CUBIC FORMULA

With considerably more algebraic effort than in the case of the quadratic formula we can establish constructive formulas for the three roots α_1, α_2 and α_3 of the 'reduced' cubic equation

$$\text{(C.1)} \qquad P(x) \;=\; x^3 - px + q \;=\; 0\,, \qquad p,\, q \,\in\, \mathbb{Q}\,.$$

Establishing these formulas will use the quadratic formula as well as the roots of a particular cubic equation:

$$\text{(C.2)} \qquad x^3 \;=\; c\,, \qquad c \,=\, re^{i\theta} \,\in\, \mathbb{C}\,.$$

Indeed, if we set $w = e^{2\pi i/3}$, and $c^{1/3} = r^{1/3}e^{i\theta/3}$, then the three roots of (C.2) are

$$\text{(C.3)} \qquad c^{1/3}\,, \qquad c^{1/3}w\,, \qquad c^{1/3}\overline{w}\,.$$

This use of the solutions of a particular cubic isn't perhaps so surprising if we recall that in deriving the quadratic formula by completing the square, we needed to know the solutions of the particular quadratic $x^2 = c$.

To start on the path leading to the cubic formula we look for solutions of the form $x = u - v$. Why? Perhaps it's because because we know that the quadratic formula expresses the roots as a sum and difference of terms! More likely the idea comes only after many other ideas have been tried and found not to work. But then,

$$\begin{aligned} P(u-v) \;&=\; (u-v)^3 - p(u-v) + q \\ &=\; (u^3 - 3u^2v + 3uv^2 - v^3) - 3p(u-v) + q \\ &=\; (u^3 - v^3 + q) - (u-v)(3uv + p) \;=\; 0\,. \end{aligned}$$

This last equation will be satisfied if u, v satisfy both of

$$u^3 - v^3 + q \;=\; 0\,, \qquad 3uv + p \;=\; 0\,.$$

The point is that we can now solve for v in the second equation, getting $v = -p/3u$, and then substitute in the first producing an equation

$$u^3 + \frac{p^3}{27u^3} + p \;=\; 0\,,$$

which in turn can be written as a *quadratic* equation

$$\left(u^3\right)^2 + qu^3 + \frac{p^3}{27} \;=\; 0$$

in u^3. Thus, by the quadratic formula,

$$u^3 \;=\; -\frac{q}{2} \pm \frac{1}{2}\sqrt{q^2 - \frac{4p^3}{27}}\,\Bigg).$$

Already this starts to look promising if we recall the *discriminant*

$$\delta^2 \;=\; (\alpha_1 - \alpha_2)^2(\alpha_2 - \alpha_3)^2(\alpha_3 - \alpha_1)^2 \;=\; 4p^3 - 27q^2$$

of the polynomial P; for then

$$u^3 \;=\; -\frac{q}{2} \pm \frac{1}{2}\sqrt{\frac{-\delta^2}{27}}\,\Bigg),$$

so that by (C.3) three choices of u are

$$\Bigg(-\frac{q}{2} + \frac{1}{2}\sqrt{\frac{-\delta^2}{27}}\,\Bigg)^{1/3}, \quad \Bigg(-\frac{q}{2} + \frac{1}{2}\sqrt{\frac{-\delta^2}{27}}\,\Bigg)^{1/3} w\,, \quad \Bigg(-\frac{q}{2} + \frac{1}{2}\sqrt{\frac{-\delta^2}{27}}\,\Bigg)^{1/3} \overline{w}\,,$$

with corresponding choices

$$-\frac{p}{3}\Bigg(-\frac{q}{2} + \frac{1}{2}\sqrt{\frac{-\delta^2}{27}}\,\Bigg)^{-1/3}, \quad -\frac{p}{3w}\Bigg(-\frac{q}{2} + \frac{1}{2}\sqrt{\frac{-\delta^2}{27}}\,\Bigg)^{-1/3}, \quad -\frac{p}{3\overline{w}}\Bigg(-\frac{q}{2} + \frac{1}{2}\sqrt{\frac{-\delta^2}{27}}\,\Bigg)^{-1/3}$$

of v because $v = -p/3u$. But the process of rationalization familiar from high school allows these values of v to be rewritten in an interesting way. Indeed, $1/w = \overline{w}$ while $1/\overline{w} = w$; on the other hand, since

$$\Big(-\frac{q}{2}\Big)^2 - \Bigg(\frac{1}{2}\sqrt{\frac{-\delta^2}{27}}\,\Bigg)^2 \;=\; \frac{1}{4}\Big(q^2 + \frac{\delta^2}{27}\Big) \;=\; \frac{p^3}{27}\,,$$

we see that

$$\frac{1}{-\frac{q}{2} + \frac{1}{2}\sqrt{\frac{-\delta^2}{27}}} \;=\; \frac{27}{p^3}\Bigg(-\frac{q}{2} - \frac{1}{2}\sqrt{\frac{-\delta^2}{27}}\,\Bigg).$$

Thus after rationalization,

$$\frac{p}{3}\Bigg(-\frac{q}{2} + \frac{1}{2}\sqrt{\frac{-\delta^2}{27}}\,\Bigg)^{-1/3} \;=\; \Bigg(-\frac{q}{2} - \frac{1}{2}\sqrt{\frac{-\delta^2}{27}}\,\Bigg)^{1/3}$$

while

$$\frac{p}{3w}\left(-\frac{q}{2}+\frac{1}{2}\sqrt{\frac{-\delta^2}{27}}\ \right)^{-1/3} \;=\; \left(-\frac{q}{2}-\frac{1}{2}\sqrt{\frac{-\delta^2}{27}}\ \right)^{1/3}\overline{w}$$

and

$$\frac{p}{3\overline{w}}\left(-\frac{q}{2}+\frac{1}{2}\sqrt{\frac{-\delta^2}{27}}\ \right)^{-1/3} \;=\; \left(-\frac{q}{2}-\frac{1}{2}\sqrt{\frac{-\delta^2}{27}}\ \right)^{1/3} w\,.$$

But the solutions of $P(x) = 0$ are $u - v$. Consequently,

$$\alpha_1 \;=\; \left(-\frac{q}{2}+\frac{1}{2}\sqrt{\frac{-\delta^2}{27}}\ \right)^{1/3} + \left(-\frac{q}{2}-\frac{1}{2}\sqrt{\frac{-\delta^2}{27}}\ \right)^{1/3},$$

$$\alpha_2 \;=\; \left(-\frac{q}{2}+\frac{1}{2}\sqrt{\frac{-\delta^2}{27}}\ \right)^{1/3} w \;+\; \left(-\frac{q}{2}-\frac{1}{2}\sqrt{\frac{-\delta^2}{27}}\ \right)^{1/3}\overline{w}\,,$$

$$\alpha_3 \;=\; \left(-\frac{q}{2}+\frac{1}{2}\sqrt{\frac{-\delta^2}{27}}\ \right)^{1/3}\overline{w} \;+\; \left(-\frac{q}{2}-\frac{1}{2}\sqrt{\frac{-\delta^2}{27}}\ \right)^{1/3} w$$

where $w = e^{2\pi i/3}$.

DISCRIMINANT a la Rick

When α_1, α_2 and α_3 are the roots of the reduced cubic

$$P(x) \;=\; x^3 - px + q \;=\; 0$$

and

$$\delta^2 \;=\; (\alpha_1 - \alpha_2)^2(\alpha_2 - \alpha_3)^2(\alpha_3 - \alpha_1)^2$$

is the discriminant of P, then

$$\delta^2 \;=\; 4p^3 - 27q^2\,. \tag{C.4}$$

The proof of this is an interesting exhibition of algebraic skills. The proof of $(C.4)$ relies heavily on symmetry relationships satisfied by α_1, α_2 and α_3. Indeed, by the Fundamental Theorem of Algebra,

$$(x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \;=\; x^3 - px + q\,.$$

But after expanding and comparing coefficients we see that

$$\alpha_1 + \alpha_2 + \alpha_3 \;=\; 0\,, \qquad \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 \;=\; -p\,, \qquad \alpha_1\alpha_2\alpha_3 \;=\; -q\,.$$

The basic idea of the proof is to use these is making substitutions in the defining expression for δ^2. From the first relation it follows that $\alpha_3 = -(\alpha_1 + \alpha_2)$, so from the second it then follows that

$$p \;=\; -\Big\{\alpha_1\alpha_2 + \alpha_3(\alpha_1 + \alpha_2)\Big\} \;=\; -\alpha_1\alpha_2 + \big(\alpha_1 + \alpha_2\big)^2 \;=\; \alpha_1^2 + \alpha_1\alpha_2 + \alpha_2^2\,.$$

Since δ^2 consists of products of terms like $(\alpha_1 - \alpha_2)^2$, the next step is to note that

$$(\alpha_1 - \alpha_2)^2 \;=\; \Big\{\alpha_1^2 + \alpha_1\alpha_2 + \alpha_2^2\Big\} - 3\alpha_1\alpha_2\,,$$

so the first two of the symmetry relations ensure that

$$(\alpha_1 - \alpha_2)^2 \;=\; p - 3\alpha_1\alpha_2$$

together with corresponding results for other pairs of roots. Substituting in the expression for δ^2 we thus see that

$$\delta^2 \;= \big(p - 3\alpha_1\alpha_2\big)\big(p - 3\alpha_2\alpha_3\big)\big(p - 3\alpha_3\alpha_1\big)\,.$$

Now we expand:

$$\begin{aligned}\delta^2 \;&=\; \big(p^2 - 3p(\alpha_1\alpha_2 + \alpha_2\alpha_3) + 9\alpha_1\alpha_2^2\alpha_3\big)\big(p - 3\alpha_3\alpha_1\big)\\ &=\; \big(p^2 - 3p(\alpha_1\alpha_2 + \alpha_2\alpha_3) - 9q\alpha_2\big)\big(p - 3\alpha_3\alpha_1\big)\,,\end{aligned}$$

where, finally, the third symmetry relation has been used as well as the second one. What seems like the final step is to expand this last expression giving:

$$\begin{aligned}\delta^2 \;&=\; p^3 - 3p^2(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1) - 27q^2 + 9p(\alpha_1\alpha_2 + \alpha_2\alpha_3)\alpha_1\alpha_3 - 9pq\alpha_2\\ &=\; 4p^3 - 27q^2 + 9p(\alpha_1\alpha_2 + \alpha_2\alpha_3)\alpha_1\alpha_3 - 9pq\alpha_2\,.\end{aligned}$$

This is exactly what we want except for the last two terms. But these we can rewrite as

$$9p\alpha_2\Big\{(\alpha_1 + \alpha_3)\alpha_1\alpha_3 - q\Big\} \;=\; 9p\alpha_1\alpha_2\alpha_3(\alpha_1 + \alpha_2 + \alpha_3) \;=\; 0\,,$$

using the first and third symmetry relations. This establishes the crucial result for δ^2.