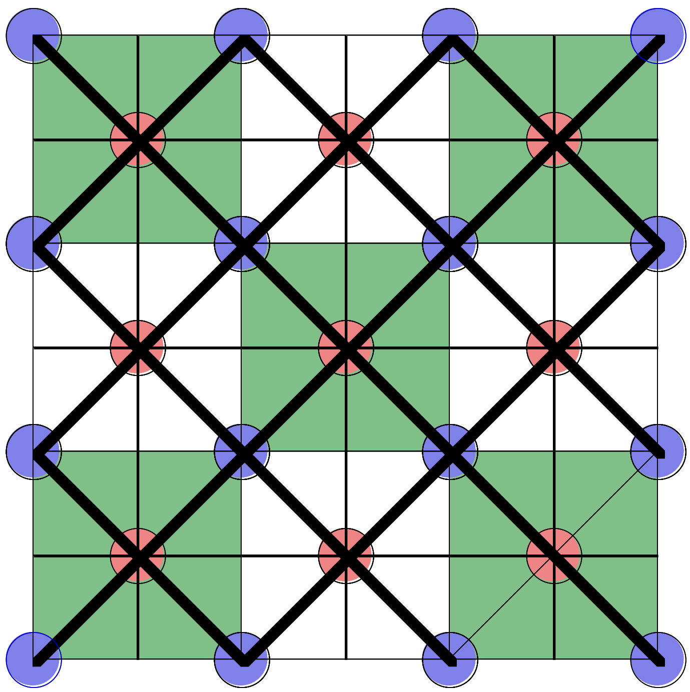
- Four which have no rotations at all;
- Five whose smallest rotation is $180°$;
- Three whose smallest rotation is $120°$;
- Three whose smallest rotation is $90°$;
- Two whose smallest rotation is $60°$;

There is a simple flowchart one can use to classify any wallpaper design. The symbols for the patterns have special meaning: *m* means mirror, *g* means glide, and a number like 2 or 4 means half-turn or quarter-turn.

Let's go back to our checkerboard design - we shall think of it as extending over the whole plane to form a tiling by congruent copies of a single square. An alternating coloring has been added for extra effect. This tiling will be left unchanged by various reflections and rotations about various points.

- Go back to the checkerboard figure and mark in the mirror lines with respect to which a reflection leaves the design unchanged. Mark the mirror lines in bold. Mark in red the centers of rotation through 90˚ that leave the design unchanged. Mark also in blue the centers of rotation through 180˚ that leave the design unchanged.

Your pattern should look like the one below.



- This tiling is classified as "p4m". The smallest rotations allowed are quarter -turns and there are reflections in four directions.
- Successive use of the reflections and rotations fixing the design would replicate the whole tiling from just one white square and one colored tile. Can the whole tiling be generated from any part smaller than theses two squares? Find the smallest piece from which the whole tiling could be generated by successive reflections and rotations. This smallest piece is called a *Fundamental Domain.*

How would the pattern of reflections and rotations differ if the tiling consisted of all white squares? What is a Fundamental domain of the new monochromatic tiling?

You can continue to examine wallpaper designs in the next set of exercises. Now we will assemble all the results and ideas developed about transformations and tilings to show how to use Sketchpad to construct figures with a prescribed symmetry. First let's see how to use Custom Tools to define our own transformations.

4.4.1 Demonstration. Custom transformations.

A custom transformation is a sequence of one or more transformations. The basic steps are given below.

- Transform an object one or more times.
- Hide any intermediate objects or format them as you wish them to appear when you apply your transformation.
- Select the pre-image and image, and select and show the labels of all marked transformation parameters.
- Create a new tool. The pre-image and transformation parameters will become given objects in the custom tool.
- In the custom tool's Script View, set each of the given transformation parameters—mirrors, centers, and so forth—to automatically match objects with the same label..

For example, let's define a rotation $\rho_{A,\theta}$ through a given angle θ about a given point *A*.

- Open a new sketch and construct a point *A* and any point *P*. Mark *A* as a center of rotation. Then construct the point P' which is the rotation of *P* about *A* through an angle θ (choose any θ).
- Next select *P* and P' and *A*. Choose "Create New Tool from the **Tools** menu. Type a name that describes the transformational sequence. In the Script View window, double click on Point A in the "Given" section and check the box "Automatically Match Sketch Object".
- You can now apply your custom transformation to any figure in your sketch. Draw any polygonal figure in your sketch and construct its interior. Select the polygon interior and apply the tool.

Repeat this process to define a reflection S_m about a given mirror line *m* and a translation T_v in a given direction.

End of Demonstration 4.4.1.

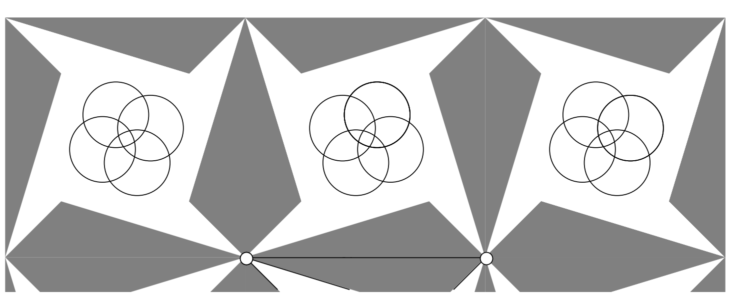
When you define a multi-step transformation, Sketchpad remembers the formatting you've applied to each step's image—whether you've colored it, or hidden it, and so forth. When you apply the transformation to new objects, Sketchpad creates intermediate images with exactly the same formatting. If you are interested only in the final image of the sequence of transformational steps, and not in the intermediate images, hide each intermediate image between your two selected objects before defining the transformation. If you want your transformed images to have a certain color, then be sure your image has the appropriate color when you define the transformation.

4.4.2 Demonstration. Producing a picture with *p4g* symmetry.

To utilize these ideas and generate the symmetries necessary for producing a picture having *p4g* symmetry:

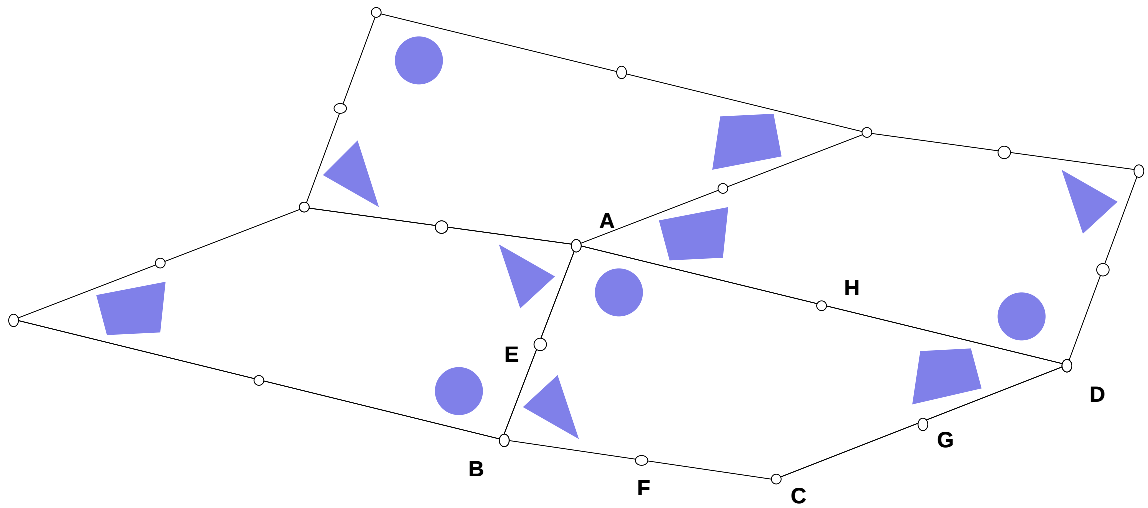
- Create a tool which performs a 4-fold rotation about *A*; call it *4-foldrot*. Construct a 2-fold rotation about *B*; call it *2-foldrot*. Finally construct a reflection about the side $\overline{BC}$ of ΔABC.
- Construct a right-angled isosceles triangle ΔABC having a right angle at *A*; this will be the fundamental domain of the figure.

Now you are free to draw any figure having p4g symmetry. Below is one example. The original Δ has been left in. The picture was constructed from one triangle inside the fundamental domain and one circle. The most interesting designs usually occur when the initial figure 'pokes' outside the fundamental domain. The vertices of the original triangle can be dragged to change the appearance of the design; the original design can be dragged too. This often results in a radical change in the design.



End of Demonstration 4.4.2

Earlier, as a consequence of the Euclidean parallel postulate, we saw that the sum of the angles of a triangle is always 180˚ no matter the shape of the triangle; similarly the sum of the angles of a quadrilateral is always 360˚ no matter the shape of the quadrilateral. Somewhat later we gave a more careful proof of this fact by determining the sum of the angles of any polygon - in fact we saw that the value depends only on the number sides of the polygon. This value was then used to show that equilateral triangles, squares and regular hexagons are the only regular polygons that tile the Euclidean plane. But nothing was said about the possibility of non-regular polygons tiling the plane. In fact, any triangle or quadrilateral can tile the plane. The figure below illustrates the case of a convex quadrilateral. *ABCD* was the original quadrilateral and *E, F, G, H* are the respective midpoints. One can obtain the figure below by rotating by 180˚ about the midpoint of each side of the quadrilateral. (You can tile the plane with any triangle by the same method – try it!)

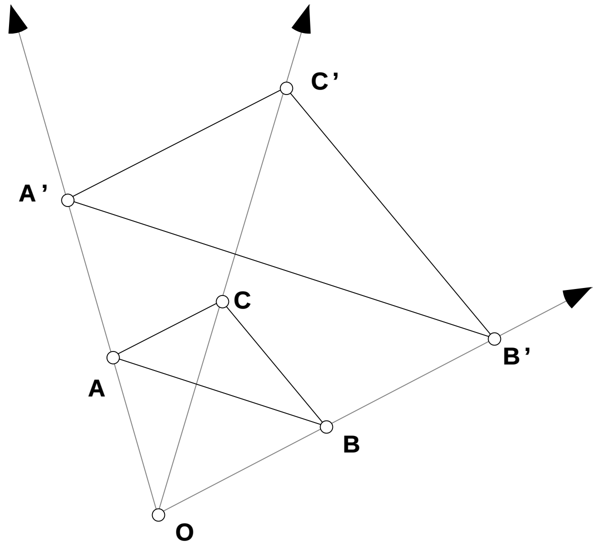


To do this for yourself, you can use custom transformations. Define a transformation for each midpoint. I've drawn a different figure in each of the corners of the chosen quadrilateral to help me distinguish among the corners. Use your four rotations to produce a tiling of the plane by congruent copies of the original quadrilateral with one copy of each of the four corners occurring at every vertex. Join neighboring images of the midpoints by line segments. What resulting repeating diagram emerges? You should see an overlay of parallelograms. Can you find a parallelogram and points so that successive rotations of the parallelogram through 180˚ about the points would produce the same tiling?

4.5 DILATIONS. In this section we would like look at another type of mapping, dilation, that is frequently used in geometry. Dilation will not be an isometry but it will have another useful property, namely that it preserves angle measure.

4.5.1 Definition. A geometric transformation of the Euclidean Plane is said to be conformal when it preserves angle measure. That is, if A', B', and C' are the images of A, B, and C then $m\ \ A'B'C' = m\ \ ABC$.

4.5.2 Definition. A dilation with center O and dilation constant $k \neq 0$ is a transformation that leaves O *fixed* and maps any other point P to the point P' on the ray OP such that $OP' = k\ \ OP$.



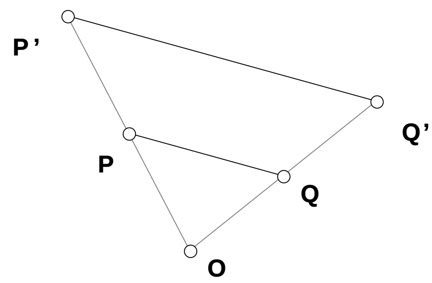
4.5.2a Demonstration. Dilation with Sketchpad.

Sketchpad has the dilation transformation built into the program.

- Open a new sketch and construct a point O and ΔABC.
- Select O and then "Mark Center O." under the **Transform** menu.
- Select ΔABC and then select "dilate" from the **Transform** menu.
- Enter the desired scale factor (dilation constant). (In the figure above the dilation constant is equal to 2. Notice that in the dialogue box, the scale factor is given as a fraction. In this case, we would either enter $\frac{2}{1}$ or $\frac{1}{0.5}$.)
- What is the image of a segment under dilation? Is the dilation transformation is conformal?
- Next construct a circle and dilate about the center O by the same constant. What is the image of a circle?

End of Demonstration 4.5.2a.

4.5.3 Theorem. The image of $\overline{PQ}$ under dilation is a parallel segment, $\overline{P'Q'}$ such that $P'Q' = |k|\, PQ$



Proof. From SAS similarity it follows that $\Delta POQ \sim \Delta P'OQ'$ and thus $P'Q' = |k|\ PQ$. The proof needs to be modified when *O,P,* and *Q* are collinear.

4.5.4 Theorem. The dilation transformation is conformal.

Proof. See **Exercise Set 4.6**.

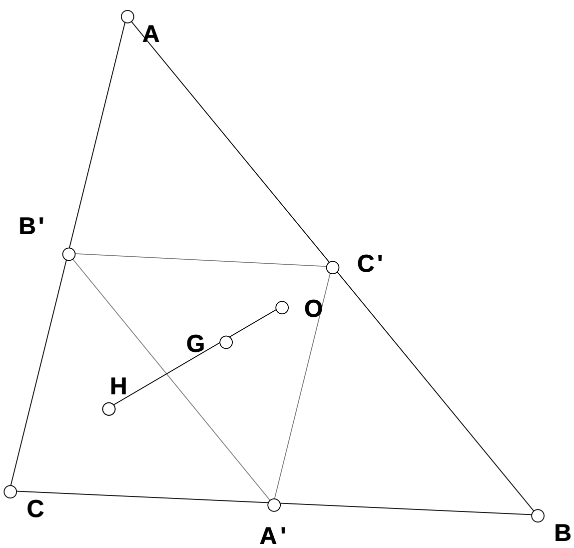
One can easily see that the following theorem is also true. The idea for the proof is to show that all points are a fixed distance from the center.

4.5.5 Theorem. The image of a circle under dilation is another circle.

Proof. Let O be the center of dilation, Q be the center of the circle, and P be a point on the circle. *Q'* will be the center of the image circle. By Theorem 4.5.3, $\frac{P'Q'}{PQ} = \frac{OQ'}{OQ}$ or

$P'Q' = \frac{PQ\ \ OQ'}{OQ}$. Now each segment in the right-hand expression has a fixed length so *P'Q'* is a constant. Thus for any position of *P, P'* lies on a circle with center *Q'*.

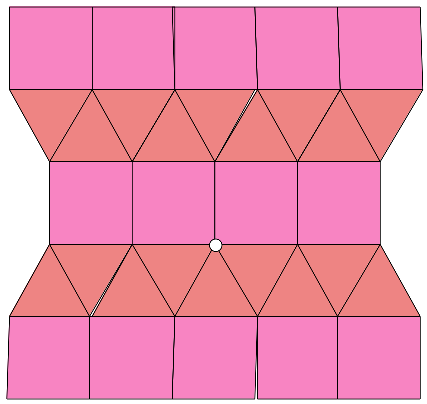
Using dilations we can provide an alternate proof for the fact that the centroid of a triangle trisects the segment joining the circumcenter and the orthocenter (The Euler Line).

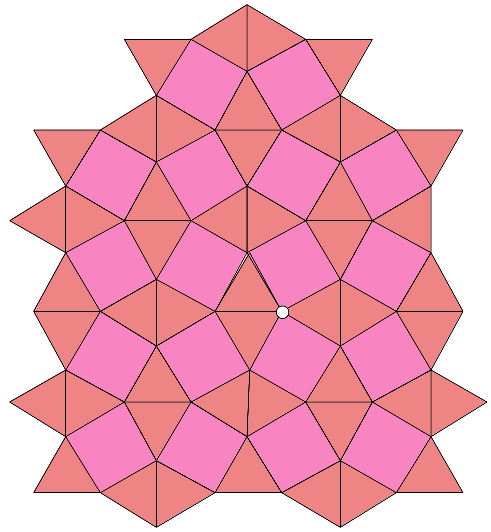


Given ΔABC with centroid G, orthocenter H, and circumcenter O. Let A', B', and C' be the midpoints of the sides. First note that O is the orthocenter of $\Delta A'B'C'$ and that G divides each median into a 2:3 ratio. Thus if we dilate ΔABC about G with a dilation constant of $-\frac{1}{2}$, ΔABC will get mapped to $\Delta A'B'C'$ and H will get mapped to O (their orthocenters must correspond). Hence O, G, and H must be collinear by the definition of a dilation and $OG = \frac{1}{2}HG$. **QED.**

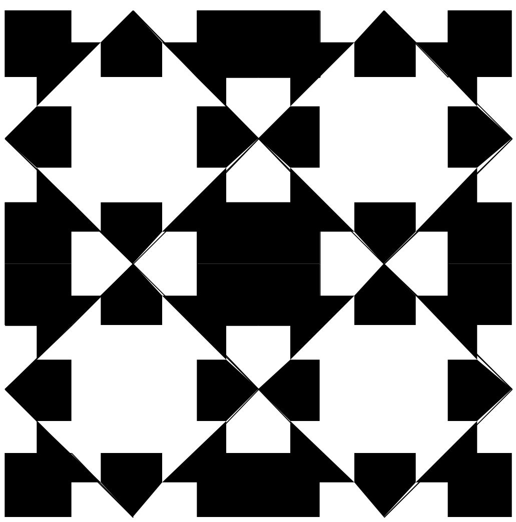
4.6 Exercises.

Exercise 4.6.1. Recall the two regular tilings of order 2 produced with squares and triangles. Classify each as a wallpaper design.

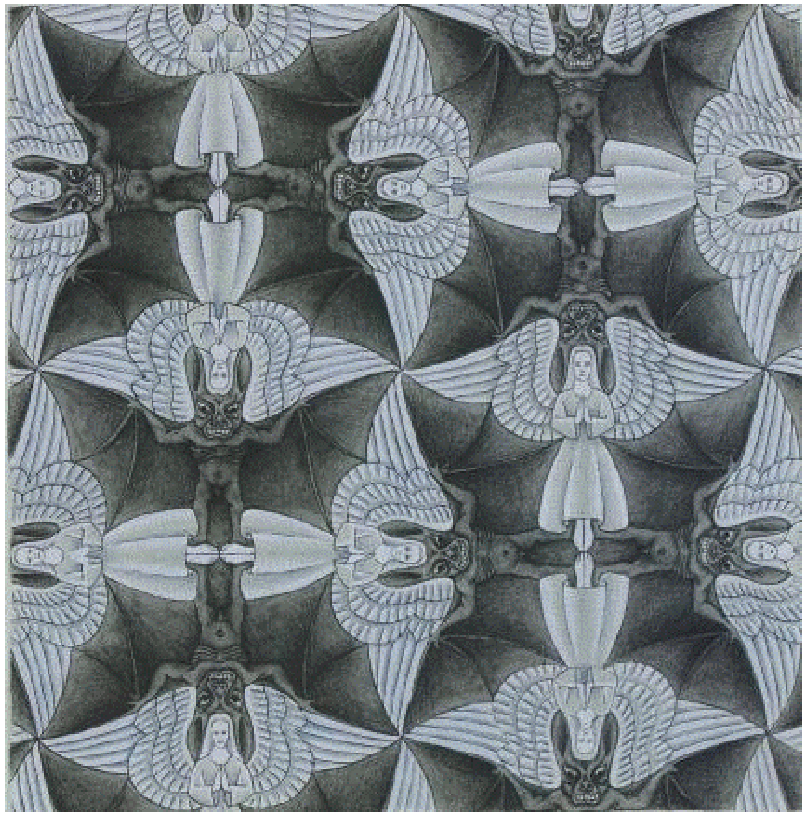




Exercise 4.6.2. Classify the following wallpaper design. Is there any relation to the checkerboard tiling?

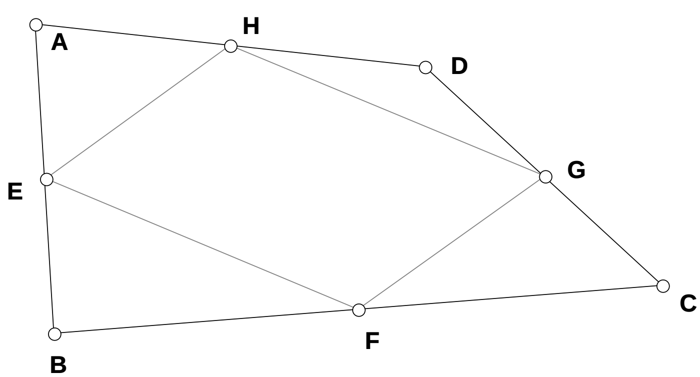


Exercise 4.6.3. What type of wallpaper design is Escher's version of 'Devils and Angels' for Euclidean geometry?



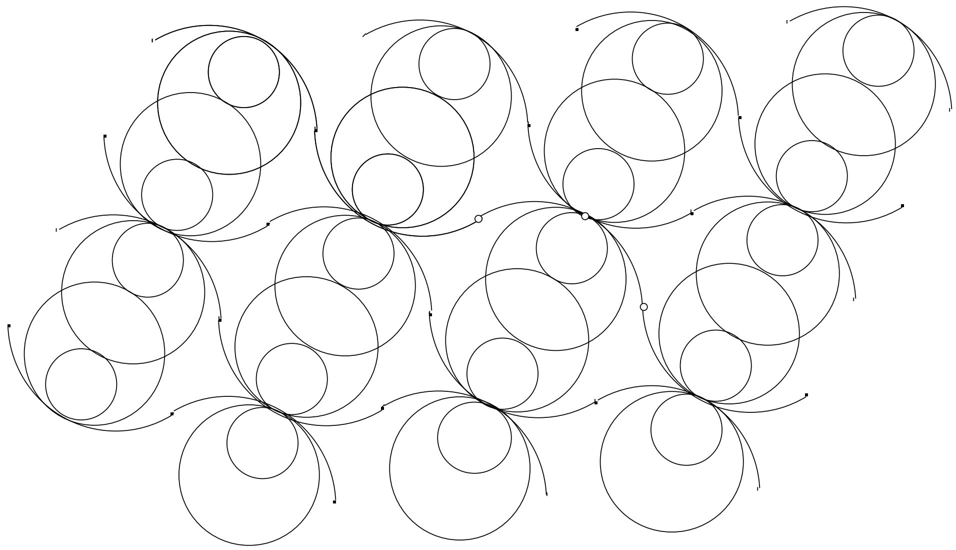
Exercise 4.6.4. On sketchpad use custom transformations to create a wallpaper design other than a p4g.

Exercise 4.6.5. Let *ABCD* be a quadrilateral. In the figure below *E,F,G,* and *H* are the midpoints of the sides. Prove that *EFGH* is a parallelogram. Hint: Similar triangles.



Exercise 4.6.6. Escher's lizard graphic is shown below. Mark all the points in the picture about which there are rotations by 180˚. What do you notice about these points? Exhibit a parallelogram and three points about which successive rotations through 180˚ would produce Escher's design. What is the wallpaper classification for the lizard design?

Exercise 4.6.7. Now pretend that you are Escher. Start with a parallelogram *PQRS.* Draw some geometric design inside this parallelogram - a combination of circles and polygons, say. Choose three points and define rotations through 180˚ about these points so that successive rotations about these three point tiles the plane with congruent copies of your design. Try making a second design allowing some of the circles and polygons to fall outside the initial parallelogram - this usually produces a more interesting picture. Here's one based on two circles and an arc of a circle

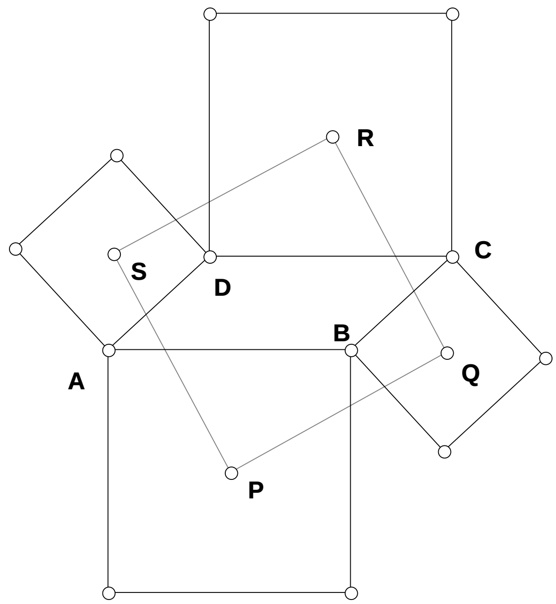


Exercise 4.6.8. Prove Theorem 4.5.4. The dilation transformation is conformal.

4.7 USING TRANSFORMATIONS IN PROOFS

Transformations can also be useful in proving certain theorems, sometimes providing a more illuminating proof than those accomplished by synthetic or analytic methods. We "discovered" Yaglom's Theorem in the second assignment and re-visited it while looking at tilings. There is an easy proof that uses transformations.

4.7.1 Theorem. Let *ABCD* be any parallelogram and suppose we construct squares externally on each side of the parallelogram. Then centers of these squares also form a square.



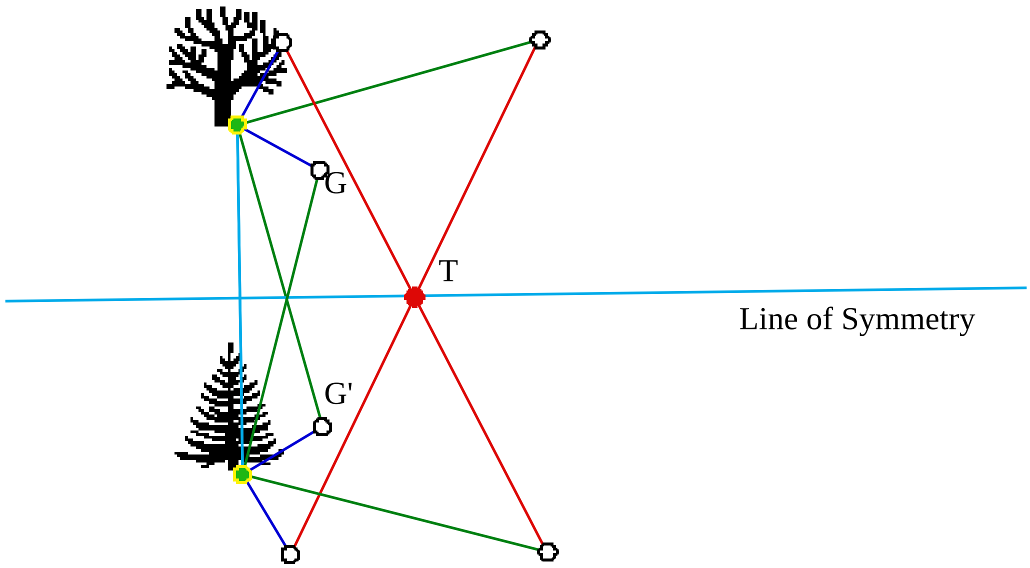
Proof. Consider the rotation about P by 90˚. (Try it on sketchpad.) The square centered at P will rotate onto its original position and $\overline{AB}$ must rotate to $\overline{A'A}$, so the square centered at Q will rotate to onto the square centered at S. Thus their centers will coincide. This tells us that the segment $\overline{PQ}$ rotates 90˚ onto the segment $\overline{PS}$, and therefore PQ=PS and $m\angle QPS$ =90. Do the same for the other centers Q, R, and S. Thus $PQRS$ is a square.

QED

Earlier in this chapter we looked at the Buried Treasure problem (**Exercise 4.3.6**). After working with the Treasure sketch one notices that the location of the treasure is likely to be independent of the position of the gallows. If we use this observation as an assumption, then perhaps we can gain an understanding as to where the treasure is buried with respect to the trees.

The map's instructions are very symmetrical. Since the only reference points are the two trees, a symmetry argument will be used with objects reflected across the perpendicular bisector of the segment joining the trees. Choose a position for the gallows (G) near the Oak tree, and its reflection (G') near the Pine tree (Figure 1).

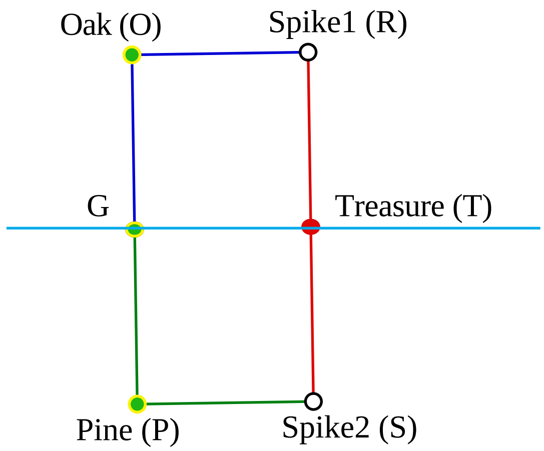
Figure 1



The treasure must lie upon the line of symmetry; or else it is in two different places. Therefore, the treasure lies upon the perpendicular bisector of the Pine Oak segment.

To calculate where upon the perpendicular bisector the treasure lies, we next choose *G* to be a point on the line of symmetry, specifically the midpoint between the Pine (*P*) and the Oak (*O*) trees (Figure 2). We will need to find *GT*. Since *G* is the midpoint of $\overline{OP}$, we see that *GO* = *GP*; in addition, by following the treasure map directions, we see that *GP* = *PS* and *GO* = *OR*.

Figure 2



OR = PS by transitivity. $\overline{OR} \parallel \overline{PS}$ since they are both perpendicular to the same line, therefore ORSP is a parallelogram, specifically a rectangle. *OP* = *RS*, and since *G* is the midpoint of $\overline{OP}$ and *T* is the midpoint of $\overline{RS}$ it follows that *GP* = *TS*. Therefore *GTSP* is a parallelogram, more specifically a square. So one solution to help José is the following: he needs to find and mark the midpoint between the Pine and the Oak. Then starting at the pine tree he should walk toward the marker while counting his steps, then make a 90° turn to the right and pace off the same number of paces. The treasure is at this point.

We can provide a proof of our result by coordinate geometry or by transformations.

1. Solution by coordinate geometry:

José should be happy now with his treasure, but in the preceding argument we made a fairly big assumption, so our conclusion is only as strong as our assumptions. Using coordinate geometry we can develop a proof of the treasure's location without making such assumptions.

- Pick convenient coordinate axes. The pine and oak trees are the only clear references. Let the pine tree be the origin and the oak tree some point on the y-axis $(0, a)$. The gallows are in an unknown position, say (x, y).
- Calculate the position of Spike 2 (S). Rotating the gallows position -90° about the pine tree gives the coordinate of S as $(y, -x)$.

- Calculate the position of Spike1 (R). Rotating the gallows position 90° about the oak tree will take a little more effort. If the oak tree were the origin then the rotation of 90° would be simple. So lets reduce our task to a more simple task. Translate the entire picture, $T_{(0, -a)}$. This will place the oak tree on the origin. Rotate the translated gallows $(x, y - a)$ 90° about the origin to $(-y + a, x)$. Now translate the picture $T_{(0, a)}$ and the picture is back where it began. The position of R is now $(-y + a, x + a)$.
- Our last task is to calculate where the treasure is located.

Use the midpoint formula to calculate the position of the treasure halfway between the spikes.

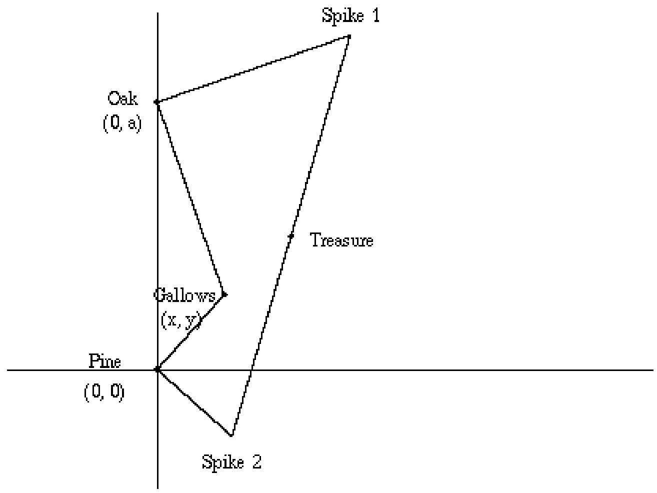
Spike 1: R $(-y + a, x + a)$

Spike 2: S $(y, -x)$

Treasure: T $(a/2, a/2)$

Coordinate geometry proves that the position of the treasure is invariant with respect to the gallows.

Figure 3



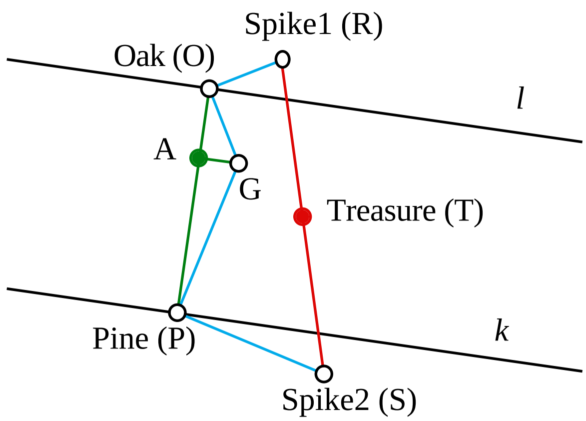
2. Explanation by Isometries:

So far the explanations have given a solution, but they haven’t given us much insight as to why the location of the treasure is independent of the position of the gallows. *Sketchpad* can assist in the explanation using transformations.

4.7.2 Demonstration. The Buried Treasure Problem using Sketchpad.

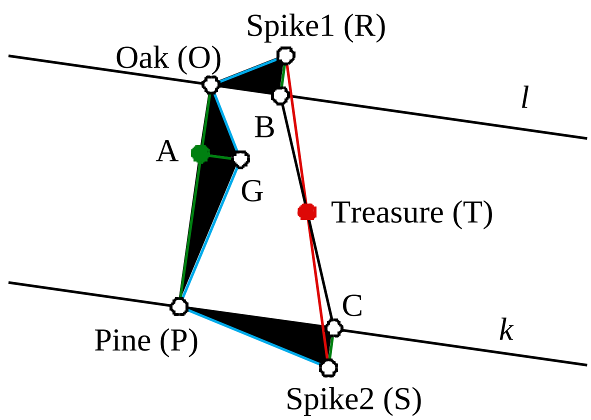
The exact position of the gallows is unknown, therefore we indicate the position of the Gallows by the letter *G* and make no more assumptions about its position. Construct the segment joining the Oak tree (*O*) and Pine tree (*P*). Construct lines *l* and *k* perpendicular to $\overline{OP}$ passing through *O* and *P* respectively. Lines *l* and *k* are parallel to each other. Construct $\overline{GA}$ as the altitude of the ΔPOG. By the instructions given in the map, construct the positions of the spikes (*R* and *S*), and the treasure (*T*). Hide all unnecessary lines and points. (Figure 4)

Figure 4



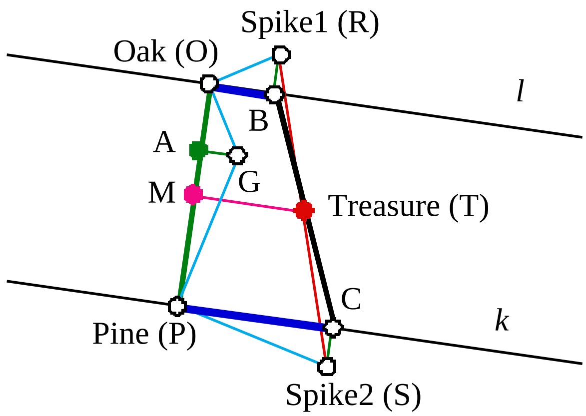
In the coordinate proof the spike positions were found by rotating the position of the gallows about the trees. We will use this technique again in this proof. Rotate ΔOAG 90° about *O*, forming ΔOBR. Rotate ΔPAG -90° about *P*, forming ΔPCS. It is simple to show *B* lies on *l* and *C* lies on *k*. Since isometries preserve distance the following congruencies hold: $\overline{GA}$ $\overline{RB}$; $\overline{GA}$ $\overline{SC}$, and by transitivity $\overline{RB}$ $\overline{SC}$. Since $\overline{RB} \| \overline{SC}$, BRT   CST. By SAS ΔRBT ΔSCT. From this we can conclude *B, T, C* are collinear, *T* is the midpoint of $\overline{BC}$ and therefore equidistant from *l* and *k*. (See **Figure 5**).

Figure 5 →



With *T* established as the midpoint of $\overline{BC}$, we will change our focus to the trapezoid *OBCP* (See Figure 6). Naming *M* the midpoint of $\overline{OP}$, yields the median $\overline{MT}$. The length of the median is the average of the two bases, thus $MT = \frac{1}{2}(OB + PC)$. But by the original rotation we know that $OB + PC = OA + AP = OP$; thus $MT = \frac{1}{2}OP$. From this we can conclude that ΔPMT is an isosceles right triangle.

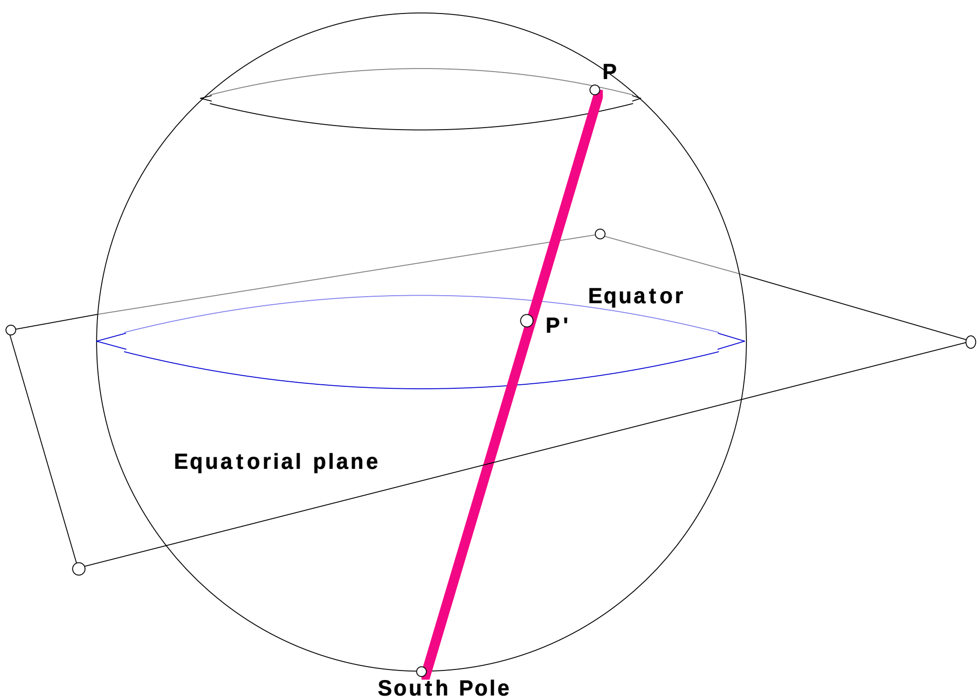
Figure 6



End of Demonstration 4.7.2.

4.8 STEREOGRAPHIC PROJECTION. In all the previous discussions the geometric transformation has mapped one model of a geometry onto the *same* model. But in map-making, for instance, the problem is to map the sphere model to a different model, in fact to a model realized as some geometry realized in the plane. One very important example of this is the transformation known as *Stereographic Projection.* We shall see this plays also a crucial role in describing the geometric transformation taking the line model of hyperbolic geometry in terms of lines and planes inside a cone in 3-space to the Poincaré model **D**.

To construct the stereographic projection of the sphere onto the plane, first draw the equatorial plane - this will serve as the plane onto which the sphere is mapped. Now take any point P on the sphere other than the South Pole and draw the ray starting at the South Pole and passing through P. Label by P' the point of intersection of this ray with the equatorial plane. For clarity in the figure below the ray has been drawn as the line segment joining the South Pole and P.



Stereographic projection is the mapping $P \to P'$ from the sphere to the equatorial plane. It has a number of important **properties**:

1. When P lies on the equator, then $P = P'$ so the image of the equator is itself. More precisely, the equator is left fixed by the transformation $P \to P'$. For convenience, let's agree to call this circle the *equatorial circle*.
2. When P lies in the Northern hemisphere then P' lies inside the equatorial circle, while if P lies in the Southern hemisphere, P' lies outside the equatorial circle.

3. Since the ray passing through the South Pole and P approaches the tangent line to the sphere at the South Pole, and so becomes parallel to the equatorial plane, as P approaches the South Pole, the image of the South Pole under stereographic projection is identified with infinity in the equatorial plane.
4. There is a 1-1 correspondence between the equatorial plane and the set of all points on the sphere excluding the South Pole.
5. The image of any line of longitude, *i.e*., any great circle passing through the North and South Poles, is a straight line passing through the center of the equatorial circle. Conversely, the pre-image of any straight line through the center of the equatorial circle is a line of longitude on the sphere.
6. The image of any line of latitude on the sphere is a circle in the equatorial plane concentric to the equatorial circle.
7. The image of any great circle on the sphere is a circle in the equatorial plane. Now every great circle intersects the equator at diametrically opposite points on the equator. On the other hand, the points on the equator are fixed by stereographic projection, so we see that the image of any great circle on the sphere is a circle in the equatorial plane passing through diametrically opposite points on the equatorial circle
8. Stereographic projection is *conformal* in the sense that it preserves angle measure. In other words, if the angle between the tangents at the point of intersection of two great circles is θ, then the angle between the tangents at the points of intersection of the images of these great circles is again θ.

Many books develop the properties of stereographic projection listed above by using the idea of inversion in 3-space. These same properties can, however, be established algebraically. This is what we'll do at this juncture because it brings in results learned earlier in calculus courses. Let Σ be the sphere in 3-space centered at the origin having radius 1. The points on Σ can described by

$$(\xi, \eta, \zeta), \quad \xi^2 + \eta^2 + \zeta^2 = 1,$$

so in the figure above, let $P = P(\xi, \eta, \zeta)$ and let $P' = P'(x, y)$ be its image in the equatorial plane under stereographic transformation where the center of the equatorial circle is taken as the origin. In particular, the equation of the equatorial circle is $x^2 + y^2 = 1$. To determine the relation between (ξ,η,ζ) and (x,y) we use similar triangles to show that

$$\text{(A)} \qquad \Pi : P(\xi,\eta,\zeta) \to P'(x,y), \qquad x = \frac{\xi}{1+\zeta}, \qquad y = \frac{\eta}{1+\zeta}.$$

This is the algebraic formulation of stereographic projection. Since $\xi^2 + \eta^2 + \zeta^2 = 1$, the coordinates of $P'(x,y)$ satisfy the relation

$$\text{(B)} \qquad x^2 + y^2 = \frac{1-\zeta^2}{(1+\zeta)^2} = \frac{1-\zeta}{1+\zeta}.$$

As illustration, consider the case first of the North Pole $P = (0,0,1)$. Under stereographic projection Π maps $P = (0,0,1)$ maps to $P' = (0,0)$ in the equatorial plane, *i.e.*, to the origin in the equatorial plane. By contrast, the South Pole is the point $P = (0,0,-1)$ and it is the only point of the sphere with $\zeta = -1$. Thus the South Pole is the only point on Σ for which the denominator $1+\zeta = 0$. Thus the south Pole maps to infinity in the equatorial plane, and it is the only point on Σ which does so. That $P(\xi,\eta,\zeta) \to P'(x,y)$ is a 1-1 mapping from $\Sigma \setminus (0,0,-1)$ onto the equatorial plane can also be shown solving the equations

$$x = \frac{\xi}{1+\zeta}, \qquad y = \frac{\eta}{1+\zeta}$$

given a point (ξ,η,ζ) in $\Sigma \setminus (0,0,-1)$ or a point (x,y) in the equatorial plane.

Now let's turn to the important question of what Π does to circles on Σ. Every such circle is the intersection with Σ of a plane; for instance, a great circle is the intersection of Σ and a plane through the origin. In calculus you learned that a plane is given by the equation

$$\text{(C)} \qquad A\xi + B\eta + C\zeta = D$$

where the vector (A, B, C) is the normal to the plane and $\dfrac{D}{\sqrt{A^2 + B^2 + C^2}}$ is the distance of the plane from the origin. The simplest case is that of a line of longitude. Algebraically, this is the intersection of Σ with a *vertical* plane through the origin, so the normal lies in the (ξ, η)-plane meaning that $C = D = 0$ in the equation above. Thus a line of longitude is the set of points (ξ, η, ζ) such that

$$A\xi + B\eta = 0, \quad \xi^2 + \eta^2 + \zeta^2 = 1.$$

The image of any such point under Π is the set of points (x, y) in the equatorial plane such that $Ax + By = 0$, which is the general equation of a straight line passing through the origin. Conversely, given any straight line l in the equatorial plane, it will be given by $Ax + By = 0$ for some choice of constants A, B. So l will be the image of the great circle defined by the plane $A\xi + B\eta = 0$. This shows that there is a 1-1 correspondence between lines of longitude and straight lines through the center of the equatorial circle, proving property **5** above.

The image of a line of latitude is easily determined also since a line of latitude is the intersection of Σ with a *horizontal* plane, *i.e.*, a plane $\zeta = D$ with $-1 < D < 1$. But then, by the general relation (B) the image of the line of latitude determined by the plane $\zeta = D$ consists of all points (x, y) in the equatorial plane such that

$$x^2 + y^2 = \frac{1 - D}{1 + D}.$$

This is the equation of a circle centered at the origin and radius $\sqrt{(1 - D)/(1 + D)}$; as D varies over the range $-1 < D < 1$, this describes the family of all circles centered at the origin. So Π defines a 1-1 mapping of the lines of latitude onto the family of all circles concentric with the equatorial circle.

The proof of property **7** is a little more tricky. Consider first the case of a plane passing through the points $(0, \pm 1, 0)$ on Σ; we could think of these as being the East and West ‘Poles’. Also, the plane need not be vertical because otherwise its intersection with Σ

would be a line of longitude dealt with earlier in property **6**. Thus we are led to considering a great circle determined by the plane

$$\zeta = \xi \tan\theta ,$$

where θ is fixed, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$; in fact, θ is the angle between the plane and the (ξ, η)-plane. By relations (A) and (B), the points (x, y) in the image of the intersection with Σ of the plane $\zeta = \xi \tan\theta$ will satisfy the equations

$$x = \frac{\xi}{1 + \xi \tan\theta} \quad , \qquad x^2 + y^2 = \frac{1-\zeta}{1+\zeta} = \frac{1-\xi \tan\theta}{1+\xi \tan\theta} \ .$$

After eliminating ξ from these equations we see that the image point (x, y) satisfies the equation

$$x^2 + y^2 = 1 - 2x\tan\theta .$$

In other words, the image of the great circle determined by the plane $\zeta = \xi \tan\theta$ is the circle

$$(x + \tan\theta)^2 + y^2 = 1 + (\tan\theta)^2 = (\sec\theta)^2$$

which is the circle centered at $(-\tan\theta, 0)$ having radius $1/\cos\theta$. As problem 7 in Assignment 6 shows, this is a circle passing though diametrically opposite points of the circle $x^2 + y^2 = 1$; in fact, it passes through the points $y = \pm 1$ which are the image of the points of intersection of the great circles determined by the plane $\zeta = \xi \tan\theta$ and the equator in Σ.

But how do we deal with a more general great circle that is not a line of longitude and does not pass through the East and West Poles? The fundamental idea we'll use is that a rotation of the sphere about the ζ-axis through an angle ϕ will fix the ζ-coordinate of a point $P(\xi, \eta, \zeta)$ on Σ while rotating the ξ, η-coordinates, but it will also rotate the x, y-coordinates of the image $P'(x, y)$ by the same angle ϕ. So the effect of rotating a great circle is to rotate its image under stereographic projection. Since a rotation is an isometry, it maps a circle to a circle. Hence the image of any great circle is a circle. Let's do the details.

4.8.1 Theorem. Under the rotation $\rho_{O,\phi}$ about the origin the point (ξ,η) is mapped to the point $(\xi',\eta') = \rho_{O,\phi}(\xi,\eta)$ where

$$\xi' = \xi\cos\phi - \eta\sin\phi, \quad \eta' = \xi\sin\phi + \eta\cos\phi .$$

More generally, the point (ξ,η,ζ) is mapped to the point (ξ',η',ζ).

Under $\rho_{O,\phi}$ the plane $\zeta = \xi\tan\theta$ is mapped to the plane $\zeta = (\xi\cos\phi + \eta\sin\phi)\tan\theta$. The angle between this plane and the (ξ,η)-plane is again θ and the intersection of the plane with Σ is a great circle passing through the equator at the points

$$(-\sin\phi,\cos\phi,0), \quad (\sin\phi,-\cos\phi,0).$$

Now by (A), the point (ξ',η',ζ) is mapped to (x',y') where

$$x' = x\cos\phi - y\sin\phi, \quad y' = x\sin\phi + y\cos\phi .$$

Consequently, stereographic projection commutes with the rotation $\rho_{O,\phi}$ in the sense that

(D) $$\Pi\circ\rho_{O,\phi} = \rho_{O,\phi}\circ\Pi .$$

Since the isometry $\rho_{O,\phi}$ will map circles to circles, we obtain the following result, completing the proof of property **7** listed above.

4.8.2 Theorem. Stereographic projection maps the great circle determined by the rotated plane $\zeta = (\xi\cos\phi + \eta\sin\phi)\tan\theta$ to the circle in the equatorial plane obtained after rotation by $\rho_{O,\phi}$ of the image of the great circle determined by the plane $\zeta = \xi\tan\theta$.

The general result of property 8 can be established using similar transformation ideas to those in the proof of Theorem 4.8.2.