Lecture 2

Introductory signal processing ideas

In this lecture we'll look at some basic properties of finite energy digital signals, in other words sequences $x = \{x_n\}_n$ in $\ell^2(X)$ for some discrete set X which almost always will be the set $\mathbb{Z}$ of all integers and so we will write ℓ^2 instead of the more formal $\ell^2(\mathbb{Z})$; if needed, we could restrict attention to sequences in which only finitely many of the x_n are non-zero, but that's usually not necessary from a strictly mathematical point of view. From a practical point of view, the values of the x_n would almost certainly be real, but it's actually more illuminating to allow complex-valued sequences, and so that's what we shall do. From a mathematical point of view what will be crucial is the fact that translation $n \longrightarrow n+m$ is defined on $\mathbb{Z}$ for all integers m, but to appreciate fully the mathematical and signal processing ideas you need to keep track of how and where the space ℓ^2 is being used. On some occasions it is a space of finite energy signals, while on others it is a space of *coefficients* of elements of an inner product space with respect to some orthonormal family. The interplay between mathematics and signal processing ideas will be a recurring theme in this and subsequent lectures!

(2.1) Examples, decompositions. 1. The simplest such signal is the *unit impulse*

$$\delta \;=\; (\ldots, 0, 1, 0, \ldots) \;=\; \{\delta_n\}, \qquad \delta_n \;=\; \begin{cases} 1, & n = 0, \\ 0, & n \neq 0, \end{cases}$$

i.e., $\delta = \varepsilon^{(0)}$ to use the notation of the previous lecture. Similarly, each of the $\varepsilon^{(n)}$ is a finite energy digital signal in which only one entry is non-zero.

2. Each of

$$\varphi^{(n)} \;=\; \frac{1}{\sqrt{2}}\Big(\varepsilon^{(2n)} + \varepsilon^{(2n+1)}\Big), \qquad \widetilde{\varphi}^{(n)} \;=\; \frac{1}{\sqrt{2}}\Big(\varepsilon^{(2n)} - \varepsilon^{(2n+1)}\Big)$$

is a finite energy signal such that $E(\varphi^{(n)}) = E(\widetilde{\varphi}^{(n)}) = 1$.

3. There are also *oscillatory* signals: fix a real number ξ_0 and define e_{ξ_0} by

$$(e_{\xi_0})_n \;=\; e^{2\pi i n \xi_0}.$$

For no value of ξ_0 does e_{ξ_0} have finite energy and hence does not belong to ℓ, however. Nonetheless,if we fix integers $k,\ N,\ \ 0 \le k < N$, and define e_w by

$$(e_w)_n \;=\; w^n, \qquad w \;=\; e^{2\pi i k/N}.$$

Then w is an N^{th} root of unity since $w^N = e^{2\pi i k} = 1$ for each choice of k. Again the digital signal e_w cannot have finite energy, but it has the important property that it is N-*periodic*, so it belongs to $\ell^2(\mathbb{Z}_N)$.

To get started let's look at a particularly simple signal

$$a \;=\; \{\ldots,\, 0,\, 5,\, 4,\, 7,\, 9,\, 0,\, \ldots\} \;=\; 5\,\varepsilon^{(0)} + 4\,\varepsilon^{(1)} + 7\,\varepsilon^{(2)} + 9\,\varepsilon^{(3)}$$

having just 4 non-zero entries; calculations show that $E(a) = 171$. It admits a decomposition

$$a \;=\; \Big\{\ldots,\, 0,\, \frac{9}{2},\, \frac{9}{2},\, 8,\, 8,\, 0,\, \ldots\Big\} \\ \qquad\qquad + \;\Big\{\ldots,\, 0,\, \frac{1}{2},\, -\frac{1}{2},\, -1,\, 1,\, 0,\, \ldots\Big\}$$

into what we might call its *coarse details*, obtained by averaging pairs of consecutive terms, and its *fine details*, obtained by 'differencing' pairs of consecutive terms. There are 8 non-zero terms in this decomposition, but each value is repeated twice, so we should be able to express this decomposition by 4 terms. To achieve this observe that we can write

$$\Big\{\ldots,\, 0,\, \frac{9}{2},\, \frac{9}{2},\, 8,\, 8,\, 0,\, \ldots\Big\} \;=\; \frac{9}{\sqrt{2}}\,\varphi^{(0)} + \frac{16}{\sqrt{2}}\,\varphi^{(1)}$$

and

$$\Big\{\ldots,\, 0,\, \frac{1}{2},\, -\frac{1}{2},\, -1,\, 1,\, 0,\, \ldots\Big\} \;=\; \frac{1}{\sqrt{2}}\,\widetilde{\varphi}^{(0)} - \frac{2}{\sqrt{2}}\,\widetilde{\varphi}^{(1)},$$

so that

$$a \;=\; \frac{9}{\sqrt{2}}\,\varphi^{(0)} + \frac{16}{\sqrt{2}}\,\varphi^{(1)} + \frac{1}{\sqrt{2}}\,\widetilde{\varphi}^{(0)} - \frac{2}{\sqrt{2}}\,\widetilde{\varphi}^{(1)}$$

is a more compact way of representing the coarse + fine detail decomposition of a, more compact because we need only 4 coefficients. But is there some 'clever' way of expressing how these 4 coefficients are obtained? Well, simple calculations show that

$$\frac{9}{\sqrt{2}} \;=\; (a,\, \varphi^{(0)}), \qquad \frac{16}{\sqrt{2}} \;=\; (a,\, \varphi^{(1)}),$$

while

$$\frac{1}{\sqrt{2}} \;=\; (a,\, \widetilde{\varphi}^{(0)}), \qquad -\frac{2}{\sqrt{2}} \;=\; (a,\, \widetilde{\varphi}^{(1)}).$$

Consequently,

$$a \;=\; \underbrace{(a,\, \varphi^{(0)})\,\varphi^{(0)} + (a,\, \varphi^{(1)})\,\varphi^{(1)}}_{coarse\ details} + \underbrace{(a,\, \widetilde{\varphi}^{(0)})\,\widetilde{\varphi}^{(0)} + (a,\, \widetilde{\varphi}^{(1)})\,\widetilde{\varphi}^{(1)}}_{fine\ details}.$$

Thus we see how the orthonormal families introduced in the previous lecture provide one decomposition of a signal into its coarse and fine details; note that a change in the orthonormal families will change the decomposition, of course! What's the point? We obtain 25% compression of a taking

$$a \approx (a, \varphi^{(0)})\, \varphi^{(0)} + (a, \varphi^{(1)})\, \varphi^{(1)} + (a, \widetilde{\varphi}^{(1)})\, \widetilde{\varphi}^{(1)},$$

with only 0.292% loss of energy and 50% compression taking

$$a \approx (a, \varphi^{(0)})\, \varphi^{(0)} + (a, \varphi^{(1)})\, \varphi^{(1)}$$

with only 1.461% loss of energy .

In this lecture the objective will be to see how such decompositions can also be achieved using *filtering*. After that the process of choosing various filters can begin.

(2.2) Delay, convolution. Various linear operators on ℓ^2 will be needed. Engineers make frequent use of the *Delay* operator

$$S : \{x_n\}_n \longrightarrow \{x_{n-1}\}_n\,;$$

mathematicians would call this a *translation* operator on the additive group $\mathbb{Z}$ of integers! Since

$$E(Sx) = \sum_n |x_{n-1}|^2 = \sum_n |x_n|^2 = E(x),$$

S is energy-preserving. The most important operator of all, however, is convolution, or *filtering* as engineers call it. Recall first that the convolution $h * x$ of two sequences is defined by

$$(h * x)_n = \sum_m h_m\, x_{n-m} = \sum_m h_{n-m}\, x_m\,;$$

in operator terms

$$h * x = \sum_m h_m\, S^m(x).$$

To check that convolution is well-defined on finite energy signals, observe that by the triangle inequality,

$$\begin{aligned} E^{1/2}(h * x) &= E^{1/2}\Big(\sum_m h_m\, S^m x\Big) \\ &\le \sum_m \Big\{|h_m|\, E^{1/2}(S^m x)\Big\} = \Big\{\sum_m |h_m|\Big\} E^{1/2}(x) \end{aligned}$$

since $E(Sx) = E(x)$. Consequently, the convolution $h * x$ of a finite energy signal x will itself have finite energy provided $\sum_n |h_n| < \infty$, *i.e.*, when the coefficients of h are absolutely convergent; in particular, the convolution $x \longrightarrow h * x$ will map finite energy signals to finite energy ones if only finitely many $h_n \neq 0$.

Now, the convolution operator $x \longrightarrow h * x$ will have an adjoint on ℓ^2. But what form will this adjoint take? Well, given sequences $x = \{x_n\}_n$ and $y = \{y_n\}_n$, we see that

$$(h * x,\, y) \;=\; \sum_n \Big(\sum_m h_{n-m}\, x_m\Big)\, \overline{y_n}$$

$$= \sum_m \Big(\sum_n h_{n-m}\, \overline{y_n}\Big)\, x_m \;=\; (x,\, h^* * y)$$

where the last convolution is defined by

$$h^* * y \;=\; \sum_n \overline{h_{n-m}}\, y_n, \qquad i.e., \quad h^* \;=\; \{\, \overline{h_{-n}}\,\}_n.$$

(2.3) Discrete-Time Fourier Transform. Given a sequence $x = \{x_n\}_n$, its *Discrete-Time Fourier Transform* (*DTFT*), $x \to \widehat{x}$, is defined by

$$\widehat{x}(\xi) \;=\; \sum_n x_n\, e^{-2\pi i n \xi}\,;$$

in signal analysis one usually writes $X(\xi)$ instead of $\widehat{x}$ and we shall often follow this convention. The sum makes good sense if only finitely many $x_n \neq 0$. In this case $\widehat{x}(\xi)$ can be interpreted as the inner product

$$\widehat{x}(\xi) \;=\; (x,\, e_\xi) \;=\; X(\xi)$$

of x with the oscillatory signal e_ξ defined in the previous section except that we have to be careful because e_ξ does not have finite energy; in addition, because each exponential function $e^{-2\pi i n\xi}$ has period 1, $X(\xi)$ has period 1.

Definition. *The Discrete-Time Fourier transform $X(\xi)$ of a finite energy signal x will be called the Frequency Response function of x.*

It is often useful to think of these period 1 functions $X(\xi)$ as functions on $[-\frac{1}{2},\, \frac{1}{2}]$. Since $\{\, e^{2\pi i n\xi} \,:\, -\infty < n < \infty \,\}$ is orthonormal in $L^2[-\frac{1}{2},\, \frac{1}{2}]$, it follows that

$$E(X) \;=\; \int_{-1/2}^{1/2} \big|X(\xi)\big|^2\, d\xi$$

$$= \int_{-1/2}^{1/2} \Big|\sum_n x_n\, e^{-2\pi i n\xi}\Big|^2\, d\xi \;=\; \sum_n |x_n|^2 \;=\; E(x).$$

Consequently, $x \to X(\xi)$ is energy-preserving as a mapping from ℓ^2 into $L^2[-\frac{1}{2}, \frac{1}{2}]$. One property of the $(DTFT)$ is that

$$\widehat{Sx}(\xi) \;=\; \sum_n x_{n-1}\, e^{-2\pi i n\xi} \;=\; e^{-2\pi i\xi}\, X(\xi),$$

i.e., the Discrete time Fourier transform *maps delay into modulation* which is a signal-processing way of talking about pointwise multiplication by the oscillatory function $e^{-2\pi i\xi}$. But a more crucial property is the following.

Theorem. *The $(DTFT)$ maps convolution into pointwise multiplication, more precisely,*

$$\widehat{h * x}(\xi) \;=\; \widehat{h}(\xi)\,\widehat{x}(\xi) \;=\; H(\xi)\, X(\xi)$$

for all real ξ.

Proof: by definition

$$\begin{aligned}\widehat{h * x}(\xi) \;&=\; \sum_n \Big(\sum_m h_{n-m}\, x_m\Big)\, e^{-2\pi i n\xi} \\ &=\; \sum_n \Big(\sum_m h_{n-m}\, x_m\Big)\, e^{-2\pi i m\xi}\, e^{-2\pi i(n-m)\xi},\end{aligned}$$

which after simplification becomes

$$\widehat{h * x}(\xi) \;=\; \Big(\sum_n h_{n-m}\, e^{-2\pi i(n-m)\xi}\Big) \sum_m x_m\, e^{-2\pi i m\xi} \;=\; H(\xi)\, X(\xi),$$

completing the proof. □

Corollary. *The Frequency Response function of the adjoint $h^* = \{\overline{h_{-n}}\}_n$ is given by*

$$\widehat{h^*}(\xi) \;=\; \overline{H(\xi)}, \qquad H(\xi) \;=\; \sum_n h_n\, e^{2\pi i n\xi}.$$

Proof: by definition,

$$\widehat{h^*}(\xi) \;=\; \sum_n \overline{h_{-n}}\, e^{-2\pi i n\xi} \;=\; \overline{\Big\{\sum_n h_{-n}\, e^{2\pi i n\xi}\Big\}} \;=\; \overline{\Big\{\sum_n h_n\, e^{-2\pi i n\xi}\Big\}} \;=\; \overline{H(\xi)},$$

completing the proof. □

The adjoint of the $(DTFT)$ defines a linear mapping from $L^2[-\frac{1}{2}, \frac{1}{2}]$ into ℓ^2. As we shall see in the next lecture this is (almost) the same as the Fourier coefficient mapping. By Parseval's theorem applied to the $(DTFT)$,

$$(x, y) = \int_{-1/2}^{1/2} X(\xi)\,\overline{Y(\xi)}\,d\xi = \int_{-1/2}^{1/2} X(\xi)\,\overline{\Big(\sum_n y_n\, e^{-2\pi i n\xi}\Big)}\,d\xi$$

$$= \sum_n \Big(\int_{-1/2}^{1/2} X(\xi)\, e^{2\pi i n\xi}\,d\xi\Big)\,\overline{y_n} = \sum_n x_n\,\overline{y_n}$$

for all finite energy sequences x, y. From this it follows that

$$x_n = \int_{-1/2}^{1/2} X(\xi)\, e^{2\pi i n\xi}\,d\xi,$$

thereby recovering x from its Frequency Response function $X(\xi)$. In fact, if we get 'clever' here and write

$$(\dagger\dagger) \qquad x = \sum_n \Big(\int_{-1/2}^{1/2} X(\xi)\, e^{2\pi i n\xi}\,d\xi\Big)\,\varepsilon^{(n)},$$

we can actually regard this as yet another way of *representing* a signal, this time in terms of its Frequency Response function. When a periodic signal is studied using Fourier analysis techniques, $(\dagger\dagger)$ is the most important way of representing this signal.

(2.4) z-transform. The (forward) z-transform, $X(z)$, of a signal $x = \{x_n\}$ is defined by

$$X(z) = \sum_n x_n\, z^{-n}, \qquad z \in \mathbb{C}.$$

As $z = e^{2\pi i\xi}$ on the unit circle in the complex plane,

$$X(e^{2\pi i\xi}) = \sum_n x_n\, e^{-2\pi i n\xi} = \widehat{x}(\xi),$$

in other words, the z-transform of x can be regarded as the extension of the Discrete Fourier transform of x from the unit circle to the whole complex plane $\mathbb{C}$. Perhaps not surprisingly, engineers make a lot of use of complex analysis (poles, zeros, Residue theorem *etc.*) in signal analysis, as does Daubechies at a crucial stage in her construction of wavelets!

Basic to all of sub-band coding are two sample rate changes; in other words, they are involved in changing resolution. They have a particularly informative description as operators on ℓ^2 as well as in terms of Discrete Time Fourier transforms.

(2.5) Sampling rate changes I. *Down-sampling* or *sub-sampling* a signal $x = \{x_n\}_n$ produces a new signal

$$(\downarrow 2)x \; = \; \{x_{2n}\}_n,$$

by discarding all *odd-indexed* terms from x and re-indexing; clearly,

$$E((\downarrow 2)x) \; = \; \sum_n |x_{2n}|^2 \; \le \; \sum_n |x_n|^2 \; = \; E(x).$$

Notice that $(\downarrow 2)x = (\downarrow 2)y$ irrespective of the values of their odd-indexed terms $x_{2n+1},\ y_{2n+1}$; thus different sequences may coincide after downsampling. On the other hand, in Fourier terms,

$$\begin{aligned}\tfrac{1}{2}\Big\{X(\tfrac{1}{2}\xi) + X(\tfrac{1}{2}\xi + \tfrac{1}{2})\Big\} \; &= \; \tfrac{1}{2}\Big\{\sum_n x_n\, e^{-\pi i n\xi} + \sum_n x_n\, e^{-\pi i n(\xi+1)}\Big\} \\ &= \; \tfrac{1}{2}\Big\{\sum_n x_n\, e^{-\pi i n\xi}(1 + (-1)^n)\Big\} \; = \; \sum_n x_{2n}\, e^{2\pi i n\xi}.\end{aligned}$$

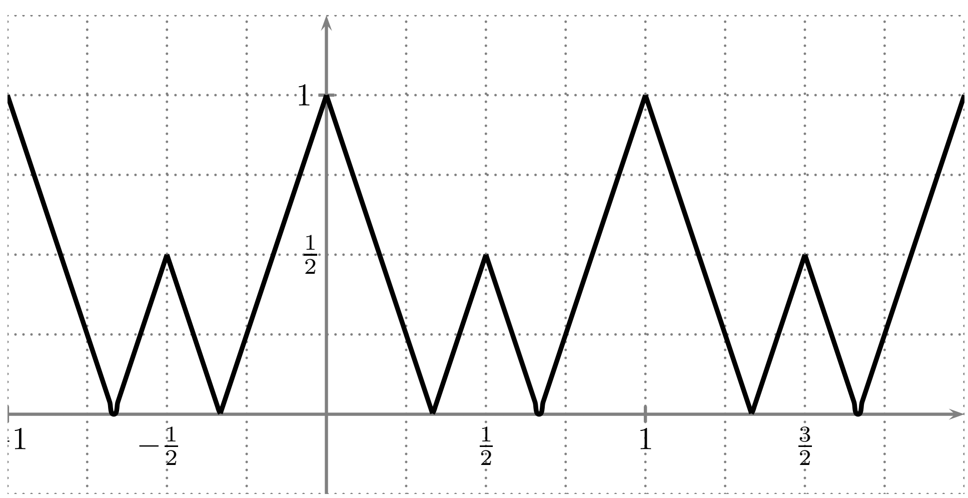
Consequently,

$$\widehat{(\downarrow 2)x}(\xi) \; = \; \tfrac{1}{2}\Big\{X(\tfrac{1}{2}\xi) + X(\tfrac{1}{2}\xi + \tfrac{1}{2})\Big\}.$$

Notice that each of the functions $X(\frac{1}{2}\xi),\ X(\frac{1}{2}\xi+\frac{1}{2})$ has period 2, but when we add them we end up with a period 1 function. As an illustration, consider the signal x whose Frequency Response function is the period 1 function

$$X(\xi) \; = \; |1 - 3|\xi||, \quad 0 \le |\xi| \le \tfrac{1}{2}, \qquad X(\xi + 1) \; = \; X(\xi);$$

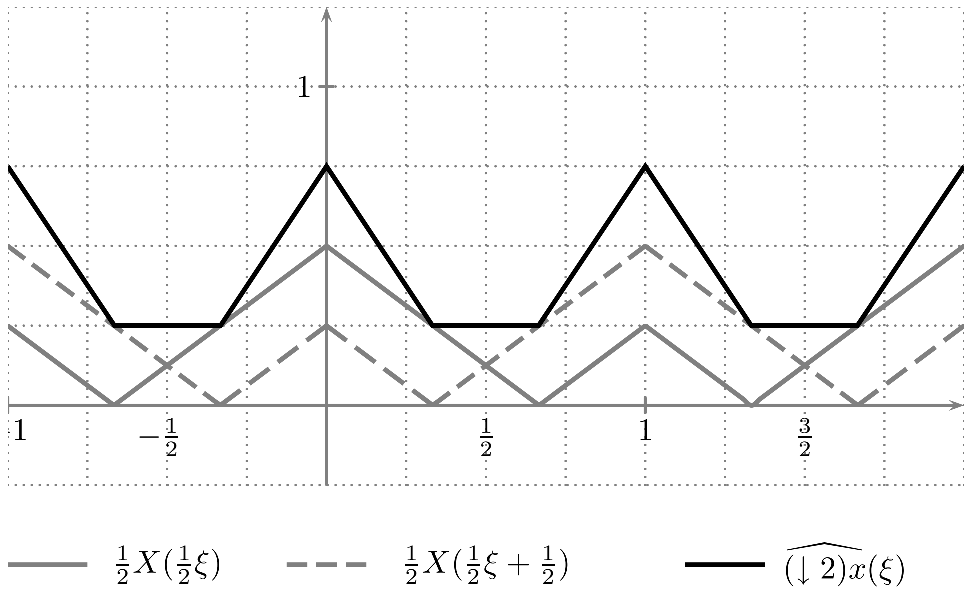
its graph is



Plotting the graphs of all three of

$$\tfrac{1}{2}X(\tfrac{1}{2}\xi), \qquad \tfrac{1}{2}X(\tfrac{1}{2}\xi+\tfrac{1}{2}), \qquad \widehat{(\downarrow 2)x}(\xi) \;=\; \tfrac{1}{2}X(\tfrac{1}{2}\xi)+\tfrac{1}{2}X(\tfrac{1}{2}\xi+\tfrac{1}{2})$$

on the same axes we thus obtain



The figure makes clear that the graph of $X(\frac{1}{2}\xi+\frac{1}{2})$ is simply the graph of $X(\frac{1}{2}\xi)$ shifted in frequency by 1; signal processing language calls $X(\frac{1}{2}\xi+\frac{1}{2})$ an *alias* of $X(\frac{1}{2}\xi)$.

(2.6) Sampling rate changes II. *Up-sampling* is the converse of down-sampling. Given a signal $y = \{y_n\}_n$, up-sampling produces a new signal

$$(\uparrow 2)y \;=\; \{v_n\}, \qquad \begin{cases} v_{2n} \;=\; y_n, \\ v_{2n+1} \;=\; 0 \end{cases}$$

by inserting zeros between consecutive terms of y and relabelling. In mathematical terms, up-sampling is the adjoint of down-sampling: indeed, on sequences x and y,

$$((\downarrow 2)x,\, y) \;=\; \sum_n x_{2n}\,\overline{y_n} \;=\; (x,\, (\uparrow 2)y), \qquad x,\, y \,\in\, \ell^2.$$

More is true, in fact: since

$$E((\uparrow 2)y) \;=\; \sum_n \Big\{|v_{2n}|^2 + |v_{2n+1}|^2\Big\} \;=\; \sum_n |y_n|^2 \;=\; E(y),$$

it is clearly energy-preserving. On the other hand, in terms of the $(DTFS)$,

$$\widehat{(\uparrow 2)y}(\xi) \;=\; \sum_n y_n\, e^{-2\pi i 2n\xi} \;=\; Y(2\xi),$$

which is now a period $\frac{1}{2}$ function, so up-sampling *decreases periodicity*; more precisely: it *halves it*.

Finally, we come to the notion that dominates the theory of wavelets whatever point of view is adopted.

(2.7) Filtering. The term 'filter' suggests 'removal' or 'selection', and this is precisely what a filter does to a signal as will be made precise shortly: think of passing water through a filter to purify it or to make a cup of coffee! We adopt a theoretical definition. A (discrete) *filter* is a *linear, time-invariant operator* acting on ℓ^2; in other words, it is a linear operator $\mathcal{H}$ mapping finite energy signals to finite energy signals and satisfying $\mathcal{H}(Sx) = S(\mathcal{H}(x))$ with respect to the delay operator S. Any such operator is given by convolution:

$$\mathcal{H}:\ x \ = \ \{x_n\}_n \ \longrightarrow \ h * x \ = \ \Big\{\sum_{\ell} h_\ell\, x_{n-\ell}\Big\}_n$$

with a given sequence $\{h_n\}_n$, the sequence of *filter coefficients*. Another way of writing this is

$$\mathcal{H} \ = \ \sum_{\ell} h_\ell\, S^\ell, \qquad \mathcal{H}(x) \ = \ \sum_{\ell} h_\ell\, S^\ell(x),$$

a form which has the advantage of making the time-invariance of $\mathcal{H}$ very clear since

$$S(\mathcal{H}x) \ = \ S\Big(\sum_{\ell} h_\ell\, S^\ell x\Big) \ = \ \sum_{\ell} h_\ell\, S^{\ell+1}x \ = \Big(\sum_{\ell} h_\ell\, S^\ell\Big)Sx \ = \ \mathcal{H}(Sx).$$

When only finitely many of the $h_m \neq 0$ it is usual to say that $\mathcal{H}$ is an FIR (Finite Impulse Response) filter, by contrast with the case of an IIR (Infinite Impulse Response) filter where infinitely many of the $h_m \neq 0$. A filter is said to be *causal* when $h_\ell = 0,\ \ell < 0$, the point being that in terms of convolution

$$(\mathcal{H}x)_n \ = \ \sum_{\ell \geq 0} h_\ell\, x_{n-\ell}$$

so $(\mathcal{H}x)_n$ depends only on x_n and *earlier* terms $x_m,\ m < n$, in the signal when $\mathcal{H}$ is causal. Most of the filters $\mathcal{H}$ we shall study are FIR filters, so there is no question that $\mathcal{H}$ is well-defined, and we don't care if the filters are causal.

But how does a filter actually 'filter' a signal? Well, the $(DTFT)$ maps convolution to pointwise multiplication, so

$$y \ = \ \mathcal{H}(x) \ = \ h * x \quad \Longrightarrow \quad Y(\xi) \ = \ H(\xi)\, X(\xi)$$

where, as before,

$$H(\xi) \;=\; \sum_{\ell} h_\ell \, e^{-2\pi i \ell \xi}.$$

is the *Frequency Response function* of the filter coefficients of $\mathcal{H}$. But then, in view of representation (††),

$$\mathcal{H}(x) \;=\; \sum_n \left(\int_{-1/2}^{1/2} H(\xi)\, X(\xi)\, e^{2\pi i n \xi}\, d\xi \right) \varepsilon^{(n)};$$

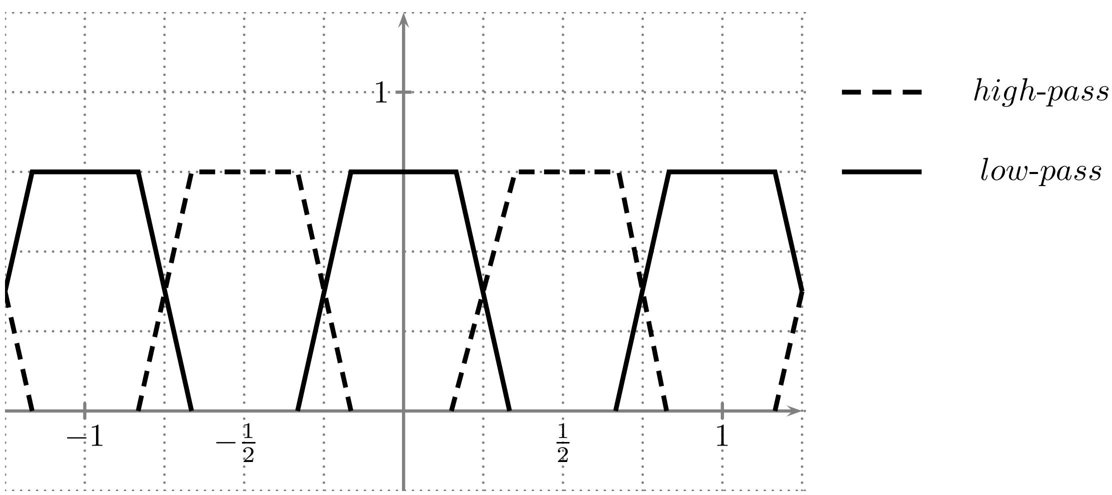
in other words, the action of $\mathcal{H}$ is to *select or reject frequencies* in a signal. Filters come in many 'flavors' depending on how these frequencies are selected or rejected:

(a) *ideal*: $H(\xi)$ takes only the values 0, 1;

(b) *low-pass*: $|\xi| > a \implies H(\xi) = 0$ for some $0 < a < \frac{1}{2}$;

(c) *high-pass*: $|\xi| < a \implies H(\xi) = 0$ for some $0 < a < \frac{1}{2}$;

Typical, but unrealistic, examples are



Thus a low-pass filter keeps only 'low' frequencies in some band about the origin, while a high-pass filter keeps only 'high' frequencies in some band not including the origin.

No FIR filters can be low or high pass in this sense defined above, however. Indeed, if $\mathcal{H}$ is a FIR filter with filter coefficients $h_0, h_1, \ldots, h_{L-1}$ for some L, say, then

$$H(z) \;=\; h_0 + \frac{h_1}{z} + \ldots + \frac{h_{L-1}}{z^{L-1}} \;=\; \frac{P(z)}{z^{L-1}}$$

for some polynomial P. So $H(\xi)$ has at most finitely many zeros, hence cannot vanish on any interval. The best that $H(\xi)$ can do is vanish at a point, so we shall adopt the following definition.

Definition. *An FIR filter $\mathcal{H}$ will be said to 'try hard' to be a low-pass filter when $H(0) \neq 0$ and $H(\pm\frac{1}{2}) = 0$; by contrast, $\mathcal{H}$ will be said to 'try hard' to be a high-pass filter when $H(0) = 0$ and $H(\pm\frac{1}{2}) \neq 0$.*

A key step in Daubechies construction of wavelets is to associate a 'high-pass' FIR filter $\widetilde{\mathcal{H}}$ to any 'low-pass' FIR filter. For suppose $\mathcal{H}$ has filter coefficients $\{h_\ell\}_\ell$ and Frequency Response function

$$H(\xi) \;=\; \sum_n h_n\, e^{-2\pi i n\xi}, \qquad H(0) \;\neq\; 0, \qquad H(\pm\tfrac{1}{2}) \;=\; 0.$$

Now define $\widetilde{\mathcal{H}}$ to be the FIR filter having filter coefficients $\{\widetilde{h}_n\}_n$,

$$\widetilde{h}_n \;=\; (-1)^n\, \overline{h_{1-n}}.$$

One of the problems for this lecture asks you to show that the corresponding Frequency Response function $\widetilde{H}(\xi)$ is given by

$$\widetilde{H}(\xi) \;=\; \sum_n (-1)^n\, \overline{h_{1-n}}\, e^{-2\pi i n\xi} \;=\; -e^{-2\pi i\xi}\, \overline{H(\xi+\tfrac{1}{2})};$$

in particular, therefore,

$$H(0) \neq 0 \;\Longrightarrow\; \widetilde{H}(\pm\tfrac{1}{2}) \;\neq\; 0, \qquad H(\pm\tfrac{1}{2}) \;=\; 0 \;\Longrightarrow\; \widetilde{H}(0) \;=\; 0,$$

so $\widetilde{\mathcal{H}}$ tries hard to be a high-pass filter when $\mathcal{H}$ tries hard to be a low-pass filter.

Examples. Let's look at three examples to make this clearer. Notice that in these examples we are allowing the number of filter coefficients to increase from 2 to 3, and then to 4. There must be some pattern to this!!

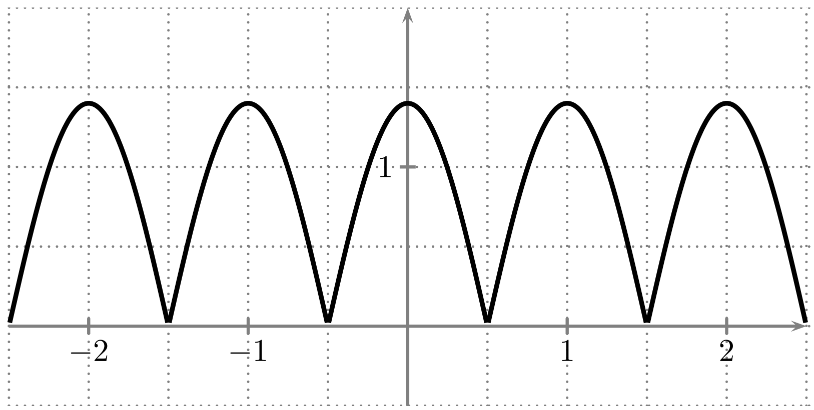
1. The Haar filters: $h = \{h_n\}_n$ *and* $\widetilde{h} = \{\widetilde{h}_n\}_n$ *are causal FIR filters defined by*

$$h_n \;=\; \begin{cases} \frac{1}{\sqrt{2}}\,, & n \;=\; 0, \\[1ex] \frac{1}{\sqrt{2}}\,, & n \;=\; 1, \\[1ex] 0, & n \;\neq\; 0,\, 1, \end{cases} \qquad \widetilde{h}_n \;=\; \begin{cases} \frac{1}{\sqrt{2}}\,, & n \;=\; 0, \\[1ex] -\frac{1}{\sqrt{2}}\,, & n \;=\; 1, \\[1ex] 0, & n \;\neq\; 0,\, 1. \end{cases}$$

(I realize I'm using the same notation for different things - you have to work out the precise meaning from context!! Shortly everything will settle down and h will mean just one thing.) Simple calculations show that their respective Frequency Response functions $H(\xi)$ and $\widetilde{H}(\xi)$ are given by

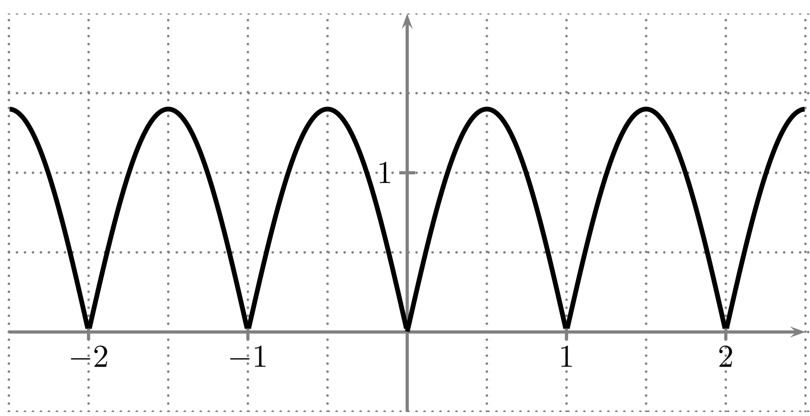
$$H(\xi) \;=\; \sqrt{2}\, e^{-\pi i\xi}\, \cos \pi\xi, \qquad \widetilde{H}(\xi) \;=\; \sqrt{2}\, i\, e^{-\pi i\xi}\, \sin \pi\xi.$$

As often happens with attempts to graph a Fourier transform, however, the presence of complex values forces us to graph *absolute values* of the particular Fourier transform. With that in mind we obtain



for the graph of $|H(\xi)|$. Thus $|H(\xi)| > 0$ in a neighborhood of $\xi = 0$, while $H(\pm\frac{1}{2}) = 0$ This is about the best a polynomial can do in behaving like a low-pass filter; the only question, and an absolutely key one for the Daubechies filters, is how many zeros does $H(\xi)$ have at $\xi = \pm\frac{1}{2}$? Does it have zeros up to order 1, or 2, or some large finite integer value? In the Haar case, the **order of the zero of $H(\xi)$ at $\xi = \pm\frac{1}{2}$ is one**.

On the other hand, the graph of $|\widetilde{H}(\xi)|$ is



so $|\widetilde{H}(\xi)| > 0$ in a neighborhood of $\xi = \pm\frac{1}{2}$, while $\widetilde{H}(0) = 0$. Again, this is the best a polynomial can do in behaving like a high-pass filter, the only question being the order of the zero at $\xi = 0$. In the Haar case, the **order of the zero of $\widetilde{H}(\xi)$ at $\xi = 0$ is one**.

2. Tent filter: it is easy and important to increase the zeros of a low-pass filter at $\xi = \pm\frac{1}{2}$. For let $\mathcal{G}$ be the filter having filter coefficients

$$g_n = \begin{cases} \frac{1}{2\sqrt{2}}, & n = 0, 2, \\ \frac{1}{\sqrt{2}}, & n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Its frequency response function

$$G(\xi) \;=\; \sqrt{2}\, e^{-2\pi i\xi}\, (\cos \pi\xi)^2$$

now has a double zero at $\xi \;=\; \pm\frac{1}{2}$, and the associated filter $\widetilde{\mathcal{G}}$,

$$\widetilde{g}_n \;=\; \begin{cases} -\frac{1}{2\sqrt{2}}, & n \;=\; -1,\, 1, \\ \frac{1}{\sqrt{2}}, & n \;=\; 0, \\ 0, & \text{otherwise}, \end{cases}$$

has Frequency response function

$$\widetilde{G}(\xi) \;=\; \sqrt{2}\, (\sin \pi\xi)^2.$$

Clearly $\widetilde{\mathcal{G}}$ is trying hard to be a high-pass filter having a double zero at the origin.

3. Daubechies $db2$-filter: even at this early stage it's impossible to resist introducing the famous Daubechies-$db2$ filter coefficients - we do resist for the moment saying where they come from, however! There are 4 coefficients and they are defined by

$$h_0 \;=\; \frac{1+\sqrt{3}}{4\sqrt{2}}, \quad h_1 \;=\; \frac{3+\sqrt{3}}{4\sqrt{2}}, \quad h_2 \;=\; \frac{3-\sqrt{3}}{4\sqrt{2}}, \quad h_3 \;=\; \frac{1-\sqrt{3}}{4\sqrt{2}}$$

with corresponding Frequency response function

$$H(\xi) \;=\; \sqrt{2} e^{-3\pi i\xi}\, (\cos \pi\xi)^2\, (\cos \pi\xi + i\sqrt{3} \sin \pi\xi).$$

(To check this expression for $H(\xi)$ it's probably easiest to start with the given expression and then use trig identities to write $H(\xi)$ as a finite sum $\sum_{n=0}^{3} h_n e^{-2\pi i n\xi}$.) Clearly $db2$ tries hard to be a low-pass filter. Notice that the presence of the $(\cos \pi\xi)^2$-term ensures that $H(\xi)$ **has a zero of order 2 at** $\xi \;=\; \pm\frac{1}{2}$. This is the reason why it is often referred to as the $db2$-filter; the Haar could well be called the $db1$-filter. Others refer to the $db2$-filter as the $D4$-filter because it has 4 coefficients. We have followed Matlab in the choice of notation because you will be making good use of its Wavelet toolbox!

(2.8) Filtering together with up/down sampling. Finally, let's put these crucial operations of filtering and up/down sampling together. Given an FIR filter $\mathcal{H}$, consider the operators

$$(\ddagger) \qquad \downarrow 2 \circ \mathcal{H}^* : x \longrightarrow \Big\{ \sum_{\ell} x_\ell\, \overline{h_{\ell-2n}} \Big\}_n, \qquad \mathcal{H} \circ \uparrow 2 : x \longrightarrow \Big\{ \sum_{\ell} x_\ell\, h_{n-2\ell} \Big\}_n$$

on a sequence $\{x_n\}_n$. What's perhaps not clear is why we use $\downarrow 2 \circ \mathcal{H}^*$ instead of $\downarrow 2 \circ \mathcal{H}$; to see why, note that

$$\big(\mathcal{H} \circ \uparrow 2\big)^* \;=\; \big(\uparrow 2\big)^* \circ \mathcal{H}^* \;=\; \downarrow 2 \circ \mathcal{H}^*;$$

thus the first operator is just the adjoint of the second. More crucial perhaps, is the question of just what the effect of these operators is on a signal. The $\mathcal{S}\circ\mathcal{S}^*$ example from chapter 1 will provide the answer. In terms of Frequency Response functions

$$((\downarrow 2\circ\mathcal{H}^*)\,x)\widehat{\ }(\xi) \;=\; \tfrac{1}{2}\Big\{\,\overline{H(\tfrac{1}{2}\xi)}\,X(\tfrac{1}{2}\,\xi)+\overline{H(\tfrac{1}{2}\xi+\tfrac{1}{2})}\,X(\tfrac{1}{2}\,\xi+\tfrac{1}{2})\,\Big\},$$

while

$$((\mathcal{H}\circ\uparrow 2)\,x)\widehat{\ }(\xi) \;=\; H(\xi)\,X(2\xi).$$

These results, though useful later, shed little light on the coarse+fine decomposition we started out with. Patience! Let's look first at the effect of these operators on sequences themselves.

Example. Consider the case of the Haar filters

$$h_n \;=\; \begin{cases} \frac{1}{\sqrt{2}}\,, & n\,=\,0,\\[1ex] \frac{1}{\sqrt{2}}\,, & n\,=\,1,\\[1ex] 0, & n\,\neq\,0,\,1,\end{cases} \qquad \widetilde{h}_n \;=\; \begin{cases} \frac{1}{\sqrt{2}}\,, & n\,=\,0,\\[1ex] -\frac{1}{\sqrt{2}}\,, & n\,=\,1,\\[1ex] 0, & n\,\neq\,0,\,1.\end{cases}$$

Now fix sequences $a\,=\,\{a_n\}_n,\ c\,=\,\{c_n\}_n$ and recall the sequences $\{\varphi^{(n)}\}_n,\ \{\widetilde{\varphi}^{(n)}\}_n,$

$$\varphi^{(n)} \;=\; \frac{1}{\sqrt{2}}\big(\varepsilon^{(2n)}+\varepsilon^{(2n+1)}\big), \qquad \widetilde{\varphi}^{(n)} \;=\; \frac{1}{\sqrt{2}}\big(\varepsilon^{(2n)}-\varepsilon^{(2n+1)}\big),$$

defined in lecture 1 as well as being discussed earlier in this lecture. Since the filter coefficients are real, we can omit complex conjugates in (‡) and compute using the convolution formulas in (‡):

(1) $\downarrow 2\circ\mathcal{H}^*$: (averaging operator)

$$(\downarrow 2\circ\mathcal{H}^*)a \;=\; \sum_n\ \frac{1}{\sqrt{2}}\big(a_{2n}+a_{2n+1}\big)\,\varepsilon^{(n)} \;=\; \big\{\,(a,\ \varphi^{(n)})\,\big\}_n;$$

(2) $\downarrow 2\circ\widetilde{\mathcal{H}}^*$: (difference operator)

$$(\downarrow 2\circ\widetilde{H}^*)a \;=\; \sum_n\ \frac{1}{\sqrt{2}}\big(a_{2n}-a_{2n+1}\big)\,\varepsilon^{(n)} \;=\; \big\{\,(a,\ \widetilde{\varphi}^{(n)})\,\big\}_n;$$

(3) $\mathcal{H}\circ\uparrow 2$: (spreading operator)

$$\sum_\ell\ c_\ell\,h_{n-2\ell} \;=\; \begin{cases} \frac{1}{\sqrt{2}}\,c_m, & n\,=\,2m,\\[1ex] \frac{1}{\sqrt{2}}\,c_m, & n\,=\,2m+1,\end{cases} \qquad (\mathcal{H}\circ\uparrow 2)c \;=\; \sum_n\ c_n\,\varphi^{(n)}\,;$$

(4) $\widetilde{\mathcal{H}} \circ \uparrow 2$: (spread and oscillate)

$$\sum_\ell c_\ell \, \widetilde{h}_{n-2\ell} \; = \; \begin{cases} \frac{1}{\sqrt{2}} c_m, & n \, = \, 2m, \\ -\frac{1}{\sqrt{2}} c_m, & n \, = \, 2m+1, \end{cases} \qquad (\widetilde{\mathcal{H}} \circ \uparrow 2)c \, = \, \sum_n c_n \, \widetilde{\varphi}^{(n)} \, ;$$

What's the point? Combining (1) and (3), we see that

$$(\mathcal{H} \circ \uparrow 2) \circ (\downarrow 2 \circ \mathcal{H}^*)a \; = \; \sum_n (a, \, \varphi^{(n)}) \, \varphi^{(n)}$$

which is just the orthonormal series expansion of a with respect to the family $\{\varphi^{(n)}\}_n$, while combining (2) and (4) we see that

$$(\widetilde{\mathcal{H}} \circ \uparrow 2) \circ (\downarrow 2 \circ \widetilde{\mathcal{H}}^*)a \; = \; \sum_n (a, \, \widetilde{\varphi}^{(n)}) \, \widetilde{\varphi}^{(n)}$$

which is the orthonormal series expansion with respect to the second family $\{\widetilde{\varphi}^{(n)}\}_n$. But together we know that the two families form a complete orthonormal family in ℓ^2. Hence we have obtained a *splitting*:

$$a \; = \; \sum_n (a, \, \varphi^{(n)}) \, \varphi^{(n)} + \sum_n (a, \, \widetilde{\varphi}^{(n)}) \, \widetilde{\varphi}^{(n)}$$

of a which was seen earlier to split a into 'coarse' and 'fine' details. The coarse details thus come from filtering with a filter trying hard to be low-pass, while the fine details come from filtering with an associated filter trying hard to be high-pass. This is exactly what sub-band coding is all about. We need to try this with different filters!

Summary. So why all this emphasis on down-sampling and up-sampling? Well, we are going to use two filters, one low-pass to single out low frequency terms (coarse details) and the other high-pass to single out the high frequencies (fine details) in a signal: the Haar $\mathcal{H}$ and $\widetilde{\mathcal{H}}$, for instance. But a convolution $h * x$ *contains just as many terms* as the original sequence x, so if we use two of them we will end up with 'twice' as many terms as we had before filtering started. We may have separated out the frequencies bands in a signal, but doubling the number of terms is hardly impressive compression! Thus the role of down-sampling is to ensure that we still have the *same number of terms after* filtering with both low-pass and high-pass filters. As down-sampling introduces aliasing, however, we've then got to up-sample and filter again to eliminate the aliasing and hence reconstruct the original signal. For digital signals all this is made precise in the notion of *filter bank* introduced in lecture 5, which then carries over to produce the decomposition of analogue signals into coarse and fine details. As always, the big problem is how to design the appropriate filters. That's the miracle achieved by the Daubechies FIR filters!

Problems.

1. The simplest FIR filter is the *Lazy Filter*

$$\mathcal{I} : x \longrightarrow \mathcal{I}(x) = x,$$

so-called because it does nothing to a signal; its filter coefficents $\{i_n\}_n$ are given by

$$i_n = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

In other words, $\mathcal{I}(x)$ is simply the convolution

$$\mathcal{I}(x) = \delta * x = \varepsilon^{(0)} * x$$

of x with the unit impulse δ. Notice that $\mathcal{I}$ 'fills in' the missing first example in the series of filters given in (2.7) because it has only *one* non-zero filter coefficient. There really must be a pattern to these filters (check problem 10(iv) also)!!

(i) Show that the Frequency Response function $I(\xi)$ of $\mathcal{I}$ is given by $I(\xi) = 1$ for all ξ. Why does this make good sense? What is the adjoint $\mathcal{I}^*$ of $\mathcal{I}$? Does $\mathcal{I}$ succeeding in trying hard to be low-pass or highpass?

(ii) Determine the associated FIR filter $\widetilde{\mathcal{I}}$. What is its Frequency Response function $\widetilde{I}(\xi)$? What is its adjoint $\widetilde{\mathcal{I}}^*$?

(iii) Determine

$$(\downarrow 2 \circ \mathcal{I}^*)x, \qquad (\mathcal{I} \circ \uparrow 2)x$$

for a sequence x.

(iv) Determine

$$(\downarrow 2 \circ \widetilde{\mathcal{I}}^*)x, \qquad (\widetilde{\mathcal{I}} \circ \uparrow 2)x$$

for a sequence x.

(v) Determine the coarse + fine detail decomposition

$$\Big\{(\mathcal{I} \circ \uparrow 2) \circ (\downarrow 2 \circ \mathcal{I}^*) + (\widetilde{\mathcal{I}} \circ \uparrow 2) \circ (\downarrow 2 \circ \widetilde{\mathcal{I}}^*)\Big\}x$$

of a sequence x.

2. Show that the adjoint, S^*, of the delay operator S is the advancing operator

$$S^* : \{x_n\}_n \longrightarrow \{x_{n+1}\}_n,$$

translating the sequence in the opposite direction to S.

3. Let $x = \{x_n\}_n$ be the sequence whose Frequency Response function is the period 1 extension of the function

$$X(\xi) = |1 - 3|\xi||, \qquad 0 \le |\xi| \le \tfrac{1}{2}.$$

By using representation (††) in (2.3), find the sequence $\{x_n\}_n$. (Hint: get rid of the outer absolute value by using the fact that X is even, *i.e.*, $X(-\xi) = X(\xi)$; then get rid of the inner absolute value by splitting the new integral into two parts.)

4. As a further attempt to explain the appearance of the adjoints $\mathcal{H}^*$ and $\widetilde{\mathcal{H}}^*$ in the coarse+fine decomposition of signals, show that

$$(\mathcal{H} \circ \uparrow 2)^* = \downarrow 2 \circ \mathcal{H}^*, \qquad (\widetilde{\mathcal{H}} \circ \uparrow 2)^* = \downarrow 2 \circ \widetilde{\mathcal{H}}^*$$

for any FIR filter $\mathcal{H}$. In particular, therefore,

$$(\mathcal{H} \circ \uparrow 2) \circ (\downarrow 2 \circ \mathcal{H}^*)x = (\mathcal{H} \circ \uparrow 2) \circ (\mathcal{H} \circ \uparrow 2)^* x$$

and

$$(\widetilde{\mathcal{H}} \circ \uparrow 2) \circ (\downarrow 2 \circ \widetilde{\mathcal{H}}^*)x = (\widetilde{\mathcal{H}} \circ \uparrow 2) \circ (\widetilde{\mathcal{H}} \circ \uparrow 2)^* x,$$

which are exactly the same as the synthesis/analysis mappings

$$S[f] = (\mathcal{S} \circ \mathcal{S}^*) f$$

associated in lecture 1 with an orthonormal family $\{\phi_n\}_n$ in a general inner product space.

5. Use problem 4 with $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ the Haar filters to establish the results

$$(\mathcal{H} \circ \uparrow 2) \circ (\downarrow 2 \circ \mathcal{H}^*)a = \sum_n (a, \varphi^{(n)})\, \varphi^{(n)}$$

and

$$(\widetilde{\mathcal{H}} \circ \uparrow 2) \circ (\downarrow 2 \circ \widetilde{\mathcal{H}}^*)a = \sum_n (a, \widetilde{\varphi}^{(n)})\, \widetilde{\varphi}^{(n)}$$

for the respective orthonormal families $\{\varphi^{(n)}\}_n$, $\{\widetilde{\varphi}^{(n)}\}_n$ stated in (2.8).

6. Show that $(\downarrow 2)(\uparrow 2)x = x$, *i.e.*, up-sampling followed by down-sampling always preserves a signal.

7. Let $\widetilde{\mathcal{H}}$ be the FIR filter having filter coefficients

$$\tilde{h}_\ell \;=\; (-1)^\ell\, \overline{h_{1-\ell}}$$

associated with an FIR filter $\mathcal{H}$. Show that Frequency Response function $\widetilde{H}(\xi)$ of $\widetilde{\mathcal{H}}$ is given by

$$\widetilde{H}(\xi) \;=\; -e^{-2\pi i\xi}\, \overline{H(\xi + \tfrac{1}{2})}.$$

Show that the FIR associated with the Haar filter in which

$$h_n \;=\; \begin{cases} \frac{1}{\sqrt{2}}, & n = 0, \\ \frac{1}{\sqrt{2}}, & n = 1, \\ 0, & n \neq 0,\, 1, \end{cases}$$

is exactly the filter $\widetilde{\mathcal{H}}$ having filter coefficients

$$\tilde{h}_n \;=\; \begin{cases} \frac{1}{\sqrt{2}}, & n = 0, \\ -\frac{1}{\sqrt{2}}, & n = 1, \\ 0, & n \neq 0,\, 1; \end{cases}$$

in other words, notation used for the two Haar filters is consistent with the definition of $\widetilde{\mathcal{H}}$ being associated with $\mathcal{H}$.

8. The companion $\widetilde{db2}$-filter in the $db2$-case, for instance, is

$$\tilde{h}_{-2} \;=\; \frac{1-\sqrt{3}}{4\sqrt{2}}, \quad \tilde{h}_{-1} \;=\; -\frac{3-\sqrt{3}}{4\sqrt{2}}, \quad \tilde{h}_0 \;=\; \frac{3+\sqrt{3}}{4\sqrt{2}}, \quad \tilde{h}_1 \;=\; -\frac{1+\sqrt{3}}{4\sqrt{2}}$$

(it's not causal I know, but that's not important!).

(i) Show that the Frequency response function for the $db2$-filter is

$$H(\xi) \;=\; \sqrt{2}\, e^{-3\pi i\xi}\, (\cos \pi\xi)^2 (\cos \pi\xi + i\sqrt{3} \sin \pi\xi)$$

(ii) Then use problem 7. to show that the Frequency Response function for $\widetilde{db2}$ is given by

$$\widetilde{H}(\xi) \;=\; \sqrt{2}\, e^{\pi i\xi}\, (\sin \pi\xi)^2 (\sqrt{3} \cos \pi\xi - i \sin \pi\xi)$$

(the $db2$-filter coefficients are *real* remember!).

9. Use some graphing facility to draw the graph of $|H(\xi)|$ and $|\widetilde{H}(\xi)|$ for the Daubechies $db2$-filter.

10. Show that the mapping $\mathcal{H} \circ \uparrow 2$ is energy-preserving on ℓ^2 for a given FIR filter $\mathcal{H}$ if and only if

$$(\ddagger) \qquad |H(\xi)|^2 + |H(\xi + \tfrac{1}{2})|^2 \; = \; 2.$$

(i) Deduce that if $\mathcal{H} \circ \uparrow 2$ is energy-preserving on ℓ^2, then so is $\widetilde{\mathcal{H}} \circ \uparrow 2$.

(ii) Deduce that $\mathcal{H} \circ \uparrow 2$ is energy-preserving on ℓ^2 when $\mathcal{H}$ is the Haar filter and the $db2$-filter.

(iii) Does your proof of (ii) for the Haar and $db2$-filters suggest how one might construct other filters $\mathcal{H}$ for which condition ($\ddagger$) holds?

(iv) Use parts (i) and (ii) to show that the families $\{\varphi^{(n)}\}_n$ and $\{\widetilde{\varphi}^{(n)}\}_n$ are orthonormal in ℓ^2, results established directly in lecture 1. Then use parts (i) and (ii) to construct new orthonormal families in ℓ^2 from the $db2$ and $\widetilde{db2}$-filters.